QUANTUM STATISTICAL MECHANICS OVER FUNCTION FIELDS

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1. Introduction

It has become increasingly evident, starting from the seminal paper of Bost and Connes [3] and continuing with several more recent developments ([8], [10], [12], [13], [22], [24]), that there is a rich interplay between quantum statistical mechanics and arithmetic. In the case of number fields, the symmetries and equilibrium states of the Bost–Connes system are closely linked to the explicit class field theory of $\mathbb{Q}$, and the system constructed in [12] extends this result to the case of imaginary quadratic fields, using the relation between the arithmetic of the modular field and a 2-dimensional analog of the Bost–Connes system introduced in [10]. This leads to a far reaching generalization to Shimura varieties as developed in [22]. Moreover, very recently Benoit Jacob constructed an interesting quantum statistical mechanical system that generalizes the Bost–Connes system for function fields, using sign normalized rank one Drinfeld modules. In all of these cases, one always works with the C^*-algebra formulation of quantum statistical machanics. In the case of number fields one can extract arithmetic information by considering a suitable subalgebra (or algebra of multipliers) which is defined over $\mathbb{Q}$ or over a finite extension thereof. In the case of positive characteristic, one needs a different approach, which implies developing a version of quantum statistical mechanics that works when the algebra of observables is an algebra over a field extension of a function field rather than being an algebra over the complex numbers.

The purpose of this paper is twofold. In the first part, we introduce a geometric construction of a noncommutative space of Drinfeld modules that generalizes the noncommutative spaces of commensurability classes of $\mathbb{Q}$-lattices considered in [10]. In the second part of the paper we develop some basics of quantum statistical mechanics in positive characteristic and we show that, in the case of rank one Drinfeld modules, one obtains a natural time evolution and KMS functionals associated to the points of the underlying classical moduli space, in clear analogy to the results of [10], [12], and [13].

The structure of the paper is the following. In Section 2, we first review briefly some well known facts about Drinfeld modules, their Tate modules and isogenies, which will be useful throughout the paper. We then introduce in §2.3 a notion of *n-pointed Drinfeld module*, consisting of a Drinfeld module of rank n together with n points in its total Tate module. This notion is formulated in such a way that it resembles closely the reformulation, given in [13], of the notion of $\mathbb{Q}$-lattices in terms of Tate modules of elliptic curves. We show that isogenies define an equivalence relation on n-pointed Drinfeld modules and that the resulting quotient is best understood as a noncommutative space, likewise as for the commensurability relation on $\mathbb{Q}$-lattices.

Throughout the paper we work strictly in the "generic characteristic case", since we need the fact that the adelic Tate module of a Drinfeld module is a free module of rank n over the maximal compact subring of the finite adeles of the function field $\mathbb{K}$.

In §2.4 we show that the space of commensurability classes of n-pointed Drinfeld modules can be understood as a noncommutative generalization of the moduli space $\mathcal{M}^n$ of Drinfeld modules, by reinterpreting the notion of n-pointed Drinfeld module as the datum of a possibly degenerate level structure on a Drinfeld module. This statement is analogous to the one obtained in the case of number fields, for the noncommutative generalization of modular curves and more general Shimura varieties ([10], [13], [22]).

In Section 3, we develop a notion of $\mathbb{K}$-rational $\mathbf{L}$-lattice, for $\mathbf{L}$ a complete subfield of $\mathbf{C}_\infty$ that contains $\mathbb{K}_\infty$. This is a more direct analog of the notion of $\mathbb{Q}$-lattice. We introduce a corresponding notion of commensurability. The resulting quotients are again noncommutative spaces. In Theorem 3.11, we show that the equivalence of categories between lattices and Drinfeld modules induces an identification between the set of n-pointed Drinfeld modules, up to isogeny, and the set of commensurability classes of $\mathbb{K}$-rational $\mathbf{C}_\infty$-lattices. This result relies on an explicit description of the commensurability classes of $\mathbb{K}$-rational $\mathbf{L}$-lattices, which we obtain using the adelic description of lattices.
We mainly focus on the rank one case, for which we give also a modular interpretation of the "adele class space" of a function field, in terms of a noncommutative generalization of the covering $\tilde{\mathcal{M}}^1$ introduced in [16] of the moduli space $\mathcal{M}^1$ of Drinfeld modules.

In Section 4, we introduce suitable analogs in the function field case of the basic notions of quantum statistical mechanics. This is done with the purpose of developing a theory for algebras with values in extensions of function fields rather than in the field of the complex numbers. Not all notions that are available in the C^*-algebra context have a direct analog in positive characteristic. Most notably, one does not have an immediate replacement for the operation of taking the adjoint of an element in the algebra, hence for the notion of positivity or extremality of states. Nonetheless, we argue that enough of the quantum statistical mechanics formalism is still avalable to provide a good notion of time evolution and of equilibrium KMS functionals.

In §4.3, we show that there is a natural time evolution on the noncommutative space of commensurability classes of 1-dimensional $\mathbb{K}$-rational lattices, which is defined in terms of the exponentiation of (fractional) ideals in function fields. The partition function of such a time evolution is the zeta function $\zeta_{\mathbf{A}}(s) = \sum_{I \subset \mathbf{A}} I^{-s}$, defined in a "half plane" of $S_\infty = \mathbf{C}_\infty^* \times \mathbb{Z}_p$ and with values in $\mathbf{C}_\infty$. We prove in Theorem 4.10 that the points of the moduli space $\mathcal{M}^1$ of Drinfeld modules define KMS functionals for this time evolution, and that the induced action of symmetries of the quantum statistical mechanical system on KMS states recovers the class field theory action of $\mathbb{A}^*_{\mathbb{K},f}/\mathbb{K}^*$ on $\mathcal{M}^1$. As for the case of 2-dimensional $\mathbb{Q}$-lattices ([10],[12]), the symmetries are given by endomorphisms of the algebra compatible with the time evolution.

In §4.6, we discuss a v-adic version of this notion of time evolution and we show that the same noncommutative space of commensurability classes of 1-dimensional $\mathbb{K}$-rational lattices admits natural v-adic time evolutions for all the places $v \neq \infty$ of $\mathbb{K}$. These group homomorphisms are defined in terms of the v-adic exponentiation of ideals. In view of our ongoing work with Connes [9], we think that it is important to consider this whole family of time evolutions. In fact, in the framework of the C^*-algebras, the analogous set provides a collection of low temperature KMS states that recovers (non-canonically) a copy of the set $C(\bar{\mathbb{F}}_q)$ of algebraic points of the curve C, sitting inside the noncommutative adeles class space of a function field, and provides a good analog of this set in the case of number fields.

Finally, in Section 5 we discuss the analog in the function field setting of the "dual system" of a quantum statistical mechanical system considered in [8]. In the framework of C^*-algebras, one can see the scaling action on the dual system as a replacement for the Frobenius action in characteristic zero. In the positive characteristic setting of function fields, we find that the scaling action is indeed induced by the Frobenius action, up to a Wick rotation that exchanges the real and the imaginary part of the time evolution.

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1.1. **Notation.**

Throughout the paper we use the following notation.

- $\mathbb{F}_q$ is a finite field of characteristic p, with $q = p^{m_0}$ elements.
- C is a smooth, projective, geometrically connected curve over $\mathbb{F}_q$.
- $\mathbb{K} = \mathbb{F}_q(C)$ is the function field of C.
- $\infty \in C$ is a *chosen* closed point of degree d_∞ over $\mathbb{F}_q$, or equivalently a *fixed* place of $\mathbb{K}$ of degree d_∞.
- v_∞ is the valuation associated to the prime ∞. $|\cdot|_\infty$ is the corresponding normalized absolute value: for $x \in \mathbb{K}$, $|x|_\infty = q^{deg(x)} = q^{-d_\infty v_\infty(x)}$.
- $\mathbb{K}_\infty$ is the completion of $\mathbb{K}$ with respect to v_∞.
- $\overline{\mathbb{K}}_\infty$ is a fixed algebraic closure of $\mathbb{K}_\infty$.
- $\mathbf{C}_\infty$ is the completion of $\overline{\mathbb{K}}_\infty$ with respect to the canonical extension of v_∞ to $\overline{\mathbb{K}}_\infty$. The field $\mathbf{C}_\infty$ is also algebraically closed.
- $\mathbf{A} \subset \mathbb{K}$ is the ring of functions regular outside ∞.
- $\mathcal{F}$ is an $\mathbf{A}$-field, that is a field $\mathcal{F}$ together with a fixed homomorphism $\iota : \mathbf{A} \to \mathcal{F}$. The prime ideal $\wp = \mathrm{Ker}(\iota)$ is called the characteristic of $\mathcal{F}$. $\mathcal{F}$ has *generic characteristic* if $\mathrm{Ker}(\iota) = (0)$.
- $\mathcal{F}\{\tau\}$ is the (non-commutative) ring of polynomials $f(\tau) = \sum_{i=0}^{\nu} a_i \tau^i$ with τ the q-th power mapping, that is, $\mathcal{F}\{\tau\}$ is endowed with the product $\tau a = a^q \tau$.
- $\mathbf{L}$ is a complete subfield of $\mathbf{C}_\infty$ that contains $\mathbb{K}_\infty$.
- $\Sigma_\mathbb{K}$ denotes the set of places $v \in \mathbb{K}$, and $\Sigma_\mathbf{A} = \{v \in \Sigma_\mathbb{K} | v \neq \infty\}$.
- For $v \in \Sigma_\mathbf{A}$, $\mathbf{A}_v$ denotes the v-adic completion of $\mathbf{A}$, with $\mathbf{A}_v \subset \mathbb{K}_v$, and $\mathbb{K}_v$ the completion of $\mathbb{K}$ at v.
- $\mathbb{A}_\mathbb{K} = \prod'_{v \in \Sigma_\mathbb{K}} \mathbb{K}_v$ the ring of adeles of $\mathbb{K}$ (restricted product), that is the set of elements $(a_v) \in \prod_{v \in \Sigma_\mathbb{K}} \mathbb{K}_v$, with $a_v \in \mathbf{A}_v$ for all but finitely many places v.
- $\mathbb{A}_{\mathbb{K},f} = \prod'_{v \in \Sigma_\mathbf{A}} \mathbb{K}_v$ the ring of finite adeles of $\mathbb{K}$.
- $R = \prod_{v \in \Sigma_\mathbf{A}} \mathbf{A}_v$ the ring of finite integral adeles (maximal compact subring of $\mathbb{A}_{\mathbb{K},f}$).

2. Pointed Drinfeld modules and isogenies

We begin by recalling some well known facts about Drinfeld modules. The main references are Drinfeld's original papers [15] and [16]. Here we follow mostly [18].

Let $\mathcal{F}$ be an $\mathbf{A}$-field. Then, it is well known that the ring of $\mathcal{F}$-endomorphisms of the additive group $\mathbb{G}_a$ is given by $\mathrm{End}_\mathcal{F}(\mathbb{G}_a) = \mathcal{F}\{\tau\}$.

A Drinfeld $\mathbf{A}$-module over $\mathcal{F}$ is a homomorphism of $\mathbb{F}_q$-algebras

$$\Phi : \mathbf{A} \to \mathrm{End}_{\mathcal{F}}(\mathbb{G}_a) = \mathcal{F}\{\tau\}, \quad a \mapsto \Phi_a(\tau) \in \mathcal{F}\{\tau\}, \tag{2.1}$$

such that $D \circ \Phi = \iota$, where D is the derivation $Df := a_0 = f'(\tau)$, for $f(\tau) = \sum_{i=0}^{\nu} a_i \tau^i \in \mathcal{F}\{\tau\}$.

One also requires that Φ is nontrivial, that is $\Phi_a \neq \iota(a)\tau^0$, for some $a \in \mathbf{A}$.

A Drinfeld $\mathbf{A}$-module over $\mathcal{F}$ is of rank $n \in \mathbb{N}$ if

$$\deg \Phi_a(\tau) = n \deg(a) = -n d_\infty v_\infty(a) \qquad \forall a \in \mathbf{A}. \tag{2.2}$$

In the case of $\mathbb{K} = \mathbb{F}_q(T)$, *i.e.* for $\mathbb{K}$ the function field of $\mathbb{P}^1_{/\mathbb{F}_q}$, and $\mathbf{A} = \mathbb{F}_q[T]$, $\deg(a)$ is the degree as a polynomial in $\mathbf{A} = \mathbb{F}_q[T]$.

If L is a field extension of $\mathcal{F}$, one can view L as an $\mathbf{A}$-module through Φ. We denote the resulting $\mathbf{A}$-module by $\Phi(L)$. Since $\mathbf{A}$ is a Dedekind domain, an ideal $I \subset \mathbf{A}$ is generated by at most two elements $\{i_1, i_2\}$. We denote by Φ_I the monic generator of the left ideal in $\mathcal{F}\{\tau\}$ generated by Φ_{i_1} and Φ_{i_2} and by $\Phi[I]$ the finite subgroup of $\Phi(\overline{\mathcal{F}})$ given by the roots of Φ_I. For $a \in \mathbf{A}$, we use the notation $\Phi[a] = \Phi[(a)]$.

2.1. **Torsion points and Tate modules.**

For $a \in \mathbf{A}$, the a-torsion points of a Drinfeld $\mathbf{A}$-module Φ of rank n over $\mathcal{F}$ are the roots in $\Phi(\overline{\mathcal{F}})$ of the polynomial Φ_a. One obtains in this way a finite $\mathbf{A}$-module $\Phi[a]$. If a is prime to the characteristic of $\mathcal{F}$, $\Phi[a] \simeq (\mathbf{A}/(a))^n$. For $v \in \Sigma_{\mathbf{A}}$, the v-adic Tate module $T_v\Phi$ of Φ is the $\mathbf{A}_v$-module

$$T_v\Phi := \mathrm{Hom}_{\mathbf{A}}(\mathbb{K}_v/\mathbf{A}_v, \Phi[v^\infty]),$$

where $\Phi[v^\infty] := \bigcup_{m \geq 1} \Phi[v^m]$. $T_v\Phi$ is characterized by the isomorphism

$$T_v\Phi \simeq \varprojlim_{m \in \mathbb{N}} \Phi[v^m]. \tag{2.3}$$

If v is prime to the characteristic of $\mathcal{F}$, and in particular if $\mathcal{F}$ is of generic characteristic, then $T_v\Phi$ is a free $\mathbf{A}_v$-module of rank n.

For Drinfeld $\mathbf{A}$-modules of rank n over $\mathcal{F}$, one can also define the adelic Tate module

$$T\Phi = \prod_{v \in \Sigma_{\mathbf{A}}} T_v\Phi,$$

which is the analog of the total Tate module of an elliptic curve. If $\mathcal{F}$ is of generic characteristic, then $T\Phi$ is a free module of rank n over R.

2.2. **Isogenies.**

Two Drinfeld $\mathbf{A}$-modules Φ and Ψ of rank n over $\mathcal{F}$ are said to be isogenous if there is a nonzero polynomial $P(\tau) \in \mathcal{F}\{\tau\}$ satisfying the condition

$$P\Phi_a = \Psi_a P, \qquad \forall a \in \mathbf{A}. \tag{2.4}$$

In the category of Drinfeld modules of rank n, whose objects are Drinfeld $\mathbf{A}$-modules over $\mathcal{F}$, the morphisms $\mathrm{Hom}_{\mathcal{F}}(\Phi, \Psi)$ are given by the isogenies. A morphism given by an isogeny P is an isomorphism iff there exists $Q \in \mathcal{F}\{\tau\}$ such that $P \cdot Q = \tau^0$, *i.e.* iff $P(\tau)$ is of degree zero.

It is known that isogenies give rise to an equivalence relation on the set of Drinfeld modules ([18], 4.7.13-14).

The operation that associates to a Drinfeld module Φ its v-adic Tate module $T_v\Phi$ defines a covariant functor from the category of Drinfeld modules with morphisms given by isogenies to the category of $\mathbf{A}_v$-modules. If v is different from the characteristic of $\mathcal{F}$, and in particular if $\mathcal{F}$ is of generic characteristic, the natural induced map

$$\mathrm{Hom}_{\mathcal{F}}(\Phi,\Psi)\otimes \mathbf{A}_v \to \mathrm{Hom}_{\mathbf{A}_v}(T_v\Phi, T_v\Psi) \tag{2.5}$$

is injective with torsion free cokernel ([18], 4.12.11). Under the above assumption, it follows from the injectivity of (2.5) that an isogeny between Drinfeld modules is determined by the induced action on the v-adic Tate modules. One defines in a similar way a covariant functor $\Phi \mapsto T\Phi$, by considering the adelic Tate module.

2.3. **Pointed Drinfeld modules.**

In this paragraph we introduce the new notion of a n-pointed Drinfeld module. We enrich the structure of a Drinfeld module by the extra datum of a finite set of points in the associated (adelic) Tate module, modulo isogeny. In the next sections we will explain how the space of n-pointed Drinfeld modules modulo isogeny is a suitable replacement, in the function field setting, for the space of $\mathbb{Q}$-lattices up to scaling and modulo the commensurability relation introduced in [3] and [10].

