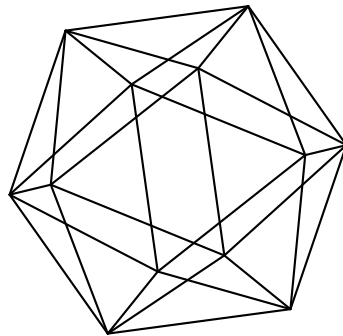


# Max-Planck-Institut für Mathematik Bonn

## Singular Fano threefolds of genus 12

by

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# SINGULAR FANO THREEFOLDS OF GENUS 12

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ABSTRACT. We study singular Fano threefolds of type  $V_{22}$ .

## 1. INTRODUCTION

All Fano varieties in this paper are supposed to be three-dimensional, to have at worst terminal Gorenstein singularities and defined over an algebraically closed field of characteristic zero. A Fano threefold  $X$  is said to be *of the main series* if the canonical class  $K_X$  generates the Picard group  $\text{Pic}(X)$ . For a Fano threefold  $X$  of the main series we can write  $-K_X^3 = 2g(X) - 2$ , where  $g(X)$  is an integer which is called the *genus* of  $X$ . It is known that  $g(X)$  takes the following values:  $g(X) \in \{2, 3, \dots, 10, 12\}$  (see [IP99] and [Na97]). Smooth Fano threefolds of the main series and genus 12 were described by Iskovskikh [Is78] and Mukai [Mu89]. In this paper we study singular Fano threefolds of genus 12. An important invariant of a Fano variety is  $r(X) := \text{rk Cl}(X)$ , the rank of the Weil divisor class group. The case  $r(X) = 1$  is already known:

**1.1. Theorem** [Mu02], [Pr15]. *Let  $X$  be a  $\mathbb{Q}$ -factorial Fano threefold of the main series with  $g(X) = 12$ . Then  $X$  is smooth.*

In §3 we prove the following result.

**1.2. Theorem.** *Let  $X$  be a Fano threefold of the main series with  $g(X) = 12$ . Assume that  $X$  satisfies one of the following equivalent conditions*

- (i) *the singular locus of  $X$  consists of one ordinary double point,*
- (ii)  $r(X) = 2$ .

*Then  $X$  is the midpoint of a Sarkisov link, i.e. it suits to the following commutative diagram*

$$(1.2.1) \quad \begin{array}{ccccc} & & Y & \overset{\chi}{\dashrightarrow} & Y^+ \\ & f \swarrow & & \searrow \pi & & \swarrow \pi^+ & & \searrow f^+ \\ & Z & & X & & & & Z^+ \end{array}$$

*where  $\pi$  and  $\pi^+$  are small  $\mathbb{Q}$ -factorializations, and  $\chi$  is a flop. The morphisms  $f$  and  $f^+$  are extremal Mori contractions described as follows*

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| No.   | $Z$            | $f$   | $Z^+$          | $f^+$   |
|-------|----------------|---|----------------|---|
| (I)   | $\mathbb{P}^3$ | the blowup of a smooth rational quintic curve $B \subset \mathbb{P}^3$ which is not contained in a quadric  | $\mathbb{P}^3$ | the blowup of a smooth rational quintic curve       |
| (II)  | $Q$            | the blowup of a smooth rational quintic curve $B \subset Q$ which is not contained in a hyperplane section  | $\mathbb{P}^2$ | a conic bundle whose discriminant curve of degree 3 |
| (III) | $V_5$          | the blowup of a smooth rational quartic curve $B \subset V_5$   | $\mathbb{P}^1$ | a del Pezzo fibration of degree 6                   |
| (IV)  | $\mathbb{P}^2$ | $\mathbb{P}_{\mathbb{P}^2}(\mathcal{E}) \rightarrow \mathbb{P}^2$ , where $\mathcal{E}$ is a stable rank-2 vector bundle on $\mathbb{P}^2$ with $c_1 = 0$ , $c_2 = 4$ | $\mathbb{P}^1$ | a del Pezzo fibration of degree 5                   |

where  $Q$  is a smooth quadric in  $\mathbb{P}^4$  and  $V_5 \subset \mathbb{P}^6$  is a smooth del Pezzo threefold of degree 5.

In principle, the proof can be extracted from the series of papers [JPR11], [Ta09], [CM13], [BL12]. However, all these papers consider the situation where  $\mathbb{Q}$ -factorializations in (1.2.1) are smooth. We prefer to give a relatively short self-contained proof in general case.<sup>1</sup> Note that all the varieties in Theorem 1.2 admit a moving decomposition [Mu02, §7]. Thus one cannot expect that Mukai's techniques [Mu89, Mu02] works in our case.

In §§4-5 we study cases (I)-(IV) in details. In particular, we show that these cases occurs and describe general members of the families. According to Mukai the moduli space  $\mathcal{M}_{\text{Fano}}^{12}$  of smooth Fano threefolds is 6-dimensional and birational to the moduli space  $\mathcal{M}_3$  of genus 3 curves. Our four families (I)-(IV) are parametrized by varieties  $\mathcal{M}_{\text{Fano}}^{(I)}, \dots, \mathcal{M}_{\text{Fano}}^{(IV)}$  of dimension 5.

The original motivation of this work was to study  $G$ -varieties and finite subgroups of the Cremona group (see [Pr12, Pr13a]). An algebraic variety is said to be a  $G$ -variety if it is equipped with an action  $G \rightarrow \text{Aut}_{\mathbb{k}}(X)$  of a finite group  $G$ . A projective  $G$ -variety  $X$  is a  $G$ -Fano variety if  $X$  has at worst terminal singularities,  $-K_X$  is an ample Cartier divisor, and the rank of the invariant part  $\text{Cl}(X)^G$  of the Weil divisor class group equals 1.  $G$ -Fano threefolds which are not of the main series were classified in [Pr13b, Pr13c].

The main result of this paper is the following.

**1.3. Theorem.** *Let  $X$  be a  $G$ -Fano threefold of the main series with  $g(X) = 12$ . Then  $X$  either is smooth or its singular locus consists of one ordinary*

<sup>1</sup>The variety (IV) was erroneously omitted in [JPR11]

double point. In the latter case  $X$  is the anticanonical image of the blowup of  $\mathbb{P}^3$  along a smooth rational curve of degree 5 (see Theorem 1.2 (I)).

In particular, the diagram (1.2.1) in the case (I) gives a well known Cremona map  $\mathbb{P}^3 \dashrightarrow \mathbb{P}^3$  of degree (3, 3) (see [SR49, Ch. VIII, Ex. 8, P. 185]).

One can modify the definition of  $G$ -varieties ( $G$ -Fano varieties etc.) for the arithmetic case: the variety  $X$  is defined over a non-closed field  $\mathbb{k}$  and  $G$  is the absolute Galois group acting on  $\bar{X} = X \otimes \bar{\mathbb{k}}$  through the second factor. Our result can be applied in this situation with small modifications.

Another motivation of for study singular Fano threefolds comes from some questions about affine and Moishezon non-projective varieties (cf. [Fu06]). In §4 we construct a new compactification of the affine space  $\mathbb{A}^3$  as a variety of type (III) (see Theorem 4.4).

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## 2. PRELIMINARIES

### Standard notation.

- $\text{Cl}(X)$  denotes the Weil divisor class group of a variety  $X$ ,
- $r(X) = \text{rk Cl}(X)$ ,
- $\text{Pic}(X)$  denotes the Picard group of a variety  $X$ ,
- $\rho(X)$  is the Picard number of  $X$ ,
- $Q = Q_d \subset \mathbb{P}^{d+1}$  is a smooth quadric of dimension  $d$ ,
- $Q' = Q'_d \subset \mathbb{P}^{d+1}$  is a singular irreducible quadric,
- $V_5 \subset \mathbb{P}^6$  is a smooth quintic del Pezzo threefold,
- $\mathcal{N}_{C/V}$  is the normal bundle of  $C$  in  $V$ ,
- $\mathbb{F}_n$  is a rational ruled surface (Hirzebruch surface),
- $\iota(X)$  is the Fano index of a (generalized) Fano variety [IP99].

A *contraction* is a morphism with connected fibers of normal varieties.

**Convention.** Throughout this paper all (generalized) Fano varieties are supposed to have at worst terminal Gorenstein singularities. A Fano threefold  $X$  of Fano index 2 is called a *del Pezzo threefold* (see e.g. [IP99, Pr13b]).

We will use systematically the following very useful observation.

**2.1. Proposition** [Kaw88, Lemma 5.1]. *Let  $U$  be a threefold with terminal Gorenstein singularities. Then any  $\mathbb{Q}$ -Cartier divisor on  $U$  is Cartier.*

**2.2. Proposition** [Kaw88, Corollary 4.5]. *Let  $U$  be a threefold with terminal singularities. Then there exists a small morphism  $\pi : \tilde{U} \rightarrow U$  such that  $\tilde{U}$  has only terminal  $\mathbb{Q}$ -factorial singularities and  $K_{\tilde{U}} = \pi^* K_U$ .*

If in the above notation the variety  $U$  is Gorenstein, then  $\tilde{U}$  has only terminal factorial singularities. In this case, we say that  $\pi$  is *factorialization* of  $U$ .

The classification of extremal contractions of terminal Gorenstein threefolds is almost the same as in the smooth case:

**2.3. Theorem** ([Cu88]). *Let  $V$  be a threefold with terminal factorial singularities and let  $f : V \rightarrow W$  be an extremal Mori contraction with  $\dim W > 0$ . Then one of the following holds.*

- (**e**<sub>1</sub>)  *$f$  is birational and contracts a surface  $E$  to an irreducible curve  $B$ . In this case  $B$  has locally planar singularities and is contained in the smooth locus of  $W$ . The contraction  $f$  is the blowup of the ideal of  $B \subset W$  and the variety  $W$  also has only terminal factorial singularities.*
- (**e**<sub>2-5</sub>)  *$f$  is birational and contracts a surface  $E$  to a point  $P$ . In this case  $f$  is the blowup of the maximal ideal of  $P \in W$  and one of the following holds:*

| (2.3.1) | ( <b>e</b> <sub>2</sub> )   | smooth                 | $\mathbb{P}^2$  | $\mathcal{O}(-1)$ | 2             | 8             | 4 |
|---------|-----------------------------|------------------------|-----------------|-------------------|---------------|---------------|---|
|         | ( <b>e</b> <sub>3-4</sub> ) | cA-point               | $Q_2$ or $Q'_2$ | $\mathcal{O}(-1)$ | 1             | 2             | 2 |
|         | ( <b>e</b> <sub>5</sub> )   | $\frac{1}{2}(1, 1, 1)$ | $\mathbb{P}^2$  | $\mathcal{O}(-2)$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 |

where  $\delta := (-K_W)^3 - (-K_V)^3$  and  $\alpha_E$  is the discrepancy of  $E$ .

- (**d**)  *$W$  is a smooth curve and  $f$  is a del Pezzo fibration;*
- (**c**)  *$W$  is a smooth surface and  $f$  is a conic bundle.*

**2.3.2.** In the case (**c**) there exists a reduced curve  $\Delta_f \subset W$  such that  $f$  is a smooth morphism over  $W \setminus \Delta_f$  and any fiber  $V_w := f^{-1}(w)$  over  $w \in \Delta_f$  is a singular (either reducible or non-reduced) conic. We say that  $\Delta_f$  is the *discriminant curve* of  $f$ . If  $\Delta_f = \emptyset$ , then  $f$  is a  $\mathbb{P}^1$ -bundle, in particular,  $V$  is smooth (see e.g. [Pr15, Prop. 5.2]). If furthermore, the surface  $W$  is rational, then this bundle is locally trivial.

**2.3.3.** In the case (**d**) the general fiber  $V_\nu$  is a smooth del Pezzo surface of degree  $d = K_{V_\nu}^2$  with  $1 \leq d \leq 9$ ,  $d \neq 7$ . If  $d = 9$ , then  $f$  is a locally trivial  $\mathbb{P}^2$ -bundle. If  $d = 8$ , then the general fiber  $V_\nu$  is a smooth quadric.

**2.3.4.** There exists the following natural exact sequence

$$0 \longrightarrow \text{Pic}(W) \xrightarrow{f^*} \text{Pic}(V) \xrightarrow{-l} \mathbb{Z}$$

where  $l$  is any curve contracted by  $f$ . To describe the cokernel of the right-side map, we denote

$$(2.3.5) \quad \mu_f := \min \{-K_V \cdot l \mid l \subset V \text{ is a curve contracted by } f\}.$$



Thus the cokernel of the map  $\text{Pic}(V) \rightarrow \mathbb{Z}$  is a cyclic group of order  $\mu_f$ . Then  $\mu_f = 1$  except for the following cases:

$$\begin{aligned} \mu_f = 2: & \text{ (e}_2\text{), (c) with } \Delta_f = \emptyset, \text{ (d) with } K_{V_\eta}^2 = 8; \\ \mu_f = 3: & \text{ (d) with } K_{V_\eta}^2 = 9. \end{aligned}$$

**2.4. Lemma** [Kal11], [Pr15]. *Let  $f : V \rightarrow W$  be the blowup of a reduced (but possibly reducible) locally planar curve  $B$  as in (e<sub>1</sub>) of Theorem 2.3. Then*

$$(2.4.1) \quad \begin{aligned} (-K_V)^3 &= (-K_W)^3 + 2K_W \cdot B + 2p_a(B) - 2, \\ (-K_V)^2 \cdot E &= -K_W \cdot B - 2p_a(B) + 2, \\ (-K_V) \cdot E^2 &= 2p_a(B) - 2. \end{aligned}$$

Therefore,

$$(2.4.2) \quad (-K_W)^3 - (-K_V)^3 = 2K_V^2 \cdot E + 2p_a(B) - 2.$$

**2.5. Lemma.** *Let  $V$  be a projective threefold with at worst terminal  $\mathbb{Q}$ -factorial singularities such that  $-K_V$  is nef. Let  $f : V \rightarrow W$  be a birational contraction such that  $-K_W$  is a nef  $\mathbb{Q}$ -Cartier divisor. Let  $S \subset V$  be an irreducible surface which is not contained in the  $f$ -exceptional locus. Then*

$$K_V^2 \cdot S \leq K_W^2 \cdot f(S).$$

*Proof.* Write

$$K_V = f^*K_W + \sum \alpha_i E_i,$$

where  $E_i$  are prime exceptional divisors, and  $\alpha_i \in \mathbb{Q}$ . Since  $-K_V$  is  $f$ -nef,  $\alpha_i \geq 0$ . Put  $E = \sum \alpha_i E_i$ . Then

$$\begin{aligned} K_V^2 \cdot S &= K_V \cdot (f^*K_W + E) \cdot S \leq K_V \cdot f^*K_W \cdot S = \\ &= (f^*K_W + E) \cdot f^*K_W \cdot S \leq (f^*K_W)^2 \cdot S = K_W^2 \cdot f(S). \end{aligned}$$

□

**2.6. Definition.** A *generalized Fano variety* is a projective variety  $X$  with at worst terminal Gorenstein singularities such that the anticanonical divisor  $-K_X$  is nef and big and the natural morphism

$$\Phi : X \longrightarrow \bar{X} := \text{Proj} \oplus_{n \geq 0} H^0(X, -nK_X)$$

to the anticanonical model does not contract divisors.

