Binary Quadratic Forms and the Fourier Coefficients of Elliptic and Jacobi Modular Forms

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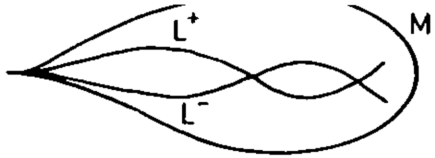


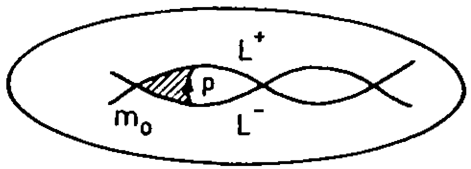
REMARKS. (1) If this approach is taken as a definition of $\lambda(Y)$ then one must show independence of the Heegaard splitting.

(2) Casson uses this approach to produce an effective algorithm for computing $\lambda(M)$, when M is described in terms of surgery on links in S^3 (another way of presenting homology 3-spheres).

Since the Floer homology groups are a refinement of the Casson invariant, it is now reasonable to ask if there is a way of computing $HF(Y)$ using the Heegaard splitting and the Riemann surface X. I shall outline an approach to this problem, which has yet to be fully worked out.

Since M is a Kähler manifold (with singularities) it is in particular symplectic. In fact, as shown in [1], the symplectic structure is canonical and independent of the metric on X. Moreover $L^{\pm}$ are *Lagrangian* submanifolds, i.e. submanifolds of middle dimension on which the symplectic 2-form ω of M is identically zero. Now Floer [6, 7] has studied, in general, the problem of intersections of Lagrangian submanifolds of compact symplectic manifolds and, for this purpose, has developed a homology theory. From an analytical point of view this is very similar to the theory leading to the HF groups described in §4. When applied to the particular case of $L^{\pm}$ in M above it is highly plausible that it should coincide with the theory of §4, as I shall indicate later. So first let me outline Floer's "symplectic Morse theory".

We start from any compact symplectic manifold M and two (connected) Lagrangian submanifolds L^+ and L^-. Consider the space Q of paths in M starting on L^- and ending on L^+. Assume for simplicity that $L^+ \cap L^-$ is not empty (otherwise the theory will be trivial) and choose a base point $m_0 \in L^+ \cap L^-$. Define a function $f(p)$ on Q as the area (integral of the symplectic 2-form ω) of a strip obtained by deforming the path p to the constant path m_0.



Since L^+ and L^- are Lagrangian, and ω is closed, this area is unchanged under continuous variations of the strip (with p fixed). However topologically inequivalent strips will differ in area by a "period" of ω. If for simplicity we

The critical points of f are easily seen to be the constant paths corresponding to the points of intersection of L^+ and L^-. The Hessian is again of Dirac type and one can define a relative Morse index as in §4. This turns out to be well-defined modulo $2N$, where $c_1(M) = N[\omega]$, $c_1(M)$ being the first Chern class of M (note that symplectic manifolds have Chern classes) and $[\omega]$ is the class of ω in $H^2(M)$.

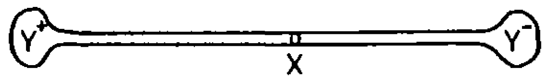
The trajectories of grad f correspond to *holomorphic* strips (with boundaries in $L^{\pm}$) in the sense of Gromov [10]. If M is actually complex Kähler then these are just holomorphic strips in the usual sense.

In this way, following Witten as in §4, Floer defines homology groups graded by Z_{2N} as intrinsic invariants of (M, L^+, L^-).

If now we take (M, L^+, L^-) to be the moduli spaces arising from a Heegaard splitting of a homology 3-sphere Y it is then reasonable to conjecture that the groups defined in the symplectic context (with care taken of the singularities of M) coincide with the groups $HF(Y)$ of §4.

Note that in both cases the representations $\pi_1(Y) \to \mathrm{SU}(2)$ give the generators of the chain group (provided these representations are nondegenerate). One has then to compare the relative Morse indices and the boundary operator ∂.

Geometrically, a path on M, i.e. a 1-parameter family of flat connections on the Riemann surface X, can be viewed as a connection on the cylinder $X \times R$. Moreover the boundary conditions (corresponding to L^+ and L^-) imply that, asymptotically as $t \to \pm\infty$, the connection extends (as a flat connection) over $Y^{\pm}$, thus giving essentially a connection on Y. In this way the symplectic theory for paths in M should be related to a limiting case of the Floer theory for the space $\mathscr{C}$ of connections on Y. Note that the limit is one in which Y is stretched out along its "neck", so that the two ends get further and further apart.



6. Donaldson invariants. Donaldson [5] has introduced certain invariants for smooth 4-manifolds which appear to be extremely powerful in distinguishing different differentiable structures. These invariants are defined in the following context. Let Z be an oriented simply connected differentiable 4-manifold and let b_2^+ and b_2^- be the number of + and − terms in a diagonalization of the quadratic (intersection) form on $H_2(Z)$. We assume b_2^+ odd and > 1. Note that, for a complex algebraic surface, we have the theorem of Hodge:

$$b_2^+ = 1 + 2p_g,$$

where p_g is the geometric genus (number of independent holomorphic 2-forms). Thus b_2^+ is odd and > 1 when $p_g \neq 0$.

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§1. Statement of results and discussion

In [S1] we described a simple arithmetical law to generate the Fourier coefficients of all modular forms of weight 2 on $\Gamma_0(m)$ and all Jacobi forms of weight 2 and index m. The aim of the present article is to generalize these results to arbitrary weight k. The final result will turn out to be a smooth and direct generalization, including the case of weight 2 as a special case. In contrast to this the method of proof used in this article is different and does not apply to the case of weight 2. A short overview of this method will be given at the end of this introductory section.

To describe the main results we have to introduce some notation which will be kept throughout this article. For numbers a, b, c the symbol $[a, b, c]$ denotes the quadratic polynomial

$$[a, b, c](X) = aX^2 + bX + c.$$

The group $GL_2(\mathbf{R})$ acts on these quadratic polynomials by

$$[a, b, c] \circ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} (X) = a(\alpha X + \beta)^2 + b(\alpha X + \beta)(\gamma X + \delta) + c(\gamma X + \delta)^2.$$

Fix a positive integer m. For any pair of integers Δ, r set

$$\mathcal{Q}_m(\Delta, r) = \left\{ [ma, b, c] | a, b, c \in \mathbf{Z}, b^2 - 4mac = \Delta, b \equiv r \bmod 2m \right\}.$$

This set is obviously invariant under the action of

$$\Gamma_0(m) = \begin{pmatrix} \mathbf{Z} & \mathbf{Z} \\ m\mathbf{Z} & \mathbf{Z} \end{pmatrix} \cap SL_2(\mathbf{Z}).$$

For $A \in SL_2(\mathbf{Z})$ we let $\mathcal{Q}_m^A(\Delta, r) := \mathcal{Q}_m(\Delta, r) \circ A$, i.e. the set of all Q such that $Q \circ A^{-1}$ lies in $\mathcal{Q}_m(\Delta, r)$. If Δ_0 is a fundamental discriminant which is a square modulo $4m$ then

$$\chi_{m,\Delta_0} : \{[ma, b, c] | a, b, c \in \mathbf{Z}\} \longrightarrow \{0, \pm 1\}$$

denotes the generalized genus character introduced in ([G-K-Z], Proposition 1), i.e.

$$\chi_{\Delta_0}([ma, b, c]) = \begin{cases} \left(\dfrac{\Delta_0}{n}\right) & \text{if } \Delta_0 \text{ divides } b^2 - 4mac \text{ such that } (b^2 - 4mac)/\Delta_0 \\ & \quad \text{is a square modulo } 4m \text{ and } \gcd(a, b, c, \Delta_0) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Here n is any integer relative prime to Δ_0 and represented by one of the quadratic forms $m_1 aX^2 + bXY + m_2 cY^2$ with $m = m_1 m_2$, $m_1, m_2 > 0$. If A is a matrix in $SL_2(\mathbf{Z})$

and Q a quadratic polynomial such that χ_{m,Δ_0} is defined for $Q \circ A^{-1}$ then we set

$$\chi^A_{m,\Delta_0}(Q) := \chi_{m,\Delta_0}(Q \circ A^{-1}).$$

Note that the function χ_{m,Δ_0} is obviously $\Gamma_0(m)$-invariant, i.e. $\chi^A_{m,\Delta_0} = \chi_{m,\Delta_0}$ for all $A \in \Gamma_0(m)$.

Finally, we introduce generalizations to the case of arbitrary level m of those zeta-functions which appear in the theory of binary quadratic forms modulo $SL_2(\mathbf{Z})$. To explain these let Δ, r and Δ_0 be as above and such that Δ_0 divides Δ and $\frac{\Delta}{\Delta_0}$ is a square modulo 4m. Let $\xi \in \mathbb{P}_1(\mathbb{Q})$. We associate to these data a Dirichlet series by setting

$$\zeta_{m,\Delta,r,\xi,\Delta_0}(s) := \sum_{Q \in \mathcal{Q}_m(\Delta,r)/\Gamma_0(m)} \frac{\chi_{m,\Delta_0}(Q)}{[SL_2(\mathbf{Z})_Q : \Gamma_0(m)_Q]} \sum_{\substack{\binom{x}{y} \in \Gamma_0(m)_Q \backslash \mathbf{Z}^2 \\ \frac{x}{y} \equiv \xi \bmod \Gamma_0(m) \\ Q(x,y) > 0}} \frac{1}{Q(x,y)^s}.$$

Here the first sum is over a complete set of representatives Q of $\mathcal{Q}_m(\Delta, r)$ modulo $\Gamma_0(m)$. For each such Q we use $\Gamma_0(m)_Q$ and $SL_2(\mathbf{Z})_Q$ for the stabilizer of Q in $\Gamma_0(m)$ and $SL_2(\mathbf{Z})$, respectively, and – if $Q = [a, b, c]$ and $x, y \in \mathbf{Z}$ – we set $Q(x, y) = ax^2+bxy+cy^2$. The innner sum is over a complete set of representatives $\binom{x}{y}$ for $\mathbf{Z}^2$ modulo the usual action of $\Gamma_0(m)_Q$ on column vectors which satify the stated conditions, i.e. which satisfy $Q(x, y) > 0$ and generate the same orbit as ξ under the usual action of $\Gamma_0(m)$ on $\mathbb{P}_1(\mathbb{Q})$. By standard arguments from the theory of quadratic forms it is easily seen that the first sum is finite and that the inner sums are convergent for $\Re(s) > 1$.

We are now able to describe the arithmetical rule to generate the Fourier coefficients of modular forms. This depends on 4 parameters (Δ_0, r_0, A, P) where Δ_0, r_0 is a pair of integers such that $\Delta_0 \equiv r_0^2 \bmod 4m$ and Δ_0 is a fundamental discriminant, where $A \in SL_2(\mathbf{Z})$, and where $P = P(a, b, c)$ is a homogeneous polynomial in three variables with complex coefficients – say of degree $k - 1$ –. To each such quadrupel (Δ_0, r_0, A, P) we shall associate a sequence $C_{m,\Delta_0,r_0,A,P}(\Delta, r)$ which is indexed by pairs of integers Δ, r with $\Delta \equiv r \bmod 4m$, $\Delta_0\Delta > 0$.

Namely, let $P_1 = P_1(b, c)$ denote a polynomial in b, c such that

$$P_1(b, c+1) - P_1(b, c) = P(0, b, c).$$

Note that such a polynomial exists: indeed, it can be obtained by replacing each power c^{n-1} in $P(0, b, c)$ by $\frac{B_n(c)}{n}$, where $B_n(c)$ is the n-th Bernoulli polynomial, i.e. that polynomial which is uniquely determined by the properties $B_n(c + 1) - B_n(c) = nc^{n-1}$

and $\int_0^1 B_n(c)dc = 0$ for $n \geq 1$, $B_0(c) = 1$. Moreover, any two such polynomials differ by a polynomial in b. We make the specific choice

$$P_1(b,c) := P\left(0, b, \frac{B_{*+1}(c)}{*+1}\right) + \int_0^b (b-t)\frac{\partial P}{\partial a}(0,t,0)dt,$$

where the first term on the right denotes the polynomial just described. Similarily, we let

$$P_2(a,b) := P\left(\frac{B_{*+1}(a)}{*+1}, b, 0\right) + \int_0^b (b-t)\frac{\partial P}{\partial c}(0,t,0)dt,$$

where the first term on the right is the polynomial obtained by replacing each power a^{n-1} in $P(a,b,0)$ by $\frac{B_n(a)}{n}$. Let $N := m|\Delta_0|$. Using P_1, P_2, N we define a function $\widetilde{P}(Q)$ by setting for any $Q = [a,b,c]$ with integral coefficients

$$\widetilde{P}(Q) := \begin{cases} \operatorname{sign}(a)P(a,b,c) & \text{if} \quad ac < 0 \\ N^{k-1}P_1\left(\frac{b}{N}, \frac{c}{N}\right) & \text{if} \quad a = 0, 0 < c < N \\ -N^{k-1}P_2\left(\frac{a}{N}, \frac{b}{N}\right) & \text{if} \quad c = 0, 0 < a < N \\ N^{k-1}\left(P_1\left(\frac{b}{N}, 0\right) - P_2\left(0, \frac{b}{N}\right)\right) & \text{if} \quad a = c = 0, \\ 0 & \text{otherwise} \end{cases}$$

($\operatorname{sign}(a) = a/|a|$ for any non-zero real a). Finally, we set

$$\mathcal{C}_{m,\Delta_0,r_0,A,P}(\Delta, r) := \sum_{Q \in \mathcal{Q}^A_m(\Delta_0\Delta, r_0 r)} \chi^A_{m,\Delta_0}(Q)\widetilde{P}(Q) \tag{1}$$

if $k = 1$, and, if $k \geq 2$, we define $\mathcal{C}_{m,\Delta_0,r_0,A,P}(\Delta, r)$ to be the right hand side of (1) plus a correction term which is given by

$$\begin{aligned} &\gamma P(1,0,0)\left[\zeta_{m,\Delta_0\Delta,r_0r,A0,\Delta_0}(k) + (-1)^k \operatorname{sign}(\Delta_0)\zeta_{m,\Delta_0\Delta,r_0r,-A0,\Delta_0}(k)\right] \\ -&\gamma P(0,0,1)\left[\zeta_{m,\Delta_0\Delta,r_0r,A\infty,\Delta_0}(k) + (-1)^k \operatorname{sign}(\Delta_0)\zeta_{m,\Delta_0\Delta,r_0r,-A\infty,\Delta_0}(k)\right], \end{aligned} \tag{2}$$

where

$$\gamma := \frac{(-1)^k(k-1)!^2}{\zeta(2k)(2k-1)!}(\Delta_0\Delta)^{k-\frac{1}{2}}.$$

($\zeta(s)$ =Riemann zeta-function).