Definition 2.1. *An n-*pointed *Drinfeld $\mathbf{A}$-module over an $\mathbf{A}$-field $\mathcal{F}$ of generic characteristic is a datum $(\Phi,\zeta_1,\ldots,\zeta_n)$, where Φ is a Drinfeld $\mathbf{A}$-module over $\mathcal{F}$ of rank n, and the ζ_i, for $i=1,\ldots,n$, are points in the associated adelic Tate module $T\Phi$. Data $(\Phi,\zeta_1,\ldots,\zeta_n)$ and $(\Psi,\eta_1,\ldots,\eta_n)$ are said to be commensurable if there exists an isogeny $P\in\mathcal{F}\{\tau\}$ connecting the Drinfeld modules Φ and Ψ such that the two sets of points $\{\zeta_i\}$ and $\{\eta_i\}$ are related through the induced map on the Tate modules, namely*

$$(\eta_i)_v = T_v(P)(\zeta_i)_v. \tag{2.6}$$

Here $(\eta_i)_v\in T_v\Psi$ and $(\zeta_i)_v\in T_v\Phi$ are the v-adic components of $\eta_i\in T\Psi$ and $\xi_i\in T\Phi$, respectively and $T_v(P): T_v\Phi\to T_v\Psi$ is the image of P under the inclusion (2.5)*.*

Lemma 2.2. *Commensurability defines an equivalence relation on the set of n-pointed Drinfeld $\mathbf{A}$-modules of Definition 2.1.*

Proof. The proof follows by applying the functorial properties of the adelic Tate modules and because isogeny gives rise to an equivalence relation on Drinfeld modules. ☐

Definition 2.3. *We denote by $\mathcal{D}^{\mathcal{F}}_{\mathbb{K},n}$ the set of commensurability classes (Φ,ξ) of n-pointed Drinfeld $\mathbf{A}$-modules over a field $\mathcal{F}$ of generic characteristic.*

In the following, we will mainly consider the case of generic characteristic, where $\mathcal{F}=\mathbf{L}$ is a complete subfield of $\mathbf{C}_\infty$ that contains $\mathbb{K}_\infty$. It is well known that, when $n=1$ and if $\mathbf{L}$ has generic characteristic, then $\mathbf{L}$ contains the Hilbert class field $\mathbb{H}$ of $\mathbb{K}$ (*cf.* [15] and [18] p.195).

2.4. **Degenerations of level structures.**

Let Φ be a Drinfeld $\mathbf{A}$-module of rank n over $\mathcal{F} = \mathbf{L}$ of generic characteristic and let $I \subset \mathbf{A}$ be a nonzero ideal. Then, a level I structure on Φ is an isomorphism (of $\mathbf{A}$-modules)

$$\rho_I : (I^{-1}\mathbf{A}/\mathbf{A})^n \overset{\simeq}{\to} \Phi[I]. \tag{2.7}$$

In [15], Drinfeld constructed a moduli scheme of Drinfeld $\mathbf{A}$-modules. In rank n and under our assumptions, this is the projective limit

$$\mathcal{M}^n = \varprojlim_I \mathcal{M}^n_I, \tag{2.8}$$

where $\mathcal{M}^n_I$ is the moduli scheme (a smooth, n-dimensional manifold over $\mathbf{A}$) of isomorphism classes of Drinfeld $\mathbf{A}$-modules of rank n over $\mathbf{L}$ and level I structure.
Thus, a point of $\mathcal{M}^n$ is given by the data of a Drinfeld module Φ of rank n and a homomorphism of $\mathbf{A}$-modules

$$\rho : (\mathbb{K}/\mathbf{A})^n \to \Phi(\mathbf{L}) \tag{2.9}$$

which induces a compatible system of level structures ρ_I for all levels I.

The notion of n-pointed Drinfeld modules simply relaxes the condition of level I structure by allowing homomorphisms $(I^{-1}\mathbf{A}/\mathbf{A})^n \to \Phi[I]$ that are not necessarily isomorphisms. The resulting implication on the classifying space is that the moduli space of Drinfeld modules is replaced by a noncommutative space, in analogy to what happens to Shimura varieties in the context of number fields ([10], [13], [22]).

Lemma 2.4. *The set $\mathcal{D}^{\mathbf{L}}_{\mathbb{K},n}$ can be identified with the set of data (Φ, ζ) with Φ a Drinfeld $\mathbf{A}$-module of rank n and $\zeta : R^n \to T\Phi \simeq R^n$ an R-module homomorphism, up to the equivalence relation by isogenies of Drinfeld modules and the induced maps on the Tate modules.*

Proof. The datum $\zeta_i \in T\Phi$, for $i = 1, \dots, n$ uniquely determines a homomorphism of R-modules

$$\zeta : R^n \to T\Phi \tag{2.10}$$

obtained by setting $\zeta(e_i) = \zeta_i$, where $\{e_1, \dots, e_n\}$ is the standard basis of R^n as an R-module. This yields an equivalent description of an n-pointed Drinfeld module $(\Phi, \zeta_1, \dots, \zeta_n)$ as a datum (Φ, ζ), with ζ as in (2.10). We still need to describe the commensurability relation of n-pointed Drinfeld modules in terms of the data (Φ, ζ). We consider the relation of commensurability on the data of n-pointed Drinfeld modules. This is implemented by the action of isogenies, so that

$$(\Phi, \zeta_1, \dots, \zeta_n) \sim (\Psi, \xi_1, \dots, \xi_n)$$

if and only if there is an isogeny $P : \Phi \to \Psi$ such that $(\xi_i) = TP(\zeta_i)$, with $TP : T\Phi \to T\Psi$. In terms of the maps (2.10), this means that we are considering commutative diagrams

$$\begin{tikzcd} & T\Phi \arrow[dd, "TP"] \\ R^n \arrow[ur, "u_\Phi"] \arrow[dr, "u_\Psi"'] & \\ & T\Psi \end{tikzcd} \tag{2.11}$$

Using the identification $T\Phi \simeq R^n$, valid in generic characteristic, we can reformulate the data $\zeta : R^n \to T\Phi$ as an R-module homomorphism $u : R^n \to R^n$, *i.e.* with an element $u \in M_n(R)$.

This shows that the set of commensurability classes of n-pointed Drinfeld modules $(\Phi, \zeta_1, \ldots, \zeta_n)$ can be identified with the set of isogeny classes of data (Φ, ζ), with ζ as in (2.10). $\square$

Corollary 2.5. *The set $\mathcal{D}^{\mathbf{L}}_{\mathbb{K},n}$ can be identified with a generalized moduli space of Drinfeld modules with possibly degenerate level structure, namely with the moduli space of isogeny classes of data (Φ, ρ), with $\rho : (\mathbb{K}/\mathbf{A})^n \to \Phi(\mathbf{L})$ a homomorphism of $\mathbf{A}$-modules as in* (2.9).

Proof. For any prime $v \in \mathbf{A}$, a homomorphism $\zeta : R^n \to T\Phi$ determines a compatible system of induced homomorphisms $\zeta_{v^m} : (\mathbf{A}/v^m\mathbf{A})^n \to \Phi[v^m]$, hence for an ideal $I \subset \mathbf{A}$, ζ determines an induced homomorphism $\zeta_I : (I^{-1}/\mathbf{A})^n \to \Phi[I]$. Thus, the data (Φ, ζ) and (Φ, ρ) with $\rho_I = \zeta_I$ in turn determine each other. $\square$

The moduli space $\mathcal{M}^n$ of Drinfeld modules is recast as the space of the "classical points" of the generalized moduli of commensurability classes of n-pointed Drinfeld modules. In fact, it is not hard to see that isogenies of $\mathbf{A}$-modules preserving the condition (2.7) are simply isomorphisms. In terms of n-pointed Drinfeld modules, this case is described as follows.

Lemma 2.6. *Consider the subset of $\mathcal{D}^{\mathbf{L}}_{\mathbb{K},n}$ made of* invertible *Drinfeld $\mathbf{A}$-modules, in the sense that the points $(\zeta_1, \ldots, \zeta_n) \in T\Phi$ form a basis of $T\Phi \simeq R^n$ as an R-module. Then the equivalence relation of commensurability is trivial on this subset. Namely, an isogeny P that describes a commensurability relation between two invertible n-pointed Drinfeld modules is in fact an isomorphism.*

Proof. The condition that the set $\{\zeta_1, \ldots, \zeta_n\}$ forms a basis is equivalent to requiring that the map $\zeta : R^n \to T\Phi \simeq R^n$ is an isomorphism, hence it induces compatible level structures $\zeta_I : (I^{-1}\mathbf{A}/\mathbf{A})^n \to \Phi[I]$.
Let P be an isogeny connecting two modules Φ and Ψ, such that $P(\tau)$ is not of degree zero, *i.e.* P is not an isomorphism. Then we can consider the scheme theoretic kernel H of P. This is a finite, $\mathbf{A}$-invariant subscheme $H \subset \mathbb{G}_a$, with $H \subseteq \Phi[a]$ for some non-zero $a \in \mathbf{A}$. This means that the composite map

$$(a^{-1}\mathbf{A}/\mathbf{A})^n \xrightarrow{\zeta_{(a)}} \Phi[a] \xrightarrow{P} \Psi[a]$$

has a nontrivial kernel, *i.e.* it is not a level structure on Ψ. Thus, the composite map $TP \circ \zeta$ (with $TP : T\Phi \to T\Psi$) cannot be an isomorphism $R^n \to T\Psi$. This means that the datum $(\Psi = P(\Phi), \xi = TP \circ \zeta)$ is not an invertible n-pointed Drinfeld module. This shows that non-isomorphic elements in the same commensurability class of an invertible element are all non-invertible. $\square$

3. Commensurability of $\mathbb{K}$-rational lattices

It is important to notice that, while the original moduli space $\mathcal{M}^n$ of Drinfeld modules is a projective limit of n-dimensional schemes of finite type over $\mathrm{Spec}(\mathbf{A})$, the quotient space obtained by implementing the commensurability relation on the set of n-pointed Drinfeld $\mathbf{A}$-modules over $\mathcal{F} = \mathbf{L}$ of generic characteristic yields instead a noncommutative space $\mathcal{M}^n_{nc} = \mathcal{D}^{\mathbf{L}}_{\mathbb{K},n}$.
In the following we describe more in detail the nature of this equivalence relation by introducing three important different notions of lattices and re-considering, in this generalized aspect, the well-known equivalence of categories of Drinfeld modules and lattices.

In the following, we assume that the $\mathbf{A}$-field $\mathcal{F}$ is of generic characteristic and that $\mathcal{F} = \mathbf{L}$ a complete subfield of $\mathbf{C}_\infty$ that contains $\mathbb{K}_\infty$.

Definition 3.1. *One can distinguish the following notions of lattices:*

(1) *A lattice Λ of rank n is a finitely generated $\mathbf{A}$-submodule of $\mathbb{K}^n$ such that $\Lambda \otimes \mathbb{K}_\infty \simeq \mathbb{K}^n_\infty$, or equivalently a finitely generated projective $\mathbf{A}$-module of rank n which is discrete in $\mathbb{K}^n_\infty$.*

(2) *A lattice Λ in $\mathbf{C}_\infty$ is a discrete $\mathbf{A}$-submodule of $\mathbf{C}_\infty$ such that $\mathbb{K}_\infty\Lambda$ is a finite dimensional $\mathbb{K}_\infty$ vector space. The rank of the lattice is the dimension of $\mathbb{K}_\infty\Lambda$ as a $\mathbb{K}_\infty$-vector space.*

(3) *If $\mathbf{L}$ is a complete subfield of $\mathbf{C}_\infty$ that contains $\mathbb{K}_\infty$, an $\mathbf{L}$-lattice Λ is a $\mathbf{A}$-submodule of $\mathbf{C}_\infty$ which satifies the following conditions:*
   (a) *Λ is finitely generated as an $\mathbf{A}$-module*
   (b) *Λ is discrete in the topology of $\mathbf{C}_\infty$*
   (c) *Λ is contained in the separable closure $\mathbf{L}^{sep}$ of $\mathbf{L}$ in $\mathbf{C}_\infty$ and it is stable under the action of $\mathrm{Gal}(\mathbf{L}^{sep}/\mathbf{L})$.*

In the following we use these different notions of lattice to have corresponding notions of $\mathbb{K}$-rational lattices (*cf.* Definition 3.2 below). In particular, the versions (2) and (3) of Definition 3.1 above are those suitable to obtain a description of n-pointed Drinfeld modules in terms of $\mathbb{K}$-rational lattices (*cf.* Theorem 3.11 below), while version (1) of Definition 3.1 has the advantage of having a simpler parameterizing space (*cf.* Theorem 3.9 below), for which it is easier to construct a quantum statistical mechanical system. In this paper we mostly restrict our attention to the rank 1 case, for which we show in Corollary 3.5 below that these choices are in fact equivalent.

On the set of lattices Λ in $\mathbf{C}_\infty$ (as in Definition 3.1 (2) above), one can impose the equivalence relation of *similarity* or *scaling*. This relation identifies two lattices $\Lambda \sim \Lambda'$ if $\Lambda' = \lambda\Lambda$, for some $\lambda \in \mathbf{C}^*_\infty$.

The main result about lattices and Drinfeld modules is summarized in the following statement (*cf.* [15] and [18], §4). Let $\mathbf{L}$ be a complete subfield of $\mathbf{C}_\infty$ that contains $\mathbb{K}_\infty$. There is an equivalence of categories between the category of rank n $\mathbf{L}$-lattices Λ in $\mathbf{C}_\infty$ and the category of Drinfeld modules Φ of rank n over $\mathbf{L}$. To a lattice Λ one associates an entire (surjective) function e_Λ on $\mathbf{C}_\infty$ satisfying the equation $e_\Lambda(az) = \Phi^\Lambda_a(e_\Lambda(z))$ for $a \in \mathbf{A}$ and $\Phi^\Lambda_a \in \mathbf{C}_\infty\{\tau\}$. The map $a \mapsto \Phi^\Lambda_a$ defines a Drinfeld module Φ^Λ. Moreover, each rank n Drinfeld module Φ over $\mathbf{L}$ arises as Φ^Λ from a uniquely determined n-lattice Λ defined as the kernel of an entire function e_Φ as above. Two Drinfeld modules Φ^Λ and $\Psi^{\Lambda'}$ corresponding to lattices Λ and Λ' are isomorphic iff their associated lattices Λ and Λ' are similar.

At the level of the lattices, the notion of isogeny is described as follows. An isogeny $P \in \mathbf{L}\{\tau\}$ connecting two Drinfeld modules Φ and Ψ as in (2.4) determines an element $e^{-1}_\Psi P e_\Phi$ which commutes with all $\iota(a)\tau^0 \in \mathbb{K}_\infty$. This defines an element $\lambda \in \mathbf{L}$ which acts as a morphism between the $\mathbf{L}$-lattices $\Lambda = Ker(e_\Phi)$ and $\Lambda' = Ker(e_\Psi)$ associated to Φ and Ψ, respectively. In fact, a morphism of $\mathbf{L}$-lattices is an element $\lambda \in \mathbf{L} \subset \mathbf{C}_\infty$ such that $\lambda\Lambda \subset \Lambda'$.

The adelic Tate module $T\Phi$ of a Drinfeld module $\Phi = \Phi^\Lambda$ can be identified with

$$T\Phi \cong \Lambda \otimes_{\mathbf{A}} R. \tag{3.1}$$

3.1. **A short digression on Drinfeld modules and noncommutative tori.**

In general, one regards a Drinfeld module of rank one as an analog, for the arithmetic of function fields, of the multiplicative group scheme $\mathbb{G}_m$ for the field of rationals. Similarly, a Drinfeld module of rank two is thought of as an analog of an elliptic curve.
There is, however, also a major difference to keep in mind. In fact, $\mathbf{C}_\infty$ is an infinite dimensional $\mathbb{K}_\infty$-vector space. This implies that one can consider $\mathbf{A}$-modules of arbitrary large rank as discrete submodules of $\mathbf{C}_\infty$. A Drinfeld module of rank n corresponds, in fact, to a lattice Λ of rank n.
In higher ranks, one would be tempted to think of Drinfeld modules Φ^Λ as analogs of abelian varieties, due to the discreteness of the associated lattice Λ in $\mathbf{C}_\infty$. However, it turns out that the correct function field analogy of the theory of abelian varieties in characteristic zero is given by the theory of T-modules as developed by Anderson in [1]. This leaves for the moment still open the interpretation of higher rank Drinfeld modules in the characteristic zero world. In the number theory literature it is frequently stated that these objects just do not have any characteristic zero analog because of the impossibility of taking quotients of $\mathbb{C}$ by abelian groups of rank higher than two, while remaining in the context of algebraic varieties. Noncommutative geometry suggests that the correct analog should be given by generalizations of noncommutative tori given by $n-1$ irrational rotations independent over $\mathbb{Q}$, namely by C^*-algebras

$$\mathcal{A}_{\bar\theta} := C(S^1) \rtimes_{\bar\theta} \mathbb{Z}^{n-1}, \tag{3.2}$$

where the $\mathbb{Z}^{n-1}$-action on $C(S^1)$ is generated by the irrational rotations by $\exp(2\pi\sqrt{-1}\theta_i)$ with $\bar\theta := \{\theta_1, \ldots, \theta_{n-1}\}$ a collection of $\mathbb{Q}$-linearly independent irrational numbers (*cf. e.g.* [25]).
In the present paper, when we discuss the quantum statistical mechanical systems associated to Drinfeld modules, we refer mostly to the rank one case, as considered by Jacob in [24]. It is a very interesting open problem to study the quantum statistical mechanical systems arising from the moduli spaces of n-pointed Drinfeld modules of rank $n \geq 2$ and possible analogs for number fields, in the case $n \geq 3$, based on the introduction of irrational rotation algebras and noncommutative tori.

3.2. $\mathbb{K}$**-rational lattices.**

In this paragraph we develop an analog in the function field case, of the notion of $\mathbb{Q}$-lattices and commensurability ([10]), and we describe the relation to the set of (commensurability classes of) n-pointed Drinfeld modules.

Definition 3.2. *An n-dimensional $\mathbb{K}$-rational lattice is a pair of a rank n lattice Λ (as in Definition 3.1, (1)) together with a homorphism ϕ of $\mathbf{A}$-modules*

$$\phi : (\mathbb{K}/\mathbf{A})^n \to \mathbb{K}\Lambda/\Lambda. \tag{3.3}$$

Similarly, an n-dimensional $\mathbb{K}$-rational $\mathbf{L}$-lattice is the datum of a rank n $\mathbf{L}$-lattice Λ (as in Definition 3.1, (3)) and a homorphism ϕ as in (3.3) *above.*
A $\mathbb{K}$-rational ($\mathbf{L}$-)lattice is invertible if ϕ is an isomorphism.

