Fake Projective spaces

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In my talk I reported on the following two papers written jointly with Sai-Kee Yeung:

(1) *Fake projective planes*, Inventiones Math. **168**(2007), 321-370.

(2) *Arithmetic fake projective spaces and arithmetic fake Grassmannians*, MPIM-Bonn preprint.

A compact complex 2-dimensional manifold $\mathcal{P}$ is said to be a *fake projective plane* if it is not isomorphic to the complex projective plane $\mathbf{P}^2_{\mathbb{C}}$ but its Betti numbers are equal to that of $\mathbf{P}^2_{\mathbb{C}}$. For such a $\mathcal{P}$, $c_1^2 = 3c_2 = 9$, and it follows from Yau's theorem on the Calabi conjecture that the open unit ball B^2 in $\mathbb{C}^2$ is the universal cover of $\mathcal{P}$; $\pi_1(\mathcal{P})$ is then a discrete cocompact torsion-free subgroup of $\mathrm{PU}(2,1)$. It was independently proved by Bruno Klingler and Sai-Kee Yeung that the fundamental group of a fake projective plane is in fact an *arithmetic* subgroup of $\mathrm{PU}(2,1)$.

A fake projective plane is a smooth complex projective algebraic surface of general type. Its geometric genus p_g is zero, and its Euler-Poincaré characteristic is 3.

A compact Kähler manifold of dimension n is called a *fake projective space*, or a *fake* $\mathbf{P}^n_{\mathbb{C}}$ if it is not isomporphic to $\mathbf{P}^n_{\mathbb{C}}$ but it has the same Betti numbers as $\mathbf{P}^n_{\mathbb{C}}$. We will call a n-dimensional fake projective space an *arithmetic fake projective space*, or an *arithmetic fake* $\mathbf{P}^n_{\mathbb{C}}$, if it is the quotient of the open unit ball B^n in $\mathbb{C}^n$ by a torsion-free cocompact *arithmetic* subgroup of $\mathrm{PU}(n,1)$. Note that B^n is the symmetric space of $\mathrm{PU}(n,1)$, and $\mathbf{P}^n_{\mathbb{C}}$ is the compact dual of B^n. So, more generally, if X is the symmetric space of a real semi-simple Lie group $\mathcal{G}$, and X_u is the compact dual of X, we shall say that $Y := X/\Gamma$ is an *"arithmetic fake X_u"* if Γ is a torsion-free cocompact *arithmetic* subgroup of $\mathcal{G}$, and the complex cohomology of Y is isomorphic to that of X_u. It is known that $H^*(X_u;\mathbb{C}) \cong H^*(\mathfrak{g},\mathcal{K};\mathbb{C})$, where $\mathfrak{g}$ is the Lie algebra of $\mathcal{G}$, and $\mathcal{K}$ is a maximal compact subgroup. Moreover, there is a natural embedding of $H^*(\mathfrak{g},\mathcal{K};\mathbb{C})$ in $H^*(X/\Gamma;\mathbb{C})$. Thus $H^*(X/\Gamma;\mathbb{C}) \cong H^*(X_u;\mathbb{C})$ if and only if $H^*(\mathfrak{g},\mathcal{K};\mathbb{C})$ maps onto $H^*(X/\Gamma;\mathbb{C})$.

Now we will state our main theorems.

Theorem 1. *There are seventeen "small" classes of fake projective planes. Besides these there can exist one more class.*

Theorem 2. (i) *Arithmetic fake* $\mathbf{P}^n_{\mathbb{C}}$ *can exist only for* $n = 2$ *and* 4. *There exist at least four distinct arithmetic fake* $\mathbf{P}^4_{\mathbb{C}}$. *The first integral homolgy of any arithmetic fake* $\mathbf{P}^4_{\mathbb{C}}$ *is nonzero.*

(ii) *There exist at least four arithmetic fake* $\mathbf{Gr}_{2,5}$*, but no arithmetic fake* $\mathbf{Gr}_{p,n}$*, with* $n > 5$*, odd.*

(iii) *There also exist at least five distinct irreducible arithmetic fake* $\mathbf{P}^2_{\mathbb{C}} \times \mathbf{P}^2_{\mathbb{C}}$.