Note that the sum in (1) is finite. In fact, if $\widetilde{P}(Q) \neq 0$ – say $Q = [a,b,c]$ – then a,b,c satisfies $a = 0, 0 \leq c < N$ or $c = 0, 0 \leq a < N$ or $ac < 0$. But obviously there are only finitly many integral (a,b,c) satisfying one of these equations and the equation $b^2 - 4ac = \Delta_0\Delta$. Also note that there is a contribution to the sum (1) from terms with $a = 0$ or $c = 0$ only if $\Delta_0\Delta$ is a perfect square. If the latter is the case then the contribution coming from $P(0, b, \frac{B_{*+1}(c)}{*+1})$ and $P(\frac{B_{*+1}(a)}{*+1}, b, 0)$ may be viewed as a natural value assigned to the (divergent) sums

$$-\frac{1}{2}\sum \chi^A_{m,\Delta_0}(Q)\mathrm{sign}(c)P(0,b,c), \quad \frac{1}{2}\sum \chi^A_{m,\Delta_0}(Q)\mathrm{sign}(a)P(a,b,0),$$

taken over all $Q = [a, b, c] \in \mathcal{Q}^A_m(\Delta_0\Delta, r_0 r)$ such that $a = 0$ and $c = 0$, respectively. Indeed, replace e.g. in the first of these sums each power c^{n-1} by $c^{n-1}|c|^{-s}$, note that the resulting expression can be analytically continued to the complex plane (since it can be written as a linear combination of Hurwitz zeta-functions), and compute its value at $s = 0$. The latter is easily done using the formula

$$\frac{1}{2}\sum_{\substack{c\in x+\mathbb{Z}\\ c\neq 0}} \frac{\mathrm{sign}(c)c^{n-1}}{|c|^s} \quad at\ s=0 \quad = -\frac{B_n(x)}{n}$$

(valid for any positive integer n and any real x with $0 \le x < 1$, except for $n = 1, a = 0$ where the left hand side of this identity is obviously 0). The equation (1) could therefore be written symbolically as

$$\mathcal{C}_{m,\Delta_0,r_0,A,P}(\Delta, r) := \sum_{Q\in\mathcal{Q}^A_m(\Delta_0\Delta, r_0 r)} \chi^A_{m,\Delta_0}(Q)\mathrm{sign}(Q)P(Q) \ + \left\{\begin{matrix}\text{certain}\\ \text{correction}\end{matrix}\right\},$$

where, for any $Q = [a, b, c]$, we use

$$\mathrm{sign}(Q) := \frac{1}{2}\left(\mathrm{sign}(a) - \mathrm{sign}(c)\right),$$

(with sign of the real number $0 := 0$), and $P(Q) = P(a, b, c)$. The 'certain correction' is given by (2) and a contribution due to the integrals in the definition of P_1 and P_2. Finally, this shows – as the reader can also verify more directly – that in the definition of $\widetilde{P}$ we could choose for N any positive integer such that the value $\chi^A_{m,\Delta_0}(Q)$ and the condition '$Q \in \mathcal{Q}^A_m(\Delta_0\Delta, r_0 r)$' for any Q of the form $[0, b, c]$ or $[a, b, 0]$ depend only on c resp. a modulo N: another choice would of course affect the definition of $\widetilde{P}$ but not the value of the sum (1).

The numbers $\mathcal{C}_{m,\Delta_0,r_0,A,P}(\Delta, r)$ represent the arithmetical rule to generate the Fourier coefficients af any elliptic modular form. To state this precisely let $\mathfrak{M}^{\mathrm{cusp}}_{2k}(m)$, for positive integers k, m denote that certain space of elliptic cusp forms f of weight $2k$ on $\Gamma_0(m)$ which was introduced in [S-Z]. By definition it is the space of modular forms spanned by all cusp forms f of weight $2k$ on $\Gamma_0(m)$ such that the standard L-series $L(f,s)=\sum_{\ell\geq 1} a(\ell)\ell^{-s}$ of $f(\tau)=\sum a(\ell)e^{2\pi i\ell\tau}$ is of the form

$$L(f,s)=\Big(\prod_{p\mid\frac{m}{m'}} Q_p(s)\Big)L(g,s)$$

for some $m'|m$, some new-form g on $\Gamma_0(m')$, and with polynomials $Q_p(s)$ in p^{-s} satisfying

$$p^{\frac{t}{2}s}Q_p(s)=p^{\frac{t}{2}(2k-s)}Q_p(2k-s)$$

for all $p^t\|\frac{m}{m'}$. Thus, $\mathfrak{M}_{2k}^{\mathrm{cusp}}(m)$ contains all new-forms of level m and a certain choice from each old-class. We then have

Theorem 1. *Let k,m be positive integers. For any $A\in SL_2(\mathbf{Z})$, for any homogeneous polynomial $P(a,b,c)$ of degree $k-1$, satisfying*

$$\left(\frac{\partial^2}{\partial b^2}-\frac{\partial}{\partial a}\frac{\partial}{\partial c}\right)P=0,$$

and for any two pairs Δ_i, r_i $(i=0,1)$ such that $\Delta_i\equiv r_i^2 \bmod 4m$, $\Delta_0\Delta_1>0$ and the Δ_i are fundamental discriminants, define a function of one variable $\tau\in\mathbf{C}, \Im(\tau)>0$ by setting

$$f_{\Delta_0,r_0,\Delta_1,r_1,A,P}(\tau):=\sum_{\ell=1}^{\infty}\left\{\sum_{a|\ell} a^{k-2}\left(\frac{\Delta_1}{a}\right)C_{m,\Delta_0,r_0,A,P}\left(\Delta_1\frac{\ell^2}{a^2},r_1\frac{\ell}{a}\right)\right\}e^{2\pi i\ell\tau}.$$

Then these functions are elements of $\mathfrak{M}_{2k}^{\mathrm{cusp}}(m)$, apart from the case $\Delta_0=k=1$, where this is true only up to an additive multiple of the series $E_2^*(d\tau)-1$ with d running through the divisors of m and with E_2^* denoting that non-holomorphic modular form on $SL_2(\mathbf{Z})$ of weight 2 which is given by $E_2^*(\tau)=1-\frac{3}{\pi\Im(\tau)}-24\sum_{\ell\geq 1}\left(\sum_{d|\ell}d\right)e^{2\pi i\ell\tau}$. Vice versa, any cusp form in $\mathfrak{M}_{2k}^{\mathrm{cusp}}(m)$ can be written as a linear combination of these functions $f_{\Delta_0,r_0,\Delta_1,r_1,A,P}$.

Actually we shall prove more and Theorem 1 will be obtained as a Corollary of this more general result. To explain this let $S_{k,m}^{-}$ and $S_{k,m}^{+}$ denote the spaces of holomorphic and skew-holomorphic Jacobi cusp forms of weight k and index m, respectively (we shall review the definition of these spaces in §2). The main theorem of this paper will be

Theorem 2. *Let k, m be positive integers. Then for any $A \in SL_2(\mathbf{Z})$, any homogeneous polynomial $P(a,b,c)$ of degree $k-1$, satisfying*

$$\left(\frac{\partial^2}{\partial b^2} - \frac{\partial}{\partial a}\frac{\partial}{\partial c}\right) P = 0,$$

and for any pair of integers Δ_0, r_0 such that $\Delta_0 \equiv r_0^2 \bmod 4m$ and Δ_0 is a fundamental discriminant the function

$$\phi_{\Delta_0,r_0,A,P}(\tau,z) := \sum_{\substack{\Delta,r\in\mathbf{Z}\\ \Delta\equiv r^2 \bmod 4m\\ \Delta_0\Delta>0}} \mathcal{C}_{m,\Delta_0,r_0,A,P}(\Delta,r)\, e^{2\pi i\left(\frac{r^2-\Delta}{4m}u+\frac{r^2+|\Delta|}{4m}iv+rz\right)}$$

($\tau = u+iv, z\in\mathbf{C}, v>0$) defines an element of $S^{\epsilon}_{k+1,m}$, where $\epsilon = \operatorname{sign}(\Delta_0)$. Moreover, any Jacobi form in $S^{-}_{k+1,m}$ and $S^{+}_{k+1,m}$ is obtained as a linear combination of these functions $\phi_{\Delta_0,r_0,A,P}$.

We remark that in the case $m=1$ and $\Delta_0<0$ the correction terms (2) of the Fourier coefficients of the series $\phi_{\Delta_0,r_0,A,P}$ can be interpreted as a contribution coming from Jacobi-Eisenstein series. In fact, for $m=1$ the series $\zeta_{m,\Delta_0\Delta,rr_0,\xi,\Delta_0}(s)$ coincide with the well-known zeta-functions appearing in the theory of binary quadratic forms. In particular one has the well-known formula

$$\zeta_{1,\Delta_0\Delta,r,\xi,\Delta_0}(s) = L_{\Delta_0}(s)L_{\Delta}(s),$$

where $L_\Delta(s)$, for any $\Delta = \Delta_1 f^2$, Δ_1 fundamental, $f\in\mathbf{Z}$, $f>0$, denotes the standard Dirichlet series

$$L_\Delta(s) := \left(\sum_{\ell=1}^{\infty}\left(\frac{\Delta_1}{\ell}\right)\ell^{-s}\right)\sum_{d\mid f}\mu(d)\left(\frac{\Delta_1}{d}\right)d^{-s}e^{1-2s}$$

(cf. [Z1], Proposition 3). Using the functional equation of the $L_{\Delta_0}(s)$ the correction term (2) can then be written in a more pleasant form as

$$\frac{L_{\Delta_0}(1-k)L_\Delta(1-k)}{\zeta(1-2k)}\cdot\left(P(1,0,0)-P(0,0,1)\right).$$

From this, we recognize first of all that it is an effectively computable rational number. Moreover, $L_\Delta(1-k)$ is just the Δ-th Fourier coefficient of the Jacob-Eisenstein series of weight $k+1$ and index $m=1$ (cf. [E-Z], Theorem 2.1). A similar reasoning should be possible for arbitrary m. However, we shall not pursue this question any further in this article.