Definition 3.3. *We denote by $\mathcal{K}_{\mathbb{K},n}$ the set of isomorphism classes of n-dimensional $\mathbb{K}$-rational lattices. Similarly, we denote by $\mathcal{K}^{\mathbf{L}}_{\mathbb{K},n}$ and $\mathcal{K}^{\mathbb{C}_\infty}_{\mathbb{K},n}$ the sets of isomorphism classes of n-dimensional $\mathbb{K}$-rational $\mathbf{L}$ and $\mathbb{C}_\infty$-lattices, for the respective notions of isomorphism.*

The homorphism $\phi : (\mathbb{K}/\mathbf{A})^n \to \mathbb{K}\Lambda/\Lambda$ in Definition 3.2 induces, for any $a \in \mathbf{A}$, maps of $\mathbf{A}$-modules on the a-torsion subgroups

$$\phi_{|_{\mathrm{a-tor}}} : (\mathbf{A}/a\mathbf{A})^n \to a^{-1}\Lambda/\Lambda. \tag{3.4}$$

In the following we give an explicit description of the spaces of isomorphism classes of n-dimensional $\mathbb{K}$-rational lattices and of n-dimensional $\mathbb{K}$-rational $\mathbf{C}_\infty$-lattices. We first need to introduce some preliminary notation (*cf.* [17] §II).
Let

$$\tilde{\Omega}^n = \{\tilde{\omega} = (\omega_1, \dots, \omega_n) \in \mathbf{C}_\infty^n | \{\omega_i\}_{i=1}^n \ \mathbb{K}_\infty - \text{lin.indep}\}$$

and let $\Omega^n = \tilde{\Omega}^n / \mathbf{C}_\infty^*$. The set Ω^n can be also described as the complement of the $\mathbb{K}_\infty$-hyperplanes in $\mathbb{P}^{n-1}(\mathbf{C}_\infty)$.
The adelic description of lattices ([17], §II.1) shows that a matrix $g \in GL_n(\mathbb{A}_{\mathbb{K},f})$ defines a rank n lattice $\Lambda = \Lambda(g) \subset \mathbb{K}^n$ with $\Lambda \cdot R = R^n g^{-1} \subset \mathbb{A}^n_{\mathbb{K},f}$. We write equivalently $\Lambda(g) = R^n g^{-1} \cap \mathbb{K}$. The matrix g acts from the right on $\mathbb{A}^n_{\mathbb{K},f}$.
A point $z \in \Omega^n$ can be identified with a $\mathbb{K}_\infty$-monomorphism $\iota_z : \mathbb{K}^n_\infty \to \mathbf{C}_\infty$, up to the action of $\mathbf{C}^*_\infty$.

Proposition 3.4. *There are identifications*

$$\begin{gathered}\mathrm{GL}_n(\mathbb{K})\backslash \mathrm{GL}_n(\mathbb{A}_{\mathbb{K},f}) \times_{\mathrm{GL}_n(R)} M_n(R) \xrightarrow{\simeq} \mathcal{K}_{\mathbb{K},n}, \\ (g, \rho) \mapsto (\Lambda, \phi) = (R^n g^{-1} \cap \mathbb{K}, \rho g^{-1})\end{gathered} \tag{3.5}$$

$$\begin{gathered}\mathrm{GL}_n(\mathbb{K})\backslash \mathrm{GL}_n(\mathbb{A}_{\mathbb{K},f}) \times \Omega^n \times_{\mathrm{GL}_n(R)} M_n(R) \xrightarrow{\simeq} \mathcal{K}^{\mathbf{C}_\infty}_{\mathbb{K},n}, \\ (g, z, \rho) \mapsto (\Lambda, \phi) = (\iota_z(R^n g^{-1} \cap \mathbb{K}), \iota_z(\rho g^{-1}))\end{gathered} \tag{3.6}$$

Proof. The set of isomorphism classes of n-dimensional lattices can be then identified with the quotient

$$\mathrm{GL}_n(\mathbb{K})\backslash \mathrm{GL}_n(\mathbb{A}_{\mathbb{K},f})/\mathrm{GL}_n(R), \tag{3.7}$$

the double coset of g corresponding to the isomorphism class of the lattice $\Lambda(g)$. Two lattices $\Lambda = R^n g^{-1} \cap \mathbb{K}$ and $\Lambda' = R^n {g'}^{-1} \cap \mathbb{K}$ are isomorphic if and only if there exists an element $\gamma \in \mathrm{GL}_n(\mathbb{K})$ such that $g' = \gamma g$.
Through the identification of $\mathbf{A}$-modules

$$R = \mathrm{Hom}(\mathbb{K}/\mathbf{A}, \mathbb{K}/\mathbf{A}) \tag{3.8}$$

we also have an equivalent description of the homomorphism ϕ of (3.3) by means of an element $\rho \in M_n(R)$. More precisely, by means of the commutative diagram

$$\begin{tikzcd} (\mathbb{K}/\mathbf{A})^n \arrow[r, "\rho"] \arrow[d, "\cong"] & (\mathbb{K}/\mathbf{A})^n \arrow[r] \arrow[d, "\cong"] & \mathbb{K}^n/(R^n g^{-1} \cap \mathbb{K}) \arrow[d, "\cong"] \\ (\mathbb{A}_{\mathbb{K},f}/R)^n \arrow[r, "\rho"] & (\mathbb{A}_{\mathbb{K},f}/R)^n \arrow[r, "g^{-1}"] & \mathbb{A}^n_{\mathbb{K},f}/R^n g^{-1} \end{tikzcd} \tag{3.9}$$

we get $\phi = \rho g^{-1}$. We obtain in this way the map (3.5). This identifies all elements of the form $(gm, m \circ \rho = \rho m)$ for $m \in \mathrm{GL}_n(R)$ to the same $(\Lambda, \phi) \in \mathcal{K}_{\mathbb{K},n}$. Conversely, if $(\Lambda, \phi) = (\Lambda', \phi')$ one has $m = g' g^{-1} \in \mathrm{GL}_n(R)$ and $\rho' = \rho m$, so the identification (3.5) follows.

To obtain the identification (3.6) one proceeds in a similar manner. The set of isomorphism classes of lattices in $\mathbf{C}_\infty$, up to scaling, is identified with the quotient

$$\mathrm{GL}_n(\mathbb{K})\backslash \mathrm{GL}_n(\mathbb{A}_{\mathbb{K},f}) \times \Omega^n / \mathrm{GL}_n(R) \tag{3.10}$$

(*cf.* [17], §II.1 and the notation above). One parameterizes the datum ϕ of (3.3) as in (3.9) and uses the identification $\Omega^n \ni z \mapsto \iota_z : \mathbb{K}_\infty^n \hookrightarrow \mathbf{C}_\infty$. The definition of the map (3.6) follows. As before, one concludes that this map descends to an identification modulo $\mathrm{GL}_n(R)$.
The identifications are, so far, bijections at the set theoretic level, since we have not yet explicitly discussed the topology on the spaces $\mathcal{K}_{\mathbb{K},n}$ and $\mathcal{K}_{\mathbb{K},n}^{\mathbf{C}_\infty}$. In fact the maps (3.5) and (3.6) induce a natural choice of topology on these spaces in which the identifications (3.5) and (3.6) become homeomorphisms. □

Corollary 3.5. *In the case of rank one lattices there are identifications*

$$\mathcal{K}_{\mathbb{K},1} \simeq R \times_{R^*} (\mathbb{A}_{\mathbb{K},f}^*/\mathbb{K}^*) \simeq \mathcal{K}_{\mathbb{K},1}^{\mathbf{C}_\infty}. \tag{3.11}$$

Proof. In the rank one case, lattices can be described in terms of ideals $I \subset \mathbf{A}$. The adelic description gives $I = sR \cap \mathbb{K}$, for some $s \in \mathrm{GL}_1(\mathbb{A}_{\mathbb{K},f})$ (*cf.* [17] §II and [12], Proposition 2.6). Using the identification of $\mathbf{A}$-modules (3.8), we write the datum $\phi : \mathbb{K}/\mathbf{A} \to \mathbb{K}\Lambda/\Lambda$ in terms of an element $\rho \in R$. If $\Lambda = \Lambda'$ then $sR \cap \mathbb{K} = s'R \cap \mathbb{K}$, which implies $s's^{-1} \in R^*$. Then one sees that the map $(\rho, s) \mapsto (\Lambda = sR \cap \mathbb{K}, \rho)$ identifies all pairs $(r^{-1}\rho, rs)$ for $r \in R^*$, to the same $\mathbb{K}$-lattice. The identifications (3.11) are a consequence of the fact that in the rank one case the domain Ω^1 is a point. □

Corollary 3.6. *Let $\tilde{\mathcal{K}}_{\mathbb{K},1}$ denote the set of data (Λ, ϕ) with $\Lambda = \xi I$, for $\xi \in \mathbb{K}_\infty^*$ and $I \subset \mathbf{A}$ an ideal, and with $\phi : \mathbb{K}/\mathbf{A} \to \mathbb{K}\Lambda/\Lambda$ an $\mathbf{A}$-module homomorphism. There is an identification*

$$\tilde{\mathcal{K}}_{\mathbb{K},1} \simeq R \times_{R^*} (\mathbb{A}_{\mathbb{K}}^*/\mathbb{K}^*). \tag{3.12}$$

Proof. The proof is analogous to that of Corollary 3.5, where one writes $\Lambda = \xi(sR \cap \mathbb{K})$ and one considers the map $(\rho, s, \xi) \mapsto (\Lambda = \xi(sR \cap \mathbb{K}), \phi = \xi\rho)$ which identifies all elements of the form $(r^{-1}\rho, rs, r\xi)$ for $r \in R^*$. Conversely, if $\Lambda = \Lambda'$, then $\xi'\xi^{-1} \in \mathbb{K}^*$ and $sR \cap \mathbb{K} = s'R \cap \mathbb{K}$, which gives $s's^{-1} \in R^*$. (3.12) follows using the description $\mathbb{A}_{\mathbb{K}}^* = \mathbb{A}_{\mathbb{K},f}^* \times \mathbb{K}_\infty^*$. □

3.3. **Commensurability.**

On the set $\mathcal{K}_{\mathbb{K},n}$ of isomorphism classes of n-dimensional $\mathbb{K}$-lattices we consider the equivalence relation of commensurability and the corresponding quotient space. The results described in this section are the function field analogs of the ones exposed in [12].

Definition 3.7. *Two elements (Λ, ϕ) and (Λ', ϕ') of $\mathcal{K}_{\mathbb{K},n}$ are commensurable if there exists $\gamma \in \mathrm{GL}_n(\mathbb{K})$ such that $\Lambda' = \Lambda\gamma$ and $\phi' = \gamma \circ \phi$ according to the commutative diagram*

$$\begin{array}{ccccc} \phi : (\mathbb{K}/\mathbf{A})^n & \xrightarrow{\rho} & (\mathbb{K}/\mathbf{A})^n & \xrightarrow{g^{-1}} & \mathbb{K}^n/(R^n g^{-1} \cap \mathbb{K}) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \gamma^{-1} \\ \phi' : (\mathbb{A}_{\mathbb{K},f}/R)^n & \xrightarrow{\rho} & (\mathbb{A}_{\mathbb{K},f}/R)^n & \xrightarrow{g^{-1}\gamma^{-1}} & \mathbb{A}_{\mathbb{K},f}^n/R^n g^{-1}\gamma^{-1} \end{array} \tag{3.13}$$

With this definition, it is immediate to verify that commensurability is an equivalence relation. One defines the commensurability relation on $\mathcal{K}_{\mathbb{K},n}^{\mathbf{C}_\infty}$ and on $\tilde{\mathcal{K}}_{\mathbb{K},1}$ in the same way.

Definition 3.8. *We denote by $\mathcal{L}_{\mathbb{K},n}$, $\mathcal{L}_{\mathbb{K},n}^{\mathbf{C}_\infty}$, and $\tilde{\mathcal{L}}_{\mathbb{K},1}$, respectively the quotients of $\mathcal{K}_{\mathbb{K},n}$, $\mathcal{K}_{\mathbb{K},n}^{\mathbf{C}_\infty}$, and $\tilde{\mathcal{K}}_{\mathbb{K},1}$ by the commensurability relation introduced in Definition 3.7.*

We have the following description of these quotient spaces.

Theorem 3.9. *The map*

$$\Theta : \mathrm{GL}_n(\mathbb{K})\backslash \mathrm{GL}_n(\mathbb{A}_{\mathbb{K},f}) \times_{\mathrm{GL}_n(R)} M_n(R) \to \mathrm{GL}_n(\mathbb{K})\backslash M_n(\mathbb{A}_{\mathbb{K},f}), \quad (g,\rho)\mapsto \rho g^{-1} \tag{3.14}$$

induces an identification of the set $\mathcal{L}_{\mathbb{K},n}$ with the quotient $\mathrm{GL}_n(\mathbb{K})\backslash M_n(\mathbb{A}_{\mathbb{K},f})$, where the action of $\gamma \in \mathrm{GL}_n(\mathbb{K})$ is given by $u \mapsto u\gamma^{-1}$, for $u \in M_n(\mathbb{A}_{\mathbb{K},f})$.
Similarly, the map

$$\begin{gathered}\Upsilon : \mathrm{GL}_n(\mathbb{K})\backslash \mathrm{GL}_n(\mathbb{A}_{\mathbb{K},f}) \times \Omega^n \times_{\mathrm{GL}_n(R)} M_n(R) \to \mathrm{GL}_n(\mathbb{K})\backslash M_n(\mathbb{A}_{\mathbb{K},f}) \times \Omega^n/\mathrm{GL}_n(R) \\ \Upsilon(g,z,\rho) = (\rho g^{-1}, z)\end{gathered} \tag{3.15}$$

induces an identification of the space $\mathcal{L}^{\mathbf{C}_\infty}_{\mathbb{K},n}$ with the quotient $\mathrm{GL}_n(\mathbb{K})\backslash M_n(\mathbb{A}_{\mathbb{K},f}) \times \Omega^n/\mathrm{GL}_n(R)$.

Proof. The map Θ is well defined on the quotient by $\mathrm{GL}_n(R)$, since the image ρg^{-1} is invariant under $\rho \mapsto \rho m$ and $g \mapsto gm$. It is surjective. It remains to show that elements (g,ρ) and (g',ρ') are mapped to the same image when they represent commensurable $\mathbb{K}$-rational lattices $(\Lambda,\phi) \sim (\Lambda',\phi')$. Indeed, one sees that if $\Lambda' = \Lambda\gamma$ and $\phi' = \gamma\circ\phi$ then the elements ρg^{-1} and $\rho g^{-1}\gamma^{-1}$ determine the same class in the quotient $\mathrm{GL}_n(\mathbb{K})\backslash M_n(\mathbb{A}_{\mathbb{K},f})$. Conversely, the condition $\rho g^{-1} = \rho' g'^{-1}\gamma^{-1}$ with $\gamma \in \mathrm{GL}_n(\mathbb{K})$ implies $\phi' = \gamma\circ\phi$. At the level of the lattices one has $\Lambda = R^n(gm)^{-1}\cap\mathbb{K}$ and $\Lambda' = R^n(gm')^{-1}\gamma^{-1}\cap\mathbb{K}$ for some $m,m' \in \mathrm{GL}_n(R)$, which yields commensurable data $(\Lambda,\phi)\sim(\Lambda',\phi')$. The proof is similar in the case of $\mathcal{L}^{\mathbf{C}_\infty}_{\mathbb{K},n}$. □

For rank one lattices, the result above specializes as follows.

Corollary 3.10. *In the rank 1 case, the map Θ as in Theorem 3.9 induces an identification of the space $\mathcal{L}_{\mathbb{K},1}$ with the quotient $\mathbb{A}_{\mathbb{K},f}/\mathbb{K}^*$. Similarly, the space $\tilde{\mathcal{L}}_{\mathbb{K},1}$ is identified with the quotient $\mathbb{A}^{\cdot}_{\mathbb{K}}/\mathbb{K}^*$ via the map*

$$\tilde\Theta : R\times_{R^*} \mathbb{A}^*_{\mathbb{K}}/\mathbb{K}^* \to \mathbb{A}^{\cdot}_{\mathbb{K}}/\mathbb{K}^*, \quad \tilde\Theta(\rho,s,\xi) = \rho s^{-1}\xi^{-1} \tag{3.16}$$

where $\mathbb{A}^{\cdot}_{K} = \mathbb{A}_{\mathbb{K},f}\times\mathbb{K}^*_\infty$.

It is important to observe that, while the spaces

$$\mathrm{GL}_n(\mathbb{K})\backslash \mathrm{GL}_n(\mathbb{A}_{\mathbb{K},f})/\mathrm{GL}_n(R) \quad \text{and} \quad \mathrm{GL}_n(\mathbb{K})\backslash \mathrm{GL}_n(\mathbb{A}_{\mathbb{K},f})\times\Omega^n/\mathrm{GL}_n(R)$$

parameterizing lattices, as well as the spaces $\mathcal{K}_{\mathbb{K},n}$, $\tilde{\mathcal{K}}_{\mathbb{K},1}$ and $\mathcal{K}^{\mathbf{C}_\infty}_{\mathbb{K},n}$ are ordinary classical quotients, this is no longer the case for the quotients $\mathcal{L}_{\mathbb{K},n}$, $\mathcal{L}^{\mathbf{C}_\infty}_{\mathbb{K},n}$ and $\tilde{\mathcal{L}}_{\mathbb{K},1}$. In fact, in such quotients the action of $\mathrm{GL}_n(\mathbb{K})$ on $M_n(\mathbb{A}_{\mathbb{K},f})$ is no longer discrete. To continue our geometric study on them we need to treat these quotients as noncommutative spaces. This means that we shall associate to each of them a suitable noncommutative algebra of coordinates obtained as a convolution algebra associated to the groupoid of the equivalence relation. This algebra will be viewed as the ring of functions of the corresponding quotient. In practice, the process of setting up the right convolution algebra can be quite involved, we refer for example to the work of E. Ha and F. Paugam for the description of similar adelic quotients in the study of noncommutative spaces arising from Shimura varieties ([22]). In this paper, we do not treat the general case of n-dimensional $\mathbb{K}$-rational lattices, rather we will mainly concentrate on the rank one case.

The following result describes the relation between isomorphism classes of $\mathbb{K}$-rational $\mathbf{L}$-lattices and pointed Drinfeld $\mathbf{A}$-modules over $\mathbf{L}$. For the rest of this section, we take $\mathbf{L} = \mathbf{C}_\infty$.