Sai-Kee Yeung has recently proved that any fake $\mathbf{P}^4_{\mathbb{C}}$ is arithmetic. This, together with Theorem 2(i), implies that a compact Kähler manifold of dimension 4 is isomorphic to $\mathbf{P}^4_{\mathbb{C}}$ if and only if its integral cohomolgy equals that of the latter.

The first fake projective plane was constructed by David Mumford in 1979 using p-adic uniformization. Our construction of fake projective planes is quite direct and explicit, so several of the geometric properties of these surfaces can be derived rather easily now. For example, we have proved that for any fake projective plane $\mathcal{P}$ belonging to any of the seventeen classes, $H^1(\mathcal{P}, \mathbb{Z})$ is nonzero. The automorphism group of $\mathcal{P}$ can be determined "easily": the automorphism group turns out to be of order 1, 3, 7, 9 or 21. We have also shown that for $\mathcal{P}$ belonging to fourteen of the seventeen classes, there is a line bundle L such that the canonical line bundle $K_{\mathcal{P}}$ equals $3L$. This property is equivalent to the assertion that the following short exact sequence

$$0 \to \mathbb{Z}/3\mathbb{Z} \to \widetilde{\Pi} \to \Pi \to 1,$$

splits, where $\widetilde{\Pi}$ is the inverse image in $\mathrm{SU}(2,1)$ of the fundamental group $\Pi\,(\subset \mathrm{PU}(2,1))$ of $\mathcal{P}$. $7L$ is very ample and provides an embedding of $\mathcal{P}$ in $\mathbf{P}^{14}_{\mathbb{C}}$ as a complex surface of degree 49.

Now I will list some problems arising from paper (1) on fake projective planes.

(1) For $\mathcal{P}$ with $\mathrm{Aut}\,(\mathcal{P})$ nontrivial, is the surface $\mathcal{P}/\mathrm{Aut}\,(\mathcal{P})$ ever simply connected?

(2) Bloch's conjecture for smooth projective surfaces with $p_g = 0$ is open. It would be of considerable interest to settle this conjecture for the fake projective planes using the geometric properties now known to us.

(3) Besides determining all the fake projective planes, in paper (1) we have determined all (compact and noncompact) complex 2-ball quotients by arithmetic subgroups of $\mathrm{PU}(2,1)$ whose orbifold Euler-Poincaré characteristic is $\leqslant 3$. Study of these surfaces, and their singularties, will be of interest.

I will now sketch the idea of the proof of Theorem 1. Let $\Pi\,(\subset \mathrm{PU}(2,1))$ be the fundamental group of a fake projective plane. Let $\widetilde{\Pi}$ be the inverse image of Π in $\mathrm{SU}(2,1)$. Then $\widetilde{\Pi}$ is a cocompact arithmetic subgroup. Its orbifold Euler-Poincaré characteristic equals $\chi(\Pi)/3 = \chi(\mathbf{P}^2_{\mathbb{C}})/3 = 1$, since the covering $\mathrm{SU}(2,1) \to \mathrm{PU}(2,1)$ is of degree 3. Therefore, for any arithmetic subgroup Γ of $\mathrm{SU}(2,1)$ containing $\widetilde{\Pi}$, $\chi(\Gamma)$ is a reciprocal integer.

As $\widetilde{\Pi}$ is a cocompact arithmetic subgroup, it determines a totally real number field k, and an *anisotropic* k-form G of $\mathrm{SU}(2,1)$. Such a k-form is described as the special unitary group of an anisotropic hermitian form h on ℓ^3, or it is described in terms of a cubic division algebra $\mathcal{D}$ with center ℓ, $\mathcal{D}$ endowed with an involution σ of the second kind, where ℓ is a totally complex quadratic extension of k. G and Π, and the corresponding fake projective plane will be said to be of *first type* if $G = \mathrm{SU}(h)$, and in the other cases they will be said to be of *second type.*