We end this section by some remarks concerning the method to derive the stated theorems, their connection to published results and the organization of this article. Theorem 2 was proved for the case of weight $k+1=2$ in [S1]. The basic idea for the proof of the general case remains in essence the same and relies on the diagrams

$$S^-_{k+1,m} \xrightarrow{\mathcal{S}_{\Delta_0,r_0}} S_{2k}(\Gamma_0(m)) \xrightarrow{\rho^-} H^1_{\text{par.}}(\Gamma_0(m), \mathbf{C}[X]_{2k-2})^- ,$$

and the corresponding diagrams with '$-$' replaced by '$+$'. Here the $\mathcal{S}_{\Delta_0,r_0}$ are certain lifting maps to $S_{2k}(\Gamma_0(m))$ (= space of cusp forms of weight $2k$ on $\Gamma_0(m)$) which were studied in [S-Z]. They are indexed by pairs of integers Δ_0, r_0 with $\Delta_0 \equiv r_0^2 \bmod 4m$, $\Delta_0 < 0$ and fundamental. The symbol $H^1_{\text{par.}}(\Gamma_0(m), \mathbf{C}[X]_{2k-2})$ denotes the first 'cuspidal' cohomology group of $\Gamma_0(m)$ acting in the natural way on the space $\mathbf{C}[X]_{2k-2}$ of complex polynomials of degree $\leq 2k-2$. The outer automorphism $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix}\alpha & -\beta\\ -\gamma & \delta\end{smallmatrix}\right)$ of $\Gamma_0(m)$ induces an involution on this cohomology group and the '$-$'-sign denotes the '$-(-1)^k$'-eigenspace of this involution. Finally ρ^- is that isomorphism which is induced by the Eichler-Shimura isomorphism. Let

$$\mathcal{H}^-_{\Delta_0,r_0} \in H^1_{\text{par.}}(\Gamma_0(m), \mathbf{C}[X]_{2k-2})^- \otimes S^-_{k+1,m}$$

denote the kernel function of $\rho^- \circ \mathcal{S}_{\Delta_0,r_0}$ (with respect to the natural Petersson scalar product on $S^-_{k+1,m}$). If λ is a linear functional on the first factor of the above tensor product then $(\lambda \otimes 1)(\mathcal{H}^-_{\Delta_0,r_0})$ is an element of $S^-_{k+1,m}$. By results of [S-Z] the intersection of all kernels of the $\mathcal{S}_{\Delta_0,r_0}$ is void, and since ρ^- is injective (actually, it is an isomorphism) the intersection of all the kernels of the $\rho^- \circ \mathcal{S}_{\Delta_0,r_0}$ is void too. From the latter it is easily deduced that the $(\lambda \otimes 1)(\mathcal{H}^-_{\Delta_0,r_0})$ span the whole space $S^-_{k+1,m}$. The miracle is that the Fourier coefficients of the $\mathcal{H}^-_{\Delta_0,r_0}$ (and the corresponding $\mathcal{H}^+_{\Delta_0,r_0}$) can be explicitely computed and are given by a finite and effective formula; suitable choices of the functionals λ produce exactly the the forms $\phi_{\Delta_0,r_0,A,P}$ introduced in Theorem 2, and mapping these forms to $S_{2k}(\Gamma_0(m))$ via the maps $\mathcal{S}_{\Delta_1,r_1}$ produces the forms $f_{\Delta_0,r_0,\Delta_1,r_1,A,P}$ appearing in Theorem 1.

In order to compute the Fourier development of $\mathcal{H}^-_{\Delta_0,r_0}$ one will first of all try to replace the above first cohomology group and the Eichler-Shimura isomorphism by a more handy space and map. In the case of level $m=1$, i.e. the $SL_2(\mathbf{Z})$-case, these more handy items are found by replacing the first cohomology by the space of period polynomials, a certain subspace of the space $\mathbf{C}[X]_{2k-2}$, which was explicitly determined in [K-Z]. Now this procedure can be generalized to arbitrary m and it turns out that the above cohomlogy group attached to a given k, m has to be replaced by a certain subspace of $\mathbb{C}[X]_{2k-2}^{\mathbf{P}_1(\mathbb{Z}/m\mathbb{Z})}$. This generalization seems to be well-known in a more or less precise formulation but the author could not find any reference in the literature (although it was communicated to the author that there should be an article on this by Henri Cohen which disappeared in some proceedings volume). Also, this subspace and the correspondingly modified Eichler-Shimura isomorphism will be studied in a forthcoming paper by J.Antoniadis [A]. We shall give a precise statement (including proofs) of the very basic items of this theory (as far as we need them in this article) in §3. Now, replacing the above first cohomology by the space of period polynomials, we may view the kernel functions $\mathcal{H}^{\pm}_{\Delta_0,r_0}$ as elements of $\mathbb{C}[X]_{2k-2}^{\mathbf{P}_1(\mathbb{Z}/m\mathbb{Z})} \otimes S^{\pm}_{k+1,m}$. We shall compute these kernel functions in §4.

For the computation of $\mathcal{H}^{-}_{\Delta_0,r_0}$ we start with a kernel function for $\mathcal{S}_{\Delta_0,r_0}$. The choice of this kernel function is the main difference to the quoted article [S1]. Here we shall use a holomorphic kernel function as it was defined and studied in [G-K-Z] whereas in [S1] we used a non-holomorphic theta function as kernel. In principle it should be possible to use such theta kernels in the general case too, and to compute directly the period polynomials associated to these kernels, considered, with fixed first argument, as (non-holomorphic) modular forms. This was the procedure in [S1] and it yielded period polynomials, the coefficients of which have been holomorphic or skew-holomorphic Jacobi forms and which are essentially identical with the $\phi_{\Delta_0,r_0,A,P}$ appearing in Theorem 2. However, for weight $k+1 \geq 3$ these coefficients are no longer holomorphic or skew-holomorphic, and thus one would have to append a holomorphic or skew-holomorphic projection. This all together would give a proof from scratch of the above theorems but the computation of the $\mathcal{H}^{\pm}_{\Delta_0,r_0}$ seems to become somewhat lengthy in such a setting. On the other hand the disadvantage of using the holomorphic kernel function is that it works only for the case of weight $k+1 \geq 3$. This is due to certain problems of convergence which could probably be circumvented (using the so called Hecke trick), but the treatment of this would spoil the whole presentation. Thus, in this paper we shall only deal with the case of weight ≥ 3; for the case of weight 2 the reader is referred to [S1].

The computation of the Fourier development of the $\mathcal{H}^{\pm}_{\Delta_0,r_0}$ in §4 is very closely related to similar computations in [K-Z]. In fact, the Fourier coefficients of the holomorphic kernel function of $\mathcal{S}_{\Delta_0,r_0}$, considered as a Jacobi form, are certain modular forms which, in the case of level $m = 1$, are simply linear combinations of the functions

$$\sum_{A \in SL_2(\mathbb{Z})_Q \backslash SL_2(\mathbb{Z})} (Q \circ A)(t)^{-k}$$

where $Q(t)$ is a quadratic polynomial with positive discriminant. These functions were introduced in [Z2] in connection with the Doi-Naganuma lifting. Its periods have been calculated in [K-Z], Theorem 5. This Theorem, in essence, is a special case ($m = \Delta_0 = 1$) of the Proposition 4 below. The calculations in [K-Z] which led to Theorem 5 loc.cit. can essentially be carried over to our situation, and we shall precisely do so. This computation is based on three key lemmas which are stated and proved in the Appendix A and B..

With respect to the '+'-case there are some remarks indispensable. The proof is completely identical with the proof of the '$-$'-case. Nevertheless, at the first glance there are two obstructions: First of all the maps $\mathcal{S}_{\Delta_0,r_0}$ ($\Delta_0 > 0$) on $S^+_{k+1,m}$ are not yet defined in the literature. Secondly, the fact that the intersection of their kernels is void seems to depend on a trace formula, and the corresponding computations (which will be given in [S2]) have not yet been published. Both problems can be solved by literally copying the corresponding facts and proofs in the '$-$'-case. The first problem will be solved in precisely this manner in §2, for the second one the reader is referred to the quoted paper. Thus, since this paper is not yet available, a suspicious reader might wish a modified formulation of the Theorems 1 and 2 with respect to the '+'-case; the correct and honest formulation can be found in §5 where we shall summarize and append some formal considerations to complete the proof of the two claimed theorems.

§2. The lifting maps from Jacobi forms to modular forms

As in the foregoing section let $S^-_{k,m}$ and $S^+_{k,m}$ denote the spaces of holomorphic and skew-holomorphic Jacobi cusp forms, respectively. Thus, $S^\epsilon_{k,m}$, for positive integers k, m and $\epsilon = \pm 1$, is by definition the space of smooth and periodic functions $\phi(\tau, z)$ with $\tau \in \mathfrak{H}$, the set of complex numbers with positive imaginary part, and $z \in \mathbf{C}$, which satisfy the following two properties:

(i) The Fourier expansion of $\phi(\tau,z)$ is of the form

$$\phi(\tau,z) = \sum_{\substack{\Delta,r\in\mathbf{Z},\,\epsilon\Delta>0\\ \Delta\equiv r^2 \bmod 4m}} C_\phi(\Delta,r)\mathrm{e}_{\Delta,r}(\tau,z),$$

where the coefficients $C_\phi(\Delta,r)$ depend on r only modulo $4m$. Here

$$\mathrm{e}_{\Delta,r}(\tau,z) = \mathrm{e}^{2\pi i\left(\frac{r^2-\Delta}{4m}u + \frac{r^2+|\Delta|}{4m}iv + rz\right)} \qquad (\tau = u+iv).$$

(ii) One has

$$\phi\left(\frac{-1}{\tau},\frac{z}{\tau}\right)\mathrm{e}^{-2\pi i m\frac{z^2}{\tau}} = \phi(\tau,z)\cdot\begin{cases} \tau^k & \text{if } \epsilon=-1\\ \overline{\tau}^{k-1}|\tau| & \text{if } \epsilon=+1\end{cases}.$$

Let $\mathcal{J}(\mathbf{Z}) = SL_2(\mathbf{Z}) \propto \mathbf{Z}^2$, where the semi direct product has to be taken with respect to the natural (right-) action of $SL_2(\mathbf{Z})$ on $\mathbf{Z}^2$, the column vectors with integral entries. The group $\mathcal{J}(\mathbf{Z})$ acts on $\mathfrak{H}\times\mathbf{C}$ by

$$\Upsilon\cdot(\tau,z) = \left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta},\frac{z+\lambda\tau+\mu}{\gamma\tau+\delta}\right), \quad (\Upsilon = \left(\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix},\lambda,\mu\right))$$

and for any given pair of integers k,m on functions $\phi(\tau,z)$ by

$$\left(\phi|^+_{k,m}\Upsilon\right)(\tau,z)$$
$$= \phi\left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta},\frac{z+\lambda\tau+\mu}{\gamma\tau+\delta}\right)\mathrm{e}^{-2\pi i m\left(\frac{(z+\lambda\tau+\mu)^2}{\gamma\tau+\delta}+\lambda^2\tau+2\lambda z\right)}(c\overline{\tau}+\delta)^{1-k}|(c\tau+\delta)|^{-1},$$

and similarily by a slash operator '$|^-_{k,m}$', which is defined by the same formula as '$|^+_{k,m}$' but with $(c\overline{\tau}+\delta)^{1-k}|(c\tau+\delta)|$ replaced by $(c\tau+\delta)^{-k}$.

It is easily verified that any element of $\phi\in S^\epsilon_{k,m}$ satisfies $\phi|^\epsilon_{k,m}\Upsilon = \phi$ for all $\Upsilon\in\mathcal{J}(\mathbf{Z})$, and that for any two functions $\phi,\psi\in S^\epsilon_{k,m}$ the function

$$|\phi(\tau,z)\overline{\psi(\tau,z)}|\mathrm{e}^{-4\pi m\frac{y^2}{\Im(\tau)}}\Im(\tau)^k$$

is invariant under $(\tau,z)\mapsto\Upsilon\cdot(\tau,z)$ for all $\Upsilon\in\mathcal{J}(\mathbf{Z})$. The *Petersson scalar product* of ϕ,ψ is defined by

$$\langle\phi|\psi\rangle := \int_{\mathcal{J}(\mathbf{Z})\backslash\mathfrak{H}\times\mathbf{C}} |\phi(\tau,z)\overline{\psi(\tau,z)}|\mathrm{e}^{-4\pi m\frac{y^2}{\Im(\tau)}}\Im(\tau)^k dV.$$

Here dV is the $\mathcal{J}(\mathbf{Z})$-invariant volume element on $\mathfrak{H} \times \mathbf{Z}$, i.e.

$$dV(\tau, z) = \frac{dudvdxdy}{v^3} \quad (\tau = u + iv, z = x + iy),$$

and the integral has to be taken over any (measurable) fundamental domain of $\mathfrak{H} \times \mathbb{C}$ modulo $\mathcal{J}(\mathbf{Z})$. As fundamental domain one can take e.g.

$$\{(\tau, \lambda\tau + \mu) \,|\, |\tau| \geq 1, -\frac{1}{2} \leq \Re(\tau) \leq \frac{1}{2}, 0 \leq \lambda, \mu \in \mathbb{R}, \lambda + \mu \leq 1\}.$$

The above integral is absolutely convergent since, for any cusp form ϕ, the function

$$|\phi(\tau, z)| e^{-2\pi m \frac{y^2}{\Im(\tau)}} \Im(\tau)^{\frac{k}{2}}$$

is exponentially decreasing as $\Im(\tau)$ tends to infinity, as it is immediate from the Fourier development of ϕ. Thus, $\langle\phi|\psi\rangle$ defines a non-degenerate scalar product on $S^{\epsilon}_{k,m}$.