Theorem 3.11. *The equivalence of categories between* $\mathbf{C}_\infty$*-lattices and Drinfeld* $\mathbf{A}$*-modules over* $\mathbf{C}_\infty$ *induces the identification*

$$\mathcal{L}^{\mathbf{C}_\infty}_{\mathbb{K},n} \simeq \mathcal{D}^{\mathbf{C}_\infty}_{\mathbb{K},n} \tag{3.17}$$

of the spaces of n*-dimensional* $\mathbb{K}$*-rational* $\mathbf{C}_\infty$*-lattices and of* n*-pointed Drinfeld modules, up to the respective commensurability relations.*

Proof. We consider the map $\mathcal{K}^{\mathbf{C}_\infty}_{\mathbb{K},n} \to \mathcal{D}^{\mathbf{C}_\infty}_{\mathbb{K},n}$ that assigns to a pair (Λ, ϕ) the class of the n-pointed Drinfeld module $(\Phi, \zeta_1, \ldots, \zeta_n)$ with $\Phi = \Phi^\Lambda$ and $\zeta_i = \hat{\phi}(e_i)$. Here, $\{e_i\}$ denotes the standard basis of R^n as an R-module and by taking $\mathrm{Hom}(-, \mathbb{K}/\mathbf{A})$ we identify the $\mathbf{A}$-homomorphism $\phi : \mathbb{K}/\mathbf{A} \to \mathbb{K}\Lambda/\Lambda$ with the datum of an R-homomorphism $\hat{\phi} : R^n \to \Lambda \otimes_{\mathbf{A}} R$. Using the description of the identification (3.6) as given in the proof of Proposition 3.4, it is not hard to check that two commensurable elements (Λ, ϕ) and (Λ', ϕ') map to the same element in $\mathcal{D}^{\mathbf{C}_\infty}_{\mathbb{K},n}$. In fact, the equivalence $(\Lambda, \phi) \sim (\Lambda', \phi')$ is described by stating that the elements (g, z, ρ) and (g', z', ρ) defining the two pairs in $\mathrm{GL}_n(\mathbb{A}_{\mathbb{K},f}) \times \Omega^n \times_{\mathrm{GL}_n(R)} M_n(R)$ are connected through the relations $g' = \gamma g$ and $z' = \gamma z \lambda$ for some $g \in \mathrm{GL}_n(\mathbb{K})$ and $\lambda \in \mathbf{C}^*_\infty$. We write the lattices as $\Lambda = \iota_z(R^n g^{-1} \cap \mathbb{K})$ and

$$\Lambda' = \iota_{z'}(R^n {g'}^{-1} \cap \mathbb{K}) = \lambda \iota_z(R^n(\gamma g)^{-1} \cap \mathbb{K}). \tag{3.18}$$

Then, the corresponding R-homomorphisms fit in the following commutative diagram

$$\begin{tikzcd}
 & R^n \arrow[r, "g^{-1}"] & R^n g^{-1} \arrow[r, "\simeq"] & \Lambda \otimes R = T\Phi \arrow[dd, "TP"] \\
R^n \arrow[ur, "\rho"] \arrow[urrr, "\hat{\phi}"'] \arrow[dr, "\rho"'] \arrow[drrr, "\hat{\phi}'"] & & & \\
 & R^n \arrow[r, "(\gamma g)^{-1}"] & R^n(\gamma g)^{-1} \arrow[r, "\lambda"'] & \Lambda' \otimes R^n = T\Psi
\end{tikzcd} \tag{3.19}$$

The element $\lambda \in \mathbf{C}^*_\infty$ determines an isogeny P of the corresponding Drinfeld modules $\Phi = \Phi^\Lambda = \Phi^{\Lambda\gamma}$ and $\Psi = \Psi^{\Lambda'}$. (3.19) shows that the elements $(\zeta_1, \ldots, \zeta_n) \in T\Phi$ and $(\xi_1, \ldots, \xi_n) \in T\Psi$ with $\zeta_i = \hat{\phi}(e_i)$ and $\xi_i = \hat{\phi}'(e_i)$ are related by the induced map $TP : T\Phi \to T\Psi$ as required. Conversely, assume that the data $(\Phi, \zeta_1, \ldots, \zeta_n)$ and $(\Psi, \xi_1, \ldots, \xi_n)$ are commensurable, namely that they are related by an isogeny $P : \Phi \to \Psi$ of Drinfeld modules. In the equivalence of categories between Drinfeld modules and lattices, such a morphism P corresponds to a morphism of lattices, that is, an element $\lambda \in \mathbf{C}_\infty$ with $\lambda\Lambda \subset \Lambda'$. If $\lambda \neq 0$ this implies that $\mathbb{K}\lambda\Lambda$ and $\mathbb{K}\Lambda'$ span the same n-dimensional $\mathbb{K}$-vector space inside $\mathbf{C}_\infty$, hence there exists an element $\gamma \in \mathrm{GL}_n(\mathbb{K})$ such that $\Lambda' = \lambda\Lambda\gamma$. This is equivalent to say that we can represent Λ and Λ' as in (3.18). Moreover, the relation $(\xi_i) = TP(\zeta_i)$ implies that $\hat{\phi}' = \lambda(\gamma \circ \hat{\phi})$, which in turn gives the expected relation of commensurability for the data (Λ, ϕ) and (Λ', ϕ'). $\square$

3.4. **A remark on the adeles class space.**

An important example of a noncommutative adelic quotient is the *adele class space* $\mathbb{A}_\mathbb{K}/\mathbb{K}^*$, introduced by Connes in [7] as the geometric space on which he formulated his result on the spectral realization of the zeros of L-functions with Grössencharakter. The crucial role of the adele class space in Connes' work is that it is on this space that the (semi-local) Lefschetz trace formula takes place. An important property of this trace formula is that it breaks up as a

sum of contributions associated to the individual places of $\mathbb{K}$, although the adele class space is essentially a product. This means that the contributions to the trace formula come separately from the strata $\mathbb{A}_{\mathbb{K},v}/\mathbb{K}^*$.
In the function field case, we sketch here briefly an interpretation of the adeles class space and of the strata $\mathbb{A}_{\mathbb{K},v}/\mathbb{K}^*$ in terms of a noncommutative moduli space for pointed Drinfeld modules analogous to what we obtained in Corollary 2.5 and the subsequent interpretation $\mathcal{D}_{\mathbb{K},n} = \mathcal{M}^n_{nc}$ of the space of commensurability classes of n-pointed Drinfeld modules as a noncommutative generalization of the moduli space $\mathcal{M}^n$ of Drinfeld modules where one allows degenerate level structures.
The group $\mathrm{GL}_n(\mathbb{A}_{\mathbb{K},f})/\mathbb{K}^*$ acts (from the left) on $\mathcal{M}^n$. In particular, in the rank one case and with the level structure divisible by two distinct primes, Drinfeld showed ([15], § 8) that the moduli scheme $\mathcal{M}^1$ is the spectrum of the ring of integers of the maximal abelian extension of $\mathbb{K}$ completely split at ∞ and the action of $\mathbb{A}^*_{\mathbb{K},f}/\mathbb{K}^*$ on $\mathcal{M}^1$ coincides with the action in class field theory. Moreover, there is an identification $\mathcal{M}^1(\mathbb{K}_\infty) = \mathcal{M}^1(\overline{\mathbb{K}}_\infty) = \mathbb{A}^*_{\mathbb{K},f}/\mathbb{K}^*$ consistent with the action of $\mathbb{A}^*_{\mathbb{K},f}/\mathbb{K}^*$. This should be compared in our noncommutative setting with the fact that $\mathcal{D}_{\mathbb{K},1} = \mathbb{A}_{\mathbb{K},f}/\mathbb{K}^*$ (Corollary 3.10 and Theorem 3.11). Moreover, as we discuss in §4.5 below, one recovers the action of $\mathbb{A}^*_{\mathbb{K},f}/\mathbb{K}^*$ as symmetries of a natural quantum statistical mechanical system associated to the noncommutative space $\mathcal{D}_{\mathbb{K},1}$.
In this subsection, we give a brief outline of an argument that relates the adeles class space $\mathbb{A}_{\mathbb{K}}/\mathbb{K}^*$ in a similar way to another moduli space introduced by Drinfeld in [16], namely the covering $\tilde{\mathcal{M}}^1$ of $\mathcal{M}^1$.

The general construction of the schemes $\tilde{\mathcal{M}}^n$ given in [16] is obtained by considering the universal Drinfeld module over $\mathcal{M}^n$. Using the property that $\mathcal{M}^n = \mathrm{Spec}(B)$ is an affine scheme, the universal rank n Drinfeld module is defined as a homomorphism $\Phi : \mathbf{A} \to B\{\tau\}$. One can eventually introduce some extra information in the moduli problem, which plays the role of a "level ∞ structure" on the universal Drinfeld module. The resulting space is a scheme $\tilde{\mathcal{M}}^n$ endowed with two actions. The first action is given by the group $\mathrm{GL}_n(\mathbb{A}_{\mathbb{K},f})$ and is induced by a corresponding action on $\mathcal{M}^n$. The second action is produced by the multiplicative group of units of a central division algebra D over $\mathbb{K}_\infty$. For any open normal subgroup $U \subset D^*$, the quotient $U\backslash\tilde{\mathcal{M}}^n$ is a Galois covering of $\mathcal{M}^n$ with Galois group $\hat{D}^*/U$, where $\hat{D}^*$ denotes the profinite completion of D^*.
In the rank $n = 1$ case Drinfeld showed that the scheme $\tilde{\mathcal{M}}^1$ is the spectrum of the ring of integers of the maximal abelian extension of $\mathbb{K}$ and that the group $D^* \times \mathbb{A}^*_{\mathbb{K},f} = \mathbb{K}^*_\infty \times \mathbb{A}^*_{\mathbb{K},f} = \mathbb{A}^*_{\mathbb{K}}$ acts on $\tilde{\mathcal{M}}^1$ with the action of class field theory.
In view of these facts we can reinterpret the identification

$$\tilde{\mathcal{L}}_{\mathbb{K},1} \simeq \mathbb{A}^{\cdot}_{\mathbb{K}}/\mathbb{K}^* \tag{3.20}$$

obtained in Corollary 3.10 as the noncommutative version of $\tilde{\mathcal{M}}^1$, which we write as $\tilde{\mathcal{M}}^1_{nc} = \tilde{\mathcal{L}}_{\mathbb{K},1}$, extending in this way the interpretation

$$\mathcal{M}^1_{nc} = \mathcal{L}_{\mathbb{K},1} \tag{3.21}$$

arising from (3.17). In terms of pointed Drinfeld modules, we can further reinterpret the noncommutative space $\tilde{\mathcal{L}}_{\mathbb{K},1}$ as the set of commensurability classes of 1-pointed Drinfeld modules together with an ∞ level structure. In particular we notice that the description of $\tilde{\mathcal{M}}^1_{nc}$ as a quotient of $\mathbb{A}^{\cdot}_{\mathbb{K}} = \mathbb{A}_{\mathbb{K},f} \times \mathbb{K}^*_\infty$ corresponds to the fact that the datum of the ∞ level structure is kept always non-degenerate even though the finite level structures may possibly degenerate. One can further enlarge this noncommutative space by also allowing for degenerate ∞ level structure corresponding to the point $0 \in \mathbb{K}_\infty$. In terms of $\mathbb{K}$-rational lattices, this means considering the space of commensurability classes of data (Λ, ρ) with $\Lambda = \xi I$, for $\xi \in \mathbb{K}_\infty$, $I \subset \mathbf{A}$ an ideal, and $\rho : \mathbb{K}/\mathbf{A} \to \mathbb{K}/\mathbf{A}$ an $\mathbf{A}$-module homomorphism. One obtains in this way the adeles class space $\mathbb{A}_{\mathbb{K}}/\mathbb{K}^*$.

We did not develop in detail here the formulation in terms of ∞ level structure and the construction of Drinfeld [16], because this is beyond the purpose of this paper and not directly needed in the following, but this certainly deserves further investigation.

One can give a similar description of the strata considered in the trace formula of [7] on the adele class space. For $v \in \Sigma_{\mathbf{A}}$, consider the $\mathbb{K}^*$-invariant subspace $\mathbb{A}_{\mathbb{K},v} \subset \mathbb{A}_{\mathbb{K}}$ defined by

$$\mathbb{A}_{\mathbb{K},v} = \{a = (a_w)_{w \in \Sigma_{\mathbb{K}}} \in \mathbb{A}_{\mathbb{K}} \,|\, a_v = 0\}. \tag{3.22}$$

In terms of $\mathbb{K}$-rational lattices, this corresponds to the space $\tilde{\mathcal{K}}_{\mathbb{K},1,v}$ of data (Λ, ρ) as above with the further property that the v-component ρ_v of the induced R-homomorphism $\rho : R \to R$ vanishes. The isomorphism

$$\tilde{\mathcal{L}}_{\mathbb{K},1,v} \simeq \mathbb{A}_{\mathbb{K},v}/\mathbb{K}^* \tag{3.23}$$

follows easily from Corollary 3.10.

4. Quantum Statistical Mechanics over function fields

The definition of a quantum statistical mechanical system usually includes a (unital) complex C^*-algebra $\mathcal{A}$ of observables and a time evolution, that is, a homomorphism $\sigma : \mathbb{R} \to \mathrm{Aut}(\mathcal{A})$ ([4], [23]). These data are complemented by the notion of states on $\mathcal{A}$, namely continuous linear functionals $\varphi : \mathcal{A} \to \mathbb{C}$ which satisfy the two properties of positivity and normalization

$$\varphi(a^* a) \geq 0, \quad \forall a \in \mathcal{A}, \qquad \varphi(1) = 1. \tag{4.1}$$

Notice that in this C^*-algebra context the continuity requirement is redundant with positivity, but not in our generalization to function fields below.
In the non-unital case the condition $\varphi(1) = 1$ is replaced by the requirement that φ is of norm one. On a quantum statistical mechanical system $(\mathcal{A}, \sigma)$ there is also a good notion of thermodynamic equilibrium states. These are states satisfying the KMS (Kubo–Martin–Schwinger) condition, which depends on a termodynamic parameter: the *inverse temperature* $\beta \in \mathbb{R}_+ \cup \{\infty\}$. In the case $\beta = 0$, KMS states are just traces, while in the case $\beta = \infty$ a good notion of KMS states is obtained as in [10] by considering weak limits

$$\varphi_\infty(a) = \lim_{\beta \to \infty} \varphi_\beta(a), \quad \forall a \in \mathcal{A},$$

of extremal KMS$_\beta$-states. For the purposes of this paper, we restrict to the case $\beta \in \mathbb{R}_+^*$.

For $\beta \in \mathbb{R}_+^*$, consider the strip $I_\beta = \{w \in \mathbb{C} : 0 < \Im(w) < \beta\}$. A state φ satisfies the KMS$_\beta$ condition if, for any $f_1, f_2 \in \mathcal{A}$ there exists a bounded continuous function $F_{f_1,f_2}(z)$ defined on the closure $z \in \overline{I_\beta}$, holomorphic on I_β and with the property that

$$F_{f_1,f_2}(t) = \varphi(f_1 \sigma_t(f_2)), \quad \text{and} \quad F_{f_1,f_2}(t + i\beta) = \varphi(\sigma_t(f_2) f_1), \quad \forall t \in \mathbb{R}. \tag{4.2}$$

One knows that there exists a norm dense $*$-subalgebra $\mathcal{A}^{an} \subset \mathcal{A}$ such that, for all $f \in \mathcal{A}^{an}$ the map $t \mapsto \sigma_t(f)$ extends to an entire function (*cf.* [4] Corollary 2.5.23).

It is also well known that the KMS$_\beta$ condition (4.2) can be restated in the following equivalent form ([5], Definition 5.3.1 and Corollary 5.3.7). Given a C^*-dynamical system $(\mathcal{A}, \sigma)$, a state φ on $\mathcal{A}$ is a KMS$_\beta$ state for the time evolution σ if the identity

$$\varphi(f_1 \, \sigma_{i\beta}(f_2)) = \varphi(f_2 f_1), \tag{4.3}$$

holds for all f_1, f_2 in a norm dense and σ-invariant $*$-subalgebra of $\mathcal{A}^{an}$. For $f_1 \in \mathcal{A}$ and $f_2 \in \mathcal{A}^{an}$, the holomorphic function $F_{f_1,f_2}(z)$ in the definition of a KMS$_\beta$ state is given by the analytic continuation

$$F_{f_1,f_2}(z) = \varphi(f_1 \sigma_z(f_2)), \quad \forall z \in \mathbb{C}. \tag{4.4}$$

In this context of C^*-algebras, one also speaks of *extremal* KMS states. In fact, one can show (*cf.* [5]) that the set of KMS$_\beta$ states is a Choquet simplex. This implies that there is a well defined notion of extremal KMS states, given by the extremal points of the simplex. The fact that one can decompose states as convex combinations of extremal ones is closely relies upon the positivity property of states, as one can see explicitly in Theorem 5.3.30 of [5].

A quantum statistical mechanical system associated to (sign-normalized) rank one Drinfeld modules was recently introduced and analyzed by Benoit Jacob in [24]. It would be interesting to re-interpret this system in terms of commensurability classes of $\mathbb{K}$-rational lattices (as for the reinterpretation of the Bost–Connes system in terms of $\mathbb{Q}$-lattices ([10])), and also to consider higher rank generalizations of it as was done in [10] in the case of 2-dimensional $\mathbb{Q}$-lattices.

In this paper we take a different viewpoint. One of the most intriguing and interesting aspects of the quantum statistical mechanical systems associated to arithmetic objects such as number fields or Shimura varieties is the "intertwining property" between the Galois action on values of extremal KMS$_\infty$ states evaluated on a suitable rational subalgebra and the symmetries of the quantum statistical mechanical system. This general principle has so far been successful at recasting, within the context and methods of quantum statistical mechanics, the explicit class field theory for $\mathbb{Q}$ and for imaginary quadratic fields ([3], [10], [12]). Other recent constructions for number fields and Shimura varieties ([22]) may lead to a successful description of other cases of explicit class field theory such as that of abelian varieties with complex multiplication. The system constructed in [24] for function fields appears also promising in this respect, mainly for the reason that sign-normalized rank one Drinfeld modules play a fundamental role in the explicit class field theory for function fields. Unfortunately though, the desired intertwining property is lost when one works with C^*-algebras and complex valued states. For this reason, it is important to try to introduce a suitable version of the quantum statistical mechanical formalism by working entirely in positive characteristic. In the following we present some steps towards this goal. In this paper we concentrate only on the case of the noncommutative space of commensurability classes of 1-pointed Drinfeld modules (equivalently of 1-dimensional $\mathbb{K}$-rational lattices), where we use the geometry of Drinfeld modules as a guiding principle in the process of building the appropriate general strategy.