Note that in this situation  $\bar{X}$  is a Fano variety (with at worst terminal Gorenstein singularities). Conversely, if  $\bar{X}$  is a Fano variety (as above) and  $\Phi : X \rightarrow \bar{X}$  is its small factorization (see Proposition 2.2), then  $X$  is a generalized Fano variety.

**2.7.** Let  $X$  be a generalized Fano threefold. Put  $g(X) := -K_X^3/2 + 1$ . By Riemann-Roch and Kawamata-Viehweg vanishing  $\dim | -K_X| = g(X) + 1$ . Hence  $g(X)$  is an integer. It is called the *genus* of  $X$ . The Picard group  $\text{Pic}(X)$  and the Weil divisor class group  $\text{Cl}(X)$  are finitely generated and torsion free [IP99]. Moreover, there exists a natural embedding  $\text{Pic}(X) \hookrightarrow \text{Cl}(X)$  as a primitive sublattice (see Proposition 2.1). The *Fano index* of  $X$  is the maximal integer  $\iota = \iota(X)$  such that  $K_X$  is divisible by  $\iota$  in  $\text{Pic}(X)$ . The *degree* of a surface  $S$  in a generalized Fano threefold  $X$  is the degree with respect to the anticanonical divisor, i.e. the (positive) number  $(-K_X)^2 \cdot S$ . We say that a surface  $S \subset X$  is a *plane* if its degree equals 1. Note that for a Fano threefold  $X$  of the main series with  $g(X) \geq 4$ , the linear system  $| -K_X|$  is base point free [JR06]. Hence, in this situation, any surface of degree 1 is isomorphic to  $\mathbb{P}^2$ , i.e. it is a plane in the usual sense.

### 3. PROOF OF THEOREM 1.2.

In this section we prove Theorem 1.2.

**3.1. Proposition.** *Let  $U$  be a Fano threefold whose singular locus consists of one ordinary double point. Then  $r(U) \leq \rho(U) + 1$ .*

*Proof.* We may assume that  $U$  is not  $\mathbb{Q}$ -factorial. Let  $\pi : \tilde{U} \rightarrow U$  be a small factorization (see Proposition 2.2). Then  $r(\tilde{U}) = r(U)$ . On the other hand,  $\rho(\tilde{U}/U) = 1$  because  $\pi$  contracts exactly one irreducible curve.  $\square$

Thus in Theorem 1.2 the implication (i)  $\implies$  (ii) holds and we may assume that  $r(X) = 2$ .

Let  $X$  be a Fano threefold with  $g(X) = 12$ ,  $\rho(X) = 1$ , and  $r(X) = 2$ . Let  $\pi : Y \rightarrow X$  be a small factorization. By our assumption  $\pi$  is not an isomorphism and  $Y$  is a *generalized* Fano threefold with  $\rho(Y) = 2$ . The existence of diagram (1.2.1) is a standard fact (see e.g. [IP99, §4.1]). Consider the possibilities for the contraction  $f$  according to Theorem 2.3. Note that our diagram (1.2.1) is symmetric. So we can interchange  $f$  and  $f^+$ . In particular, if one of contractions  $f$  and  $f^+$  is birational, then we may assume that this holds for  $f$ .

**3.2. Lemma.** *Neither  $f$  nor  $f^+$  is of type  $(\mathbf{e}_2)$ - $(\mathbf{e}_{3-4})$ .*

*Proof.* Assume that  $f$  is of type  $(\mathbf{e}_2)$  or  $(\mathbf{e}_{3-4})$ . Let  $E \subset Y$  be the exceptional divisor. Then  $Z$  is a  $\mathbb{Q}$ -factorial Fano threefold with  $\rho(Z) = 1$ . By (2.3.1) we have  $-K_Z^3 = 30$  or  $24$ . Hence,  $\iota(Z) \geq 2$ . Note that  $-K_Z^3$  must be divisible by  $\iota(Z)^3$ . The only possibility is  $\iota(Z) = 2$  and  $-K_Z^3 = 24$ , i.e.  $Z$  is a del Pezzo threefold of degree 3 and  $f$  is of type  $(\mathbf{e}_{3-4})$ . In other words,  $Z = Z_3 \subset \mathbb{P}^4$  is a cubic threefold and  $f(E) \in Z$  is a cA-point. In this case,  $Z^+ \simeq \mathbb{P}^3$ , the map  $Z \dashrightarrow Z^+$  is just a projection from  $f(E) \in Z$ , and  $Y$  is a Fano threefold (see [MM82, Table 2, n° 15°]), a contradiction.  $\square$

**3.3. Type (e<sub>1</sub>).** Let  $E \subset Y$  be the exceptional divisor and let  $E^+ := \chi_* E$ . Then  $Z$  is a  $\mathbb{Q}$ -factorial Fano threefold with  $\rho(Z) = 1$  and  $B := f(E) \subset Z$  is an irreducible curve. Let  $H$  be the ample generator of  $\text{Pic}(Z)$ , let  $H^* := f^* H$ , and  $H^+ := \chi_* H^*$ .

**3.3.1. Lemma.** *In the notation of 3.3 one of the following holds:*

- (I)  $Z \simeq \mathbb{P}^3$ ,  $p_a(B) = 0$ ,  $\deg B = 5$ ;
- (II)  $Z \simeq Q \subset \mathbb{P}^4$ ,  $p_a(B) = 0$ ,  $\deg B = 5$ ;
- (III)  $Z \simeq V_5 \subset \mathbb{P}^6$ ,  $p_a(B) = 0$ ,  $\deg B = 4$ .

*Proof.* By Lemma 2.4 we have

$$(3.3.2) \quad \begin{aligned} (-K_Z)^3 &= 22 + 2K_Y^2 \cdot E + 2p_a(B) - 2 > 24, \\ (-K_Z)^3 &= 22 - 2K_Z \cdot B - 2p_a(B) + 2. \end{aligned}$$

Hence  $\iota(Z) \geq 2$  and by Corollary 7.2.2 we have the following possibilities:

*Subcase:  $\iota(Z) = 4$  and  $Z \simeq \mathbb{P}^3$ .* Then the relations (3.3.2) give us

$$p_a(B) = 4k, \quad \deg B = 5 + k, \quad K_Y^2 \cdot E = 22 - 4k$$

If  $k = 1$ , the divisor  $-K_Y$  is in fact ample [MM82, Table 2, n° 15°] and  $\pi$  is an isomorphism, a contradiction. Thus we may assume that  $k \geq 2$ . Since the linear system  $|-K_Y|$  is base point free, the curve  $B$  is an intersection of quartics. In particular,  $B$  is not contained in a plane. We get a contradiction by the Castelnuovo genus bound (see e.g. [Ha77, ch. 4, Theorem 6.4]).

*Subcase:  $\iota(Z) = 3$  and  $Z$  is a smooth quadric  $Q \subset \mathbb{P}^4$ .* As above we have

$$p_a(B) = 3k, \quad \deg B = 5 + k, \quad K_Y^2 \cdot E = 17 - 3k, \quad 0 \leq k \leq 5.$$

If  $k = 0$ , we get the case (II). Let  $k \geq 1$ . Then according to the Castelnuovo genus bound the curve  $B$  is contained in a hyperplane. Since  $|-K_Y|$  is base point free,  $B$  is intersection of cubics. Hence,  $\deg B \leq 6$ . Then  $k = 1$ ,  $\deg B = 6$ , and  $B$  is a complete intersection of a quadric and a cubic. But in this case  $p_a(B) = 4$ , a contradiction.

*Subcase:  $\iota(Z) = 2$  and  $Z = Z_d \subset \mathbb{P}^{d+1}$  is a del Pezzo threefold of degree  $d = 4$  or  $5$ .* Note that  $\mathbb{Q}$ -factorial del Pezzo threefold of degree 5 is smooth (and isomorphic to  $V_5$ , see [Pr13b, Corollary 5.4]). As above

$$p_a(B) = 2k, \quad \deg B = 2d - 6 + k, \quad K_Y^2 \cdot E = 4d - 10 - 2k, \quad 0 \leq k < 2d - 5.$$

If  $k = 0$  and  $d = 5$ , we get the case (III). If  $k = 0$  and  $d = 4$ ,  $Y$  is a Fano threefold as in [MM82, Table 2, n° 16°], so it does not admit a small contraction, a contradiction. Let  $k \geq 1$ . Then  $p_a(B) \geq 2$  and  $\deg B \geq 3$ . Note that the curve  $B$  is an intersection of quadrics because  $|-K_Y|$  is base point free. Hence  $\dim \langle B \rangle \geq 4$ . Moreover, if  $\dim \langle B \rangle = 4$ , then  $\deg B = 5$ ,  $d = 5$ , and  $B = V_5 \cap \langle B \rangle$  is a curve of arithmetic genus 1 with contradicts

our relation  $p_a(B) = 2k \geq 2$ . Therefore,  $\dim\langle B \rangle \geq 5$ . Then, we get a contradiction by the Castelnuovo genus bound.  $\square$

**3.4. Lemma.** *Neither  $f$  nor  $f^+$  is of type  $(\mathbf{e}_5)$ .*

*Proof.* By symmetry we may assume that  $f$  is of type  $(\mathbf{e}_5)$ . Let  $E \subset Y$  be the exceptional divisor. We have

$$(3.4.1) \quad (-K_Y)^2 \cdot E = 1, \quad (-K_Y) \cdot E^2 = -2, \quad E^3 = 4.$$

The variety  $Z$  is a  $\mathbb{Q}$ -factorial non-Gorenstein threefold with  $\rho(Z) = 1$ , and  $f(E) \in Z$  is a point of type  $\frac{1}{2}(1, 1, 1)$ . (Thus  $Z$  is so-called  $\mathbb{Q}$ -Fano threefold). By (2.3.1) we have  $-K_Z^3 = -K_Y^3 + 1/2 = 45/2$ . Assume that  $\text{Cl}(Z)$  contains a torsion element, say  $T$ . Since  $f(E)$  is the only non-Gorenstein point of  $Z$  and it is of type  $\frac{1}{2}(1, 1, 1)$ , in its neighborhood we have  $K_Z + T \sim 0$ . This means that  $K_Z + T$  is a Cartier divisor. On the other hand,  $(K_Z + T)^3 = K_Z^3$  is not an integer, a contradiction. Therefore, the group  $\text{Cl}(Z)$  is torsion free. Let  $A$  be the ample generator of  $\text{Cl}(Z)$ . We can write  $-K_Z \sim qA$ ,  $q \in \mathbb{Z}$ . Then  $45/2 = -K_Z^3 = q^3 A^3$ . By Proposition 2.2 the divisor  $2A$  is Cartier. Hence,  $2A^3$  is an integer and so  $q = 1$ , i.e.  $K_Z$  generates the group  $\text{Cl}(Z)$ . Since  $f_*K_Y = K_Z$ , the group  $\text{Pic}(Y)$  is generated by  $K_Y$  and  $E$  (see 2.3.4). Thus, for any effective divisor  $D \neq E$  on  $Y$ , we can write

$$D \sim -\alpha K_Y - \beta E, \quad \alpha \in \mathbb{Z}_{>0}, \quad \beta \in \mathbb{Z}_{\geq 0}.$$

Then

$$(-K_Y) \cdot D^2 = 22\alpha^2 - 2\alpha\beta - 2\beta^2.$$

We may assume that the contraction  $f^+$  is of type  $(\mathbf{c})$ ,  $(\mathbf{d})$ , or  $(\mathbf{e}_5)$ . Let  $D^+ \subset Y^+$  be the pull-back of a line, the fiber, and the exceptional divisor in these cases, respectively. Let  $D := \chi_*^{-1}D^+$ . Then  $(-K_Y) \cdot D^2 = 2, 0$ , or  $-2$ , respectively. Hence,

$$(3.4.2) \quad 11\alpha^2 - \alpha\beta - \beta^2 = \delta, \quad \delta \in \{1, 0, -1\}.$$

This equation has solutions only for  $\delta = -1$ , i.e. when  $f^+$  is of type  $(\mathbf{e}_5)$ . But then, as above,  $\text{Pic}(Y)$  is generated by  $K_Y$  and  $D$ . Hence,  $\beta = 1$ . In this case, (3.4.2) has no solutions, a contradiction.  $\square$

**3.5. Lemma.** *Both contractions  $f$  and  $f^+$  cannot be of type  $(\mathbf{d})$ .*

*Proof.* Assume that both  $f$  and  $f^+$  are of type  $(\mathbf{d})$ . Let  $F$  be a fiber of  $f$ . Clearly,

$$(-K_Y) \cdot F^2 = 0, \quad (-K_Y)^2 \cdot F = K_F^2.$$

Let  $L^+$  be a fiber of  $f^+$  and let  $L$  be the proper transform of  $L^+$  under the flopping map. Write  $L \sim -\alpha K_Y - \beta F$ ,  $\alpha, \beta \in \frac{1}{2}\mathbb{Z} \cup \frac{1}{3}\mathbb{Z}$  (see 2.3.4). Then

$$\begin{aligned} 0 &= (-K_Y) \cdot L^2 = 22\alpha^2 - 2(K_F^2)\alpha\beta, \\ K_{F^+}^2 &= (-K_Y)^2 \cdot L = 22\alpha - (K_F^2)\beta. \end{aligned}$$

This gives us

$$11\alpha = (K_F^2)\beta = K_{F^+}^2.$$

But then  $K_{F^+}^2$  must be divisible by 11, a contradiction.  $\square$

**3.6. Lemma.** *If both contractions  $f$  and  $f^+$  are not birational, then up to permutation we may assume that  $f$  is a  $\mathbb{P}^1$ -bundle over  $\mathbb{P}^2$  and  $f^+$  is a del Pezzo fibration of degree 5.*

*Proof.* By symmetry we may assume that  $f$  is of type (c). Let  $\Delta_f \subset Z$  be the discriminant curve of our conic bundle  $f$ . Let  $l \subset Z = \mathbb{P}^2$  be a general line and let  $F := f^{-1}(l)$ . Then  $F$  is a smooth rational surface having a conic bundle structure with degenerate fibers over the points  $l \cap \Delta_f$ . Hence,

$$(3.6.1) \quad F^3 = 0, \quad K_Y \cdot F^2 = -2, \quad K_Y^2 \cdot F = 12 - \deg \Delta_f.$$

We may assume that the contraction  $f^+$  is of type (c) or (d). Consider these subcases separately.