Fix a pair of positive integers k, m. Let $\Delta_0, r_0 \in \mathbf{Z}, \Delta_0 \equiv r_0^2 \bmod 4m, \Delta_0$ fundamental; let $\epsilon = \operatorname{sign}(\Delta_0)$. For any $\phi \in S^{\epsilon}_{k,m}$ we set

$$(\mathcal{S}_{\Delta_0, r_0}\phi)(t) := \sum_{\ell=1}^{\infty} \left(\sum_{a|\ell} a^{k-2} \left(\frac{\Delta_0}{a}\right) C_\phi \left(\Delta_0 \frac{\ell^2}{a^2}, r_0 \frac{\ell}{a}\right)\right) e^{2\pi i \ell t} \quad (t \in \mathfrak{H}),$$

and

$$\Omega_{\Delta_0, r_0}(t; \tau, z) := \sum_{\substack{\Delta, r \in \mathbf{Z}, \Delta_0\Delta > 0 \\ \Delta \equiv r^2 \bmod 4m}} \left(|\Delta_0 \Delta|^{k - \frac{3}{2}} \sum_{Q \in \mathcal{Q}_m(\Delta_0\Delta, r_0 r)} \frac{\chi_{m,\Delta_0}(Q)}{Q(t)^{k-1}} \right) e_{\Delta, r}(\tau, z) \qquad (3)$$

($\tau, t \in \mathfrak{H}, z \in \mathbb{C}$). For $\epsilon = \pm 1$ and an integer k let $S^{\epsilon}_{k}(\Gamma_0(m))$ denote the subspace of cusp forms f of weight k on $\Gamma_0(m)$ satisfying $f\left(\frac{-1}{m\tau}\right) = (-1)^{\frac{k}{2}} \epsilon (\sqrt(m)\tau)^k) f(\tau)$, i.e. the space of cusp forms f of weight k on $\Gamma_0(m)$ such that $L^*(f, s) := (2\pi)^{-s} m^{\frac{s}{2}} \Gamma(s) L(f, s) = L^*(f, k - s)$.

Proposition 1. *Assume $k \geq 3$. The series (3) is normally convergent on $\mathfrak{H} \times \mathfrak{H} \times \mathbb{C}$. For fixed t it defines an element of $S^{\epsilon}_{k,m}$, and for fixed τ, z it defines an element of $S^{\epsilon}_{2k-2}(\Gamma_0(m))$, where $\epsilon = \operatorname{sign}(\Delta_0)$. For any $\phi \in S^{\epsilon}_{k,m}$ one has*

$$\mathcal{S}_{\Delta_0, r_0}\phi = \langle \phi | c^{\epsilon}_{k,m} \Omega_{\Delta_0, r_0}(-\bar{t}; \cdot) \rangle, .$$

Here $c^{\epsilon}_{k,m} = (\frac{2\epsilon i}{m})^{k-2} \frac{\sqrt{-\epsilon}}{\pi} \binom{2k-4}{k-2}^{-1}$ ($\sqrt{-1} := i$) is a constant depending only on k, m and ϵ.

Remark. Note that the Proposition implies in particular that $\mathcal{S}_{\Delta_0,r_0}$ maps $S^{\epsilon}_{k,m}$ into $S^{\epsilon}_{2k-2}(\Gamma_0(m))$. This was proved for the '−'-case in [S-Z] and for the '+'-case and $k=2$ in [S1]. The kernel function Ω_{Δ_0,r_0} was introduced in [G-K-Z], and the above Proposition was proved loc.cit. ('Theorem' in II.3) for the '−'-case .

The proof of the above Proposion for the '−'-case extends almost without change to the "+"-case. The only new ingredient which has to be inserted is the *skew-holomorphic Poincaré series.*

Proposition 2. *Let $k \geq 3$. Let $\Delta_0, r_0 \in \mathbf{Z}, \Delta_0 \equiv r_0^2 \bmod 4m, \Delta_0 \neq 0$. Set*

$$P_{\Delta_0,r_0} := \sum_{\Upsilon \in \mathcal{J}(\mathbf{Z})_\infty \backslash \mathcal{J}(\mathbf{Z})} \mathrm{e}_{\Delta_0,r_0}|^{\epsilon}_{k,m}\Upsilon,$$

where $\epsilon = \mathrm{sign}(\Delta_0)$, where $\mathcal{J}(\mathbf{Z})_\infty = \left(\left(\begin{smallmatrix}1 & \mathbf{Z}\\ 0 & 1\end{smallmatrix}\right), 0, \mathbf{Z}\right)$, and where the sum is over a complete set of representatives for $\mathcal{J}(\mathbf{Z})$ modulo $\mathcal{J}(\mathbf{Z})_\infty$. Then this sum is well-defined (i.e. does not depend on the choice of the representives Υ) and normally convergent on $\mathfrak{H} \times \mathbf{C}$, and it defines an element of $S^{\epsilon}_{k,m}$. For any $\phi \in S^{\epsilon}_{k,m}$ one has

$$\langle \phi | P_{\Delta_0,r_0} \rangle = d_{k,m} \frac{C_\phi(\Delta_0, r_0)}{|\Delta_0|^{k-\frac{3}{2}}},$$

where $d_{k,m} = \frac{m^{k-2}\Gamma(k-\frac{3}{2})}{2\pi^{k-\frac{3}{2}}}$. Moreover, P_{Δ_0,r_0} has the Fourier expansion

$$\begin{aligned} P_{\Delta_0,r_0} = & \sum_{\substack{r \in \mathbf{Z} \\ r \equiv r_0 \bmod 4m}} \left(\mathrm{e}_{\Delta_0,r} - \epsilon(-1)^k \mathrm{e}_{\Delta_0,-r}\right) \\ & + \sum_{\substack{\Delta, r \in \mathbf{Z}, \Delta_0\epsilon > 0, \\ \Delta \equiv r^2 \bmod 4m}} g_{\Delta_0,r_0}(\Delta, r) \left(\mathrm{e}_{\Delta,r} - \epsilon(-1)^k \mathrm{e}_{\Delta,-r}\right), \end{aligned} \qquad (4).$$

where

$$g_{\Delta_0,r_0}(\Delta, r) = (-\epsilon i)^k \sqrt{-\epsilon}\pi \sqrt{\frac{2}{m}} \left(\frac{\Delta}{\Delta_0}\right)^{\frac{k}{2}-\frac{3}{4}} \sum_{\gamma=1}^{\infty} H_\gamma(\Delta_0, r_0; \Delta, r) J_{k-\frac{3}{2}}\left(\frac{\pi\sqrt{\Delta_0\Delta}}{m\gamma}\right).$$

Here $J_{k-\frac{3}{2}}(z) = \sum_{n=0}^{\infty}(-1)^n \frac{(z/2)^{k+2n-\frac{3}{2}}}{n!\Gamma(k+n-\frac{1}{2})}$ is the Bessel function of order $k-\frac{3}{2}$, we use $\sqrt{-1} = i$, and

$$H_\gamma(\Delta_0, r_0; \Delta, r) = \gamma^{-\frac{3}{2}} \sum_{\substack{\lambda, \alpha, \delta \bmod \gamma \\ \alpha\delta \equiv 1 \bmod \gamma}} \mathrm{e}^{2\pi i \left(\frac{(r_0+2m\lambda)^2 - \Delta_0}{4m\gamma}\alpha + \frac{r^2-\Delta}{4m\gamma}\delta + \frac{(r_0+2m\lambda)r}{2m\gamma}\right)}.$$

Remark. For the "$-$"-case the above Proposition was proved in [G-K-Z] (Proposition in II.2).

Proof. We show that the arguments given in [G-K-Z] for the '$-$'-case remain valid in the '+'-case by shortly reviewing the computation given loc.cit. and thereby including the case of positive discriminants. The arguments for normally convergence, well-definednes and correct invariance under $\mathcal{J}(\mathbf{Z})$ are literally the same as loc.cit.. Thus, in view of the asserted Fourier development, which we shall deduce below, it is in fact an element of $S^{\epsilon}_{k,m}$.

The scalar product $\langle\phi|P_{\Delta_0,r_0}\rangle$ equals by the usual argument of 'unfolding the integral'

$$\int_{\mathcal{J}(\mathbf{Z})_\infty\backslash\mathfrak{H}\times\mathbf{C}} \phi(\tau,z)\overline{e_{\Delta_0,r_0}(\tau,z)}e^{-4\pi m\frac{y^2}{v}}v^k dV,$$

where as always $\tau = u+iv, z = x+iy$. Inserting the Fourier development of ϕ and choosing $[0,1]\times\mathbf{R}_{>0}\times[0,1]\times\mathbf{R}$ as a fundamental domain for $\mathfrak{H}\times\mathbf{C}$ modulo $\mathcal{J}(\mathbf{Z})_\infty$ one thus finds

$$\sum_{\Delta,r} \mathrm{C}_\phi(\Delta,r)\int_0^1 du\int_0^\infty dv\int_0^1 dx\int_{-\infty}^{+\infty} dy\ e_{\Delta,r}(\tau,z)\overline{e_{\Delta_0,r_0}(\tau,z)}e^{-4\pi m\frac{y^2}{v}}v^k.$$

Integrating with respect to u and x yields 0, unless $(\Delta,r)=(\Delta_0,r_0)$. Thus the scalar product in question equals

$$C_\phi(\Delta_0,r_0)\int_0^\infty dv\int_{-\infty}^{+\infty} v^{k-3}e^{-\pi\left(\frac{r_0^2+|\Delta_0|}{m}v+4r_0y+4m\frac{y^2}{v}\right)} = C_\phi(\Delta_0,r_0)\frac{m^{k-2}\Gamma(k-\frac{3}{2})}{2\pi^{k-\frac{3}{2}}},$$

as claimed.

To compute the Fourier development we choose as representatives Υ the elements $(1,\lambda,0)\cdot(A_{\gamma,\delta},0)(U,0)$, where λ runs through $\mathbf{Z}$, where $U\in\{\pm\text{unit matrix}\}$, where (γ,δ) runs through all coprime pairs of integers with $\gamma\geq 1$ or $(\gamma,\delta)=(0,1)$, and where for each such pair $A_{\gamma,\delta}$ denotes an element of $SL_2(\mathbf{Z})$ with second row equal to γ,δ. We then obtain

$$P_{\Delta_0,r_0}(\tau,z) = S_0(\tau,z)+\sum_{\gamma=1}^{\infty}\left(S_\gamma(\tau,z)-\epsilon(-1)^k S_\gamma(\tau,-z)\right),$$

where

$$S_\gamma(\tau,z) = \sum_{\substack{\lambda,\delta\in\mathbf{Z}\\(\delta,\gamma)=1}}\{\gamma\tau+\delta\}e_{\Delta_0,r_0}\left(A\tau,\frac{z}{\gamma\tau+\delta}+\lambda A\tau\right)e^{2\pi i m\left(\frac{-\gamma z^2}{\gamma\tau+\delta}+\lambda^2 A\tau+2\lambda\frac{z}{\gamma\tau+\delta}\right)}$$

Here $A = A_{\gamma,\delta}$, and for any complex number w we use $\{w\} = w^{-k}$ if $\Delta_0 < 0$, and $\{w\} = \overline{w}^{1-k}|w|^{-1}$ if $\Delta_0 > 0$. Now S_0 is just the first sum of the right hand side of (4). To compute S_γ for $\gamma \geq 1$ we rewrite it, using $A\tau = \frac{\alpha\tau+\beta}{\gamma\tau+\delta} = \frac{\alpha}{\gamma} + \frac{-1}{\gamma(\gamma\tau+\delta)}$, as

$$S_\gamma(\tau,z) = \sum_{\substack{\lambda,\delta\in\mathbb{Z}\\ (\delta,\gamma)=1}} \{\gamma\tau+\delta\} \mathrm{e}_{\Delta_0,r_0}\left(\frac{\alpha}{\gamma}, \lambda\frac{\alpha}{\gamma}\right) \mathrm{e}^{2\pi i m\lambda^2\frac{\alpha}{\gamma}}$$

$$\times\, \mathrm{e}_{\Delta_0,r_0}\left(\frac{-1}{\gamma(\gamma\tau+\delta)}, \frac{z-\frac{\lambda}{\gamma}}{\gamma\tau+\delta}\right) \mathrm{e}^{2\pi i m\left(\frac{-\gamma\left(z-\frac{\lambda}{\gamma}\right)^2}{\gamma\tau+\delta}\right)}$$

This in turn can be rewritten as

$$S_\gamma(\tau,z) = \sum_{\substack{\lambda,\alpha,\delta \bmod \gamma\\ \alpha\delta\equiv 1 \bmod \gamma}} \mathrm{e}^{2\pi i\left(\frac{(r_0+2m\lambda)^2-\Delta_0}{4m\gamma}\alpha\right)} \gamma^{-k} F_\gamma\left(\tau+\frac{\delta}{\gamma}, z-\frac{\lambda}{\gamma}\right), \tag{5}$$

$$F_\gamma(\tau,z) = \sum_{s,t\in\mathbb{Z}} \{\tau+s\} \mathrm{e}_{\Delta_0,r_0}\left(\frac{-1}{\gamma^2(\tau+s)}, \frac{z+t}{\gamma(\tau+s)}\right) \mathrm{e}^{2\pi i m\left(\frac{-(z+t)^2}{\tau+s}\right)}.$$

But $F_\gamma(\tau,z)$ is periodic; thus we have

$$F_\gamma = \sum_{\substack{\Delta,r\in\mathbb{Z}\\ \Delta\equiv r^2 \bmod 2m}} c_{\Delta,r}\mathrm{e}_{\Delta,r}\ ,$$

$$c_{\Delta,r} = \int_{\Im(\tau)=C_1(>0)} d\tau \int_{\Im(z)=C_2} dz \{\tau\} \mathrm{e}_{\Delta_0,r_0}\left(\frac{-1}{\gamma^2\tau}, \frac{z}{\gamma\tau}\right) \mathrm{e}^{2\pi i m\left(\frac{-z^2}{\tau}\right)} \mathrm{e}_{\Delta,r}(\tau,z)^{-1}.$$