4.1. **Convolution algebras.**

In this section we define an algebra of observables that is naturally associated to the space we are interested in, that is, the quotient $\mathbb{A}_{\mathbb{K},f}/\mathbb{K}^*$ of the set of 1-pointed Drinfeld modules by the relation of commensurability. As we remarked already, this quotient should be regarded as a noncommutative space, hence a natural candidate for its algebra of continuous functions is a convolution algebra associated to the groupoid of the equivalence relation of commensurability.

Definition 4.1. *Let* $\mathbf{L}$ *be a complete subfield of* $\mathbf{C}_\infty$ *that contains* $\mathbb{K}_\infty$*. For* $L = (\Lambda, \phi) \sim L' = (\Lambda', \phi')$ *commensurable* $\mathbb{K}$*-lattices in* $\mathcal{K}_{\mathbb{K},1}$*, we denote by* $\mathcal{A}_{\mathbf{L}}(\mathcal{L}_{\mathbb{K},1})$ *the algebra of continuous, compactly supported,* $\mathbf{L}$*-valued functions* $f(L, L')$ *with the convolution product*

$$(f_1 * f_2)(L, L') = \sum_{L \sim L'' \sim L'} f_1(L, L'') f_2(L'', L'). \tag{4.5}$$

The definition of the algebras $\mathcal{A}_{\mathbf{L}}(\mathcal{L}^{\mathbf{C}_\infty}_{\mathbb{K},1})$ *and* $\mathcal{A}_{\mathbf{L}}(\tilde{\mathcal{L}}_{\mathbb{K},1})$ *is analogous. To simplify notation, in the following we often write* $\mathcal{A}(\mathcal{L}_{\mathbb{K},1})$ *for* $\mathcal{A}_{\mathbf{C}_\infty}(\mathcal{L}_{\mathbb{K},1})$.

Similarly, we can form a convolution algebra $\mathcal{A}_{\mathbf{L}}(\mathcal{D}_{\mathbb{K},1})$ of compactly supported, $\mathbf{L}$-valued functions evaluated on pairs of commensurable 1-pointed Drienfeld modules

$$f((\Phi,\zeta),(\Psi,\xi)), \qquad (\Phi,\zeta)\sim(\Psi,\xi). \tag{4.6}$$

The algebra $\mathcal{A}_{\mathbf{L}}(\mathcal{D}_{\mathbb{K},1})$ is endowed with the convolution product

$$(f_1 * f_2)((\Phi,\zeta),(\Psi,\xi)) = \sum_{(\Phi,\zeta)\sim(\Xi,\eta)\sim(\Psi,\xi)} f_1((\Phi,\zeta),(\Xi,\eta)) f_2((\Xi,\eta),(\Psi,\xi)). \tag{4.7}$$

Definition 4.2. *For* $L=(\Lambda,\phi)\in\mathcal{K}_{\mathbb{K},1}$ *a* $\mathbb{K}$*-rational lattice, we denote by* $c(L)=\{L'\in\mathcal{K}_{\mathbb{K},1}\,|\,L'\sim L\}$ *be the commensurability class of* L*. We let* $\mathcal{V}_L$ *denote the vector space of compactly supported* $\mathbf{L}$*-valued functions on* $c(L)$.

There is no good analog of the theory of Hilbert spaces in the non-archimedean setting, but one can work with Banach spaces (*cf.* [2], [28]). For instance, the completion of $\mathcal{V}_L$ in the norm $\|\xi\|=\sup_{L'\in c(L)}|\xi(L')|$, with $|\cdot|$ the non-archimedean absolute value, is a non-archimedean Banach space. For simplicity of notation we still denote such a norm completion by $\mathcal{V}_L$.

Lemma 4.3. *There is a representation* $\pi_L:\mathcal{A}_{\mathbf{L}}(\mathcal{L}_{\mathbb{K},1})\to\mathrm{End}(\mathcal{V}_L)$ *given by*

$$\pi_L(f)(\xi)(L') = \sum_{L''\in c(L)} f(L',L'')\xi(L''). \tag{4.8}$$

Proof. It follows from the definition (4.8) that π_L is compatible with the convolution product (4.5), hence it defines a representation of $\mathcal{A}_{\mathbf{L}}(\mathcal{L}_{\mathbb{K},1})$ on $\mathcal{V}_L$. ☐

The elements of $\mathcal{A}_{\mathbf{L}}(\mathcal{L}_{\mathbb{K},1})$ act on the normed vector space $\mathcal{V}_L$ by bounded linear operators, hence we can make $\mathcal{A}_{\mathbf{L}}(\mathcal{L}_{\mathbb{K},1})$ into a topological algebra by the operator norm

$$\|f\|_{\pi_L} = \sup_{\xi\neq 0\in\mathcal{V}_L}\frac{\|\pi_L(f)(\xi)\|}{\|\xi\|}. \tag{4.9}$$

The completion in this norm is a non-archimedean Banach algebra (*cf.* [2], [28]), which we still denote by $\mathcal{A}_{\mathbf{L}}(\mathcal{L}_{\mathbb{K},1})$ for simplicity.

4.2. **A comment on involutions.**

In the context of C^*-algebras, one introduces a similar type of convolutions algebra, by considering compactly supported $\mathbb{C}$-valued functions on the equivalence classes, with the convolution product dictated by the equivalence relation. This algebra is endowed with the natural involution $f^*(L,L')=\overline{f(L',L)}$ obtained by considering complex conjugation on the values of the function. The involution property of the algebra plays a crucial role in all the issues related to positivity, and in particular when one takes convex combinations of states. In the function field case (in positive characteristic), the problem of finding a suitable replacement for the positivity properties is a nontrivial problem.

We consider a choice of a sign function, defined as follows (*cf.* [18], Definition 7.2.1).

Definition 4.4. *A sign function is a homomorphism* $\mathrm{sign}:\mathbb{K}_\infty^*\to\mathbb{F}^*_{q^{d_\infty}}$ *that restricts to the identity on* $\mathbb{F}^*_{q^{d_\infty}}$*. One sets* $\mathrm{sign}(0)=0$.

Such a choice gives rise to a notion of positivity in the function field setting. An element $x \in \mathbb{K}_\infty^*$ is positive if $\mathrm{sign}(x) = 1$.
There are $\#\mathbb{F}_{q^{d_\infty}}^* = q^{d_\infty} - 1$ choices of the sign function and they differ from one another by $\mathrm{sign}'(x) = \mathrm{sign}(x)\xi^{\deg(x)/d_\infty}$, for some $\xi \in \mathbb{F}_{q^{d_\infty}}^*$. (*cf.* [18], Proposition 7.2.3.)

Let u_∞ be a uniformizer at ∞. Notice that a choice of a uniformizer implies a choice of a sign function as in Example 7.2.2 of [18]. This is obtained by writing $x \in \mathbb{K}_\infty^*$ as $x = u_\infty^m \zeta\gamma$ with $m \in \mathbb{Z}$, $\zeta \in \mathbb{F}_{q^{d_\infty}}^*$ and $\gamma \in U_1$, where U_1 is the group of 1-units, that is, the set of elements $u \in \mathcal{O}_\infty$ with $u \equiv 1$ modulo the maximal ideal m_∞ of the ring of integers $\mathcal{O}_\infty$ of $\mathbb{K}_\infty$. One then sets $\mathrm{sign}(x) = \zeta$.
Having made such a choice of uniformizer and of the corresponding sign function as above, we are considering the decomposition of $\mathbb{K}_\infty = \mathbb{F}_{q^{d_\infty}}((u_\infty))$ given as above in the form

$$\mathbb{K}_\infty^* = \mathbb{F}_{q^{d_\infty}}^* \times u_\infty^{\mathbb{Z}} \times U_1. \tag{4.10}$$

The multiplicative decomposition (4.10) of $\mathbb{K}_\infty^*$ makes it possible to define an involution on $\mathbb{K}_\infty^*$. Upon decomposing elements $x \in \mathbb{K}_\infty^*$ as

$$x = \mathrm{sign}(x)\, u_\infty^{v_\infty(x)}\, \langle x\rangle, \tag{4.11}$$

One can define an involution on elements $x \in \mathbb{K}_\infty^*$ as the group (anti)homomorphism

$$x \mapsto \bar{x} = \mathrm{sign}(x)^{-1}\, u_\infty^{v_\infty(x)}\, \langle x\rangle. \tag{4.12}$$

Although the decomposition (4.11) can be seen as an analog of the decomposition $z = |z|e^{i\theta}$ in $\mathbb{C}$, the corresponding involution (4.12) does not satisfy the additivity property $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ that the involution $z \mapsto \bar{z} = |z|e^{-i\theta}$ has. Thus, (4.12) cannot be used as a replacement of complex conjugation to define an adjoint on the algebras.

We shall discuss later how one can to some extent bypass this problem in the process of defining extremal KMS states for the particular time evolution that we construct on the algebra of coordinates of the noncommutative space of commensurability classes of 1-dimensional $\mathbb{K}$-lattices up to scaling. In fact, in this case we can use presence of an underlying classical moduli problem (in our case, the moduli space of Drinfeld modules) in order to construct KMS states. At present, there is however no general theory relating KMS states to classical moduli problems.

4.3. **Time evolution.**

In this section we introduce a notion of time evolution on the convolution algebra of commensurability classes of $\mathbb{K}$-rational lattices, which generalizes, in the function field context, the time evolution on $\mathbb{Q}$-lattices and on $\mathbb{Q}(\sqrt{-d})$-lattices introduced in [10] and [12].

We start off by recalling that in function field arithmetic, while the analog of $\mathbb{C}$ as a field is provided by the field $\mathbf{C}_\infty$, a good analog of the complex line (where for instance the domain of definition of L-functions lies) is played by the topological group

$$S_\infty = \mathbf{C}_\infty^* \times \mathbb{Z}_p. \tag{4.13}$$

For $t_1 = (x_1, y_1)$ and $t_2 = (x_2, y_2)$ in S_∞, the group operation in S_∞ is given by the rule $t_1 + t_2 = (x_1x_2, y_1 + y_2)$.
Inside S_∞ one considers the "line"

$$S_\infty \supset \{s = (1, y) \in S_\infty \,:\, y \in \mathbb{Z}_p\} \cong \mathbb{Z}_p. \tag{4.14}$$

This set inherits the group structure from that of S_∞, which is just the additive group structure of $\mathbb{Z}_p$.

Definition 4.5. *Let* $\mathbf{L}$ *be a complete subfield of* $\mathbf{C}_\infty$ *that contains* $\mathbb{K}_\infty$ *and let* $\mathcal{A}$ *be a Banach algebra over* $\mathbf{L}$*. An algebraic time evolution* σ *on* $\mathcal{A}$ *is a group homomorphism*

$$\sigma : \mathbb{Z}_p \to \mathrm{Aut}(\mathcal{A}). \tag{4.15}$$

As above, let $\mathcal{A}$ *be a Banach algebra and let* π *be a representation of* $\mathcal{A}$ *as bounded operator on a Banach space* $\mathcal{V}$*. A continuous time evolution is a homomorphism* (4.15) *such that the map* $y \mapsto \pi(\sigma_y(a))\xi$ *is continuous, for all* $a \in \mathcal{A}$ *and for all* $\xi \in \mathcal{V}$*.*

Notice that here we take as our working notions of continuity for a time evolution an analog of the weak continuity of time evolutions in [4], [5]. This suffices for our purposes below. More generally, one could also introduce a stronger notion of continuity by requiring that the map $y \mapsto \|\sigma_y(a)\|$ is continuous, for all $a \in \mathcal{A}$.

Next, we construct a time evolution on the convolution algebra $\mathcal{A}_{\mathbf{L}}(\mathcal{L}_{\mathbb{K},1})$ of the commensurability relation on 1-dimensional $\mathbb{K}$-lattices. We first need to recall some well-known facts on exponentiation of ideals. We refer the reader to [18], §8.2 for the details.

In the function field case, there is a formula that replaces the classical exponentiation of a positive real number by a complex number as follows. For $\lambda \in \mathbb{K}_\infty^*$ positive and $s = (x, y) \in S_\infty = \mathbf{C}_\infty^* \times \mathbb{Z}_p$, one has

$$\lambda^s = x^{\deg(\lambda)}\, \langle \lambda \rangle^y, \tag{4.16}$$

with $\deg(\lambda) = -d_\infty v_\infty(\lambda)$ and $\langle \lambda \rangle^y = \sum_{j=0}^{\infty} \binom{y}{j}(\langle \lambda \rangle - 1)^j$. The exponentiation $s \mapsto \lambda^s$ defined by (4.16) is an entire function $S_\infty \to \mathbf{C}_\infty^*$. It satisfies the rule $\lambda^{s+t} = \lambda^s \lambda^t$, for all $s, t \in S_\infty$.

The exponentiation (4.16) extends to fractional ideals $I \subset \mathbb{K}$

$$I^s := x^{\deg(I)} \langle I \rangle^y, \qquad s = (x, y) \in S_\infty. \tag{4.17}$$

Here $\langle I \rangle^y$ is the unique extension of $a \mapsto \langle a \rangle$ from principal ideals $I = (a)$ generated by positive elements to fractional ideals. For an ideal $I = (a)$ one has $I^s = x^{-v_\infty(a) d_\infty} \langle a \rangle^y$.

Let $(\Lambda, \phi) \sim (\Lambda', \phi')$ be a pair of commensurable $\mathbb{K}$-lattices in $\mathcal{K}_{\mathbb{K},1}$ and let I, J be the corresponding ideals in $\mathbf{A}$ (Definition 3.1, (1)).

Proposition 4.6. *Let* $\mathcal{A}(\mathcal{L}_{\mathbb{K},1}) = \mathcal{A}_{\mathbf{C}_\infty}(\mathcal{L}_{\mathbb{K},1})$ *as in Definition 4.1. The expression*

$$(\sigma_y\, f)(L, L') = \frac{\langle I \rangle^y}{\langle J \rangle^y} f(L, L'), \quad \forall y \in \mathbb{Z}_p, \tag{4.18}$$

defines a continuous time evolution on the algebra $\mathcal{A}(\mathcal{L}_{\mathbb{K},1})$*, with the norm* (4.9)*. This extends analytically to*

$$(\sigma_s\, f)(L, L') = \frac{I^s}{J^s} f(L, L'), \quad \text{for} \quad s = (x, y) \in S_\infty. \tag{4.19}$$

Proof. We need to show that σ_y is an automorphism of $\mathcal{A}(\mathcal{L}_{\mathbb{K},1})$ for all $y \in \mathbb{Z}_p$ and that $\sigma_{y_1+y_2} = \sigma_{y_1}\sigma_{y_2}$. We check these properties more in general for σ_s, with $s \in S_\infty$. One can then specialize to the case $s = (1, y) \in \mathbb{Z}_p$. One sees that

$$\begin{aligned} \sigma_s(f_1 * f_2)(L, L') &= \tfrac{I^s}{J^s} \sum_{L \sim L'' \sim L'} f_1(L, L'') f_2(L'', L') = \\ &= \sum_{L \sim L'' \sim L'} \frac{I^s}{\tilde{I}^s} f_1(L, L'') \frac{\tilde{I}^s}{J^s} f_2(L'', L') = \sigma_s(f_1) * \sigma_s(f_2), \end{aligned} \tag{4.20}$$

where $\tilde{I}$ is the ideal representing the lattices Λ'' of $L'' = (\Lambda'', \phi'')$. Moreover, since $(IJ)^s = I^s J^s$ and $I^{s_1+s_2} = I^{s_1} I^{s_2}$ one sees that $\sigma_{s_1+s_2} = \sigma_{s_1}\sigma_{s_2}$.

To see that the time evolution (4.18) is continuous in the sense of Definition 4.5, first notice that, for any given pair of ideals I,J, the map $s \mapsto I^s J^{-s}$ is continuous. Thus, so is $s \mapsto I^s J^{-s} f(L', L'')\xi(L'')$, for given $L', L'' \in c(L)$, with I, J the corresponding ideals. We fix I and let $F_J(s) = I^s J^{-s} f(L', L'')\xi(L'')$, with ξ compactly supported on $c(L)$. The function $s \mapsto F(s) = \sum_J F_J(s)$ is then also continuous. ☐

A general treatment of traces of linear operators can be done in the context of nuclear operators on locally convex topological vector spaces (*cf.* [21], [2], [28]). Here we consider a simpler setting, which is sufficient for what we need below.
Let V be a $\mathbf{L}$-vector space. Suppose given a linear basis $\{\epsilon_\alpha\}$ for V and let $\bar{V}$ be a completion of V in a non-archimedean norm. Let $T : V \to V$ be a linear operator that extends to a bounded linear operator on $\bar{V}$. One can then consider the matrix elements $\langle \epsilon_\beta, T\epsilon_\alpha\rangle \in \mathbf{L}$. We write

$$\mathrm{Tr}_V(T) = \sum_\alpha \langle \epsilon_\alpha, T\epsilon_\alpha\rangle \tag{4.21}$$

provided the sum converges in $\mathbf{L}$.
This definition, in principle, depends on the choice of the basis $\{\epsilon_\alpha\}$. In all our explicit applications below we indeed have a preferred choice of a basis for V, so we will not discuss this issue in detail. We refer the reader to §IV.21 of [28] for a detailed discussion of the intrinsic formulation of the trace and the independence of the basis in the context of nuclear spaces.