*Subcase:  $f^+$  is of type (c).* Let  $\Delta_{f^+} \subset Z^+$  be the discriminant curve of  $f^+$  and let  $L^+ := f^{+^{-1}}(l^+)$ , where  $l^+ \subset Z^+ = \mathbb{P}^2$  is a line. Let  $L$  be the proper transform of  $L^+$  under the flopping map. By 2.3.4 we can write

$$L \sim -\alpha K_Y - \beta F, \quad \alpha, \beta \in \frac{1}{2}\mathbb{Z}, \quad \alpha, \beta > 0.$$

Similar to (3.6.1) we have

$$(3.6.2) \quad 2 = (-K_{Y^+}) \cdot L^{+2} = (-K_Y) \cdot L^2 = 22\alpha^2 - 2(12 - \deg \Delta_f)\alpha\beta + 2\beta^2.$$

If  $\Delta_f = \emptyset$ , then the last relation can be rewritten as follows

$$1 + 25\alpha^2 = (6\alpha - \beta)^2.$$

It is easy to see that this equation has no solutions in  $\alpha, \beta \in \frac{1}{2}\mathbb{Z}$ . Therefore we may assume that  $\Delta_f \neq \emptyset$  and  $\Delta_{f^+} \neq \emptyset$ . In this case, the divisors  $K_Y$  and  $F$  generate  $\text{Pic}(Y)$ . Hence,  $\alpha, \beta \in \mathbb{Z}$ . By symmetry  $K_Y$  and  $L$  generate  $\text{Pic}(Y)$ . Hence,  $\beta = 1$ . Then (3.6.2) has the form

$$12 = 11\alpha + \deg \Delta_f$$

Hence,  $\deg \Delta_f = 1$ , i.e.  $\Delta_f$  is a line on  $Z = \mathbb{P}^2$ . In this case  $f^{-1}(\Delta_f)$  must be a reducible surface which contradicts the extremal property of  $f$ .

*Subcase:  $f^+$  is of type (d).* Let  $L^+$  be a fiber of  $f^+$  and let  $L$  be the proper transform of  $L^+$  under the flopping map. Write  $L \sim -\alpha K_Y - \beta F$ ,  $\alpha, \beta \in \frac{1}{2}\mathbb{Z}$ . Then

$$\begin{aligned} 0 &= (-K_Y) \cdot L^2 = 22\alpha^2 - 2(12 - \deg \Delta_f)\alpha\beta + 2\beta^2, \\ K_{L^+}^2 &= (-K_Y)^2 \cdot L = 22\alpha - (12 - \deg \Delta_f)\beta. \end{aligned}$$

The first equation has a rational solution only for  $\deg \Delta_f = 0$  and then

$$K_{L^+}^2 = 2\beta(11\alpha/\beta - 6\beta), \quad \alpha/\beta = 1, \quad K_{L^+}^2 = 10\beta = 5.$$

Since  $\Delta_f = \emptyset$ , the variety  $Y$  is smooth and  $f$  is a  $\mathbb{P}^1$ -bundle (see 2.3.2). By Corollary 3.7.4 below we get the case (IV).  $\square$

**3.7. Lemma.** *Let  $\mathcal{E}$  be a rank-2 vector bundle on a smooth surface  $Z$ , let  $Y := \mathbb{P}_Z(\mathcal{E})$ , and let  $M$  be the tautological divisor on  $Y$ . Then the following relations hold*

$$(3.7.1) \quad -K_Y = 2M + f^*(-K_Z - c_1(\mathcal{E})),$$

$$(3.7.2) \quad M^2 = M \cdot f^*c_1(\mathcal{E}) - f^*c_2(\mathcal{E}), \quad M^3 = c_1(\mathcal{E})^2 - c_2(\mathcal{E}).$$

$$(3.7.3) \quad -K_Y^3 = 6K_Z^2 + 2c_1(\mathcal{E})^2 - 8c_2(\mathcal{E}).$$

*Proof.* Use the relative Euler exact sequence and the Hirsch formula.  $\square$

**3.7.4. Corollary.** *In the notation of 3.7 assume that  $Z = \mathbb{P}^2$  and  $Y := \mathbb{P}_{\mathbb{P}^2}(\mathcal{E})$  is a generalized Fano threefold with  $g(Y) = 12$ . Then  $\mathcal{E}$  is stable with  $c_1(\mathcal{E})^2 + 16 = 4c_2(\mathcal{E})$ .*

*Proof.* By (3.7.3) up to twisting  $\mathcal{E}$  by a line bundle, we may assume that  $c_1(\mathcal{E}) = 0$ ,  $c_2(\mathcal{E}) = 4$ . Thus  $-K_Y = 2M + 3F$ . Since

$$2(-K_Y)^2 \cdot M = (-K_Y)^3 - 3(-K_Y)^2 \cdot F = 22 - 36 < 0,$$

we have  $|M| = \emptyset$  and so  $H^0(\mathcal{E}) = 0$ . Then  $\mathcal{E}$  must be stable (see e.g. [Fr98, Ch. 4, Prop. 14]).  $\square$

To finish our proof of Theorem 1.2 it remains to describe the right hand side of the diagram (1.2.1) in cases (I), (II), and (III).

**3.8. Notation.** Let  $Z$  be a smooth Fano threefold with  $\rho(Z) = 1$  and  $B \subset Z$  be a smooth curve. Let  $\iota := \iota(Z)$  be the Fano index of  $Z$  and  $d := \iota - 1$ . Let  $H = -\frac{1}{\iota}K_Z$  be the ample generator of  $\text{Pic}(Z)$ . Let  $f : Y \rightarrow Z$  be the blowup of  $B$ , let  $E \subset Y$  be the  $f$ -exceptional divisor, and  $H^* := f^*H$ .

We assume that the pair  $(Z, B)$  is one of that described in Lemma 3.3.1. Thus  $(Z, B)$  appears in the left hand side of (1.2.1) in the cases (I)-(III). Denote by  $|dH - B|$  the subsystem of  $|dH|$  consisting of all divisors passing through  $B$ .

**3.8.1. Lemma.** *In the above notation we have  $\dim |dH - B| \geq d$ .*

*Proof.* We have  $h^0(\mathcal{I}_B(d)) \geq h^0(\mathcal{O}_Z(d)) - h^0(\mathcal{O}_B(d)) = d + 1$ .  $\square$

**3.8.2. Lemma.** *If  $Y$  is a generalized Fano threefold, then any member  $\tilde{S} \in |dH^* - E|$  is irreducible.*

*Proof.* Let  $\tilde{S}_1 \subsetneq \tilde{S}$  be an irreducible component other than  $E$ . Then  $\tilde{S}_1 \sim d'H^* - kE$ , where  $d' \leq d$ ,  $k \geq 1$  and at least one of these inequalities is strict. Then easy computations give us  $(-K_Y)^2 \cdot \tilde{S}_1 \leq 0$ . This contradicts our assumption.  $\square$

**3.8.3. Corollary.** *In the assumptions of 3.8.2 any member  $S \in |dH - B|$  is irreducible. In particular,  $B$  is not contained in a quadric in the case  $Z = \mathbb{P}^3$  and  $B$  is not contained in a hyperplane section in the case  $Z = Q$ .*

**3.9.** Now assume that  $Y$  is a generalized Fano threefold, i.e.  $|-K_Y|$  is nef and does not contract divisors. Let  $\chi : Y \dashrightarrow Y^+$  be the flop, let  $H^+ := \chi_*H^*$ , and  $E^+ := \chi_*E$ . The description of the right hand side of the diagram (1.2.1) in cases (I), (II), and (III) is an immediate consequence of the following.

**3.10. Proposition.** *In the above notation,  $dH^+ - E^+$  is a supporting divisor for the contraction  $f^+$ .*

*Proof.* Let  $\tilde{S} \in |dH^* - E|$  and  $\tilde{S}^+ := \chi_*\tilde{S}$ . Since  $-\tilde{S} \sim K_Y + H^*$  is  $\pi$ -ample,  $\tilde{S}^+$  must be  $\pi^+$ -ample. Since the linear system  $|\tilde{S}^+|$  has no fixed components and the contraction  $f^+$  is not small,  $\tilde{S}^+$  is non-negative on the fibers of  $f^+$ . Therefore,  $\tilde{S}^+$  is nef. It remains to show that  $\tilde{S}^+$  is not ample. Assume the converse. Let  $l$  be a curve in a fiber of  $f^+$  that attains the minimum of  $\mu_{f^+}$  (see (2.3.5)). Since  $|H^+|$  has no fixed components and  $H^+$  is negative on flipped curves, we have  $H^+ \cdot l > 0$ . Then  $\mu_{f^+} = -K_{Y^+} \cdot l = \tilde{S}^+ \cdot l + H^+ \cdot l \geq 2$ . Moreover, if  $\mu_{f^+} = 2$ , then  $\tilde{S}^+ \cdot l = H^+ \cdot l = 1$ . In this case the supporting divisor for  $f^+$  has the form  $\tilde{S}^+ - H^+$  and so  $|\tilde{S} - H^*| \neq \emptyset$ . On the other hand,  $(-K_Y)^2 \cdot (\tilde{S} - H^*) \leq 0$ . This contradicts our assumption that  $Y$  is a generalized Fano threefold. Hence,  $\mu_{f^+} = 3$ . Then  $Z^+ \simeq \mathbb{P}^1$  and  $f^+$  is a  $\mathbb{P}^2$ -bundle. As above, the only possibility is the following:  $\tilde{S}^+ \cdot l = 1$ ,  $H^+ \cdot l = 1$ , and the supporting divisor of  $\varphi^+$  is  $2\tilde{S}^+ - H^+$ . Since  $\dim Z^+ = 1$ ,  $(2\tilde{S}^+ - H^+)^2 = 0$ . On the other hand,  $(2\tilde{S}^+ - H^+)^2 \cdot K_{Y^+} = (2\tilde{S} - H^*)^2 \cdot K_Y < 0$  by (2.4.1), a contradiction.  $\square$

This concludes our proof of Theorem 1.2.

#### 4. FANO THREEFOLDS OF TYPES (I), (II), AND (III).

First we show that the possibilities (I), (II) and (III) of Theorem 1.2 really occur. More precisely, we show that any pair  $(Z, B)$  satisfying the conditions of (I)-(III) of Theorem 1.2 generates diagram (1.2.1).

**4.1.** Let  $(Z, B)$  be as in 3.8. By Lemma 3.8.1  $\dim |dH - B| > 0$ . Assume additionally that  $B$  is not contained in a quadric in the case  $Z = \mathbb{P}^3$  and  $B$  is not contained in a hyperplane section in the case  $Z = Q$ . This implies that any member  $S \in |dH - B|$  is irreducible.

**4.1.1. Lemma.** *Any member  $S \in |dH - B|$  is del Pezzo surface of degree  $d$  with at worst Du Val singularities.*

*Proof.* First we claim that  $S$  is normal. Assume that  $S$  is singular along a curve  $J$ . Then  $J$  is contained in  $\text{Bs}|dH - B|$  by Bertini's theorem. The

intersection of two members  $S, S' \in |(\iota - 1)H - B|$  contains  $J$  with multiplicity  $\geq 4$  and the curve  $B$ . Then  $(\iota - 1)^2 H^3 = S \cdot S' \cdot H \geq H \cdot B + 4H \cdot B$ . In the cases (II) and (III) this gives a contradiction. In the case (I) we have an equality  $S \cdot S' = B + 4J$ . Since  $\dim |(\iota - 1)H - B| = 3$  in this case, we can take  $S$  and  $S'$  so that they have a common point  $P \notin B \cap J$ . Then  $S \cdot S' \not\supseteq B + 4J$ . This means that  $S$  and  $S'$  have a common component and  $S$  is reducible. The contradiction proves our claim.

Further, by the adjunction formula  $-K_S = H|_S$  is ample, i.e.  $S$  is a del Pezzo surface. If the singularities of  $S$  are worse than Du Val, then it is a cone over an elliptic curve [HW81]. But this is impossible because  $S$  contains a rational curve  $B$  of degree  $> 1$ .  $\square$

**4.1.2. Corollary.** *In the above notation any member  $\tilde{S} \in |dH^* - E|$  is a del Pezzo surface of degree  $d$  with at worst Du Val singularities and the restriction map  $f_S : \tilde{S} \rightarrow f(\tilde{S}) = S$  is crepant.*

*Proof.* By the adjunction formula  $K_{\tilde{S}} = -f^*H|_{\tilde{S}} = f^*K_S$ .  $\square$

**4.2. Proposition.** *Notation as in 3.8. Let  $S \in |dH - B|$  be a general member and let  $\tilde{S} \subset Y$  be the proper transform of  $S$ . Then  $S$  is smooth and  $f_S : \tilde{S} \rightarrow S$  is an isomorphism. The blowup  $f : Y \rightarrow Z$  can be completed to the diagram (1.2.1). The  $\pi$ -exceptional locus consists of exactly one smooth rational curve  $\Upsilon$  which is a  $(-1)$ -curve on  $\tilde{S}$  and  $\mathcal{N}_{\Upsilon/Y} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ .*

*Proof.* We claim that  $Y$  is a generalized Fano threefold. Assume that  $K_Y \cdot \Upsilon \geq 0$  for some irreducible curve  $\Upsilon$ . Clearly,  $\Upsilon$  is not a fiber of the ruling  $E \rightarrow B$ . Hence,  $H^* \cdot \Upsilon > 0$  and  $\tilde{S} \cdot \Upsilon < 0$ . In particular,  $\Upsilon \subset \tilde{S}$ ,  $\Upsilon \subset \text{Bs}|dH^* - E|$ , and there are at most a finite number of such curves. On the minimal resolution  $S_{\min}$  of  $\tilde{S}$ , the Mori cone is generated by  $(-1)$ - and  $(-2)$ -curves (because  $S_{\min}$  is a weak del Pezzo surface). Hence we may assume that  $-K_{\tilde{S}} \cdot \Upsilon = 1$  and so  $H \cdot f(\Upsilon) = -K_S \cdot f(\Upsilon) = 1$ , i.e.  $f(\Upsilon)$  is a line on  $Z$ . Since  $K_Y \cdot \Upsilon \geq 0$ , we have  $E \cdot \Upsilon \geq d + 1$  and so  $f(\Upsilon)$  is a  $(d + 1)$ -secant line to  $B \subset S \subset \mathbb{P}^d$ . It is easy to see that  $B$  has no  $(d + 2)$ -secant lines. Therefore,  $E \cdot \Upsilon = d + 1$  and  $K_Y \cdot \Upsilon = 0$ , i.e.  $-K_Y$  is nef. Obviously,  $-K_Y$  is big. This proves our claim.