Note that the Fourier coefficients $c_{\Delta,r}$ are a priori functions of the imaginary parts of the arguments of F_γ, so that one would have to fix such arguments and to choose C_1, C_2 equal to their imaginary parts. However, as the following computation will show, the above double integral does not depend on $C_1(>0), C_2$, so that we allow ourself the above notational shortcut. Now the inner integral equals

$$\int_{\Im(z)=C_2} \mathrm{e}^{2\pi i\left(\frac{r_0 z}{\gamma\tau} - \frac{mz^2}{\tau} - rz\right)} dz = \left(\frac{\tau}{2im}\right)^{\frac{1}{2}} \mathrm{e}^{2\pi i\left(\frac{r_0^2}{4m\gamma\tau} + \frac{r^2}{4m}\tau - \frac{r_0 r}{2m\gamma}\right)}.$$

Here, for any complex numbers w, r, we use $w^r := \mathrm{e}^{i\theta r}$ $(-\pi < \theta := \mathrm{Arg}(w) \leq +\pi)$. Inserting the last formula in the double integral we find

$$c_{\Delta,r} = \frac{\mathrm{e}^{2\pi i\left(-\frac{r_0 r}{2m\gamma}\right)}}{\sqrt{2mi}} \int_{-\infty}^{+\infty} (u+ic_1)^{\frac{1}{2}-k} \mathrm{e}^{\frac{\pi i}{2m}\left(\frac{-|\Delta_0|}{u+ic_1} + \Delta(u+\sigma i c_1)\right)} du$$

for negative Δ_0, and

$$c_{\Delta,r} = \frac{e^{2\pi i(-\frac{r_0 r}{2m\gamma})}}{\sqrt{2mi}} \int_{-\infty}^{+\infty} (u - ic_1)^{\frac{1}{2}-k} e^{\frac{\pi i}{2m}\left(\frac{|\Delta_0|}{u+ic_1} + \Delta(u-\sigma ic_1)\right)} du$$

for positive Δ_0. Here $\sigma = 1$ if Δ_0 and Δ have the same sign, and $\Sigma = -1$ otherwise. Note that the second integral is the complex conjugate of the first but with Δ replaced by $-\Delta$. Thus it suffices to evaluate the first one. It vanishes if Δ_0 and Δ have opposite signs since we then can shift the path of integration to $i\infty$. If Δ_0 and Δ have the same sign substitute $\tau = i\sqrt{|\frac{\Delta_0}{\gamma^2\Delta}|}w$. Then the first integral becomes

$$-i^{\frac{3}{2}-k} \left|\frac{\Delta_0}{\gamma^2\Delta}\right|^{\frac{3}{4}-\frac{k}{2}} \int_{c_1-i\infty}^{c_1+i\infty} w^{\frac{1}{2}-k} e^{\rho(\frac{-1}{w}+w)} dw,$$

where $\rho = \frac{\pi\sqrt{\Delta_0\Delta}}{\gamma m}$. But the integral here equals $2\pi i J_{k-\frac{3}{2}}(\rho)$. Inserting this in the formula for $c_{\Delta,r}$, and then inserting the resulting Fourier expansion of F_γ into the formula (5) for S_γ, we finally find that the Δ, r-th Fourier coefficient of S_γ equals $-(-1)^k \epsilon g_{\Delta_0,r_0}(\Delta, r)$. From this we find the asserted Fourier expansion of P_{Δ_0,r_0}. This concludes the proof of the Proposition.

Proof of Proposition 1. The convergence properties and the modular behaviour of $\Omega_{\Delta_0,r_0}(t;\tau,z)$ as function of t are proved as in [G-K-Z] (for the behaviour under $t \mapsto \frac{-1}{mt}$ apply (loc.cit., I. Proposition 1 (P2))). For the remaining assertions it is obviously enough to prove the identity

$$\sum_{\ell=1}^{\infty} \left(\sum_{a|\ell} a^{k-2} \left(\frac{\Delta_0}{a}\right) \frac{|\Delta_0|^{k-\frac{3}{2}}}{d_{k,m}} P_{\Delta_0 \frac{\ell^2}{a^2}, r_0 \frac{\ell}{a}}(\tau, z) \right) e^{2\pi i \ell t} = c_{k,m}^{\epsilon} \Omega_{\Delta_0,r_0}(t;\tau,z). \quad (6)$$

But this can be checked as in [K-Z] by just comparing the Fourier coefficients (in the Fourier development with respect to τ, z, t) on both sides. The Fourier development of the left hand side is immediately obtained by inserting the developments of the P_{Δ_0,r_0} computed in Proposition 2. The Fourier development of the right hand side is obtained by inserting the Fourier developments of the

$$\sum_{Q \in \mathcal{Q}_m(\Delta_0\Delta, r_0 r)} \frac{\chi_{m,\Delta_0}(Q)}{Q(t)^{k-1}}$$

These latter Fourier developments have been computed in [K-Z] (II.1 Proposition 1) for positive *and* negative Δ_0 (however, there is a tiny mistake in the formula given loc. cit.: the term $\epsilon_N(m, \Delta, \rho, D_0)$ on p.517, second line from the bottom, has to me mutiplied by the factor $(-\text{sign } D_0)^k$). Finally, in that paper the Fourier coefficients of both sides of (6) have been compared in the case of negative Δ_0 and this comparison can literally be copied for the case of positive Δ_0. This completes the proof of Proposition 1.

§3. A variation of the Eichler-Shimura isomorphism

Let f be a cusp form of weight k on $\Gamma_0(m)$, and $A \in SL_2(\mathbf{Z})$. We define a complex polynomial in the indeterminate X by setting

$$\rho_{k,A}(f) = \int_0^{i\infty} (f|_k A)(t)(X-t)^{k-2} dt.$$

Here the integral has to be taken along the line $t = i\eta$ $(0 \leq \eta)$, and for any function f, defined on the upper half plane, any $A = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in SL_2(\mathbf{R})$, and any integer k we use

$$(f|_k A)(t) = f\left(\frac{\alpha t + \beta}{\gamma t + \delta}\right)(\gamma t + \delta)^{-k}.$$

Since f is a cusp form $f|_k A(t)$ is exponentially decreasing for $t \to 0, i\infty$ and any A, and hence the above integral is absolutely convergent. Note also that $\rho_{k,A}(f)$ depends only on the left coset of A in $\Gamma_0(m)\backslash SL_2(\mathbf{Z})$. Let $g := \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, thus, $g\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)g = \left(\begin{smallmatrix} \alpha & -\beta \\ -\gamma & \delta \end{smallmatrix}\right)$. For $\epsilon \in \{\pm 1\}$ we set

$$\rho_{k,A}^{\epsilon}(f) := \rho_{k,A}(f) + (-1)^{k/2}\epsilon\rho_{k,gAg}(f)|_{2-k}g,$$

i.e.

$$\rho_{k,A}^{\epsilon}(f) = \int_0^{i\infty} \left((f|_k A)(t)(X-t)^{k-2} + (-1)^{k/2}\epsilon(f|_k gAg)(t)(X+t)^{k-2}\right) dt \qquad (7).$$

Identifying $\Gamma_0(m)\backslash SL_2(\mathbf{Z})$ with $\mathbf{P}_1(\mathbf{Z}/m\mathbf{Z})$ via $\Gamma_0(m)\left(\begin{smallmatrix} * & * \\ \gamma & \delta \end{smallmatrix}\right) \mapsto (\gamma \bmod m : \delta \bmod m)$ we may view

$$\rho_k^{\epsilon}(f) := \left\{\rho_{k,A}^{\epsilon}(f)\right\}_{A \in \Gamma_0(m)\backslash SL_2(\mathbf{Z})}$$

as an element of $\mathbf{C}[X]_{k-2}^{\mathbf{P}_1(\mathbf{Z}/m\mathbf{Z})}$, where $\mathbf{C}[X]_{k-2}$ is the space of complex polynomials in X of degree less or equal to $k-2$. The correspondences $f \mapsto \rho_k^{\epsilon}(f)$ thus define maps

$$\rho_k^{\epsilon}\colon S_k(\Gamma_0(m)) \longrightarrow \mathbf{C}[X]_{k-2}^{\mathbf{P}_1(\mathbf{Z}/m\mathbf{Z})},$$

respectively.

Proposition 3. *For each integer k the maps ρ_k^+ and ρ_k^- are injective.*

Proof. Let $\sigma \in \{\pm 1\}$, $f \in S_k(\Gamma_0(m))$ and $A \in SL_2(\mathbf{Z})$. For $t = i\eta$ one has $t = -\bar{t}$ and $gAgt = -A\bar{t}$, and thus

$$\left(f|_k A(t)(X-t)^{k-2} + \sigma f|_k gAg(t)(X+t)^{k-2}\right) dt$$
$$= f(At)(ct+d)^{-k}(X-t)^{k-2}dt - \sigma f(-\overline{At})(c\bar{t}+d)^{-k}(X-\bar{t})^{k-2}d\bar{t}.$$

Therefore, decomposing

$$f = f_+ + if_-,$$
$$f_+(t) = \frac{1}{2}(f(t) + \overline{f(-\bar{t})}),\ f_-(t) = \frac{1}{2i}(f(t) - \overline{f(-\bar{t})}),$$

we have

$$\left(f|_k A(t)(X-t)^{k-2} - f|_k gAg(t)(X+t)^{k-2}\right) dt$$
$$= 2\Re\left(f_+|_k A(t)(X-t)^{k-2}dt\right) + 2i\Re\left(f_-|_k A(t)(X-t)^{k-2}dt\right),$$

and the same for $\sigma = +1$ but with $\Re$ replaced by $i\Im$.

Thus, to show that ρ_k^+ and ρ_k^- are injective it obviously suffices to show that for any $f \in S_k(\Gamma_0(m))$ the equations

$$\int_0^{i\infty} \Re\left(f|_k A(t)(X-t)^{k-2}dt\right) = 0 \quad (A \in SL_2(\mathbf{Z})) \tag{8}$$

imply $f = 0$. But the latter statement is easily reduced via the Manin trick to the fact that the usual period mapping of the Eichler-Shimura isomorphism is injective. Namely, let $B \in \Gamma_0(m)$ and $t_0 \in \mathfrak{H}$. Write $B = \pm T^{n_1}ST^{n_2}S\dots T^{n_r}S$ with $n_j \in \mathbf{Z}$ and T, S denoting the generators $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$ of $SL_2(\mathbf{Z})$ respectively, and set $B_j := \pm T^{n_1}ST^{n_2}S\dots T^{n_j}S$, $B_0 := 1$. Then

$$\int_{t_0}^{Bt_0} \Re\left(f(t)(X-t)^{k-2}dt\right) = \left(\int_{t_0}^{B_1t_0} + \int_{B_1t_0}^{B_2t_0} + \dots + \int_{B_{r-1}t_0}^{B_rt_0}\right) \Re\left(f(t)(X-t)^{k-2}dt\right),$$

and

$$\int_{B_jt_0}^{B_{j+1}t_0} \Re\left(f(t)(X-t)^{k-2}dt\right) = \int_{t_0}^{T^{n_{j+1}}St_0} \Re\left(f(B_jt)(X-B_jt)^{k-2}dB_jt\right)$$
$$= \int_{t_0}^{T^{n_{j+1}}St_0} \Re\left(f|_k B_j(t)(B_j^{-1}X-t)^{k-2}dt\right)(\gamma_jX+\delta_j)^{k-2} \quad \left(B_j^{-1} = \begin{pmatrix} * & * \\ \gamma_j & \delta_j \end{pmatrix}\right).$$

Note that one has $T^{n_{j+1}}St_0 = -\frac{1}{t_0} + n_{j+1}$. Thus, setting $t_0 = i\eta$ and letting η tend to 0, it follows

$$\int_0^{B0} \Re\left(f(t)(X-t)^{k-2}dt\right) = \sum_{j=0}^{r-1} \int_0^{i\infty} \Re\left(f|_k B_j(t)(B_j^{-1}X-t)^{k-2}dt\right)(\gamma_j X + \delta_j)^{k-2},$$

where the first integral has to be taken along the semicircle in $\mathfrak{H}$ joining 0 and $B0$. Hence (8) implies

$$\int_0^{B0} \Re\left(f(t)(X-t)^{k-2}dt\right) = 0.$$

That this equation is true for any $B \in \Gamma_0(m)$ means that f is in the kernel of the Eichler-Shimura isomorphism, i.e. $f = 0$ ([Sh], Theorem 8.4), and this was to be shown.

§4. The map from Jacobi forms into the space of period polynomials

The aim of this section is to compute the kernel function of the composed maps

$$\rho_{2k}^{\epsilon} \circ \mathcal{S}_{\Delta_0,r_0} : S_{k+1,m}^{\epsilon} \longrightarrow \mathbb{C}[X]_{2k-2}^{\mathbb{P}_1(\mathbb{Z}/m\mathbb{Z})} \qquad (\epsilon = \operatorname{sign}(\Delta_0)\ ). \tag{9}$$

We shall assume throughout this section that $k \geq 2$ since various expressions occuring in the following would not converge for smaller k. For $k = 1$ and $\Delta_0 = 1$ the composed map (9) is not even a priori well defined since in this case certain Jacobi cusp forms map to Eisenstein series and the integral (7) defining the period map will not in general converge for non cusp forms. However, as was shown in [S1], the integral (7) does in fact converge absolutely for those modular forms of weight 2 occuring as images of Jacobi cusp forms under all the lifting maps $\mathcal{S}_{\Delta_0,r_0}$. Hence we could speak of a composed map even in that quoted special case. The result of this section is as follows.