Definition 4.7. *Consider the data of a $\mathbf{L}$-algebra $\mathcal{A}$ with a time evolution $\sigma : \mathbb{Z}_p \to \mathrm{Aut}(\mathcal{A})$ that extends to $\sigma : S_\infty \to \mathrm{Aut}(\mathcal{A})$ and a representation $\pi : \mathcal{A} \to \mathrm{End}(V)$ on a $\mathbf{L}$-vector space. If there exists a homomorphism $U : S_\infty \to \mathrm{Aut}(V)$ such that*

$$\pi(\sigma_s(f)) = U(s)\pi(f)U(s)^{-1} \qquad \forall f \in \mathcal{A}\ \forall s \in S_\infty, \tag{4.22}$$

then the element $U(s_0) \in \mathrm{Aut}(V)$, with $s_0 = (0, 1)$, is called the exponential of the Hamiltonian of $(\mathcal{A}, \sigma, \pi)$.
In the setting above, suppose that $\mathcal{A}$ is a topological (Banach) algebra with a continuous time evolution σ that extends analytically to S_∞ and suppose given a representation of $\mathcal{A}$ by bounded operators on a topological (Banach) space $\bar{V}$. Let $U : S_\infty \to \mathcal{B}(\bar{V})$ be a continuous homomorphism to the algebra $\mathcal{B}(\bar{V})$ of bounded operators, satisfying the identity (4.22) above in $\mathcal{B}(\bar{V})$. Consider the function

$$Z(s) = \mathrm{Tr}_V(U(s)^{-1}) \tag{4.23}$$

whenever it is defined. The function $Z(x)$ obtained by restricting $Z(s)$ to $s = (x, 0)$, for $x \in \mathbf{C}_\infty^*$, is called the partition function of $(\mathcal{A}, \sigma, \pi)$.

In the case of the algebra $\mathcal{A}_{\mathbf{C}_\infty}(\mathcal{L}_{\mathbb{K},1})$, with the time evolution σ of Proposition 4.6 and the representation π_L of Lemma 4.3, we have the following result.

Theorem 4.8. *For $\mathbf{L} = \mathbf{C}_\infty$ and for $L = (\Lambda, \phi) \in \mathcal{K}_{\mathbb{K},1}$ an invertible $\mathbb{K}$-rational lattice, let $\mathcal{V}_L$ be the vector space of Lemma 4.3. Consider on $\mathcal{V}_L$ the operator*

$$(U(s)\xi)(L') = J^s\,\xi(L'), \tag{4.24}$$

for $L' = (\Lambda', \phi') \in c(L)$ and the lattice Λ' corresponding to the ideal $J \subset \mathbf{A}$. This defines a homomorphism $U : S_\infty \to \mathrm{Aut}(\mathcal{V}_L)$ satisfying the property

$$\pi_L(\sigma_s(f)) = U(s)\pi_L(f)U(s)^{-1}, \tag{4.25}$$

for $s \in S_\infty$ and $f \in \mathcal{A}_{\mathbf{C}_\infty}(\mathcal{L}_{\mathbb{K},1})$. Thus,

$$(U(s_0)\xi)(L') = \langle J\rangle\,\xi(L') \tag{4.26}$$

is the exp of the Hamiltonian for $(\mathcal{A}_{\mathbf{C}_\infty}(\mathcal{L}_{\mathbb{K},1}),\sigma,\pi_L)$ *and the partition function is given by the expression*

$$Z(x)=\sum_{I\subset\mathbf{A}} I^{-x}, \tag{4.27}$$

where the function $Z(s)$ *converges on the "half plane"*

$$\{s=(x,y)\in S_\infty : |x|_\infty > q\}\subset S_\infty. \tag{4.28}$$

Proof. We first show that, if $L=(\Lambda,\phi)$ is an invertible $\mathbb{K}$-rational lattice, then the commensurability class $c(L)$ can be identified with the set of ideals $J\subset\mathbf{A}$.
Recall that, by the result of Proposition 3.4, an invertible $\mathbb{K}$-rational lattice $L=(\Lambda,\phi)$ is represented by a pair (s,ρ) with $s\in\mathbb{A}^*_{\mathbb{K},f}$ and $\rho\in R^*$.
Recall also that, by the result of Theorem 3.9, two $\mathbb{K}$-rational lattices $L=(\Lambda,\phi)$ and $L'=(\Lambda',\phi')$ are commensurable if the corresponding data (s,ρ) and (s',ρ') have the same image under the map Θ of (3.14).
This implies that any data of the form $(su,\rho u)$, with $u\in R\cap\mathbb{A}^*_{\mathbb{K},f}$ determines (under the identification (3.5) of Proposition 3.4) an element $L'=(\Lambda',\phi')$ commensurable to L. In fact, these data have the same image ρs^{-1} in $\mathbb{A}_{\mathbb{K},f}/\mathbb{K}^*$.
Conversely, suppose one is given $L'\sim L$, represented as in Proposition 3.4 by a pair (s',ρ') with $s'\in\mathbb{A}^*_{\mathbb{K},f}$ and $\rho'\in R$.
We have $\rho' s'^{-1}=r\rho s^{-1}$ for some $r\in\mathbb{K}^*$, hence $\rho'=rs'\rho s^{-1}\in R\cap\mathbb{A}^*_{\mathbb{K},f}$. Thus, it follows that $(s'=su,\rho'=\rho u)$ mod $\mathbb{K}^*$, with $u=\rho'\rho^{-1}\in R\cap\mathbb{A}^*_{\mathbb{K},f}$.
This shows that the commensurability class $c(L)$ consists of elements $L'=(\Lambda',\phi')$ with $\Lambda'=R(su)^{-1}\cap\mathbb{K}$ and $\rho'=\rho u$ for some $u\in R\cap\mathbb{A}^*_{\mathbb{K},f}$.
Each such element in fact defines an ideal $J=Ru\cap\mathbb{K}$ and we can write the elements in the commensurability class in the form $L'=J^{-1}L$, with $J=Ru\cap\mathbb{K}$, and J^{-1} understood as a fractional ideal. (Notice that $J^{-1}L$ is only a convenient notation and the use of J^{-1} in this expression should not be confused with the notation J^s used for the exponentiation of ideals!)
Thus, we have obtained in this way an identification $L'\mapsto J$ of $c(L)$ with the set of ideals $J\subset\mathbf{A}$.
Thus, we can identify the space $\mathcal{V}_L$ with the $\mathbf{C}_\infty$-vector space spanned by the ideals $J\subset\mathbf{A}$. We denote by ϵ_J the basis element of $\mathcal{V}_L$ corresponding to the ideal J.
Then the representation π_L of $\mathcal{A}(\mathcal{L}_{\mathbb{K},1})$ has matrix elements

$$\langle\epsilon_{J'},\pi_L(f)\epsilon_J\rangle = f(J'^{-1}L,J^{-1}L), \tag{4.29}$$

and the operator $U(s)$ acts as

$$U(s)\epsilon_J = J^s\,\epsilon_J. \tag{4.30}$$

The time evolution then satisfies

$$\begin{aligned}&\langle\epsilon_{J'},\pi_L(\sigma_s(f))\epsilon_J\rangle = \tfrac{J^s}{J'^s} f(J'^{-1}L,J^{-1}L) = \\ &J^s f(J'^{-1}L,J^{-1}L)J'^{-s} = \langle\epsilon_{J'},U(s)\pi_L(f)U(s)^{-1}\epsilon_J\rangle.\end{aligned} \tag{4.31}$$

Finally, we have

$$Z(s)=\mathrm{Tr}_{\mathcal{V}_L}(U(s)^{-1}) = \sum_{J\subset\mathbf{A}}\langle\epsilon_J,U(s)^{-1}\epsilon_J\rangle = \sum_{J\subset\mathbf{A}} J^{-s}. \tag{4.32}$$

This is the L-function of $\mathbf{A}$ which is known to converge in the "half-plane" (4.28) (*cf.* [18] §8). $\square$

4.4. **KMS functionals.**

We introduce a notion of thermodynamical equilibrium states for a system $(\mathcal{A}, \sigma)$, which is modeled on the notion of KMS states in the C^*-algebra context. In the theory of $\mathbb{Q}$-lattices of rank 1 and 2 and more in general in the quantum statistical mechanical systems associated to Shimura varieties, the points of the underlying *classical* moduli space (a Shimura variety) determine extremal KMS states at sufficiently small temperatures (large values of β). For $\mathbb{Q}$-lattices of rank 1 and 2, one can prove that all the low temperature extremal KMS states arise in this way. In this section we show that the $\mathbb{K}_\infty$-points of the classical moduli scheme $\mathcal{M}^1$, corresponding to invertible $\mathbb{K}$-rational lattices, provide KMS functionals of the system $(\mathcal{A}(\mathcal{L}_{\mathbb{K},1}), \sigma)$ in the following sense.

Definition 4.9. *Let* $\mathbf{L}$ *be a complete subfield of* $\mathbf{C}_\infty$ *that contains* $\mathbb{K}_\infty$. *Let* $\sigma : \mathbb{Z}_p \to \mathrm{Aut}(\mathcal{A})$ *be a continuous time evolution on a Banach* $\mathbf{L}$*-algebra* $\mathcal{A}$*, which extends analytically to* $\sigma : S_\infty \to \mathrm{Aut}(\mathcal{A})$*. A continuous linear functional* $\varphi : \mathcal{A} \to \mathbf{L}$ *is a KMS functional at inverse temperature* $x \in \mathbf{C}^*_\infty$ *if it satisfies the condition*

$$\varphi(f_1 \sigma_x(f_2)) = \varphi(f_2 f_1), \qquad \forall f_1, f_2 \in \mathcal{A}, \qquad \sigma_x = \sigma_{s=(x,0)}. \tag{4.33}$$

If $\mathcal{A}$ *is unital,* KMS$_x$ *functionals are also required to satisfy the normalization condition* $\varphi(1) = 1$*, while in the non-unital case one requires* $\|\varphi\| = 1$.

Here σ_s is the analytic extension of $\sigma : \mathbb{Z}_p \to \mathrm{Aut}(\mathcal{A})$ for $s = (x, y) \in S_\infty$.

Notice that the normalization condition for states will play a role in having inner symmetries acting trivially on KMS states in Lemma 4.13 below.

Theorem 4.10. *For* L *an invertible* $\mathbb{K}$*-lattice, and for* $x \in \mathbf{C}^*_\infty$ *with* $|x|_\infty > q$*, the functional*

$$\varphi_{x,L} : \mathcal{A}(\mathcal{L}_{\mathbb{K},1}) \to \mathbf{C}_\infty, \quad \varphi_{x,L}(f) = Z(x)^{-1} \sum_{J \subset \mathbf{A}} f(J^{-1}L, J^{-1}L)\, J^{-x} \tag{4.34}$$

is a KMS$_x$*-functional.*

Proof. The convergence of $Z(x) = \sum_{J\subset \mathbf{A}} J^{-x}$ in the range $|x|_\infty > q$ ensures that (4.34) is well defined. To check the KMS$_x$ condition one computes

$$\begin{aligned} &Z(x)\varphi_{x,L}(f_1 * \sigma_x(f_2)) = \textstyle\sum_J \sum_{\tilde J} f_1(J^{-1}L, \tilde J^{-1}L)\sigma_x(f_2)(\tilde J^{-1}L, J^{-1}L)\, J^{-x} = \\ &\textstyle\sum_J \sum_{\tilde J} f_1(J^{-1}L, \tilde J^{-1}L) f_2(\tilde J^{-1}L, J^{-1}L) \frac{J^x}{\tilde J^x} J^{-x} = \\ &\textstyle\sum_{\tilde J} f_2 * f_1(\tilde J^{-1}L, \tilde J^{-1}L)\tilde J^{-x} = Z(x)\varphi_{x,L}(f_2 * f_1). \end{aligned} \tag{4.35}$$

The KMS$_x$ property for functionals of the form (4.37) then follows by linearity. The KMS-functionals obtained in this way are clearly continuous with respect to the norm $\|\cdot\|_{\pi_L}$ on the algebra. In fact, one has an estimate

$$|f(J^{-1}L, J^{-1}L)| \le \sup_{L' \in c(L)} |(\pi_L(f)\epsilon_J)(L')| \le \sup_{\xi\neq 0} \frac{\|\pi_L(f)\xi\|}{\|\xi\|}, \tag{4.36}$$

where

$$\|\pi_L(f)\epsilon_J\| = \sup_{L'\in c(L)} |(\pi_L(f)\epsilon_J)(L')|$$

and we used the fact that $\|\epsilon_J\| = \sup_{L'\in c(L)} |\epsilon_J(L')| = 1$, so that the second estimate of (4.36) follows form

$$\frac{\|\pi_L(f)\epsilon_J\|}{\|\epsilon_J\|} \le \sup_{\xi\neq 0} \frac{\|\pi_L(f)\xi\|}{\|\xi\|}.$$

Thus, we obtain from (4.36) the estimate

$$\left| \sum_{J \subset \mathbf{A}} f(J^{-1}L, J^{-1}L)\, J^{-x} \right| \leq \|\pi_L(f)\|\, |Z(x)|.$$

□

Corollary 4.11. *Any choice of a normalized $\mathbf{C}_\infty$-valued non-archimedean measure μ on the set of isomorphism classes of invertible $\mathbb{K}$-lattices determines a KMS$_x$-functional for $x \in \mathbf{C}_\infty^*$ with $|x|_\infty > q$, of the form*

$$\varphi_{x,\mu} : \mathcal{A}(\mathcal{L}_{\mathbb{K},1}) \to \mathbf{C}_\infty, \quad \varphi_{x,\mu}(f) = \int \varphi_{x,L}(f)\, d\mu(L) \tag{4.37}$$

Proof. We refer the reader to [27], §7, for an introduction to non-archimedean measures and integration. Here it suffices to show that the integral preserves continuity. One knows that a σ-additive non-archimedean measure on a σ-algebra is purely atomic ([27] Lemma 4.19), hence for such a measure (4.37) defines a continuous linear functional on $\mathcal{A}(\mathcal{L}_{\mathbb{K},1})$ with respect to the norm $\|f\| = \sup_L \|f\|_{\pi_L}$. If we consider more general types of measures, on rings of sets that are not σ-algebras as in §7 of [27], we proceed in the following way. We know, by Lemma 7.2 of [27] (see p.252-253 of [27]), that there exists a function $N_\mu : \mathbb{A}_{\mathbb{K},f}^*/\mathbb{K}^* \to \mathbb{R}_+^*$ such that

$$|\varphi_{x,\mu}(f)| \leq \sup_L |\varphi_{x,L}(f)| N_\mu(L).$$

Thus (4.37) defines a continuous functional with respect to the norm $\|f\| = \sup_L \|f\|_{\pi_L} N_\mu(L)$.

□

4.5. **Symmetries.**

We now consider the symmetries of the system $(\mathcal{A}(\mathcal{L}_{\mathbb{K},1}), \sigma)$ introduced in Proposition 4.6. First we introduce the general definition of symmetries of a system $(\mathcal{A}, \sigma)$ and describe their induced action on KMS states.

Definition 4.12. *Suppose given a time evolution $\sigma : \mathbb{Z}_p \to \mathrm{Aut}(\mathcal{A})$, such that for all $f \in \mathcal{A}$ the function $y \mapsto \sigma_y(f)$ extends analytically to $s \mapsto \sigma_s(f)$, for $s = (x, y) \in S_\infty$. A symmetry of a system $(\mathcal{A}, \sigma)$ is an algebra homomorphism $U : \mathcal{A} \to \mathcal{A}$, which commutes with the time evolution*

$$U\sigma_s = \sigma_s U, \quad \forall s \in S_\infty. \tag{4.38}$$

Consider an element $u \in \mathcal{A}$ that has a left inverse $v \in \mathcal{A}$, $vu = 1$. This defines a homomorphism

$$U(f) = ufv \tag{4.39}$$

for all $f \in \mathcal{A}$. An inner symmetry U of $(\mathcal{A}, \sigma)$ is a homomorphism (4.39) as above, such that u is an eigenvector of the time evolution, that is, it satisfies

$$\sigma_s(u) = \lambda^s u, \tag{4.40}$$

for all $s \in S_\infty$ and for some $\lambda \in \mathbb{K}_{\infty,+}^*$.

Notice that we do not require that the homomorphism $U : \mathcal{A} \to \mathcal{A}$ sends 1 to 1. In general, the element $U(1)$ will just be an idempotent in $\mathcal{A}$.

Lemma 4.13. *The symmetries of the system* $(\mathcal{A},\sigma)$ *induce a (partially defined) action on* KMS_x *functionals by*

$$U^*:\varphi\mapsto\varphi(U(1))^{-1}\,\varphi\circ U. \tag{4.41}$$

The action (4.41) *is defined, provided the value* $\varphi(U(1))\neq 0$. *The inner symmetries act trivially on* KMS_x *states.*

Proof. Let $U(f)=ufv$ be an inner symmetry, with $\sigma_s(u)=\lambda^s u$. Then $\sigma_s(v)=\lambda^{-s}v$. We check that inner symmetries act, namely that $\varphi(uv)\neq 0$. The KMS_x condition gives

$$\varphi(uv)=\varphi(v\sigma_x(u))=\lambda^x\neq 0.$$

Moreover, we have

$$U^*(\varphi)(f)=\varphi(U(f))/\varphi(U(1))=\lambda^{-x}\,\varphi(ufv)=\lambda^{-x}\,\varphi(fv\sigma_x(u))=\varphi(f).$$

Thus we see that the induced action of inner symmetries on KMS_x functionals is trivial. □

Notice that in analogy with the cases of lattices of rank 1 and 2 ([10], [12], [13], [22]), we consider the action of symmetries by endomorphisms, not just by automorphisms.
In the next statement we identify an important arithmetic group of symmetries of the system $(\mathcal{A}(\mathcal{L}_{\mathbb{K},1}),\sigma)$.