Thus the blowup  $f : Y \rightarrow Z$  can be completed to the diagram (1.2.1) and by Theorem 1.2 we have cases (I), (II), (III). Let  $\tilde{S}^+ = \chi_*\tilde{S} \subset Y^+$  be the proper transform of  $\tilde{S}$ . Then  $\tilde{S}^+$  is a smooth del Pezzo surface of degree  $d + 1$ . Since the linear system  $|\tilde{S}^+|$  is base point free,  $\tilde{S}^+$  meets flopped curves  $\Upsilon_i^+$  transversely. Hence  $\tilde{S}$  contains all the flopping curves  $\Upsilon_i$ . By the Zariski main theorem  $\chi$  induces a morphism  $\chi_S : \tilde{S} \rightarrow \tilde{S}^+$  that contracts  $\cup \Upsilon_i$ . By the Noether formula  $\rho(\tilde{S}) \leq 10 - d$  and  $\rho(\tilde{S}^+) = 9 - d$ . Since there exists at least one flopping curve,  $\rho(\tilde{S}) = \rho(\tilde{S}^+) + 1$  and  $\rho(\tilde{S}) =$



$10 - d$ . Hence  $\tilde{S}$  is smooth and  $\chi_S : \tilde{S} \rightarrow \tilde{S}^+$  is a blowup of a single point. Thus the  $\pi$ -exceptional locus consists of exactly one smooth rational curve  $\Upsilon = \Upsilon_1$  which is a  $(-1)$ -curve on  $\tilde{S}$ . Then  $\mathcal{N}_{\Upsilon/Y}$  contains a subbundle  $\mathcal{N}_{\Upsilon/\tilde{S}} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)$ . Since  $\deg \mathcal{N}_{\Upsilon/Y} = -2$ , we have the desired splitting. If the restriction  $f_S : \tilde{S} \rightarrow S$  is not an isomorphism, then the intersection  $\tilde{S} \cap E$  is reducible: it contains a section and a fiber of the ruling  $E \rightarrow B$ . But this contradicts Bertini's theorem applied to  $|\tilde{S}|_E$ .  $\square$

**4.2.1. Corollary.** *In the cases (I), (II), (III) of Theorem 1.2 the singular locus of  $X$  consists of exactly one ordinary double point.*

**4.3. Construction.** Let  $Z$  be either  $\mathbb{P}^3$ , a (possibly singular) quadric in  $\mathbb{P}^4$  or a (possibly singular) quintic del Pezzo threefold in  $\mathbb{P}^6$ . Put  $d := \iota(Z) - 1$ . Let  $H$  be the ample generator of  $\text{Pic}(Z)$  and let  $S = S_d \in |dH|$  be a smooth member. Thus  $S = S_d \subset \mathbb{P}^d$  is a smooth del Pezzo surface of degree  $d = 3, 4$ , or  $5$ . Regard  $S$  as the blowup  $\sigma : S \rightarrow \mathbb{P}^2$  of  $9 - d$  points in general position. Let  $\mathbf{h}$  be the class of line on  $\mathbb{P}^2$ ,  $\mathbf{h}^* := \sigma^*\mathbf{h}$  and let  $\mathbf{e}_1, \dots, \mathbf{e}_{9-d}$  be the  $\sigma$ -exceptional divisors. Let  $B \subset S$  be a reduced curve such that  $B \sim 2\mathbf{h}^* - \mathbf{e}_1$  in the cases  $d = 3, 4$  and  $B \sim 2\mathbf{h}^* - \mathbf{e}_1 - \mathbf{e}_2$  in the case  $d = 5$ . Then  $p_a(B) = 0$ . Moreover,  $-K_S \cdot B = 5$  in the cases  $d = 3, 4$  and  $-K_S \cdot B = 4$  in the case  $d = 5$ .

Let  $f : Y \rightarrow Z$  be the blowup of  $B$  and let  $\tilde{S} \subset Y$  be the proper transform of  $S$ . Then the singularities of  $Y$  are at worst terminal Gorenstein (cf. Theorem 2.3( $\mathbf{e}_1$ )). Clearly,  $\tilde{S} \simeq S$ . Up to this identification we can write  $-K_Y|_{\tilde{S}} \sim -(d+1)K_S - B$ . It is easy to check that this divisor is nef and big. Moreover,  $-K_Y|_{\tilde{S}}$  has positive intersection number with all curves except for one, which is a line on  $S$ . Denote it by  $\Lambda$ . If  $-K_Y \cdot \Upsilon \leq 0$  for some curve  $\Upsilon$ , then  $\tilde{S} \cdot \Upsilon < 0$ . Hence  $\Upsilon \subset \tilde{S}$  and  $\Upsilon = \Lambda$ . This shows that the divisor  $-K_Y$  is nef and  $\Lambda$  is the only  $K$ -trivial curve on  $Y$ . By using (2.4.1) one can compute  $-K_Y^3 = 22$ . So,  $Y$  is a generalized Fano threefold of genus 12.

Thus the possibilities (I), (II) (III) of Theorem 1.2 occurs.

In the above construction the curve  $B$  can be reducible and then  $r(Y) - r(Z)$  equals to the number of component of  $B$ .

**4.3.1. Example.** Let  $Z = V'_5$  be the the quintic del Pezzo threefold with  $r(Z) = 4$  (see [Pr13b]) and let  $B$  be a combinatorial chain of four lines contained in a smooth hyperplane section  $S$ . Then the above construction give us a generalized Fano threefold  $Y$  with  $r(Y) = 8$ .

Another application of our construction 4.3 is the following.

**4.4. Theorem (cf. [Fu06]).** *There exists a Fano threefold  $X = X_{22} \subset \mathbb{P}^{13}$  of type (III) and a reducible hyperplane section  $A = A_1 \cup A_2$  such that  $X \setminus A \simeq \mathbb{A}^3$ .*

*Proof.* Recall that the normal bundle of a line on  $V_5$  has the form  $\mathcal{O}_{\mathbb{P}^1}(-a) \oplus \mathcal{O}_{\mathbb{P}^1}(a)$  with  $a = 0$  or  $1$  (see e.g. [IP99] or [FN89]). Let  $\Lambda \subset V_5$  be a line with  $\mathcal{N}_{\Lambda/V_5} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$  and let  $R$  be the ruled surface swept out by lines meeting  $\Lambda$ . Then  $R$  is a hyperplane section of  $V_5$  and its normalization  $R'$  is isomorphic to  $\mathbb{F}_3$  [FN89]. Moreover,  $V_5 \setminus R \simeq \mathbb{A}^3$  (loc. sit.). The map  $\nu : R' \rightarrow R$  is an isomorphism outside of  $\Lambda$  and  $\nu^{-1}(\Lambda)$  is the union of the negative section  $\Sigma \subset R' = \mathbb{F}_3$  and a fiber  $\Gamma_1$ . Further, the map  $\nu : R' \rightarrow R \subset \mathbb{P}^5$  is given by a codimension 1 subsystem of the linear system  $|\Sigma + 4\Gamma_1|$ . Let  $B' \subset R'$  be a general member of  $|\Sigma + 3\Gamma_1|$ . Then  $B := \nu(B')$  is a smooth rational curve of degree 4 and the line  $\Upsilon := \nu(\Gamma_0)$  is a 2-secant of  $B$  for some fiber  $\Gamma_0 \subset R'$ . Thus  $V_5 \cap \langle B \rangle = B \cup \Upsilon$ . Then we can apply Proposition 4.2 and get a Fano threefold  $X_{22}$ . By our construction  $X_{22} \setminus (\pi(f^{-1}(R))) \simeq V_5 \setminus R \simeq \mathbb{A}^3$ .  $\square$

## 5. FANO THREEFOLDS OF TYPE (IV)

In this section we investigate the case (IV) of Theorem 1.2. The relation between singular Fano threefolds  $X_{22}$  and certain rank-2 vector bundles was noticed by Mukai [Mu89, Remark 5]. This observation is based on the explicit description of stable rank-2 vector bundles on  $\mathbb{P}^2$  with even  $c_1$  [Ba77].

**5.1.** Let  $\mathcal{E}$  be a stable rank-2 vector bundle on  $\mathbb{P}^2$  with  $c_1(\mathcal{E}) = 0$ ,  $c_2(\mathcal{E}) = 4$  [Ba77]. Let  $Y := \mathbb{P}_{\mathbb{P}^2}(\mathcal{E})$  and let  $f : Y \rightarrow \mathbb{P}^2$  be the projection. Let  $F$  be the pull-back of a line  $l \subset \mathbb{P}^2$  and let  $M$  be the tautological divisor. Then

$$(5.1.1) \quad M^3 = -4, \quad M^2 \cdot F = 0, \quad M \cdot F^2 = 1, \quad F^3 = 0$$

(see (3.7.2)). For  $n \leq 2$ , by the Serre duality we have

$$H^2(\mathcal{E}(n)) = H^0(\mathcal{E}(n) \otimes \det \mathcal{E}(n)^\vee \otimes \omega_{\mathbb{P}^2})^\vee = H^0(\mathcal{E}(n-3))^\vee = 0.$$

Then, by the Riemann-Roch theorem

$$h^0(\mathcal{E}(n)) \geq \frac{1}{2} (c_1(\mathcal{E}(n))^2 - 2c_2(\mathcal{E}(n)) - K_{\mathbb{P}^2} \cdot c_1(\mathcal{E}(n))) + 2 = n^2 + 3n - 2.$$

In particular,

$$(5.1.2) \quad \begin{aligned} \dim |M + F| &= h^0(\mathcal{E}(1)) - 1 \geq 1, \\ \dim |M + 2F| &= h^0(\mathcal{E}(2)) - 1 \geq 7. \end{aligned}$$

Since  $\mathcal{E}$  is stable,  $H^0(\mathcal{E}) = 0$  (see e.g. [Fr98, Ch. 4, Prop. 14]) and so  $|M| = \emptyset$ . Hence any member  $S \in |M+F|$  is irreducible. Let  $S, S' \in |M+F|$  be general members and let  $\Gamma := S \cap S'$  (scheme-theoretically).

**5.1.3. Claim.** *In the above notation, we have  $-K_Y \cdot \Gamma = 0$ , in particular,  $Y$  is not a Fano threefold. Moreover, the image  $f(\Gamma)$  is a (possibly degenerate) conic.*

*Proof.* Using (5.1.1) we obtain  $(M + F)^2 \cdot K_Y = 0$  and  $(M + F)^2 \cdot F = 2$ .  $\square$

**5.2. Proposition.** *In the above notation the following conditions are equivalent:*

- (i)  $Y$  is a generalized Fano threefold,
- (ii)  $\mathcal{E}(2)$  is ample (or, equivalently,  $M + 2F$  is ample),
- (iii)  $\Gamma$  is reduced and irreducible,
- (iv)  $\Gamma$  is a smooth rational curve,
- (v) a general member  $S \in |M + F|$  is a smooth quartic del Pezzo surface.

*Proof.* (i)  $\implies$  (ii) Since  $\rho(Y) = 2$ , the Mori cone  $\overline{\text{NE}}(Y)$  is generated by two extremal rays  $\mathcal{R}_0$  and  $\mathcal{R}$ . We may assume that  $\mathcal{R}_0$  is generated by curves in the fibers of the projection  $f : Y \rightarrow \mathbb{P}^2$ . Clearly, all effective divisors are non-negative on  $\mathcal{R}_0$ . Since  $Y$  is a generalized Fano and  $-K_Y \cdot \Gamma = 0$ , the ray  $\mathcal{R}$  is generated by the class of  $\Gamma$ . Hence  $(M + 2F) \cdot \mathcal{R} > 0$ . By the Kleiman ampleness criterion,  $M + 2F$  is ample and so  $\mathcal{E}(2)$  is.

(ii)  $\implies$  (iii) By the assumption  $M + 2F$  is ample. Since  $(M + 2F) \cdot \Gamma = (M + 2F) \cdot (M + F)^2 = 1$ ,  $\Gamma$  is reduced and irreducible.

(iii)  $\implies$  (i) Suppose that  $-K_Y \cdot C \leq 0$  for some irreducible curve  $C$ . Then  $(M + F) \cdot C < 0$ . Hence  $C$  is contained in the base locus of  $|M + F|$ . Since  $\Gamma$  is reduced and irreducible,  $C = \Gamma$ . In this case,  $-K_Y \cdot C = 0$  and so the divisor  $-K_Y$  is nef. Since  $-K_Y^3 = 22$ , it is big. Moreover,  $\Gamma$  is the only curve contracted by  $|-K_Y|$ .

(iii)  $\implies$  (iv) follows by the adjunction formula, (iv)  $\implies$  (iii) is obvious, and the implication (iv) & (ii)  $\implies$  (v) follows by Bertini's theorem and adjunction. Let us show (v)  $\implies$  (iii). By adjunction on  $S$  we have

$$-K_S \cdot \Gamma = (M + 2H) \cdot (M + H)^2 = 1.$$

Since  $-K_S$  is ample,  $\Gamma$  is reduced and irreducible.

The assertion is proved.  $\square$

**5.3.** From now on we assume that  $\mathcal{E}$  satisfies the equivalent conditions of Proposition 5.2. Let  $\pi : Y \rightarrow X = X_{22} \subset \mathbb{P}^{13}$  be the anticanonical map. It follows from the proof of 5.2 that  $\Gamma$  is the only curve contracted by  $|-K_Y|$ . Therefore, the singular locus of  $X$  consists of one point  $P := \pi(\Gamma)$ . Let  $\Gamma^+ \subset Y^+$  be the flopped curve. Thus  $\pi^+(\Gamma^+) = P$ . Then one can restore the diagram (1.2.1) with the map  $f^+$  being a degree 5 del Pezzo fibration by Theorem 1.2. The right hand side of (1.2.1) is explicitly described by K. Takeuchi [Ta09].

### 5.3.1. Remarks.

- (i) By Claim 5.1.3 and (iv) the image  $\Omega := f(\Gamma)$  is a smooth conic.
- (ii) Since  $\mathcal{N}_{\Gamma/Y}$  contains a subbundle  $\mathcal{N}_{\Gamma/S} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)$ , we have  $\mathcal{N}_{\Gamma/Y} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ . In particular, the image  $\pi(\Gamma) \in X$  is an ordinary double point.

- (iii) By the Kawamata-Viehweg vanishing theorem  $H^i(\mathcal{O}_Y(M+F)) = 0$  for  $i > 0$ . Hence,  $\dim |M+F| = 1$ .
- (iv) Since the base locus of the pencil  $|M+F|$  is a smooth curve  $\Gamma$ , *any* member  $S \in |M+F|$  has at worst isolated singularities and  $\text{Sing}(S) \cap \Gamma = \emptyset$ . Moreover,  $\Gamma$  is a  $(-1)$ -curve on  $S$  and the singularities of  $S$  are at worst Du Val [HW81].

**5.4.** The map  $f_S : S \rightarrow \mathbb{P}^2$  is the blowup of the ideal sheaf of some zero-dimensional scheme  $\Xi \subset \Omega \subset \mathbb{P}^2$  of length 5. If  $S \in |M+F|$  is general, then  $S$  is a smooth del Pezzo surface of degree 4 and  $f_S : S \rightarrow \mathbb{P}^2$  is the blowup of five points in general position.