Proposition 4. *Let k, m be positive integers, $k \geq 2$. For $A \in SL_2(\mathbf{Z})$ and integers Δ_0, r_0, Δ, r such that $\Delta_0 \equiv r_0^2 \bmod 4m$, $\Delta \equiv r^2 \bmod 4m$, $\Delta_0\Delta > 0$ and Δ_0 is fundamental, define*

$$\begin{aligned}
\mathcal{C}^A_{\Delta_0,r_0}(\Delta, r; X) := & \sum_{\substack{Q\in\mathcal{Q}^A_m(\Delta_0\Delta, r_0 r)\\ Q=[a,b,c],\, ac<0}} \chi^A_{m,\Delta_0}(Q)\operatorname{sign}(a)Q(X)^{k-1} \\
& + \sum_{\substack{Q=[0,b,c]\in\mathcal{Q}^A_m(\Delta_0\Delta,r_0 r)\\ 0\leq c<N}} \chi^A_{m,\Delta_0}(Q)\frac{N^{k-1}}{k}B_k\left(\frac{bX+c}{N}\right) \\
& - \sum_{\substack{Q=[a,b,0]\in\mathcal{Q}^A_m(\Delta_0\Delta,r_0 r)\\ 0\leq a<N}} \chi^A_{m,\Delta_0}(Q)\frac{N^{k-1}}{k}B_k\left(\frac{a+b/X}{N}\right)X^{2k-2} \\
& +\gamma\left[\zeta_{m,\Delta_0\Delta,r_0 r,A0,\Delta_0}(k) + (-1)^k\epsilon\zeta_{m,\Delta_0\Delta,r_0 r,-A0,\Delta_0}(k)\right]X^{2k-2} \\
& -\gamma\left[\zeta_{m,\Delta_0\Delta,r_0 r,A\infty,\Delta_0}(k) + (-1)^k\epsilon\zeta_{m,\Delta_0\Delta,r_0 r,-A\infty,\Delta_0}(k)\right].
\end{aligned}$$

Here $\epsilon = \operatorname{sign}(\Delta_0)$, and N is any positive integer such that for any $Q = [a, b, 0]$ or $Q = [0, b, c]$ the value of $\chi^A_{m,\Delta_0}(Q)$ and the condition '$Q \in \mathcal{Q}^A_m(\Delta_0\Delta, r_0 r)$' depend only on a, c modulo N. Moreover $B_k(X)$ is the k-th Bernoulli polynomial, and

$$\gamma := \frac{(-1)^k(k-1)!^2}{\zeta(2k)(2k-1)!}(\Delta_0\Delta)^{k-\frac{1}{2}}.$$

Set

$$\mathcal{L}^A_{\Delta_0,r_0}(\tau, z; X) = b_{k,m} \sum_{\substack{\Delta, r\in\mathbf{Z},\, \Delta_0\Delta>0\\ \Delta\equiv r^2 \bmod 4m}} \mathcal{C}^A_{\Delta_0,r_0}(\Delta, r; X) e^{2\pi i\left(\frac{r^2-\Delta}{4m}u + \frac{r^2+|\Delta|}{4m}v + rz\right)}$$

($\tau, t \in \mathfrak{H}, z \in \mathbf{C}$), where $b_{k,m} = \frac{2}{\sqrt{\epsilon}}(\frac{2\epsilon}{mi})^{k-1}$ ($\sqrt(-1) = i$). Then this defines the kernel function for the composed map $\rho^\epsilon_{2k,A} \circ \mathcal{S}_{\Delta_0,r_0}$. More precisely, $\mathcal{L}^A_{\Delta_0,r_0}$ is an element of $S^\epsilon_{k+1,m}$, and for any $\phi \in S^\epsilon_{k+1,m}$ one has

$$\langle\phi|\mathcal{L}^A_{\Delta_0,r_0}(\cdot; -\overline{X})\rangle = (\rho^\epsilon_{2k,A}\mathcal{S}_{\Delta_0,r_0}\phi)(X).$$

Remark. 1. In the statement of the proposition we are tacitly identifying polynomials and polynomial functions, i.e. for fixed τ, z we view $\mathcal{L}^A_{\Delta_0,r_0}(\tau, z; X)$ as a function in the complex variable X rather than as a polynomial in the indeterminate X. 2. Note that the statement of the proposition implies that the coefficients $\mathcal{C}^A_{\Delta_0,r_0}(\Delta, r; X)$ do

not depend on the special choice of N. However, this follows also from the well-known equations
$$n^{k-1}\sum_{\nu=0}^{n-1} B_k(\frac{X+\nu}{n}) = B_k(X) \qquad (n \in \mathbf{Z}, n>0).$$

Proof. Using the kernel function for $\mathcal{S}_{\Delta_0,r_0}$ given in Proposition 1 (but with k replaced by $k+1$) we can write the polynomial $\rho_{2k,A}\mathcal{S}_{\Delta_0,r_0}\phi$ as

$$\int_0^{i\infty}\int_{SL_2(\mathbf{Z})^J\backslash(\mathfrak{H}\times\mathbf{C})} \phi(\tau,z)\overline{c^\epsilon_{k+1,m}\Omega_{\Delta_0,r_0}|_{2k}A(-\bar{t};\tau,z)}(X-t)^{2k-2}e^{-4\pi m\frac{y^2}{v}}v^{k+1}dV\,dt.$$

It is easily checked that we can interchange the order of integration, and from this we recognize that the Jacobi form

$$-\rho_{2k,A}\left(c^\epsilon_{k+1,m}\Omega_{\Delta_0,r_0}(\cdot;\tau,z)\right)(-\overline{X})$$

is the kernel function of the map $\rho_{2k,A}\circ\mathcal{S}_{\Delta_0,r_0}$ (To conclude this one also needs that $-\bar{t}=t$ for $t\in i\mathbf{R}$.). Inserting here the Fourier expansion (3) of Ω_{Δ_0,r_0} and interchanging summation (over Δ,r) and integration we find for $C(\Delta,r;-\overline{X})$, the Δ,r-th coefficient of the kernel function for $\rho^\epsilon_{2k,A}\circ\mathcal{S}_{\Delta_0,r_0}$, the formula

$$C(\Delta,r;X) = -c^\epsilon_{k+1,m}|\Delta_0\Delta|^{k-\frac{1}{2}}\int_0^{i\infty}\left(\sum_{Q\in\mathcal{Q}}\chi_{m,\Delta_0}(Q)\frac{(X-t)^{2k-2}}{(Q\circ A)(t)^k}\right.$$
$$\left.+(-1)^k\epsilon\sum_{Q\in\mathcal{Q}}\chi_{m,\Delta_0}(Q)\frac{(X+t)^{2k}}{(Q\circ gAg)(t)^k}\right)dt.$$

Here $\mathcal{Q}=\mathcal{Q}_m(\Delta_0\Delta,r_0r)$. As a first simplification of this formula we note that the set $\mathcal{Q}$ is invariant under $Q\mapsto -Q\circ g$, i.e. under $[a,b,c]\mapsto[-a,b,-c]$. Hence we can replace Q by $-Q\circ g$ in the second sum. It is easily verified that

$$\chi_{m,\Delta_0}(-Q\circ g)=\mathrm{sign}(\Delta_0)\chi_{m,\Delta_0}(Q) \quad -(Q\circ ggAg)(t) = -(Q\circ A)(-t).$$

Thus the second sum equals $(-1)^k\epsilon$ times the first but with t replaced by $-t$, i.e. with t replaced by $\bar{t}$ (since $t\in i\mathbf{R}$). Hence, after substituting $t\mapsto\bar{t}$ in the integral of the second sum we can write

$$C(\Delta,r) = -c^\epsilon_{k+1,m}|\Delta_0\Delta|^{k-\frac{1}{2}}\int_{-i\infty}^{i\infty}\sum_{Q\in\mathcal{Q}\circ A}\chi^A_{m,\Delta_0}(Q)\frac{(X-t)^{2k}}{Q(t)^k}dt.$$

Note that, though we have studied the infinite sum occuring here only for $t\in\mathfrak{H}$, it is normally convergent in the lower half plane $\Im(t)<0$ as well, and along the imaginary axis the function defined by it can even be continuously continued to $t = 0$. Thus, the above integral makes sense.

To compute this integral we decompose it as

$$C(\Delta, r) = -c^{\epsilon}_{k+1,m} |\Delta_0 \Delta|^{k-\frac{1}{2}} (I + K), \tag{10}$$

where I and K denote that contributions to the last integral from all $Q = [a, b, c]$ such that $ac \neq 0$ and $ac = 0$, respectively.

To simplify I we proceed as in ([Ko-Za]; pp.223). From the following computation it will be clear that we can in general not interchange summation and integration. Instead we write

$$\int_{-i\infty}^{i\infty} \sum \cdots = \lim_{\substack{\lambda \to 0 \\ \lambda > 0}} \sum \left(\int_{-i\infty}^{i\infty} - \int_{-i\lambda}^{i\lambda} - \int_{i/\lambda}^{i\infty} - \int_{-i\infty}^{-\frac{i}{\lambda}} \right).$$

Here interchanging integral and sum is allowed since the series is uniformly convergent on the compact pathes joining $i\lambda$ and i/λ and $-i/\lambda$ and $-i\lambda$, respectively. In the third and fourth integral we substitute $t \mapsto \frac{-1}{t}$, and $Q \mapsto Q \circ S^{-1}$, where $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Then we obtain

$$I = I_1 + I_2 + I_3, \tag{11}$$

$$I_1 = \sum_{\substack{Q=[a,b,c] \in \mathcal{Q} \circ A \\ ac \neq 0}} \chi^A_{m,\Delta_0}(Q) \int_{-i\infty}^{i\infty} \frac{(X-t)^{2k-2}}{Q(t)^k} dt,$$

$$I_2 = -\lim_{\substack{\lambda \to 0 \\ \lambda > 0}} \sum_{\substack{Q=[a,b,c] \in \mathcal{Q} \circ A \\ ac \neq 0}} \chi^A_{m,\Delta_0}(Q) \int_{-i\lambda}^{i\lambda} \frac{(X-t)^{2k-2}}{Q(t)^k} dt,$$

$$I_3 = \lim_{\substack{\lambda \to 0 \\ \lambda > 0}} \sum_{\substack{Q=[a,b,c] \in \mathcal{Q} \circ AS \\ ac \neq 0}} \chi^{AS}_{m,\Delta_0}(Q) \int_{-i\lambda}^{i\lambda} \frac{(Xt+1)^{2k-2}}{Q(t)^k} dt.$$

The inner integrals in I_1 are absolutely convergent and can easily be evaluated (cf. Lemma A2. of the Appendix). One has

$$I_1 = C_{k,\Delta_0\Delta} \sum_{\substack{Q=[a,b,c] \in \mathcal{Q} \circ A \\ ac < 0}} \chi^A_{m,\Delta_0}(Q) \mathrm{sign}(a) Q(X)^{k-1}, \tag{12}$$

with a constant $C_{k,\Delta_0\Delta}$, depending only on k and $\Delta_0\Delta$, as given in the appendix. To simplify I_2 we apply Proposition A1. The contribution to I_2 from those $Q = [a, b, c]$ with positive c can immediately read off from this lemma. To treat the contribution from those Q with negative c we have to substitute $Q \mapsto -Q \circ g$, i.e. $[a,b,c] \mapsto [-a,b,-c]$, to put it into a form such that the Proposition A1 can be applied. We obtain

$$I_2 = \frac{-2\pi i X^{2k-2}}{(2k-1)\zeta(2k)} \left[\zeta_{m,\Delta_0\Delta,r_0r,A0,\Delta_0}(k) + (-1)^k \epsilon \zeta_{m,\Delta_0\Delta,r_0r,-A0,\Delta_0}(k)\right]. \tag{13}$$

Similarily we find

$$I_3 = \frac{2\pi i X^{2k-2}}{(2k-1)\zeta(2k)} \left[\zeta_{m,\Delta_0\Delta,r_0r,A\infty,\Delta_0}(k) + (-1)^k \epsilon \zeta_{m,\Delta_0\Delta,r_0r,-A\infty,\Delta_0}(k)\right]. \tag{14}$$