Theorem 4.14. *The expression* (4.42) *below defines a non-trivial action of the semigroup* $R\cap\mathbb{A}^*_{\mathbb{K},f}$ *by endomorphisms of the algebra* $\mathcal{A}(\mathbb{A}_{\mathbb{K},f}/\mathbb{K}^*)$. *The sub-semigroup of non-zero elements of* $\mathbf{A}$ *acts by inner endomorphisms. This induces an action of* $\mathbb{A}^*_{\mathbb{K},f}/\mathbb{K}^*$ *on* KMS_x *states.*

Proof. The argument is similar to the proof of Proposition 2.14 in [12]. Given an ideal $J\subset\mathbf{A}$, adelically described by an element $u\in R\cap\mathbb{A}^*_{\mathbb{K},f}$, one says that a $\mathbb{K}$-rational lattice $L=(\Lambda,\phi)$ is divisible by J if the corresponding data (s,ρ) as in Proposition 3.4 satisfy $s=s_u u\in\mathbb{A}^*_{\mathbb{K},f}$ and $\rho=\rho_u u\in R$, for some (ρ_u,s_u) in $R\times_{R^*}(\mathbb{A}^*_{\mathbb{K},f}/\mathbb{K}^*)$, as in Corollary 3.5.
Let us assume then that $L=(\Lambda,\phi)$ is divisible by J. In this case, let $L_u=(\Lambda_u,\phi_u)$ denote the $\mathbb{K}$-rational lattice represented by the data $(s,u^{-1}\rho=\rho_u)$.
We define an action of $u\in R\cap\mathbb{A}^*_{\mathbb{K},f}$ by symmetries of $(\mathcal{A},\sigma)$ by setting

$$\theta_u(f)(L,L')=\begin{cases} f(L_u,L'_u) & L,L' \text{ divisible by } J\\ 0 & \text{otherwise.}\end{cases} \tag{4.42}$$

It is immediate to see that θ_u is an endomorphism of $\mathcal{A}$. Moreover, it is also compatible with the time evolution, since one has

$$\sigma_s(\theta_u(f))(L,L')=\begin{cases} \frac{I_u^s}{J_u^s}f(L_u,L'_u) & L,L' \text{ divisible by } J\\ 0 & \text{otherwise}\end{cases}$$

and this is the same as

$$\theta_u(\sigma_s(f))(L,L')=\begin{cases} \frac{I^s}{J^s}f(L_u,L'_u) & L,L' \text{ divisible by } J\\ 0 & \text{otherwise.}\end{cases}$$

We check when this action is implemented by inner endomorphisms. Consider the element $\mu_J\in\mathcal{A}(\mathcal{L}_{\mathbb{K},1})$ of the form

$$\mu_J(L,L')=\begin{cases} 1 & L=L'_u\\ 0 & \text{otherwise.}\end{cases} \tag{4.43}$$

μ_J has a left inverse $\tilde{\mu}_J$ given by

$$\tilde{\mu}_J(L, L') = \begin{cases} 1 & L' = L_u \\ 0 & \text{otherwise,} \end{cases} \tag{4.44}$$

hence it gives rise to an inner action $U_J : \mathcal{A} \to \mathcal{A}$ as in (4.39), of the form

$$U_J(f) = \mu_J * f * \tilde{\mu}_J, \tag{4.45}$$

where, as usual, $*$ denotes the convolution product (4.5) in the algebra $\mathcal{A}(\mathcal{L}_{\mathbb{K},1})$. The action (4.45) is a symmetry of the system $(\mathcal{A}, \sigma)$, in fact, one has

$$\sigma_s(\mu_J) = J^s \, \mu_J, \quad \forall s \in S_\infty. \tag{4.46}$$

The equality $\theta_u(f) = U_J(f)$ holds for all $f \in \mathcal{A}$ if and only if $u \in \mathbf{A} \smallsetminus \{0\}$. In fact, it is only for $u \in \mathbf{A} \smallsetminus \{0\}$ that the $\mathbb{K}$-lattices $L = (u\Lambda, \phi)$ and $L_u = (\Lambda, u^{-1}\phi)$ are isomorphic in $\mathcal{K}_{\mathbb{K},1}$. This means that, for $u \in \mathbf{A} \smallsetminus \{0\}$ the action θ_u is inner and given by $\theta_u(f) = U_J(f)$. □

This shows that the class field theory action of $\mathbb{A}^*_{\mathbb{K},f}/\mathbb{K}^*$ on $\mathcal{M}^1$ is carried over to an action by symmetries of the quantum statistical mechanical system $(\mathcal{A}(\mathcal{L}_{\mathbb{K},1}), \sigma)$. In fact, on the one hand we know by [15], §8, that $\mathcal{M}^1(\overline{\mathbb{K}}_\infty) = \mathbb{A}^*_{\mathbb{K},f}/\mathbb{K}^*$ with the action of $\mathbb{A}^*_{\mathbb{K},f}/\mathbb{K}^*$ corresponding to the class field theory action. On the other hand, by Theorem 4.14 above, this is indeed the same as the action induced by the symmetries $R \cap \mathbb{A}^*_{\mathbb{K},f}$ of the system on the set of KMS$_x$ states of Theorem 4.10, given by the set $\mathbb{A}^*_{\mathbb{K},f}/\mathbb{K}^*$ of isomorphism classes of invertible $\mathbb{K}$-lattices.

4.6. v-adic time evolutions.

The time evolution considered in the Bost–Connes system, in the context of C^*-algebras and for 1-dimensional $\mathbb{Q}$-lattices is that associated to the archimedean valuation. For algebras over function fields, the construction of the time evolution associated to a *chosen* point ∞ has been developed in §4.3. More in general, it is possible to define time evolutions associated to non-archimedean valuations (of a number-field) and consider them simultaneously in the study of the set of the classical points (extremal zero temperature KMS states) of the corresponding system. This technique can be very useful. In fact, in the number field case, the resulting set of extremal zero temperature KMS states can be seen as an analog of the algebraic points $C(\bar{\mathbb{F}}_q)$ of a curve C. This analogy is based on the fact that, in the function field case, the same set is indeed identified (though only through a non-canonical identification of orbits of the Frobenius action) with the set $C(\bar{\mathbb{F}}_q)$. We refer to [9] for the details of an interesting application of this technique. In the following, we shall review first (from [9]), the definition of a time evolution associated to a non-archimedean valuation. Then we will introduce a similar notion in the function field setting, that describes a time evolution associated to a place different from ∞.

Let v be a non-archimedean place of $\mathbb{Q}$. On the convolution C^*-algebra $\mathcal{A}(\mathbb{A}_{\mathbb{Q},v}/\mathbb{Q}^*)$ associated to the noncommutative space $\mathbb{A}_{\mathbb{Q},v}/\mathbb{Q}^*$, one considers the time evolution

$$\sigma_t^v(f)(L, L') = \left| \frac{cov(\Lambda')}{cov(\Lambda)} \right|_v^{it} f(L, L'), \tag{4.47}$$

where $(\Lambda, \phi) = L \sim L' = (\Lambda', \phi')$ are commensurable 1-dimensional $\mathbb{Q}$-lattices defining a class in $\mathbb{A}_{\mathbb{Q},v}/\mathbb{Q}^*$. The ratio $cov(\Lambda')/cov(\Lambda)$ denotes the ratio of the covolumes of the two lattices in $\mathbb{R}$ and $|\cdot|_v$ is the valuation associated to the chosen non-archimedean place. If one parameterizes 1-dimensional $\mathbb{Q}$-lattices in terms of data (ρ, λ), with $\rho \in \hat{\mathbb{Z}}$ and $\lambda \in \mathbb{R}^*_+$, and the commensurability relation is implemented by a partially defined action of $\mathbb{Q}_+^*$, then the time evolution (4.47) can be written as

$$\sigma_t^v(f)(r,\rho,\lambda) = |r|_v^{it}\, f(r,\rho,\lambda). \tag{4.48}$$

The original time evolution of the Bost–Connes system is obtained by using $|\cdot|_\infty$ for the archimedean valuation on $\mathbb{Q}$ and the convolution algebra of $\mathbb{A}_{\mathbb{Q},f}/\mathbb{Q}^*$. The analogous time evolution in the case of imaginary quadratic fields is expressed in terms of ratio of norms of ideals $r = \frac{n(I)}{n(J)}$ ([12]).

The set of zero temperature extremal KMS states for the time evolution (4.47) can be identified with the $C_{\mathbb{Q}}(= \mathbb{A}_{\mathbb{Q},f}^*/\mathbb{Q}^*)$-orbit of the adele $a^{(v)} \in \mathbb{A}_{\mathbb{Q}}$

$$a_w^{(v)} = \begin{cases} 1 & w \neq v \\ 0 & w = v. \end{cases} \tag{4.49}$$

When $\mathbb{K} = \mathbb{F}_q(C)$, time evolutions involving the norms $|\cdot|_v : \mathbb{K} \to \mathbb{R}$, and associated to the different valuations $v \in \Sigma_{\mathbf{A}}$ can be defined in a form analogous to (4.48). One considers two commensurable $\mathbb{K}$-rational lattices $L = (\Lambda,\phi) \sim L' = (\Lambda',\phi')$ that represent a class in $\mathbb{A}_{\mathbb{K},v}/\mathbb{K}^*$. Here, the lattices Λ and Λ' correspond respectively to ideals I, J in $\mathbf{A}$. As for $\mathbb{Q}$, the zero temperature KMS states for the time evolution σ_t^v on $\mathcal{A}(\mathbb{A}_{\mathbb{K},v}/\mathbb{K}^*)$ can be identified with the orbit of the action of $C_{\mathbb{K}} = A_{\mathbb{K},f}^*/\mathbb{K}^*$ on the adele $a^{(v)} \in \mathbb{A}_{\mathbb{K}}$ defined as in (4.49). One obtains a (non-canonical) identification between the set $\cup_{v\in\Sigma_{\mathbf{A}}} C_{\mathbb{K}}\, a^{(v)}$ union of the zero temperature KMS states, for all the time evolutions σ_t^v on the various $\mathcal{A}(\mathbb{A}_{\mathbb{K},v}/\mathbb{K}^*)$ and the set $C(\bar{\mathbb{F}}_q)$ of algebraic points of the curve C. This identification is defined by mapping the orbit of the Frobenius action on $C(\bar{\mathbb{F}}_q)$ at a place w to the orbit of $\mathbb{K}_w^* \subset C_{\mathbb{K}}$ on $\cup_{v\in\Sigma_{\mathbf{A}}} C_{\mathbb{K}}\, a^{(v)}$. One deduces clearly the analogy between the locus of zero temperature KMS states in the number field case and the algebraic points $C(\bar{\mathbb{F}}_q)$ of the smooth, projective curve C in the function field case.

In the following we shall transport these ideas from the C^*-algebra context to the function field setting and define a good notion of v-adic time evolution. We will consider function field valued algebras, which correspond to v-adic rather than ∞-adic completions. We will resort to the theory of v-adic L-functions in function field arithmetic (*cf.* [18] §8.6) for the necessary notions.

For a place v of $\mathbb{K}$, with $v \neq \infty$, the analog of the decomposition (4.11) is now given by the canonical decomposition of elements $\alpha \in \mathbf{A}_v^*$ as

$$\alpha = \omega_v(\alpha)\langle\alpha\rangle_v, \tag{4.50}$$

where $\omega_v(\alpha) \in \mu_{(q^{d_v}-1)}$ is a $(q^{d_v}-1)$-st root of unity in $\mathbf{A}_v^*$, and $\langle\alpha\rangle_v$ is a 1-unit at v. Recall that the value field $\mathbb{V}$ is the smallest subfield of $\mathbf{C}_\infty$ containing $\mathbb{K}$ and the values I^{s_1}, where $s_1 = (\tilde{u}_\infty^{-1}, 1) \in S_\infty$ and $\tilde{u}_\infty^{-1}$ is a fixed d_∞ root of u_∞. For $I = (a)$, one has $I^{s_1} = a/\mathrm{sign}(a)$. A choice of an embedding $\tau : \mathbb{V} \to \overline{\mathbb{K}_v}$ (*i.e.* a choice of a finite place for $\mathbb{V}$) determines a finite extension $\mathbb{K}_{\tau,v} = \mathbb{K}_v(\tau(\mathbb{V}))$ of $\mathbb{K}_v$. Let $\mathbb{A}_{\tau,v}$ be the ring of integers of $\mathbb{K}_{\tau,v}$. Then, (4.50) determines a corresponding decomposition of $\alpha \in \mathbb{A}_{\tau,v}^*$

$$\alpha = \omega_{\tau,v}(\alpha)\langle\alpha\rangle_{\tau,v}, \tag{4.51}$$

where $\omega_{\tau,v}(\alpha)$ is a $(q^{d_v f_\tau}-1)$-st root of unity in $\mathbb{A}_{\tau,v}^*$ and $\langle\alpha\rangle_{\tau,v}$ is a 1-unit. Here f_τ is the residue degree of $\mathbb{K}_{\tau,v}$ over $\mathbb{K}_v$.

In order to define v-adic time evolutions, we first recall some well-known facts about v-adic exponentiation of ideals (*cf.* [18] §8.5). The v-adic exponentiation of ideals is defined on the v-adic analog of the "complex plane" S_∞, namely the group

$$S_v = \mathbf{C}_v^* \times \mathbb{Z}_p \times \mathbb{Z}/(q^{d_v f_\tau}-1)\mathbb{Z}. \tag{4.52}$$

We write elements of S_v as $s_v = (x_v, y_v)$ with $y_v = (y_{v,0}, y_{v,1}) \in \mathbb{Z}_p \times \mathbb{Z}/(q^{d_v f_\tau} - 1)\mathbb{Z}$. For a fractional ideal $I = (a)$ of $\mathbb{K}$ prime to the ideal of v one has

$$I^{s_v} = I^{(x_v,y_v)} = x_v^{\deg(I)}(\tau(I^{s_1})^{y_v} = x_v^{\deg(a)}\omega_{\tau,v}(\tau(a/\mathrm{sign}(a)))^{y_{v,1}}\langle\tau(a/\mathrm{sign}(a))\rangle_{\tau,v}^{y_{v,0}}, \tag{4.53}$$

for all $s_v = (x_v, y_v) = (x_v, (y_{v,0}, y_{v,1})) \in S_v$. If $I = (a)$ with a positive then (4.53) simplifies to $I^{s_v} = x_v^{\deg(a)} a^{y_v}$.

Definition 4.15. *Let* $\mathbf{L}$ *be a complete subfield of* $\mathbf{C}_v$ *that contains* $\mathbb{K}_{\tau,\infty}$*. A* v*-adic time evolution on a (topological)* $\mathbf{L}$*-algebra* $\mathcal{A}$ *is a (continuous) group homomorphism*

$$\sigma^v : \mathbb{Z}_p \times \mathbb{Z}/(q^{d_v f_\tau} - 1)\mathbb{Z} \to \mathrm{Aut}(\mathcal{A}). \tag{4.54}$$

We construct v-adic time evolutions on the convolution algebras $\mathcal{A}_{\mathbf{C}_v}(\mathbb{A}_{\mathbb{K},v}/\mathbb{K}^*)$ associated to the noncommutative spaces $\mathbb{A}_{\mathbb{K},v}/\mathbb{K}^*$. The convolution algebra is obtained by restriction of the convolution algebra associated to $\tilde{\mathcal{L}}_{\mathbb{K},1}$ to the commensurability classes of $\mathbb{K}$-rational $\mathbf{C}_\infty$-lattices that define elements in $\mathbb{A}_{\mathbb{K},v}/\mathbb{K}^*$.

Proposition 4.16. *Let* $\mathcal{A} = \mathcal{A}_{\mathbf{C}_\infty}(\mathbb{A}_{\mathbb{K},v}/\mathbb{K}^*)$ *be the convolution algebra described above. The expression*

$$(\sigma_y^v f)(L, L') = \frac{I^y}{J^y} f(L, L'), \quad \forall y \in \mathbb{Z}_p \times \mathbb{Z}/(q^{d_v f_\tau} - 1)\mathbb{Z}, \tag{4.55}$$

defines a v*-adic time evolution on* $\mathcal{A}$*, which extends analytically to*

$$(\sigma_s f)(L, L') = \frac{I^s}{J^s} f(L, L'), \quad \text{for} \quad s = (x, y) \in S_v. \tag{4.56}$$

Proof. The proof is analogous to that of Proposition 4.6. ☐

5. Frobenius, scaling, and the dual system

In [8] a general construction was introduced that provides an analog of Frobenius action in characteristic zero. This is obtained by considering a scaling action on the *dual* of a quantum statistical mechanical system. This idea can, to some extent, be transported in the function field setting and shows a relation between scaling action and Frobenius in this setting.

5.1. The dual system: archimedean case.

We start by reviewing shortly the fundamental steps of this construction in [8].

In the C^*-algebra context, one considers data $(\mathcal{A}, \sigma)$ consisting of a C^*-algebra $\mathcal{A}$ and a time evolution $\sigma : \mathbb{R} \to \mathrm{Aut}(\mathcal{A})$. If $\mathcal{B}$ is a dense subalgebra of $\mathcal{A}$ which is preserved by σ, the dual system is given by the crossed product algebra $\hat{\mathcal{B}} = \mathcal{B} \rtimes_\sigma \mathbb{R}$. The elements of $\hat{\mathcal{B}}$ can be written in the form

$$f = \int \ell(t) U_t dt \in \hat{\mathcal{B}}, \tag{5.1}$$

where $\ell \in \mathcal{S}(\mathbb{R}, \mathcal{B})$ is a rapidly decaying function (in the Schwartz space) with values in $\mathcal{B}$ and the U_t are unitaries implementing the $\mathbb{R}$-action. For eleemnts of the form (5.1), the associative product on the algebra $\hat{\mathcal{B}}$ is just given by the composition

$$\int_{\mathbb{R}} \ell_1(t) U_t dt \int_{\mathbb{R}} \ell_2(r) U_r dr = \int_{\mathbb{R}^2} \ell_1(t) \sigma_t(\ell_2(r)) U_{t+r}\, dt\, dr. \tag{5.2}$$

Since the measure dt on $\mathbb{R}$ is translation invariant, one can equivalently describe the dual system as the space $\mathcal{S}(\mathbb{R},\mathcal{B})$ endowed with the algebra structure given by the associative product

$$(\ell_1 \star \ell_2)(s) = \int_{\mathbb{R}} \ell_1(t)\sigma_t(\ell_2(s-t))\, dt. \tag{5.3}$$

Notice that the translation invariance of the measure is needed in showing that (5.3) is associative. One has in this case

$$\int_{\mathbb{R}} \ell_1(t) U_t dt \int_{\mathbb{R}} \ell_2(r) U_r dr = \int_{\mathbb{R}} (\ell_1 \star \ell_2)(s) U_s\, ds. \tag{5.4}$$

Consider a representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$, where the time evolution is implemented by a Hamiltonian H, that is, a (usually unbounded) self-adjoint operator on $\mathcal{H}$ such that

$$\pi(\sigma_t(x)) = e^{itH}\pi(x)e^{-itH}.$$

This determines a corresponding representation of $\hat{\mathcal{B}}$ of the form

$$\pi(f) = \int \pi(\ell(t))\, e^{itH}\, dt, \tag{5.5}$$

for f as in (5.1).