**5.4.1.** Denote  $T := f^{-1}(\Omega)$ . It is easy to show that  $T \simeq \mathbb{F}_6$  and  $\Gamma \subset T$  is the negative section. Since  $(M+F)^2 \cdot T = 4$ , any member of the restriction  $|M+F|_T$  has the form  $\Gamma + \Upsilon_1 + \dots + \Upsilon_5$ , where  $\Upsilon_i$  are fibers. In other words,  $|M+F|_T = \Gamma + f_T^* \mathfrak{g}$ , where  $\mathfrak{g} = \mathfrak{g}_5^1$  is a pencil (linear series) on  $\Omega$  of degree 5. Thus  $\mathcal{E}$  defines a  $\mathfrak{g}_5^1$  on  $\Omega \simeq \mathbb{P}^1$ .

**5.5.** In general, a stable rank-2 vector bundle  $\mathcal{E}$  on  $\mathbb{P}^2$  with  $c_1(\mathcal{E}) = 0$ ,  $c_2(\mathcal{E}) = 4$  suits to the following exact (non-unique) sequence

$$(5.5.1) \quad 0 \longrightarrow \mathcal{O}_{\mathbb{P}^2} \longrightarrow \mathcal{E}(1) \longrightarrow \mathcal{O}_{\mathbb{P}^2}(2) \otimes \mathcal{I}_{\Xi} \longrightarrow 0,$$

where  $\mathcal{I}_{\Xi}$  is the ideal sheaf of a zero-dimensional subscheme  $\Xi \subset \mathbb{P}^2$  of length 5 which is not contained in a line (see e.g. [Fr98, Ch. 4, Prop. 14]). Conversely, for  $\Xi = \{P_1, \dots, P_5\}$  the extension (5.5.1) corresponds to an element  $s$  of the 5-dimensional vector space

$$\text{Ext}^1(\mathcal{I}_{\Xi}(2), \mathcal{O}_{\mathbb{P}^2}) \simeq \mathcal{O}_{\Xi}(1) = \bigoplus \mathcal{O}_{P_i}(1).$$

and for general choice of  $s$  the sheaf  $\mathcal{E}(1)$  is locally free [Ba77, §5]. This shows that for a general choice of  $\mathcal{E}$  (in the sense of moduli) the vector bundle  $\mathcal{E}$  satisfies the conditions of 5.2.

**5.6.** We say that an irreducible curve  $\Lambda$  on a generalized Fano threefold  $U$  is a *line* if  $-K_U \cdot \Lambda = 1$ . Denote by  $\mathfrak{L}(U)$  the union of those components the Hilbert scheme  $\text{Hilb}(U)$  that contain classes of lines. Roughly speaking, we may regard  $\mathfrak{L}(U)$  as the scheme parametrizing lines on  $U$ .

**5.6.1. Lemma.** *Let  $\Lambda \subset Y$  be a line. Then  $F \cdot \Lambda = 1$  and  $\Gamma \cap \Lambda = \emptyset$ .*

*Proof.* Since  $M+2F$  is ample, we have  $(M+2F) \cdot \Lambda = 1$  and  $(M+F) \cdot \Lambda = 0$ . Hence  $\Lambda$  is disjoint from a general member of the pencil  $|M+F|$ .  $\square$

**5.6.2. Corollary.** *Any line on  $X$  is contained in a smooth locus and so  $\mathfrak{L}(Y) \simeq \mathfrak{L}(X) \simeq \mathfrak{L}(Y^+)$ . In particular,  $\mathfrak{L}(X)$  parametrizes lines in the fibers of  $f^+$ .*

**5.6.3. Corollary.** *For a general line  $\Lambda \subset X$  there are exactly three other lines meeting  $\Lambda$ .*

**5.7.** Now we give another description of varieties  $X_{22}$  of type (IV) which does not use the vector bundle techniques.

**5.7.1. Lemma.** *Let  $g : \hat{X} \rightarrow X$  be the blowup of the singular point  $P \in X$ . Then  $\hat{X}$  is a (smooth) Fano threefold with  $\rho(\hat{X}) = 3$  and  $(-K_{\hat{X}})^3 = 20$ .*

*Proof.* By Corollary 5.6.2 the variety  $X = X_{22} \subset \mathbb{P}^{13}$  contains no lines passing through  $P$  and so  $T_{P,X} \cap X = \{P\}$ .  $\square$

**5.7.2.** Clearly, the exceptional divisor  $R \subset \hat{X}$  is isomorphic to  $\mathbb{P}^1 \times \mathbb{P}^1$  and  $\hat{X}$  admits two contractions  $\tau : \hat{X} \rightarrow Y$  and  $\tau^+ : \hat{X} \rightarrow Y^+$  such that  $\tau(R) = \Gamma$  and  $\tau^+(R) = \Gamma^+$ . Thus we have the product map

$$h = (f \circ \tau) \times (f^+ \circ \tau^+) : \hat{X} \rightarrow \mathbb{P}^2 \times \mathbb{P}^1.$$

We get the following extension of the diagram (1.2.1):

$$(5.7.3) \quad \begin{array}{ccccc} & & \mathbb{P}^2 \times \mathbb{P}^1 & & \\ & \swarrow p_2 & \uparrow h & \searrow p_1 & \\ & Y & \hat{X} & Y^+ & \\ & \swarrow f & \xleftarrow{\tau} & \xrightarrow{\tau^+} & \searrow f^+ \\ & \mathbb{P}^2 & \downarrow \pi & X & \downarrow \pi^+ & \mathbb{P}^1 \end{array}$$

Since  $\hat{X}$  is a Fano threefold,  $h$  is an extremal contraction. Since  $b_3(\hat{X}) = 0$  and  $(-K_{\mathbb{P}^2 \times \mathbb{P}^1})^3 - (-K_{\hat{X}})^3 = 34$ , the contraction  $h$  is the blowup of a smooth rational curve  $C \subset \mathbb{P}^2 \times \mathbb{P}^1$  with  $-K_{\mathbb{P}^2 \times \mathbb{P}^1} \cdot C = 16$  (see (2.4.1)). Then it is easy to see that  $C$  is a curve of bidegree  $(2, 5)$  such that the restriction  $p_2|_C$  is an embedding (see [MM82, Table 3, n<sup>o</sup> 5<sup>o</sup>]). Clearly,  $p_2(C) = \Omega$  and  $h(R) = p_2^{-1}(\Omega)$ . The  $h$ -exceptional divisor is the proper transform of  $f^{-1}(\Omega)$ . Let  $G := p_2^{-1}(\Omega) = h(R)$ . Then  $G \simeq \mathbb{P}^1 \times \mathbb{P}^1$  and  $C \subset G$  is a curve of bidegree  $(5, 1)$ . The projection  $C \rightarrow \mathbb{P}^1$  to the first factor is defined by a linear system  $\mathfrak{g}_5^1$ . Conversely, a  $\mathfrak{g}_5^1$  on  $\mathbb{P}^1$  defines an embedding  $\mathbb{P}^1 \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1 = G$  as a divisor of bidegree  $(5, 1)$  and therefore it defines the whole diagram (5.7.3), where  $X$  is a Fano threefold with  $g(X) = 12$ .

**5.7.4. Corollary.** *The Fano threefolds  $X_{22}$  of type (IV) are parametrized by the set of  $\mathfrak{g}_5^1$  on  $\mathbb{P}^1$  modulo  $\text{Aut}(\mathbb{P}^1)$ .*

**5.7.5. Lemma.** *Let  $l \subset \mathbb{P}^2$  be a line. The following conditions are equivalent:*

- (i)  $l$  is a jumping line of  $\mathcal{E}$ ;
- (ii)  $l = f(\Lambda)$  for some line  $\Lambda \subset Y$ ;
- (iii)  $l$  cuts on  $\Omega$  an element  $\Xi \in \mathfrak{g}_5^1$ .

*Proof.* (i) $\Leftrightarrow$ (ii) Let  $F := f^{-1}(l)$  and let  $\Lambda$  be the minimal section of  $F \simeq \mathbb{F}_n$ . By the adjunction formula,  $K_F = 2(M + F)|_F$ . Hence  $n$  is even. Further,

$$-K_Y \cdot \Lambda = -K_F \cdot \Lambda + 1 = 3 - n.$$

In the general case we have  $-K_Y \cdot \Lambda = 3$ ,  $n = 0$ . Thus, the only possibility for the jumping case is the following:  $-K_Y \cdot \Lambda = 1$ ,  $n = 2$ .

The implications (ii) $\Leftrightarrow$ (iii) are easy and left to the reader.  $\square$

Note that any line  $\Lambda \subset Y$  is contained in a unique member  $S_\Lambda \in |M + F|$ .

**5.7.6. Lemma.** *Let  $\Lambda \subset Y$  be a line. Then  $\mathcal{N}_{\Lambda/Y} \simeq \mathcal{O}_{\mathbb{P}^1}(-a) \oplus \mathcal{O}_{\mathbb{P}^1}(a-1)$ , where  $a = 0$  or  $-1$ . If  $\Lambda$  is contained in the smooth locus of  $S_\Lambda$ , then  $a = 0$ .*

*Proof.* The first assertion is standard (see e.g. [IP99, Lemma 4.2.1]). For the second one, note that  $\Lambda$  is a  $(-1)$ -curve on  $S_\Lambda$  and so  $\mathcal{N}_{\Lambda/S_\Lambda} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)$  is a subbundle of  $\mathcal{N}_{\Lambda/Y}$ .  $\square$

**5.7.7. Corollary.**

- (i) *The scheme  $\mathfrak{L}(Y)$  is purely one-dimensional and generically reduced.*
- (ii) *The natural map  $\mathfrak{L}(Y) \rightarrow (\mathbb{P}^2)^\vee$ ,  $\Lambda \mapsto f(\Lambda)$  is bijective.*
- (iii) *The scheme  $\mathfrak{L}(Y)$  is connected.*

*Proof.* For the first (see e.g. [IP99, Proposition 4.2.2]). The rest is easy.  $\square$

**5.7.8.** Recall that a *Lüroth quartic* is a plane quartic curve containing the 10 vertices of a complete 5-side. All Lüroth quartics form an irreducible subvariety of codimension one in the space of all quartics.

**5.7.9. Corollary** [Ba77]. *Let  $J(\mathcal{E}) \subset \mathbb{P}^{2^\vee}$  be the curve of jumping lines. Then  $J(\mathcal{E})$  is a Lüroth quartic curve.*

*Proof.* Note that  $J(\mathcal{E})$  is the image of the map from Corollary 5.7.7(ii). Take a general member  $S \in |M + F|$ . Then  $S$  is a smooth del Pezzo surface and the projection  $f_S : S \rightarrow \mathbb{P}^2$  contracts 5 exceptional curves  $\Upsilon_1, \dots, \Upsilon_5$  (fibers of  $f$ ). Let  $q_1, \dots, q_5 \in \Omega \subset \mathbb{P}^2$  be images of these curves and let  $l_{i,j}$  be the line joining  $q_i$  with  $q_j$  for  $i \neq j$ . The proper transform  $\Lambda_{i,j} \subset S$  of  $l_{i,j}$  is a  $(-1)$ -curve meeting  $\Upsilon_i$  and  $\Upsilon_j$ . Since  $\Lambda_{i,j} \cap \Gamma = \emptyset$  and  $F \cdot \Lambda_{i,j} = 1$ , we have  $-K_Y \cdot \Lambda_{i,j} = 1$ , i.e.  $\Lambda_{i,j}$  is a line on  $Y$ . The lines  $l_{i,j}$  can be regarded as vertices of a complete 5-side on the dual plane  $\mathbb{P}^{2^\vee}$  and the curve of jumping lines  $J(\mathcal{E}) \subset \mathbb{P}^{2^\vee}$  passes through all these vertices  $l_{i,j}$  by Lemma 5.7.5.

Now take the pencil  $l_t$  of lines on  $\mathbb{P}^2$  passing through  $q_1$ . Since  $\text{Bs } |M + F| = \Gamma$ , the surface  $S$  is the only member of  $|M + F|$  containing the fiber  $f^{-1}(q_1)$ . Then  $l_t$  is a jumping line if and only if there is a line  $\Lambda \subset Y$  such that  $f(\Lambda) = l_t$ . This  $\Lambda$  is contained in some member  $S' \in |M + F|$ . Since  $S' \cap f^{-1}(q_1)$  contains two distinct points  $\Gamma \cap f^{-1}(q_1)$  and  $\Lambda \cap f^{-1}(q_1)$ , the surface  $S'$  must contain the fiber  $f^{-1}(q_1)$ . Thus,  $S' = S$  and so  $S \supset \Lambda$ . This

shows that  $\Lambda = \Lambda_{1,j}$  for  $j = 1, \dots, 4$ . Hence, the pencil  $l_t$  contains exactly four jumping lines, i.e.  $\deg J(\mathcal{E}) = 4$ .  $\square$

## 6. MMP ON GENERALIZED FANO THREEFOLDS

Our proof of Theorem 1.3 essentially uses the following result.

**6.1. Theorem** [Pr15]. *Let  $X$  be a  $G$ -Fano threefold of the main series with  $g(X) \geq 6$ . Then  $X$  contains no any planes.*

It turns out that the absence of planes is important for application of the minimal model program (MMP) in our situation. We explicitly describe steps of the MMP on generalized Fano threefolds. This techniques was developed [Pr05] and [Kal11].

**6.2. Set-up.** Let  $X$  be a Fano threefold (with at worst terminal Gorenstein singularities). Assume that  $X$  does not contain planes and  $r(X) > 1$ . Let  $\pi : \tilde{X} \rightarrow X$  be a factorialization (see Proposition 2.2). Then  $K_{\tilde{X}} = \pi^*K_X$  and so  $\tilde{X}$  is a generalized Fano threefold with  $\rho(\tilde{X}) = r(X) > 1$ .

The following important lemma shows that the class of generalized  $\mathbb{Q}$ -factorial Fano threefold containing no planes is closed under the MMP.

**6.2.1. Lemma** [Pr05], [Kal11]. *Let  $V$  be a generalized  $\mathbb{Q}$ -factorial Fano threefold. Assume that  $V$  does not contain planes. Let  $\varphi : V \dashrightarrow W$  be either a divisorial Mori contraction or a flop. Then  $W$  is also a generalized  $\mathbb{Q}$ -factorial Fano threefold and  $W$  does not contain planes.*

*Proof.* The assertion is obvious if  $f$  is a flop, so we assume that  $f$  is a divisorial Mori contraction. Since  $V$  contains no planes, the contraction  $f$  is not of type  $(\mathbf{e}_5)$ . Then by Theorem 2.3 the variety  $W$  has only factorial terminal singularities. Moreover,  $-K_W$  is nef and big by [Pr05, Prop. 4.5] (note that  $f$  cannot be a “bad” contraction of type  $(2, 0)_o^-$ ,  $(2, 1)_{o0}^-$ ,  $(2, 1)_{o1}^-$  because the exceptional divisor is not a plane). Finally,  $W$  contains no planes by Lemma 2.5.  $\square$

**6.3.** Let  $\Theta$  be any divisor on  $\tilde{X}$ . According to [PS09, 2.6-2.8] we can run the  $\Theta$ -MMP:

$$\varphi : \tilde{X} = \tilde{X}_0 \dashrightarrow \cdots \dashrightarrow \tilde{X}_N.$$

By Lemma 6.2.1 on each step  $\varphi_k : \tilde{X}_k \dashrightarrow \tilde{X}_{k+1}$  the following assertions hold.