To compute K we choose a positive integer N as in the statement of the Proposition (e.g. $N = m|\Delta_0|$). Using it we can write

$$\begin{aligned} K = N^{-k} \Bigg\{ & \sum_{\substack{Q=[0,b,c]\in\mathcal{Q}\circ A \\ 0<c<N}} \chi^A_{m,\Delta_0}(Q) \int_{-i\infty}^{i\infty} C_k\left(\frac{bt+c}{N}\right)(X-t)^{2k-2}dt \\ & + \sum_{\substack{Q=[a,b,0]\in\mathcal{Q}\circ A \\ 0<a<N}} \chi^A_{m,\Delta_0}(Q) \int_{-i\infty}^{i\infty} C_k\left(\frac{a+b/t}{N}\right)t^{-2k}(X-t)^{2k-2}dt \\ & + \sum_{Q=[0,b,0]\in\mathcal{Q}\circ A} \chi^A_{m,\Delta_0}(Q) \int_{-i\infty}^{i\infty} \left(C_k\left(\frac{bt}{N}\right) + C_k\left(\frac{b/t+c}{N}\right)t^{-2k} \right. \\ & \qquad \left. - \frac{1}{(bt)^k}\right)(X-t)^{2k-2}dt \Bigg\}, \end{aligned}$$

where we use

$$C_k(t) = \sum_{c\in\mathbf{Z}} \frac{1}{(t+c)^k} = \frac{2\pi i(-1)^{k-1}}{(k-1)!}\left(\frac{d}{dt}\right)^{k-1} \frac{e^{2\pi i t}}{e^{2\pi i t}-1}.$$

These integrals have been evaluated in the appendix (cf. Lemma A3), and inserting the values computed in the appendix we find

$$\begin{aligned} K = C_{k,\Delta_0\Delta} \Bigg\{ & \sum_{\substack{Q=[0,b,c]\in\mathcal{Q}\circ A \\ 0\le c<N}} \chi^A_{m,\Delta_0}(Q) \frac{N^{k-1}}{k} B_k\left(\frac{bX+c}{N}\right) \\ & - \sum_{\substack{Q=[a,b,0]\in\mathcal{Q}\circ A \\ 0\le a<N}} \chi^A_{m,\Delta_0}(Q) \frac{N^{k-1}}{k} B_k\left(\frac{a+b/X}{N}\right) X^{2k-2}, \Bigg\} \end{aligned} \tag{15}$$

Inserting the formulas (12),(13), (14), (15) into (11) and (10), picking up the constants

$$-c^{\epsilon}_{k+1,m} \times |\Delta_0\Delta|^{k-\frac{1}{2}} \times C_{k,\Delta_0\Delta}$$
$$= -\left(\frac{2\epsilon i}{m}\right)^{k-1} \frac{\sqrt{-\epsilon}}{\pi} \binom{2k-2}{k-1}^{-1} \times |\Delta_0\Delta|^{k-\frac{1}{2}} \times \frac{2\pi i(-1)^{k-1}\binom{2k-2}{k-1}}{(\Delta_0\Delta)^{k-\frac{1}{2}}}$$
$$= \frac{2}{\sqrt{\epsilon}}\left(\frac{2\epsilon}{mi}\right)^{k-1} = b_{k,m}$$

and

$$-c^{\epsilon}_{k+1,m} \times |\Delta_0\Delta|^{k-\frac{1}{2}} \times \frac{-2\pi i}{(2k-1)\zeta(2k)}$$
$$= \frac{2}{\sqrt{\epsilon}}\left(\frac{2\epsilon}{mi}\right)^{k-1} \times \frac{(-1)^k(k-1)!^2}{\zeta(2k)(2k-1)!}(\Delta_0\Delta)^{k-\frac{1}{2}} = b_{k,m} \times \gamma,$$

we find that $C(\Delta, r; X)$ equals $\mathcal{C}^A_{\Delta_0,r_0}(\Delta, r; X)$. This proves the proposition.

§5. The proof of the main result

In this last section we collect the facts of the previous discussions to complete the proof of Theorem 1 and 2. We assume throughout that $k \geq 2$. For the case of weight $k+1=2$ the reader is referred to [S1].

For $\epsilon \in \{\pm 1\}$ let $S^{\epsilon}_{per.}$ denote the subspace of $S^{\epsilon}_{k+1,m}$ spanned by the Jacobi forms $\mathcal{L}^A_{\Delta_0,r_0}(\tau, z; X)$ ($A \in SL_2(\mathbf{Z})$, $\text{sign}(\Delta_0) = \epsilon$) appearing in Proposition 4, and let $S^{\epsilon}_{spher.}$ be the space of functions spanned by all the $\phi_{\Delta_0,r_0,A,P}(\tau, z)$ with Δ_0, r_0, A, P as in Theorem 2, $\text{sign}(\Delta_0) = \epsilon$. Finally, let K^{ϵ} be the intersection of all kernels $\ker(\mathcal{S}_{\Delta_0,r_0})$ with $\text{sign}(\Delta_0) = \epsilon$. We shall show in a moment

$$S^{\epsilon}_{per.} = (K^{\epsilon})^{\perp} \tag{16}$$

$$S^{\epsilon}_{per.} = S^{\epsilon}_{spher.} \tag{17}$$

Here $(\cdot)^{\perp}$ means the orthogonal complement with respect to the Petersson scalar product. For $\epsilon = -1$ it was shown in [S-Z], Theorem 3, that $K^{\epsilon} = 0$. Thus the equations (16),(17) clearly imply Theorem 2 for $\epsilon = -1$. Moreover, it was shown loc.cit. that the sum of all the images of the $\mathcal{S}_{\Delta_1,r_1}$ with negative Δ_1 is just the subspace spanned by all modular forms f in $\mathfrak{M}^{\text{cusp}}_{2k}(m)$ whose L-series $L(f,s)$ satisfies $L^{\star}(f,s) :=$

$(2\pi/m)^{-s}\Gamma(s)L(f,s) = -L^*(f,2k-s)$. Thus, the '$-$'-*part* of Theorem 1 is a consequence of Theorem 2 by noticing that

$$f_{\Delta_0,r_0,\Delta_1,r_1,A,P}(\tau) = \mathcal{S}_{\Delta_1,r_1}\left(\phi_{\Delta_0,r_0,A,P}(\tau,z)\right).$$

For the '+'-case we can so far only deduce that $S^+_{spher.} = (K^+)^\perp$, and that the subspace spanned by all $f_{\Delta_0,r_0,\Delta_1,r_1,A,P}(\tau)$ $(\Delta_1 > 0)$ is just the sum of all the images of the $\mathcal{S}_{\Delta_1,r_1}$ with positive Δ_1. However, it will be shown in [S2] that this image is precisely that part of $\mathfrak{M}^{\mathrm{cusp}}_{2k}(m)$ spanned by all f such that $L^*(f,s) := (2\pi/m)^{-s}\Gamma(s)L(f,s) = +L^*(f,2k-s)$, and that $K^+ = 0$. Thus, the equations (16),(17) imply the '+'-parts of Theorem 1 and 2 as well.

The equation (16) follows from the logical equivalences

$$\langle\phi|\mathcal{L}^A_{\Delta_0,r_0}(\cdot;X)\rangle = 0 \iff \rho^\epsilon_{2k}\mathcal{S}_{\Delta_0,r_0}\phi = 0 \iff \phi \in \ker\left(\mathcal{S}_{\Delta_0,r_0}\right),$$

valid for any Δ_0 with $\Delta_0 = \epsilon$ and any $\phi \in S^\epsilon_{k+1,m}$. Here the first '$\iff$' follows from Proposition 4, and the second one from Proposition 3.

To prove (17) note first of all that for any X the polynomial

$$P_0(a,b,c) = \left(aX^2 + bX + c\right)^{k-1}$$

satisfies $\left(\frac{\partial^2}{\partial b^2} - \frac{\partial}{\partial a}\frac{\partial}{\partial c}\right)P_0 = 0$, and that

$$\phi_{\Delta_0,r_0,A,P_0}(\tau,z) = \mathcal{L}^A_{\Delta_0,r_0}(\cdot;X).$$

The latter is easily checked using the characteristic properties of the Bernoulli polynomials which were recalled in §1 when we defined the associated polynomials P_1 and P_2. Thus, $S^\epsilon_{per.} \subset S^\epsilon_{spher.}$. To prove the converse inclusion note that, for any fixed Δ_0, r_0, A, the map $P \mapsto \phi_{\Delta_0,r_0,A,P}$ is linear. Hence, it suffices to prove the following Lemma.

Lemma. *Let $P(a,b,c)$ be a homogeneous polynomial of degree k in the three variables a,b,c. Then the following two statements are equivalent: (i) One has $\left(\frac{\partial^2}{\partial b^2} - \frac{\partial^2}{\partial a\partial c}\right)P = 0$. (ii) The polynomial P can be written as a linear combination of polynomials of the form $\left(aX^2+bX+c\right)^k$ $(X \in \mathbf{C})$.*

Proof. That (ii) implies (i) is verified by direct computation. So assume (i). We prove (ii) by induction on the degree k of P. If $k = 0$ then (ii) is trivially true. So assume $k > 0$ and that (ii) is true for all polynomials with degree strictly smaller than k. Now, since P satisfies (i), the polynomial $\frac{\partial}{\partial b}P$ does so too, and by induction hypothesis it can thus be written as a linear combination of suitable $\left(aX^2+bX+c\right)^{k-1}$. Integrating with respect to b now shows that P is a linear combination of polynomials $\left(aX^2+bX+c\right)^k$ up to the addition of a polynomial P_0 which is independent of b. Clearly P_0 satisfies (i), and hence $\frac{\partial^2}{\partial a \partial c} P_0 = 0$. But this means that $P_0 = \alpha a^k + \gamma c^k$ for suitable constants α, γ. From this it is clear that P_0 satisfies (ii) (use e.g.

$$a^k = \frac{1}{2k} \sum_{\nu \bmod 2k} \left(a(e^{\frac{\pi i}{k}\nu})^2 + b(e^{\frac{\pi i}{k}\nu}) + c\right)^k - c^k \qquad)$$

and therefore P does so too. This concludes the proof of the lemma and the proof of Theorem 1 and 2 as well.

Appendix A: Zetafunctions associated to binary quadratic forms modulo $\Gamma_0(m)$

In this part of the appendix is we prove the following Proposition A1 which was used in the computations of §4.

Let $Q = [a, b, c]$ be a polynomial of degree ≤ 2, and let $\xi \in \mathbb{P}_1(\mathbb{Q}) (= \mathbb{Q} \cup \{\infty\})$. To the pair Q, ξ we associate a Dirichlet series $\zeta_{Q,\xi}(s)$ by setting

$$\zeta_{Q,\xi}(s) = \sum_{n=1}^{\infty} \frac{R(Q, \xi; n)}{n^s},$$

where, for any positive integer n, we use

$$R(Q, \xi; n) = \sharp \Gamma_0(m)_Q \backslash \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbf{Z}^2 \,\middle|\, \begin{array}{l} ax^2 + bxy + cy^2 = n \\ \wedge \ \frac{x}{y} \equiv \xi \bmod \Gamma_0(m) \end{array} \right\}.$$

Recall that $\frac{x}{y} \equiv \xi \bmod \Gamma_0(m)$ means that $\frac{x}{y}$ and ξ lie in the same orbit of the natural action of $\Gamma_0(m)$ on $\mathbf{P}_1(\mathbb{Q})$, and that $\Gamma_0(m)_Q$ denotes the stabilizer of Q in $\Gamma_0(m)$. Clearly, this zeta function depends only on the $\Gamma_0(m)$-equivalence classes of Q and ξ. These zeta-functions are connected to the zeta function defined in §1 by the formula

$$\zeta_{m,\Delta_0\Delta,r_0r,\xi,\Delta_0}(s) = \sum_{Q \in \mathcal{Q}_m(\Delta,r)/\Gamma_0(m)} \frac{\chi_{m,\Delta_0}(Q)}{[SL_2(\mathbf{Z})_Q : \Gamma_0(m)_Q]} \zeta_{Q,\xi}(s), \tag{18}$$

where Q runs through a complete set of representatives for $\mathcal{Q}_m(\Delta, r)$ modulo $\Gamma_0(m)$.

Lemma. *Let Q be a polynomial of degree ≤ 2 with integral coefficients, and let $\mathcal{C}$ denote the $\Gamma_0(m)$-equivalence class of Q. Let $\xi \in \mathbf{P}_1(\mathbf{Q})$, $\xi = \frac{x}{y}$, $gcd(x,y) = 1$. Then*

$$\zeta_{Q,\xi}(s) = \zeta(2s) \sum_{\substack{Q' \in \mathcal{C}/\Gamma_0(m)_\xi \\ Q'(x,y)>0}} Q'(x,y)^{-s}.$$

Here $\zeta(s)$ is the Riemann zeta function, and the sum is over a complete set of representatives for $\mathcal{C}$ modulo $\Gamma_0(m)_\xi$ (=stabilizer of ξ in $\Gamma_0(m)$). (Recall that we use $Q(x,y) = ax^2 + bxy + cy^2$ for any $Q = [a,b,c]$.)

Proof. Denote by $R^{pr.}(Q,\xi;n)$ the number of all coprime pairs of integers x, y modulo $\Gamma_0(m)_Q$ such that $Q(x,y) = n$, and $\frac{x}{y}$ is $\Gamma_0(m)$-equivalent to ξ. Clearly

$$R(Q,\xi;n) = \sum_{d^2|n} R^{pr.}(Q,\xi;\frac{n}{d^2}).$$

Now the maps $M \mapsto M\xi$ and $M \mapsto Q \circ M$ induce bijections

$$\Gamma_Q \backslash \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbf{Z}^2 \middle| \begin{array}{c} gcd(x,y) = 1 \\ \wedge\ \frac{x}{y} \equiv \xi \bmod \Gamma \end{array} \right\} \xleftarrow{\approx} \Gamma_Q \backslash \Gamma / \Gamma_\xi \xrightarrow{\approx} \mathcal{C}/\Gamma_\xi,$$

respectively, where we used $\Gamma = \Gamma_0(m)$. But via these isomorphisms we find

$$R^{pr.}(Q,\xi;n) = \sharp \left\{ Q \in \mathcal{C} | Q(\xi) = n \right\} / \Gamma_\xi.$$

Inserting this into the above equation for $R(Q,\xi;n)$, and rewriting the resulting equations in terms of Dirichlet series we obtain the asserted identity.