For any nonarchimedean place v of k which is unramified in ℓ, we fix a parahoric subgroup P_v of $G(k_v)$ which is is minimal among the parahoric subgroups normalized by Π. On the other hand, if v ramifies in ℓ, let P_v be a maximal parahoric subgroup of $G(k_v)$ normalized by Π. Then $\prod_v P_v$ is a compact-open subgroup of the group $G(A_f)$ of finite adèles of G. Let $\Lambda = G(k) \cap \prod_v P_v$. Let Γ be the normalizer of Λ in $G(k_{v_o})$, where v_o is the unique real place of k such that $G(k_{v_o}) \cong SU(2,1)$. Then Γ contains Π, and

$$\chi(\Gamma) = \frac{\chi(\Lambda)}{[\Gamma : \Lambda]} = \frac{3\mu(G(k_{v_o})/\Lambda)}{[\Gamma : \Lambda]},$$

where μ is the normalized Haar measure on $G(k_{v_o})$ used in my paper on covolumes of S-arithmetic subgroups (in Publ. Math. IHES, No. **69**(1989)). It follows from the computation in that paper that

$$\mu(G(k_{v_o})/\Lambda) = \frac{D_\ell^{5/2}}{D_k} \frac{\zeta_k(2)L_{\ell|k}(3)}{(16\pi^5)^d} \prod e'(P_v) = 2^{-2d}\zeta_k(-1)L_{\ell|k}(-2) \prod e'(P_v),$$

where $e'(P_v)$ is an integer whose value is given in section 2 of paper (1), the product is over finitely many v which are unramified in ℓ and P_v is *not* a hyperspecial parahoric subgroup, $d = [k : \mathbb{Q}]$.

Also, using Galois-cohomolgy, $[\Gamma : \Lambda]$ can be determined. Its value involves the order $h_{\ell,3}$ of the subgroup of the class group of ℓ consisting of elements of order dividing 3.

We use number theoretic computations and estimates, the Brauer-Siegel theorem, theory of Hilbert class fields, Zimmert's bound for the class number of a number field, Odlyzko's bounds for the absolute values of the discriminants D_k and D_ℓ of k and ℓ respectively, the values of $\zeta_k(-1)$ and $L_{\ell|k}(-2)$, and the fact that $\chi(\Gamma)$ is a reciprocal integer to determine all possible (k, ℓ), hermitian forms h, cubic division algebras $\mathcal{D}$, and the parahoric subgroups P_v. Some of the candidates are eliminated by proving that in the image $\overline{\Gamma}$ of Γ in $\mathrm{PU}(2,1)$, any subgroup whose Euler-Poincaré characteristic is 3 must contain a nontrivial torsion–this argument is quite tricky. If G is of second type, we use a theorem of Rogawski to say that $H^1(\Lambda, \mathbb{C})$, and then by Poincaré duality $H^3(\Lambda, \mathbb{C})$ also, vanishes.

The fake projective planes which arise from a given collection (P_v) of parahoric subgroups of $G(k_v)$ (P_v determined up to conjugacy by an element of $\overline{G}(k_v)$) constitute a (finite, and "small") class in the statement of Theorem 1. I note that we have shown in (1) that if (P_v) and (P'_v) are two collections of parahoric subgroups such that for all v, P_v is conjugate to P'_v under an element of $\overline{G}(k_v)$, then there exists a $g \in \overline{G}(k)$ such that $gP_vg^{-1} = P'_v$ for all v. Also, if v ramifies in ℓ, then $G(k_v)$ contains two distinct conjugacy classes of maximal parahoric subgroups.

The seventeen classes of fake projective planes mentioned in Theorem 1 are *all* of second type. In paper (1), we listed three possible pairs (k, ℓ), and for each of them a k-form G of $\mathrm{SU}(2,1)$ of first type which may give rise to a fake projective plane. Quite recently, Martin Deraux and Sai-Kee Yeung have eliminated two of them. The only case of the first type which now remains to be eliminated is the case where $(k, \ell) = (\mathbb{Q}(\sqrt{6}), \mathbb{Q}(\sqrt{6}, \zeta_3))$.