- (i)  $\tilde{X}_k$  has only factorial terminal singularities;
- (ii)  $\tilde{X}_k$  is a generalized Fano threefold;
- (iii)  $\tilde{X}_k$  does not contain planes;
- (iv)  $\varphi_k$  is either a flop or an extremal Mori contraction of type  $(\mathbf{e}_1)$ - $(\mathbf{e}_4)$ .

Note that in the “classical” case  $\Theta = K_{\tilde{X}}$  all the steps  $\varphi_k$  must be divisorial contractions of type  $(\mathbf{e}_1)$ - $(\mathbf{e}_4)$ , in particular,  $\varphi$  is a morphism.

**6.3.1.** We get the following diagram:

$$(6.3.2) \quad \begin{array}{ccccc} \varphi : \tilde{X} = \tilde{X}_0 & \xrightarrow{-\varphi_1} & \tilde{X}_1 & \xrightarrow{-\varphi_2} & \dots & \xrightarrow{-\varphi_N} & \tilde{X}_N \\ \downarrow \pi & & \downarrow \pi_1 & & & & \downarrow \pi_N \\ X = X_0 & & X_1 & & & & X_N \end{array}$$

where  $X_k$  is the pluri-anticanonical image of  $\tilde{X}_k$ . Denote by  $\Theta^{(k)}$  the proper transform of  $\Theta$  on  $\tilde{X}_k$ .<sup>2</sup> At the end we get one of the following possibilities:

- There exists a  $\Theta^{(N)}$ -negative Mori contraction  $v : \tilde{X}_N \rightarrow Z$  to a variety  $Z$  of dimension  $< 3$ . In particular,  $\rho(\tilde{X}_N/Z) = 1$  and  $-K_{\tilde{X}_N}$  is  $v$ -ample. If  $\dim Z > 0$ , then  $v$  is a contraction of type **(c)** or **(d)**.
- The divisor  $\Theta^{(N)}$  is nef.

**6.3.3. Definition.** A *weak del Pezzo surface* is a projective surface with at worst Du Val singularities such that the anticanonical divisor  $-K_X$  is nef and big.

**6.3.4. Remark.** It is easy to see that the class of weak del Pezzo surfaces is closed under birational contractions.

**6.4. Lemma.** *Let  $X$  be a generalized Fano threefold and let  $v : X \rightarrow Y$  be a contraction to a surface. Assume that  $X$  does not contain planes. Then the following assertions hold:*

- (i)  $Y$  is a weak del Pezzo surface.
- (ii) If  $X$  is  $\mathbb{Q}$ -factorial and  $v$  is an extremal Mori contraction, then  $Y$  is a smooth del Pezzo surface.
- (iii) If under the assumptions of (ii) the surface  $Y$  contains a  $(-1)$ -curve  $\Gamma$ , then there exists the following diagram:

$$\begin{array}{ccc} X & \xrightarrow{\chi} & X^+ & \xrightarrow{\varphi} & X' \\ \downarrow v & & & & \downarrow v' \\ Y & \longrightarrow & & & Z \end{array}$$

where  $Z$  is a smooth surface,  $Y \rightarrow Z$  is the contraction of  $\Gamma$ ,  $\chi$  is either an isomorphism or a flop,  $\varphi$  is an extremal divisorial contraction, and  $v'$  is an extremal contraction of type **(c)**.

<sup>2</sup> [Kal11] asserts that  $\rho(X_i) = 1$  for all  $i$ . This is wrong in general. Indeed, let  $X$  be a quintic del Pezzo threefold with  $r(X) = 4$  [Pr13b, §7]. For a suitable choice of factorization, we have  $N = 3$ ,  $\tilde{X}_3 = X_3 = \mathbb{P}^3$ , and  $\varphi_i$  are contractions of type **(e<sub>2</sub>)**. Then  $X_1$  is a sextic del Pezzo threefold with  $r(X) = 3$  and  $\rho(X_1) = 2$ .



*Proof.* (i) As in (6.3.2), run the  $K$ -MMP on  $X$  over  $Y$ :

$$(6.4.1) \quad \begin{array}{ccccc} X & \xleftarrow{\pi} & \tilde{X} & \xrightarrow{\varphi} & \hat{X} \\ \downarrow v & & & & \downarrow \tau \\ Y & \xleftarrow{\hspace{4cm}} & & & \hat{Y} \end{array}$$

Here  $\pi$  is a suitable factorization,  $\varphi$  is a composition of divisorial contractions, and  $\tau$  is an extremal contraction of type (c). By Lemma 6.2.1  $\hat{X}$  is a generalized Fano threefold. Then by [Pr05, Proposition 5.2 (i)]  $\hat{Y}$  is a *smooth* weak del Pezzo surface. This implies that  $Y$  is a weak del Pezzo surface (see Remark 6.3.4).

(ii) and (iii) Again by Theorem 2.3(c) and [Pr05, Proposition 5.2 (i)]  $Y$  is a smooth weak del Pezzo surface. Let  $\Gamma \subset Y$  be a curve with  $\Gamma^2 < 0$  and let  $\tau : Y \rightarrow U$  be its contraction. Clearly,  $\rho(X/U) = 2$  and so the relative Mori cone  $\overline{\text{NE}}(X/U)$  has two extremal rays, say  $\mathcal{R}_1$  and  $\mathcal{R}_2$ . Put  $F := v^*(\Gamma)$ . We may assume that  $\mathcal{R}_1$  is generated by the fibers of  $v$  and so  $F \cdot \mathcal{R}_1 = 0$ . Since  $\Gamma^2 < 0$ , for any curve  $\Gamma' \subset X$  dominating  $\Gamma$  we have  $F \cdot \Gamma' < 0$ . Hence the divisor  $F := v^*(\Gamma)$  is not nef and  $F \cdot \mathcal{R}_2 < 0$ . Thus we can run the  $F$ -MMP over  $U$  starting from  $\mathcal{R}_2$ :

$$\begin{array}{ccccc} X & \xrightarrow{\chi} & X^+ & \xrightarrow{\varphi} & X' \\ \downarrow v & & & & \downarrow v' \\ Y & \xrightarrow{\tau} & U & \xleftarrow{\tau'} & Y' \end{array}$$

Here  $\chi$  is either an isomorphism or a flop in  $\mathcal{R}_2$  and  $\varphi$  is either an isomorphism or a divisorial contraction. By Lemma 6.2.1  $X'$  is a generalized Fano threefold. Thus  $v'$  is an extremal contraction to a smooth surface  $Y'$  (see Theorem 2.3(c)). Note that  $\rho(X^+) = \rho(X) = \rho(U) + 2$ .

Assume that  $U$  is singular. Then  $\rho(Y') > \rho(U)$  and so  $\rho(X^+) = \rho(X') = \rho(Y') + 1$ , i.e.  $\varphi$  is an isomorphism. Moreover,  $Y'$  is a minimal resolution of  $U$  and so  $Y \simeq Y'$ . Since both  $-K_X$  and  $-K_{X'}$  are ample over  $Y = Y'$ , the map  $\varphi \circ \chi$  must be an isomorphism, a contradiction.

Thus  $\Gamma$  is a  $(-1)$ -curve and  $U$  is smooth. In particular, this implies that  $Y$  is a del Pezzo surface. If  $\varphi$  is an isomorphism, then  $\tau'$  is the blowup of the (smooth) point  $\tau(\Gamma) \in U$  and  $Y' \simeq Y$ . We get a contradiction as above. Hence  $\varphi$  is a (single) divisorial contraction.  $\square$

**6.5. Lemma.** *One can run the MMP (6.3.2) so that  $\rho(Z) \leq 2$ .*

*Proof.* Assume that  $\rho(Z) \geq 3$ . Then  $Z$  must be a smooth rational surface (see Lemma 6.4(ii)) and it contains a  $(-1)$ -curve. Then by Lemma 6.4(iii) we can run the MMP further until we get a surface  $Z_M$  with  $\rho(Z_M) \leq 2$ .  $\square$

**6.6. Lemma.** *Let  $X$  be a generalized Fano threefold and let  $f : X \rightarrow Y$  be a birational contraction. Assume that  $X$  does not contain planes. Then  $Y$  is also a generalized Fano threefold.*

*Proof.* As in (6.3.2), run the  $K$ -MMP on  $X$  over  $Y$ : we start with a suitable factorization  $\pi : \tilde{X} \rightarrow X$  and after a number of divisorial contraction we obtain a minimal model  $\tilde{Y}$  over  $Y$ . By Lemma 6.2.1  $\tilde{Y}$  is a generalized Fano threefold. Hence the morphism  $\tilde{Y} \rightarrow Y$  is small and crepant. So,  $Y$  is a generalized Fano threefold as well.  $\square$

**6.6.1. Corollary.** *In the notation of Lemma 6.6 we have*

$$r(X) - r(Y) \leq \frac{1}{2}K_X^2 \cdot E,$$

where  $E$  is the  $f$ -exceptional divisor.

*Proof.* Since  $X$  contains no planes,  $E$  has at most  $(K_X^2 \cdot E)/2$  components.  $\square$

## 7. DEFORMATIONS OF FANO THREEFOLDS

**7.1. Theorem** [Na97]. *Let  $X$  be a generalized Fano threefold. Then  $X$  is smoothable, that is, there exists a flat family  $\mathfrak{X} \rightarrow (\mathfrak{U} \ni 0)$  over a small disc  $(\mathfrak{U} \ni 0) \subset \mathbb{C}$  such that  $\mathfrak{X}_0 \simeq X$  and a general fiber  $\mathfrak{X}_u$ ,  $u \in \mathfrak{U} \setminus \{0\}$ , is a smooth generalized Fano threefold.*

**7.2. Theorem** [JR11]. *Let  $X$  be a generalized Fano threefold and let  $\mathfrak{X} \rightarrow (\mathfrak{U} \ni 0)$  be its smoothing as above. Then  $\mathfrak{X}$  is normal and has at worst isolated terminal factorial singularities. Moreover, there are natural identifications*

$$\text{Pic}(X) = \text{Pic}(\mathfrak{X}_u) = \text{Pic}(\mathfrak{X})$$

so that  $K_{\mathfrak{X}_s} = K_X$ .

**7.2.1. Remark.** If in the above notation for  $u \neq 0$  the fiber  $\mathfrak{X}_u$  contains a plane, then the same holds for the central fiber.

**7.2.2. Corollary.** *Let  $X$  be a Fano threefold with terminal Gorenstein singularities such that  $\rho(X) = 1$  and  $-K_X^3 > 22$ . Then  $\iota(X) > 1$ . If furthermore  $-K_X^3 \geq 40$ , then  $X$  is either  $\mathbb{P}^3$  or a quadric in  $\mathbb{P}^4$ , or a quintic del Pezzo threefold in  $\mathbb{P}^6$ . If  $-K_X^3 \geq 40$  and  $X$  is  $\mathbb{Q}$ -factorial, then  $X$  is in fact smooth.*

*Proof.* Let  $\mathfrak{X} \rightarrow \mathfrak{U} \ni 0$  be a smoothing as above. A general fiber  $\mathfrak{X}_u$  is a smooth Fano threefold with  $\rho(\mathfrak{X}_u) = \rho(X) = 1$  and  $-K_{\mathfrak{X}_u}^3 = -K_X^3 > 22$ . By the classification of smooth Fano threefolds with  $\rho = 1$  we have  $\iota(\mathfrak{X}_u) > 1$  (see [IP99]) and by Theorem 7.2  $\iota(X) = \iota(\mathfrak{X}_u) > 1$ . The case  $\iota(X) \geq 3$  is well-known (see e.g. [IP99, §3.1]). For the case  $\iota(X) = 2$  we refer to [Pr13b, Corollary 8.7].  $\square$

**7.3. Notation.** Below in this section we assume that  $X$  is a Fano threefold with terminal Gorenstein singularities and  $X$  does not contain planes. Let  $\mathfrak{X} \rightarrow (\mathfrak{U} \ni 0)$  be its 1-parameter smoothing as above.

**7.4. Proposition** (cf. [Pr13c, Proposition 6.3]). *Notation as in 7.3. Let  $\mathfrak{L}$  be a divisor on  $\mathfrak{X}$ . Then  $\mathfrak{L}$  is nef (resp. ample) if and only if the restriction  $\mathfrak{L}|_{\mathfrak{X}_u}$  is nef (resp. ample) for some  $u \in \mathfrak{U}$ .*

**7.5. Proposition** (cf. [Pr13c, Corollary 6.4]). *Notation as in 7.3. Let  $f_u : \mathfrak{X}_u \rightarrow \mathfrak{Y}_u$  be an extremal contraction. Then there exists an extremal contraction  $f : \mathfrak{X} \rightarrow \mathfrak{Y}$  over  $\mathfrak{U}$  such that the restriction  $f|_{\mathfrak{X}_u}$  coincides with  $f_u$ , where the variety  $\mathfrak{Y}$  is  $\mathbb{Q}$ -factorial. Let  $f : X = \mathfrak{X}_0 \rightarrow Y = \mathfrak{Y}_0$  be the restriction of  $f$  to  $X = \mathfrak{X}_0$ . Then  $Y$  is normal,  $\mathbb{Q}$ -Gorenstein and  $f$  has connected fibers. Moreover, the following assertions hold:*

- (i) *If  $\dim \mathfrak{Y} = 3$ , then  $Y$  is a weak del Pezzo surface.*
- (ii) *If  $f$  is birational, then  $Y$  is a generalized Fano threefold. If moreover,  $\mathfrak{Y}_u$  is smooth, then there are natural identifications  $\text{Pic}(Y) = \text{Pic}(\mathfrak{Y}_u)$  so that  $K_{\mathfrak{Y}_s} = K_Y$ , in particular,  $\rho(Y) = \rho(\mathfrak{Y}_s)$  and  $\iota(Y) = \iota(\mathfrak{Y}_s)$ .*

*Proof.* The existence of  $f$  immediately follows from Proposition 7.4 (see [Pr13c, Corollary 6.4]). By our assumptions  $\mathfrak{X}$  contains no planes. Hence by [Kac98, Theorem 1.1] there are no flipping contractions on  $\mathfrak{X}$ . This implies that the variety  $\mathfrak{Y}$  is  $\mathbb{Q}$ -factorial (see [KMM, Lemma 5-1-5, Proposition 5.1.6]) and so  $Y = \mathfrak{Y}_0$  is  $\mathbb{Q}$ -Gorenstein. By the projection formula and Kawamata-Viehweg vanishing theorem

$$R^1 f_* \mathcal{O}_{\mathfrak{X}}(-X) = R^1 f_* \mathcal{O}_{\mathfrak{X}} \otimes f^* \mathcal{O}_{\mathfrak{Y}}(-Y) = R^1 f_* \mathcal{O}_{\mathfrak{X}} \otimes \mathcal{O}_{\mathfrak{Y}}(-Y) = 0.$$