The following Proposition was proved in the case $m = 1$, D not a square, in [K-Z], Lemma on p.226.

Proposition A1. *Let $\mathcal{C}$ be the $\Gamma_0(m)-$equivalence class of a quadratic polynomial $Q_0(t) = at^2 + bt + c$. Assume that $Q_0(t)$ has real coefficients and that $D := b^2 - 4ac > 0$. Let $A \in SL_2(\mathbf{R})$. Then, for any integer $k \geq 2$ and any integer $\nu \geq 0$, the limit*

$$\lim_{\substack{\lambda \to 0 \\ \lambda > 0}} \sum_{\substack{Q = [a,b,c] \in \mathcal{C} \circ A \\ a \neq 0, c > 0}} \int_{-i\lambda}^{i\lambda} \frac{t^\nu \, dt}{Q(t)^k}$$

equals

$$[SL_2(\mathbf{Z})_Q : \Gamma_0(m)_Q]^{-1} \frac{2\pi i}{(2k-1)} \frac{\zeta_{Q_0, A0}(k)}{\zeta(2k)}$$

when $\nu = 0$, and it equals 0 otherwise.

Remark. Note that the Proposition together with (18) yields the formula

$$\lim_{\substack{\lambda\to 0\\ \lambda>0}} \sum_{\substack{Q\in\mathcal{Q}_m^A(\Delta_0\Delta, r_0 r)\\ Q=[a,b,c]\\ a\neq 0, c>0}} \chi_{m,\Delta_0}^A(Q)\int_{-i\lambda}^{i\lambda}\frac{(X-t)^{2k-2}dt}{Q(t)^k}$$
$$=\frac{2\pi i X^{2k-2}}{(2k-1)\zeta(2k)}\zeta_{m,\Delta_0\Delta,r_0 r,A0,\Delta_0}(s).$$

Proof. To compute the limit we choose a positive integer N such that $\mathcal{C}\circ A$ is invariant under $Q\mapsto Q\circ\left(\begin{smallmatrix}1&0\\N&1\end{smallmatrix}\right)$, i.e. under $[a,b,c]\mapsto[a+bN+cN^2, b+2Nc, c]$. Then we arrange the terms of the sum so that we first sum over those $Q=[a,b,c]$ with $0\le b<2cN$ and for each such Q over all $[a+bNn+cNn^2, 2Ncx, c]$ ($x=\frac{b}{2cN}+n,\ n\in\mathbf{Z}$, $(2cNx)^2\neq D$). Moreover, we write

$$[a+bNn+cNn^2, 2Ncx, c](t)=c(1+Nxt)^2-\frac{D}{4c}t^2.$$

Finally we substitute $t\mapsto\lambda t$. Thus, we have to compute

$$\lim_{\lambda\to 0}\lambda^\nu \sum_{\substack{Q=[a,b,c]\in\mathcal{C}\circ A,\\ c>0\\ 0\le b<2cN}}\frac{1}{c^k}\sum_{\substack{x\in\frac{b}{2cN}+\mathbf{Z}\\ (2cNx)^2\neq D}}\lambda\int_{-i}^{i}\frac{t^\nu dt}{\left[(1+Nx\lambda t)^2-\frac{D}{4c^2}(\lambda t)^2\right]^k}. \tag{19}$$

Now the inner sum tends to

$$\int_{-\infty}^{+\infty}\int_{-i}^{i}\frac{t^\nu dt dx}{(1+Nxt)^{2k}}$$

for $\lambda\to 0$, and it does so uniformly in b and c. Thus we see that the expression (19) equals 0 for positive ν. So let $\nu=0$. Then we may interchange in (19) the limit and the first sum, and insert the value of the last integral, which is $\frac{2\pi i}{(2k-1)N}$. Now, for N we can choose $N=[SL_2(\mathbf{Z})_Q:\Gamma_0(m)_Q]$, and then the condition '$0\le b<2cN$' means that we sum over a set of representatives for $\mathcal{C}\circ A/(A^{-1}\Gamma_0(m)A)_0$. Hence, after making the substitution $Q\mapsto Q\circ A$ we find for (19) the expression

$$\frac{2\pi i}{(2k-1)N}\sum_{\substack{Q\in\mathcal{C}/\Gamma_0(m)_{A0}\\ Q\circ A(0)>0}}(Q\circ A)(0)^{-k}.$$

Applying to this the above Lemma we recognize the asserted formula.

Appendix B: Computation of some integrals

In this part of the appendix we calculate some integrals which have been used when we computed the period polynomials of the kernel functions of the Jacobi forms – ellliptic modular forms correspondences. The following two lemmas can in principle be read off from corresponding calculations in [Ko–Za]. However, because of slightly different normalisations and the need of slightly more general formulations we include them here with independent proofs. For the following lemma compare [Ko-Za], pp. 224,225.

Lemma A2. *Let $Q(t) = at^2 + bt + c$ be a quadratic polynomial with real coefficients, and assume that $ac \neq 0$ and $D = b^2 - 4ac > 0$. Then for any complex number X and any positive integer k the integral*

$$\int_{-i\infty}^{+i\infty} \frac{(X-t)^{2k-2}}{Q(t)^k} dt$$

is absolutely convergent; it equals 0 when ac is positive, and for negative ac it equals

$$C_{k,D} \mathrm{sign}(a) Q(X)^{k-1},$$

where

$$C_{k,D} = \frac{2\pi i (-1)^{k-1} \binom{2k-2}{k-1}}{D^{k-\frac{1}{2}}}.$$

Proof. Using $ac \neq 0$ and $D > 0$ it is easily seen that $\frac{(X-t)^{2k-2}}{Q(t)^k}$ has no singularities on the imaginary axis and that it is an $\mathcal{O}(t^{-2})$ for $t \to i\infty$. Thus the integral in question is absolutely convergent.

To compute this integral note that for any sufficiently small $\lambda \in \mathbb{C}$ and for all $t \in i\mathbb{R}_{>0}$ we have $|\lambda||X-t|^2 < |Q(t)|$. Thus we can write

$$\sum_{k=1}^{\infty} \lambda^{k-1} \int_{-i\infty}^{+i\infty} \frac{(X-t)^{2k-2}}{Q(t)^k} dt = \int_{-i\infty}^{+i\infty} \sum_{k=1}^{\infty} \lambda^{k-1} \frac{(X-t)^{2k-2}}{Q(t)^k} dt$$

$$= \int_{-i\infty}^{+i\infty} \frac{dt}{Q(t) - \lambda(X-t)^2} = 2\pi i \mathrm{sign}(Q_\lambda) D_\lambda^{\frac{1}{2}},$$

where $Q_\lambda(t) = Q(t) - \lambda(X-t)^2$, and where D_λ denotes the discriminant of $Q_\lambda(t)$. Recall that $\mathrm{sign}(R) = \frac{1}{2}(\mathrm{sign}(a) - \mathrm{sign}(c))$ for any $R = [a,b,c]$. The interchanging of summation and integration is easily justified by doing the above computation with $X - t$ and $Q(t)$ replaced by its absolute values, noticing that the resultant integrals are finite and applying Lebesgue's theorem. For the last equality we used that for any real quadratic polynomial $R(t)$ with positive discriminant, the integral of $R(t)^{-1}dt$ along the imaginary axis equals $2\pi i$ times the sum of the residues of the integrand in the right half plane, which in turn equals $\operatorname{sign}(R)\cdot(\text{discriminant of } R)^{\frac{1}{2}}$. Now, by a simple calculation and by continuity

$$D_\lambda = D + 4\lambda Q(X), \quad \operatorname{sign}(Q_\lambda) = \operatorname{sign}(Q),$$

respectively. Thus

$$\sum_{k=1}^{\infty} \lambda^{k-1} \int_{-i\infty}^{+i\infty} \frac{(X-t)^{2k-2}}{Q(t)^k} dt = 2\pi i \operatorname{sign}(Q)\,(D + 4\lambda Q(X))^{\frac{1}{2}}$$

$$= 2\pi i \operatorname{sign}(Q) \sum_{k=1}^{\infty} \binom{\frac{1}{2}}{k-1} \frac{(4\lambda Q(X))^{k-1}}{D^{k-\frac{1}{2}}}.$$

Equating coefficients of these power series in λ finally proves the asserted formula of the lemma.

To state the following lemma we recall that for any positive integer k and any complex $t \in \mathbb{C} \setminus \mathbb{Z}$ from the upper half plane we use

$$C_k(t) = \frac{2\pi i(-1)^{k-1}}{(k-1)!} \frac{d^{k-1}}{dt^{k-1}} \frac{e^{2\pi i t}}{e^{2\pi i t} - 1}. \tag{20}$$

Lemma A3. *Let a, b, c be real numbers such that $b \neq 0$ and $0 < a, c < 1$. Then for any complex number X and any integer $k \geq 2$ the following integrals are absolutely convergent and the following equalities hold:*

$$\int_{-i\infty}^{+i\infty} C_k(bt+c)(X-t)^{2k-2} dt = \frac{C_{k,b^2}}{k} B_k(bX+c),$$

$$\int_{-i\infty}^{+i\infty} C_k(a+b/t)t^{-2k}(X-t)^{2k-2} dt = -\frac{C_{k,b^2}}{k} B_k(a+b/X)X^{2k-2},$$

$$\int_{-i\infty}^{+i\infty} \left(C_k(bt) + \frac{C_k\left(\frac{b}{t}\right)}{t^{2k}} - \frac{1}{(bt)^k} \right) (X-t)^{2k-2} dt = \frac{C_{k,b^2}}{k} \left(B_k(bX) - B_k\left(\frac{b}{X}\right)X^{2k-2} \right).$$

Here $B_k(X)$ denotes the k-th Bernoulli polynomial and

$$C_{k,b^2} = \frac{2\pi i(-1)^{k-1}\binom{2k-2}{k-1}}{|b|^{2k-1}}$$

(as in Lemma A2).

Proof. Immediately from the definition (20) it is clear that $C_k(t)$ is holomorphic in $\mathbb{C} \setminus \mathbb{Z}$, that it is exponentially decreasing for $\Re(t)$ fixed and $\Im(t) \to \pm\infty$, and that it has a pole at $t = 0$ with polar part t^{-k}. These statements immediately imply the absolute convergence of the integrals in the lemma.

To prove the listed identities note that the second one follows from the first by substituting $t \mapsto \frac{1}{t}$, $X \mapsto \frac{1}{X}$. Moreover, the third one follows by adding the first two, setting $a = c = \lambda$ and letting λ tend to 0. We leave the details to the reader. Finally, to prove the first equation, note that it suffices to prove it for $b = 1$: multiplying both sides of the first equation by $(-1)^k$, if necessary, and using $C_k(-t) = (-1)^k C_k(t)$ and $B_k(1-X) = (-1)^k B_k(X)$ we may assume first of all that $b > 0$; then substitute $t \mapsto \frac{t}{b}$, $X \mapsto \frac{X}{b}$. Thus, writing

$$\int_{-i\infty}^{+i\infty} C_k(t+c)(X-t)^{2k-2}dt = \int_{c-i\infty}^{c+i\infty} C_k(t)(X+c-t)^{2k-2}dt$$

we recognize that we have to prove

$$\int_{\Re(t)=c} C_k(t)(Y-t)^{2k-2}dt = 2\pi i(-1)^{k-1}\binom{2k-2}{k-1}\frac{B_k(Y)}{k}. \tag{21}$$

To do this call the left hand side of this equation $f(Y)$. Shifting the path of integration to the left crossing 0 but not -1 gives

$$f(Y+1) - f(Y) = -2\pi i \operatorname{Res}_{t=0} C_k(t)(Y-t)^{2k-2},$$

i.e.

$$f(Y+1) - f(Y) = 2\pi i(-1)^{k-1}\binom{2k-2}{k-1} Y^{k-1}.$$

Since $f(Y)$ is clearly a polynomial the last equation determines it up to a constant. On the other hand side the polynomial on the right hand side of (21) is a solution to the last equation. Thus, to conclude the proof it suffices to check for instance that $\int_0^1 f(y)dy = 0$. But indeed,

$$\int_0^1 f(y)dy = \int_{\Re(t)=c} C_k(t)\frac{t^{2k-1}-(t-1)^{2k-1}}{2k-1}dt$$

$$= \left(\int_{\Re(t)=c} - \int_{\Re(t)=c-1}\right) C_k(t)\frac{t^{2k-1}}{2k-1}dt = 2\pi i \operatorname{Res}_{t=0} C_k(t)\frac{t^{2k-1}}{2k-1} = 0$$

.

This proves the Lemma.

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