Applying  $f_*$  to the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathfrak{X}}(-X) \longrightarrow \mathcal{O}_{\mathfrak{X}} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

we obtain

$$0 \longrightarrow \mathcal{O}_{\mathfrak{Y}}(-Y) \longrightarrow \mathcal{O}_{\mathfrak{Y}} \longrightarrow f_* \mathcal{O}_X \longrightarrow 0.$$

Therefore,  $f_* \mathcal{O}_X = \mathcal{O}_Y$ ,  $Y$  is normal, and  $f$  has connected fibers, i.e.  $f$  is a contraction. Then apply Lemmas 6.4, 6.6 and Theorem 7.2.  $\square$

**7.6. Lemma.** *Notation as in 7.3. Let  $f : \mathfrak{X} \rightarrow \mathfrak{Y}$  be an extremal contraction of relative dimension one. Assume that a general fiber  $\mathfrak{X}_u$  has a contraction  $f'_u : \mathfrak{X}_u \rightarrow \mathfrak{Y}'_u \simeq \mathbb{P}^1$  which is not passed through  $\mathfrak{Y}_u$ . Then  $f$  has no two-dimensional fibers.*

*Proof.* Assume that  $f_0 : X = \mathfrak{X}_0 \rightarrow \mathfrak{Y}_0$  has a two-dimensional fiber  $F \subset X$ . By Proposition 7.4 there exists a contraction  $f' : \mathfrak{X} \rightarrow \mathfrak{Y}'$  that extends  $f'_u$ . Clearly,  $\mathfrak{Y}' \rightarrow \mathfrak{U}$  is a  $\mathbb{P}^1$ -bundle. Take ample divisors  $\mathfrak{H}$  and  $\mathfrak{H}'$  on  $\mathfrak{Y}$  and  $\mathfrak{Y}'$ , respectively. Let  $\mathfrak{L} := f^* \mathfrak{H}$  and  $\mathfrak{L}' := f'^* \mathfrak{H}'$ . Then the divisor  $\mathfrak{L}_u + \mathfrak{L}'_u$  is ample by our assumption. By Proposition 7.4 the same holds for  $\mathfrak{L} + \mathfrak{L}'$ . On the other hand,  $\mathfrak{L}|_F = 0$  and  $\dim f'(F) = 1$ . Hence, the linear system  $|\mathfrak{L} + \mathfrak{L}'|$  contracts  $F$ , a contradiction.  $\square$

**7.7. Lemma.** *Notation as in 7.3. Let  $f : \mathfrak{X} \rightarrow \mathfrak{Y}$  be a contraction (over  $\mathfrak{U} \ni 0$ ) all whose fibers are one-dimensional. Then  $\mathfrak{Y}$  is smooth,  $f$  is a conic bundle, and its discriminant locus  $\mathfrak{C} \subset \mathfrak{Y}$  is either empty or flat of relative dimension one over  $\mathfrak{U}$ . Moreover, the fiber  $\mathfrak{Y}_0$  is a smooth weak del Pezzo surface of degree  $K_{\mathfrak{Y}_0}^2 = K_{\mathfrak{Y}_u}^2$  and the discriminant locus of  $f_0$  coincides with  $\mathfrak{C}_0 \subset \mathfrak{Y}_0$ . If  $\mathfrak{C} \subset \mathfrak{Y}$  is empty, then  $X$  is smooth and  $f_0$  is a  $\mathbb{P}^1$ -bundle.*

*Proof.* Similar to [Cu88, Theorem 7] one can show that  $\mathfrak{Y}$  is smooth and  $f$  is a conic bundle. By Lemma 6.4  $\mathfrak{Y}_0$  is a smooth weak del Pezzo surface. The rest is obvious.  $\square$

**7.8. Lemma.** *Notation as in 7.3. Assume that a general fiber  $\mathfrak{X}_u$  is isomorphic to a product  $\mathfrak{X}_u \simeq Y_u \times \mathbb{P}^1$ . Then the special fiber  $X$  is smooth.*

*Proof.* Consider the extremal contraction  $f : \mathfrak{X} \rightarrow \mathfrak{Y}$  that corresponds to the projection  $f_u : \mathfrak{X}_u \simeq Y_u \times \mathbb{P}^1 \rightarrow Y_u$ . By Lemma 7.6 the contraction  $f : X \rightarrow Y$  has no two-dimensional fibers and by Lemma 7.7 the variety  $X$  is smooth.  $\square$

**7.9. Lemma.** *Notation as in 7.3. Assume that a general fiber  $\mathfrak{X}_u$  is isomorphic to a divisor of bidegree  $(1, 2)$  in  $\mathbb{P}^2 \times \mathbb{P}^2$ . Then the special fiber  $X$  has the same form and  $r(X) \leq 5$ .*

*Proof.* Two projections  $p_{u,i} : \mathfrak{X}_u \rightarrow \mathbb{P}^2$  induce two contractions  $p_i : X \rightarrow \mathbb{P}^2$ . The divisors  $H_i := p_i^* \mathcal{O}_{\mathbb{P}^2}(1)$  generate the Picard group  $\text{Pic}(X)$ . Moreover,

$$-K_X = 2H_1 + H_2, \quad H_1^3 = H_2^3 = 0, \quad H_1^2 \cdot H_2 = 2, \quad H_1 \cdot H_2^2 = 1.$$

Since  $\rho(X) = 2$ , the product map  $\pi = p_1 \times p_2 : X \rightarrow \mathbb{P}^2 \times \mathbb{P}^2$  is finite. It is easy to see that  $\pi$  is birational and the image  $Y := \pi(X)$  is a divisor of bidegree  $(1, 2)$ . Then by the adjunction formula  $K_X = \pi^* K_Y$ . Hence the map  $\pi$  is an isomorphism onto its image. If the projection  $p_2 : X \rightarrow \mathbb{P}^2$  has a two-dimensional fiber, say  $F$ , then the anticanonical image of  $F$  must be a plane. This contradicts our assumptions. Hence  $p_2$  is an equidimensional conic bundle with discriminant curve  $\Delta_f \subset \mathbb{P}^2$  of degree 3. Then  $\Delta_f$  has at most 3 components and so  $r(X) \leq r(\mathbb{P}^2) + 1 + 3 = 5$ .  $\square$

**7.10. Lemma.** *Let  $X$  be a Fano threefold. Assume that  $X$  does not contain planes, has no any birational contractions, and  $-K_X^3 > 30$ . Then either  $\iota(X) \geq 2$  or  $X \simeq \mathbb{P}^1 \times \mathbb{P}^2$ .*

*Proof.* Assume that  $\iota(X) = 1$ . Let  $\mathfrak{X} \rightarrow \mathfrak{U} \ni 0$  be a smoothing as in Theorem 7.1. Then for  $0 \neq u \in \mathfrak{U}$ , the fiber  $\mathfrak{X}_u$  is a smooth Fano threefold with  $\rho(\mathfrak{X}_u) = \rho(X)$ ,  $\iota(\mathfrak{X}_u) = 1$ , and it has no any birational contractions. Then by [MM82] we have only one possibility:  $\mathfrak{X}_u \simeq \mathbb{P}^1 \times \mathbb{P}^2$ . By Lemma 7.8  $X \simeq \mathbb{P}^1 \times \mathbb{P}^2$ .  $\square$

**7.11. Proposition.** *Let  $X$  be a Fano threefold with at worst terminal Gorenstein singularities and  $-K_X^3 = 24$ . Assume that  $X$  contains no planes. Then  $r(X) \leq 9$ .*

*Proof.* If  $\rho(X) = 1$ , then  $\iota(X) > 1$  and  $X$  is a del Pezzo threefold (see Corollary 7.2.2). In this case,  $r(X) \leq 6$  by [Pr13b, Corollary 3.13]. Thus we assume that  $\rho(X) > 1$ . Consider a 1-parameter smoothing  $\mathfrak{X} \rightarrow (\mathfrak{U} \ni 0)$  as in Theorem 7.1. A general fiber  $\mathfrak{X}_u$  is a smooth Fano threefold with  $\rho(\mathfrak{X}_u) = \rho(X) > 1$  and  $-K_{\mathfrak{X}_u}^3 = -K_X^3 = 24$ . Now we apply the classification [MM82] to  $\mathfrak{X}_u$ .

If  $\rho(\mathfrak{X}_u) > 4$ , then  $\mathfrak{X}_u$  is a product  $\mathfrak{X}_u \simeq Y_u \times \mathbb{P}^1$ , where  $Y_u$  is a del Pezzo surface of degree 4 (see [MM82, Table 5]). Then by Lemma 7.8 the variety  $X$  has the same form and so  $r(X) = 7$ . Thus we may assume that  $2 \leq \rho(\mathfrak{X}_u) \leq 4$ .

If  $\mathfrak{X}_u$  is not isomorphic to a blowup of a smooth Fano threefold, then  $\mathfrak{X}_u$  is a double cover of  $\mathbb{P}^1 \times \mathbb{P}^2$  whose branch locus is a divisor of bidegree  $(2, 2)$  ([MM82, Table 2, n° 18°]). In this case  $\mathfrak{X}_u$  has two extremal contractions: a conic bundle  $\mathfrak{f}_u : \mathfrak{X}_u \rightarrow \mathbb{P}^2$  with discriminant curve  $D_u \subset \mathbb{P}^2$  of degree 4 and a del Pezzo fibration  $\mathfrak{f}'_u : \mathfrak{X}_u \rightarrow \mathbb{P}^1$ . Consider the extremal contraction  $\mathfrak{f} : \mathfrak{X} \rightarrow \mathfrak{Y}$  that extends  $\mathfrak{f}_u$ . Then  $Y = \mathfrak{Y}_0$  is a del Pezzo surface with at worst Du Val singularities and  $\rho(Y) = 1$ . Since  $K_Y^2 = K_{\mathfrak{Y}_0}^2 = 9$ , we have  $Y \simeq \mathbb{P}^2$ . By Lemma 7.6 the contraction  $f : X \rightarrow Y$  has no two-dimensional fibers and  $\mathfrak{f}$  is a conic bundle. Let  $\mathfrak{D} \subset \mathfrak{Y}$  be the discriminant curve of  $\mathfrak{X}$ . Thus  $D_u = \mathfrak{Y}_u \cap \mathfrak{D}$ . Moreover,  $D_0 = \mathfrak{Y}_0 \cap \mathfrak{D}$  is the discriminant curve of  $\mathfrak{X}_0 \rightarrow \mathfrak{Y}_0 \simeq \mathbb{P}^2$  with  $\deg D_0 = \deg D_u = 4$ . Then as in the proof of Lemma 7.9 we have  $r(X) \leq r(Y) + 1 + 4 \leq 6$ .

In the remaining cases  $\mathfrak{X}_u$  is isomorphic to a blowup of a smooth Fano threefold  $\mathfrak{Y}_u$  along a smooth curve  $B_u$ . Consider the extremal contraction  $\mathfrak{f} : \mathfrak{X} \rightarrow \mathfrak{Y}$  that extends  $\mathfrak{X}_u \rightarrow \mathfrak{Y}_u$ . Let  $Y := \mathfrak{Y}_0$ . By Proposition 7.5  $Y$  is a generalized Fano threefold with  $\rho(Y) = \rho(\mathfrak{Y}_u)$ ,  $-K_Y^3 = -K_{\mathfrak{Y}_u}^3$ , and  $\iota(Y) = \iota(\mathfrak{Y}_u)$ . Let  $\mathfrak{E} \subset \mathfrak{X}$  be the exceptional divisor, let  $E_u := \mathfrak{E} \cap \mathfrak{X}_u$ , and let  $E := \mathfrak{E}_0$ . Obviously, the morphism  $\mathfrak{E} \rightarrow \mathfrak{U}$  is flat and so  $K_X^2 \cdot E = K_{\mathfrak{X}_u}^2 \cdot E_u$ . By Corollary 6.6.1,

$$(7.11.1) \quad r(X) \leq r(Y) + \frac{1}{2} K_{\mathfrak{X}_u}^2 \cdot E_u.$$

By [MM82] there are the following possibilities:

|    | $\rho(\mathfrak{X}_u)$ | $\mathfrak{Y}_u$                                       | $r(Y)$   | $p_a(B_u)$ | $K_{\mathfrak{X}_u}^2 \cdot E_u$ |
|----|------------------------|--|----------|------------|----------------------------------|
| 1° | 2                      | a quadric $Q \subset \mathbb{P}^4$                     | $\leq 2$ | 1          | 15                               |
| 2° | 3                      | a del Pezzo 3-fold $V_6 \subset \mathbb{P}^7$          | $\leq 3$ | 1          | 12                               |
| 3° | 3                      | $Y_{2,1} \subset \mathbb{P}^2 \times \mathbb{P}^2$     | $\leq 5$ | 0          | 4                                |
| 4° | 4                      | $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ | 3        | 1          | 12                               |

where  $Y_{2,1} \subset \mathbb{P}^2 \times \mathbb{P}^2$  is a divisor of bidegree  $(2, 1)$ . In the above table the value of  $K_{\tilde{x}_u}^2 \cdot E_u$  is computed by using (2.4.2). To estimate  $r(Y)$ , we note that  $Y$  is a (possibly singular) quadric in the case  $1^\circ$ ,  $r(Y) \leq 3$  in cases  $2^\circ$  and  $4^\circ$  by [Pr13b, Corollary 3.13], and  $r(Y) \leq 5$  in the case  $3^\circ$  by Lemma 7.9. Combining the above table with (7.11.1) we get the desired estimate  $r(X) \leq 9$ .  $\square$

## 8. PROOF OF THEOREM 1.3

To show that in the notation of Theorem 1.3 the case  $r(X) > 2$  does not occur, we need some upper bounds of the rank  $r(X)$ . We start with the following result.

**8.1. Proposition.** *Let  $X = X_{22} \subset \mathbb{P}^{13}$  be a Fano threefold of the main series with  $g(X) = 12$ . Assume that  $X$  does not contain planes. Then  $r(X) \leq 10$ .*

Note that proofs of similar estimates of  $r(X)$  in [Kal11, Sect. 3.2] contain some gaps (see footnote <sup>(2)</sup> on page 20).

*Proof.* Assume that  $r(X) \geq 11$ . Let  $\pi : \tilde{X} \rightarrow X$  be a factorization. Run the  $K$ -MMP on  $\tilde{X}$ . We get diagram (6.3.2), where all the maps  $\varphi_i$  are divisorial contractions.

**8.1.1. Claim.**  *$\varphi$  contracts a surface of degree 2.*

*Proof.* Let the minimal degree of a surface contracted by  $\varphi$  equals  $d$ . Assume that  $d \geq 3$ . We also may assume that  $\tilde{X}_N$  does not admit birational contractions and  $\rho(\tilde{X}_N) \leq 3$  by Lemma 6.5. Hence,  $N \geq r(X) - 3 \geq 8$ . By (2.4.2)

$$(8.1.2) \quad -K_{\tilde{X}_N}^3 \geq 22 + (2d - 2)N \geq 54.$$

If  $-K_{\tilde{X}_N}^3 = 54$ , then by Lemma 7.10 we have  $\rho(\tilde{X}_N) \leq 2$  and so  $N \geq 9$ . In this case,  $-K_{\tilde{X}_N}^3 \geq 22 + 4N \geq 56$ , a contradiction. Therefore,  $-K_{\tilde{X}_N}^3 > 54$  and again by Lemma 7.10 we have only one possibility:  $\tilde{X}_N \simeq \mathbb{P}^3$ . Then  $N = 10$  and  $d = 3$ . Moreover, similar to (8.1.2) we have  $-K_{\tilde{X}_{N-1}}^3 \geq 58$ . By the classification [MM82] the only possibility for the smoothing of  $X_{N-1}$  is [MM82, Table 2, n<sup>o</sup> 36<sup>o</sup>]. But in this case  $X_{N-1}$  must contain a plane, a contradiction.  $\square$

Let  $S \subset \tilde{X}$  be an irreducible surface of degree 2 contracted by  $\varphi$ . Run the  $S$ -MMP on  $\tilde{X}$ . We claim that  $S$  is contracted during this process. Indeed, otherwise after a number of flops we get a model  $\tilde{X}'$  such that the proper transform  $S' \subset \tilde{X}'$  of  $S$  is nef. Since  $\tilde{X}'$  is a generalized Fano threefold, some multiple of  $S'$  must be movable. Hence the image  $\bar{S} \subset X$  of  $S'$  under the anticanonical morphism is movable either. On the other hand,  $\bar{S}$  is a

quadric and the intersection  $\bar{S} \cap H$  with a general member  $H \in |-K_X|$  is a smooth rational curve. Since  $H$  is a smooth K3 surface, the curve  $\bar{S} \cap H$  cannot be movable. The contradiction shows that after a number of flops we must contract the proper transform of  $S$ . Replacing  $\tilde{X}$  with another factorization, we may assume that  $\varphi_1$  contracts a surface  $E_0 = S$  of degree 2. We claim that  $B_1 = \varphi_1(E_0)$  is either a point or a smooth rational curve. Indeed, otherwise  $B_1$  is a rational curve which is singular at some point  $P$ . Since  $\varphi_1$  is the blowup of a locally planar curve  $B_1$  contained in the smooth locus of  $\tilde{X}_1$  (see Theorem 2.3), easy local computations show that the exceptional divisor  $E_1$  is singular along the fiber  $C := \varphi_1^{-1}(P)$ . Since the quadric  $\pi(E_1)$  is normal,  $C$  must be contracted by  $\pi$ . This contradicts the fact that  $\varphi_1$  is a Mori contraction and proves our claim. Thus by (2.3.1) and (2.4.2)  $-K_{\tilde{X}}^3 = 24$ . Taking Proposition 7.11 into account we obtain

$$r(X) = \rho(\tilde{X}) = \rho(\tilde{X}_1) + 1 = r(X_1) + 1 \leq 10.$$

This proves the proposition.  $\square$

**8.2. Lemma.** *Let  $X = X_{22} \subset \mathbb{P}^{13}$  be a Fano threefold of the main series with  $g(X) = 12$ . Assume that  $r(X) > 2$ . Then  $X$  contains a surface of degree  $d$  with  $d \not\equiv 0 \pmod{11}$ .*

*Proof.* Assume that the degree of any surface  $S \subset X$  is divisible by 11 (in particular,  $X$  does not contain planes). Apply the construction (6.3.2). By Lemma 2.5, on each step, the variety  $\tilde{X}_k$  does not contain surfaces of degree  $\leq 10$ . Since the degree of the exceptional divisor of contractions of types  $(\mathbf{e}_2)$ - $(\mathbf{e}_4)$  is at most 4 by (2.3.1), all the birational contractions in (6.3.2) are of type  $(\mathbf{e}_1)$ . We may assume that  $\tilde{X}_N$  does not admit birational contractions.

First consider the case  $\rho(\tilde{X}_N) = 1$ . Then  $N \geq 2$  and by (2.4.2) we have

$$64 \geq -K_{\tilde{X}_N}^3 \geq 22 + 20N.$$

Hence,  $N = 2$  and  $X_2 = \tilde{X}_2 \simeq \mathbb{P}^3$ . The morphism  $X_1 \simeq \tilde{X}_1 \rightarrow \tilde{X}_2 \simeq \mathbb{P}^3$  is the blowup of an irreducible curve  $B_2 \subset \mathbb{P}^3$ . By (2.4.1)

$$K_{\tilde{X}_1}^2 \cdot E_1 = -K_{\mathbb{P}^3} \cdot B_2 + 2 - 2p_a(B_2)$$

is even. Hence,  $K_{\tilde{X}_1}^2 \cdot E_1 \geq 12$  and

$$42 = (-K_{\mathbb{P}^3})^3 - (-K_{X_0})^3 \geq 20 + 2K_{\tilde{X}_1}^2 \cdot E_1 + 2p_a(B_2) - 2$$

One can see that the above inequality is in fact an equality and so  $p_a(B_2) = 0$ ,  $K_{\tilde{X}_1}^2 \cdot E_1 = 12$ , and

$$-K_{\mathbb{P}^3} \cdot B_2 = 10 \not\equiv 0 \pmod{4},$$

a contradiction.

Therefore,  $\rho(\tilde{X}_N) \geq 2$  and  $Z$  is not a point. If  $\tilde{X}_N$  has a contraction to a curve, then a general fiber  $F$  is a del Pezzo surface. Hence,  $(-K_{\tilde{X}})^2 \cdot F = K_F^2 \leq 9$ . This contradicts our assumptions. Thus we may assume that  $Z$  is a surface with  $\rho(Z) = 1$ , i.e.  $Z \simeq \mathbb{P}^2$ . Then  $\rho(\tilde{X}_N) = 2$  and so  $N \geq 1$ . By (2.4.2)

$$(8.2.1) \quad -K_{\tilde{X}_N}^3 \geq 22 + 2 \cdot K_{\tilde{X}}^2 \cdot E_0 - 2 \geq 42.$$

Since  $\tilde{X}_N$  has no contractions to a curve and  $\rho(\tilde{X}_N) \geq 2$ , the variety  $X_N$  is not a quadric. By Lemma 7.10  $\iota(\tilde{X}_N) = 2$ . In this case, the (anticanonical) degree of any curve on  $\tilde{X}_N$  is even. Hence,  $(-K_{\tilde{X}})^2 \cdot E_0 \equiv 0 \pmod{22}$  by (2.4.1). This contradicts (8.2.1).  $\square$

**8.3. Proposition.** *Let  $X = X_{22} \subset \mathbb{P}^{13}$  be a  $G$ -Fano threefold of the main series with  $g(X) = 12$  and  $r(X) > 2$ . Let  $G_\bullet \subset G$  be an 11-Sylow subgroup. Then  $G_\bullet$  acts non-trivially on the lattice  $\text{Cl}(X)$ . In particular,  $r(X) \geq 11$ .*

*Proof.* Let  $S_1 \subset X$  be a surface of degree  $d$ . By Lemma 8.2 we can take  $S_1$  so that  $d \not\equiv 0 \pmod{11}$ . Let  $O = G \cdot S_1$  be the  $G$ -orbit of  $S_1$  and let  $n := \text{card}(O)$ . Denote  $D := \sum_{S \in O} S$ . We can write  $D \sim -aK_X$  for some  $a$ . Comparing the degrees we get

$$nd = 22a, \quad n \equiv 0 \pmod{11}.$$

Consider the following natural homomorphism of  $G$ -modules

$$\varsigma : \bigoplus_{S \in O} \mathbb{Z} \cdot S \longrightarrow \text{Cl}(X).$$

Clearly,  $\varsigma(O)$  is a  $G$ -orbit. Let  $\Theta_1, \dots, \Theta_m$  be all elements of  $\varsigma(O)$ . Take representatives  $S_i \in \varsigma^{-1}(\Theta_i) \cap O$  and put  $D' := S_1 + \dots + S_m$ . Since all the elements in the preimage  $\varsigma^{-1}(\Theta) \cap O$  are linearly equivalent,  $m$  divides  $n$  and  $D \sim \frac{n}{m}D'$ . Hence,

$$22D' = \frac{nd}{a}D' \sim \frac{md}{a}D \sim -mdK_X.$$

Since  $\iota(X) = 1$ , we have  $m \equiv 0 \pmod{11}$  and so  $G_\bullet$  acts on  $\text{Cl}(X)$  non-trivially.  $\square$

Now we are in position to prove Theorem 1.3.

*Proof of Theorem 1.3.* First assume that  $r(X) = 2$ . Since  $\text{rk Cl}(X)^G = 1$ , the action of  $G$  on  $\text{Cl}(X) \simeq \mathbb{Z}^2$  is non-trivial. Then some element  $\tau \in G$  switches two extremal rays  $\mathcal{R}$  and  $\mathcal{R}^+$  of the cone of movable divisors  $\text{Mov}(X)$ . Hence, two corresponding rational maps  $\Phi : X \dashrightarrow Z$  and  $\Phi^+ : X \dashrightarrow Z^+$  are also switched by  $\tau$ . In particular,  $\dim Z = \dim Z^+$ . This happens only in the case (I) of Theorem 1.2.



Now assume that  $r(X) > 2$ . Then by Proposition 8.3 we have  $r(X) \geq 11$ . On the other hand, the variety  $X$  does not contain planes (see Theorem 6.1) and so we have a contradiction by Proposition 8.1.  $\square$

## REFERENCES

- [Ba77] W. Barth. Moduli of vector bundles on the projective plane. *Invent. Math.*, 42:63–91, 1977.
- [BL12] J. Blanc and S. Lamy. Weak Fano threefolds obtained by blowing-up a space curve and construction of Sarkisov links. *Proc. Lond. Math. Soc. (3)*, 105(5):1047–1075, 2012.
- [CM13] J. W. Cutrone and N. A. Marshburn. Towards the classification of weak Fano threefolds with  $\rho = 2$ . *Cent. Eur. J. Math.*, 11(9):1552–1576, 2013.
- [Cu88] S. Cutkosky. Elementary contractions of Gorenstein threefolds. *Math. Ann.*, 280(3):521–525, 1988.
- [Fu06] M. Furushima. On the singular Fano threefold  $V_{22}^*$  with a small Gorenstein singularity. (An example). *Kumamoto J. Math.*, 19:37–57, 2006.
- [FN89] M. Furushima and N. Nakayama. The family of lines on the Fano threefold  $V_5$ . *Nagoya Math. J.*, 116:111–122, 1989.
- [Fr98] R. Friedman. *Algebraic surfaces and holomorphic vector bundles*. Universitext. Springer-Verlag, New York, 1998.
- [Ha77] R. Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.
- [HW81] F. Hidaka and K. Watanabe. Normal Gorenstein surfaces with ample anticanonical divisor. *Tokyo J. Math.*, 4(2):319–330, 1981.
- [IP99] V. A. Iskovskikh and Y. Prokhorov. *Fano varieties. Algebraic geometry V., Encyclopaedia Math. Sci.* **47** Springer, Berlin, 1999.
- [Is78] V. A. Iskovskikh. Fano threefolds. II. *Izv. Akad. Nauk SSSR Ser. Mat.*, 42(3):506–549, 1978.
- [JPR11] P. Jahnke, T. Peternell, and I. Radloff. Threefolds with big and nef anticanonical bundles II. *Cent. Eur. J. Math.*, 9(3):449–488, 2011.
- [JR06] P. Jahnke and I. Radloff. Gorenstein Fano threefolds with base points in the anticanonical system. *Compos. Math.*, 142(2):422–432, 2006.
- [JR11] P. Jahnke and I. Radloff. Terminal Fano threefolds and their smoothings. *Math. Z.*, 269(3-4):1129–1136, 2011.
- [Kac98] Y. Kachi. Flips from 4-folds with isolated complete intersection singularities. *Amer. J. Math.*, 120(1):43–102, 1998.
- [Kal11] A.-S. Kaloghiros. The defect of Fano 3-folds. *J. Algebraic Geom.*, 20(1):127–149, 2011.
- [Kaw88] Y. Kawamata. Crepant blowing-up of 3-dimensional canonical singularities and its application to degenerations of surfaces. *Ann. of Math. (2)*, 127(1):93–163, 1988.
- [KMM] Y. Kawamata, K. Matsuda, and K. Matsuki. Introduction to the minimal model problem. In *Algebraic geometry, Sendai, 1985, Adv. Stud. Pure Math.* **10**, 283–360. North-Holland, Amsterdam, 1987.
- [MM82] S. Mori and S. Mukai. Classification of Fano 3-folds with  $B_2 \geq 2$ . *Manuscripta Math.*, 36(2):147–162, 1981/82. Erratum: *Manuscripta Math.* 110 (2003), 407.
- [Mu89] S. Mukai. Biregular classification of Fano 3-folds and Fano manifolds of coindex 3. *Proc. Nat. Acad. Sci. U.S.A.*, 86(9):3000–3002, 1989.

- [Mu02] S. Mukai. New developments in the theory of Fano threefolds: vector bundle method and moduli problems. *Sugaku Expositions*, 15(2):125–150, 2002.
- [Na97] Y. Namikawa. Smoothing Fano 3-folds. *J. Algebraic Geom.*, 6(2):307–324, 1997.
- [Pr05] Y. Prokhorov. On the degree of Fano threefolds with canonical Gorenstein singularities. *Russian Acad. Sci. Sb. Math.*, 196(1):81–122, 2005.
- [Pr12] Y. Prokhorov. Simple finite subgroups of the Cremona group of rank 3. *J. Algebraic Geom.*, 21(3):563–600, 2012.
- [Pr13a] Y. Prokhorov. On birational involutions of  $\mathbf{P}^3$ . *Izv. Math.*, 77(3):627–648, 2013.
- [Pr13b] Y. Prokhorov. G-Fano threefolds, I. *Adv. Geom.*, 13(3):389–418, 2013.
- [Pr13c] Y. Prokhorov. G-Fano threefolds, II. *Adv. Geom.*, 13(3):419–434, 2013.
- [Pr15] Y. Prokhorov. On G-Fano threefolds. *Izv. Ross. Akad. Nauk Ser. Mat.*, 79(4), 2015.
- [PS09] Y. Prokhorov and V. V. Shokurov. Towards the second main theorem on complements. *J. Algebraic Geom.*, 18(1):151–199, 2009.
- [SR49] J. G. Semple and L. Roth. *Introduction to Algebraic Geometry*. Oxford, at the Clarendon Press, 1949.
- [Ta09] K. Takeuchi. Weak Fano threefolds with del Pezzo fibration. *ArXiv e-print*, 0910.2188, 2009.

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