The algebra $\hat{\mathcal{B}}$ has a dual action of $\mathbb{R}^*_+$ by scaling, associated to the pairing of $\mathbb{R}$ and $\mathbb{R}^*_+$ through the character

$$\mathbb{R}^*_+ \times \mathbb{R} \ni (\lambda, t) \mapsto \langle \lambda, t\rangle = \lambda^{it}. \tag{5.6}$$

The *scaling action* on $\hat{\mathcal{B}}$ is given by

$$\theta : \mathbb{R}^*_+ \to \mathrm{Aut}(\hat{\mathcal{B}}), \qquad \theta_\lambda \left(\int \ell(t)\, U_t\, dt \right) = \int \lambda^{it}\, \ell(t)\, U_t\, dt, \quad \forall \lambda \in \mathbb{R}^*_+. \tag{5.7}$$

One refers of $(\hat{\mathcal{B}}, \theta)$ as the dual system of $(\mathcal{B}, \sigma)$.

The term "dual" here refers to the role played by the crossed product algebra $\mathcal{A} \rtimes_\sigma \mathbb{R}$ in the theory of factors for von Neumann algebras, where passing to the crossed product by the time evolution determines the fundamental duality between type II and type III factors introduced in Connes thesis [6], which plays a crucial role in the classification of type III factors.

In the setting of C^*-algebras and $\mathbb{Q}$-lattices, one then has an algebra homomorphism that relates the dual system to the algebra of the noncommutative space of commensurability classes of 1-dimensional $\mathbb{Q}$-lattices, considered not up to scaling. This is given (*cf.* [8], [9], [11] §4) by the map

$$\iota(f)(\lambda L, \lambda L') = \int_{\mathbb{R}} \ell(t)(L, L')\lambda^{it}\, dt, \tag{5.8}$$

where (L, L') is a pair of commensurable 1-dimensional $\mathbb{Q}$-lattices and f and ℓ are related as in (5.1). The right hand side defines a function over the groupoid of the commensurability relation on 1-dimensional $\mathbb{Q}$-lattices. The fact that this is an algebra homomorphism is a consequence of the compatibility of the associative products

$$\begin{aligned} \iota(f_1 f_2)(\lambda L, \lambda L') &= \textstyle\int_{\mathbb{R}^2} (\ell_1(t)\sigma_t(\ell_2(s-t)))(L, L')\lambda^{is} dt ds \\ &= \textstyle\sum_{L''} \iota(f_1)(\lambda L, \lambda L'')\iota(f_2)(\lambda L'', \lambda L') = \iota(f_1)\iota(f_2)(\lambda L, \lambda L'). \end{aligned} \tag{5.9}$$

The scaling action (5.7) on the dual system corresponds, under this algebra homomorphism to the scaling action on the space of 1-dimensional $\mathbb{Q}$-lattices.

In [8] one then considers a "distilled system" given by a Λ-module (module over the cyclic category) defined as a cokernel of a map from the Λ-module of sufficiently regular elements in the algebra $\hat{\mathcal{B}}$ to the Λ-module of the commutative algebra of rapidly decaying functions on the set of extremal low temperature KMS states of $(\mathcal{A},\sigma)$. One obtains in this way an induced action of $\mathbb{R}_+^*$ on the "distilled system" and on its cyclic homology. In the case of the noncommutative space of 1-dimensional $\mathbb{Q}$-lattices, a trace formula for this action yields both a cohomological intepretation of the Riemann Weil explicit formula and the spectral realization of the zeros of the Riemann zeta function and more generally of L-functions with Grössencharakter. Because of this result, it makes sense to interpret the scaling action of $\mathbb{R}_+^*$ as a substitute of Frobenius in characteristic zero. We argue here that this interpretation is also well motivated by comparison with the function field case.

5.2. **The dual system for function fields.**

In the case of function fields, suppose given a statistical mechanical system $(\mathcal{A},\sigma)$, with a Banach algebra $\mathcal{A}$ and a continuous time evolution $\sigma : \mathbb{Z}_p \to \mathrm{Aut}(\mathcal{A})$. Let $\sigma : S_\infty \to \mathrm{Aut}(\mathcal{A})$ be an analytic (in the sense of Definition 8.5.1 of [18]) extension of σ.

As in §4.2, we suppose given a choice of a uniformizer u_∞ and a corresponding decomposition (4.10). As before, we let $\mathbb{K}_{\infty,+}^* \subset \mathbb{K}_\infty^*$ denote the subgroup $\{1\}\times u_\infty^{\mathbb{Z}}\times U_1$ in this decomposition.

Moreover, suppose given a subgroup $H = G\times\mathbb{Z}_p \subset S_\infty$, with G a totally disconnected compact topological subgroup of $\mathbf{C}_\infty^*$, and a $\mathbb{K}_\infty$-valued non-archimedean measure on $H = G\times\mathbb{Z}_p \subset S_\infty$.

Unlike what happens in the archimedean case, when working with characteristic p-valued measures, one does not have Haar measures (*cf.* [18], §8.22, [27], §8). In fact, to be more precise, one can have Haar measures on a p-free compact abelian group, *i.e.* one that does not admit any surjective homomorphism to finite groups of order multiple of p, *cf.* [27] §8. However, this is clearly not the case for the group $\mathbb{Z}_p$ where our time evolutions are defined.
The lack of translation invariance of the measure implies that the product

$$f_1 \bullet_\sigma f_2 = \int_{H^2} \ell_1(s)\sigma_s(\ell_2(s'))U_{s+s'}\, d\mu(s)d\mu(s') \tag{5.10}$$

can no longer be written in a form like (5.4), (5.3). (We introduce here the notation $\bullet_\sigma$ to distinguish (5.10) from the corresponding product of the archimedean case.)

Lemma 5.1. *Let $H \subset \mathbf{C}_\infty^*$ be as above. Let $\ell(s)$ be functions in $C(H,\mathcal{A})$ and let U_s be symbols satisfying $U_{s+s'} = U_s U_{s'}$ and the relation*

$$U_s\,\ell(s') = \sigma_s(\ell(s'))U_s, \quad \forall s, s' \in H. \tag{5.11}$$

Then the expression (5.10) *defines an associative product.*

Proof. We have

$$\begin{aligned}\textstyle\int_H \ell_1(s)U_s d\mu(s)\int_H \ell_2(s')U_{s'}d\mu(s') &= \textstyle\int_{H^2}\ell_1(s)U_s\ell_2(s')U_{s'}d\mu(s)d\mu(s')\\ &= \textstyle\int_{H^2}\ell_1(s)\sigma_s(\ell_2(s'))U_{s+s'}d\mu(s)d\mu(s').\end{aligned}$$

We then obtain

$$(f_1\bullet_\sigma f_2)\bullet_\sigma f_3 = \textstyle\int_{H^3}\ell_1(s)\sigma_s(\ell_2(s'))U_{s+s'}\ell_3(s'')d\mu(s)d\mu(s')d\mu(s'')$$
$$= \textstyle\int_{H^3}\ell_1(s)\sigma_s(\ell_2(s'))\sigma_{s+s'}(\ell_3(s''))U_{s+s'+s''}d\mu(s)d\mu(s')d\mu(s'') = f_1\bullet_\sigma(f_2\bullet_\sigma f_3).$$

This shows associativity. □

We define a dual system $(\hat{\mathcal{A}}_H, \theta)$, for H as above.

Definition 5.2. *Let $\hat{\mathcal{A}}_H$ denote the algebra generated (as algebra) by elements of the form*

$$f = \int_H \ell(s)\, U_s \, d\mu(s), \tag{5.12}$$

with $\ell \in C(H, \mathcal{A})$, endowed with the product (5.10). Here the symbols $\{U_s\}_{s\in H}$ satisfy $U_{s+s'} = U_s U_{s'}$ and (5.11). The transformation $\theta_\lambda(U_s) = \lambda^s U_s$ induces a scaling action of $\mathbb{K}^*_{\infty,+}$ on $\hat{\mathcal{A}}_H$ given by

$$\theta_\lambda(f) = \int_H \ell(s)\, \lambda^s\, U_s \, d\mu(s). \tag{5.13}$$

The pair $(\mathcal{A}_H, \theta)$ is the H-dual system of $(\mathcal{A}, \sigma)$.

We have the following results about the scaling action.

Lemma 5.3. *The scaling action* (5.13) *satisfies*

$$\theta_{\lambda_1\lambda_2}(f) = \theta_{\lambda_1}(\theta_{\lambda_2}(f)), \qquad \textit{and} \qquad \theta_\lambda(f_1 \bullet_\sigma f_2) = \theta_\lambda(f_1) \bullet_\sigma \theta_\lambda(f_2), \tag{5.14}$$

for all $f, f_1, f_2 \in \hat{\mathcal{A}}_H$ and for all $\lambda, \lambda_1, \lambda_2 \in \mathbb{K}^*_{\infty,+}$.

Proof. We have $\theta_{\lambda_1\lambda_2}(f) = \int_H \ell(s)\lambda_1^s\lambda_2^s U_s d\mu(s)$, which gives the first property. The scaling action is induced by the action $U_s \mapsto \lambda_s U_s$. This gives $\theta_\lambda(U_{s+s'}) = \lambda^{s+s'} U_{s+s'}$, so that

$$\theta_\lambda(f_1 \bullet_\sigma f_2) = \int_{H^2} \ell_1(s)\sigma_s(\ell_2(s'))\lambda^{s+s'} U_{s+s'} d\mu(s) d\mu(s').$$

This gives the second property, using the fact that $\lambda^s \sigma_s(\ell_2(s')\lambda^{s'}) = \sigma_s(\ell_2(s'))\lambda^{s+s'}$. ◻

We now discuss the function field analog of the algebra homomorphism (5.8) from the dual system to the algebra of functions on commensurability classes of 1-dimensional $\mathbb{Q}$-lattices.

For $\mathbf{L} = \mathbf{C}_\infty$, consider the algebras $\mathcal{A}(\mathcal{L}_{\mathbb{L},1})$ and $\mathcal{A}(\tilde{\mathcal{L}}_{\mathbb{K},1})$. Let $H \subset S_\infty$ be a topological subgroup as above. Let $\hat{\mathcal{A}}_H(\mathcal{L}_{\mathbb{K},1})$ be the H-dual system of $(\mathcal{A}(\tilde{\mathcal{L}}_{\mathbb{K},1}), \sigma)$, with the time evolution of Proposition 4.6.

Let $(\mathcal{V}, \pi)$ be a representation of $\mathcal{A}(\tilde{\mathcal{L}}_{\mathbb{K},1})$ and let $U(y)$, for $y \in \mathbb{Z}_p$, be the operators implementing the time evolution in the representation. Assume that these extend to operators $U(s)$, for $s \in H$, satisfying $\pi(\sigma_s(a)) = U(s)\pi(a)U(-s)$, where σ_s is the analytic continuation of σ_y.

Definition 5.4. *Let $\mathcal{A}_\pi(\tilde{\mathcal{L}}_{\mathbb{K},1}, \mathcal{U})$ denote the algebra generated by elements of the form*

$$X(\lambda L, \lambda L') = \int_H \xi(s)(\lambda L, \lambda L')\, U(s)\, d\mu(s),$$

with $\xi(s)$ in $C(H, \mathcal{A}(\tilde{\mathcal{L}}_{\mathbb{K},1}))$ and with $L = (\Lambda, \phi)$ and $L' = (\Lambda', \phi')$ commensurable $\mathbb{K}$-lattices in $\mathcal{K}_{\mathbb{K},1}$. The product on $\mathcal{A}_\pi(\tilde{\mathcal{L}}_{\mathbb{K},1}, \mathcal{U})$ is given by the convolution product

$$X_1 * X_2(\lambda L, \lambda L') = \sum_{L\sim L''\sim L'} X_1(\lambda L, \lambda L'')X_2(\lambda L'', \lambda L'). \tag{5.15}$$

On the right hand side of (5.15) the product is of the form

$$X_1(\lambda L, \lambda L'')X_2(\lambda L'', \lambda L') = \int_{H^2} \xi_1(s)(\lambda L, \lambda L'')(U(s)\xi_2(s')(\lambda L'', \lambda L')U(-s))\, U(s+s') d\mu(s) d\mu(s'). \tag{5.16}$$

We then have the following result.

Lemma 5.5. *The map*

$$f = \int_H \ell(s) U_s d\mu(s) \mapsto X_f(\lambda L, \lambda L') = \int_H \ell(s)(L, L')\, \lambda^s\, U(s)\, d\mu(s) \tag{5.17}$$

gives an algebra homomorphism from $\hat{\mathcal{A}}_H(\mathcal{L}_{\mathbb{K},1})$ *to the algebra* $\mathcal{A}_\pi(\tilde{\mathcal{L}}_{\mathbb{K},1}, \mathcal{U})$.

Proof. This follows from the compatibility of the products (5.10) and (5.15). In fact, one has

$$\begin{aligned} X_{f_1 \bullet_\sigma f_2}(\lambda L, \lambda L') &= \textstyle\int_{H^2} (\ell_1(s)\sigma_s(\ell_2(s')))(L, L')\, \lambda^{s+s'} U_{s+s'}\, d\mu(s) d\mu(s') \\ &= \textstyle\sum_{L \sim L'' \sim L'} X_{f_1}(\lambda L, \lambda L'') X_{f_2}(\lambda L'', \lambda L') = X_{f_1} * X_{f_2}(\lambda L, \lambda L'), \end{aligned}$$

where we have used (5.16) ☐

5.3. Scaling of $\mathbb{K}$-lattices, dual system and Frobenius.

We have seen that, in the function field case, we have an analog of the algebra of functions on commensurability classes of 1-dimensional $\mathbb{Q}$-lattices (not up to scale), which is given by the algebra $\mathcal{A}(\tilde{\mathcal{L}}_{\mathbb{K},1})$. We show that the action of $\mathbb{A}^*_{\mathbb{K}}/\mathbb{K}^*$ on $\tilde{\mathcal{L}}_{\mathbb{K},1}$ induces in particular a scaling action of $\mathbb{K}^*_\infty$ on $\tilde{\mathcal{L}}_{\mathbb{K},1}$, which admits a description in terms of Frobenius and inertia groups.

Remark 5.6. *By local class field theory, one knows that, for a non-archimedean local field* K*, the Artin homomorphism* $\Theta : K^* \to \mathrm{Gal}(K^{ab}/K)$ *is injective. With the identification* $K = \mathcal{O}^* \times u^{\mathbb{Z}}$*, with* $\mathcal{O}$ *the ring of integers of* K *and* $u \in \mathcal{O}$ *a chosen uniformizer, the image* $\Theta(\mathcal{O}^*)$ *is isomorphic to the inertia subgroup* $\mathrm{Gal}(K^{ab}/K^{un})$ *and* $\Theta(u)$ *corresponds to a fixed lifting in* $\mathrm{Gal}(K^{ab}/K)$ *of the Frobenius automorphism. This gives an identification* $u^{\hat{\mathbb{Z}}} \simeq \mathrm{Gal}(K^{un}/K) \simeq \mathrm{Gal}(k^s/k)$*, for* k *the residue field.*

Let Fr denote the generator of $\mathrm{Gal}(k^s/k)$, which corresponds to the uniformizer u under the above identification.

Proposition 5.7. *Suppose given a choice of a uniformizer* u_∞ *as above. The subgroups* $u_\infty^{\mathbb{Z}}$ *and* U_1 *of* $\mathbb{K}^*_{\infty,+}$ *are mapped, by the local Artin homomorphism* Θ*, to the group* $Fr^{\mathbb{Z}}$ *of integer powers of the Frobenius and to the inertia group, respectively.*

Proof. The statement of the proposition is just a particular case of the local class field theory statement recalled in Remark 5.6 above, in the case where the non-archimedean local field is $K = \mathbb{K}_\infty$, with $\mathcal{O} = \mathbf{A}_\infty$ and with the Artin homomorphism Θ as in Remark 5.6. In particular, Remark 5.6 implies directly that the subgroup $u_\infty^{\mathbb{Z}}$ of $\mathbb{K}^*_{\infty,+}$ is identified with the group of integer powers of the Frobenius in $\mathrm{Gal}(k^s/k)$ where $k = \mathbb{F}_{q^{d_\infty}}$ is the residue field. Moreover, since we have $U_1 \subset \mathbf{A}^*_\infty$, again Remark 5.6 implies that the image of the subgroup U_1 under the Artin homomorphism Θ lies in the inertia group. ☐

For simplicity of notation, in the following we no longer write explicitly the Artin homomorphism Θ and we speak loosely of $u_\infty^{\mathbb{Z}}$ as integer powers of the Frobenius and of U_1 as inertia. The injectivity of Θ ensures that we are not losing information by doing so.

We can then use the result of Lemma 5.5 to reinterpret the relation between the scaling action $L \mapsto \lambda L$ of $\lambda \in \mathbb{K}^*_{\infty,+}$ on commensurability classes of $\mathbb{K}$-lattices $L \in \tilde{\mathcal{L}}_{\mathbb{K},1}$ and the Frobenius and inertia groups in terms of the dual system $(\hat{\mathcal{A}}_H(\mathcal{L}_{\mathbb{K},1}), \theta)$.

Recall that the scaling action θ on $\hat{\mathcal{A}}_H(\mathcal{L}_{\mathbb{K},1})$ is given by (5.13), for $\lambda \in \mathbb{K}^*_{\infty,+}$. Under the decomposition $\mathbb{K}^*_{\infty,+} = u_\infty^{\mathbb{Z}} \times U_1$, we can write λ^s, for $s = (x, y) \in H = G \times \mathbb{Z}_p$, in the form $\lambda^s = x^{\deg(\lambda)} \langle\lambda\rangle^y$, where $\lambda = u_\infty^m \langle\lambda\rangle$ and $\deg(\lambda) = -d_\infty m$. The restriction to $\mathbb{Z}_p$ of the scaling action θ is the $\mathbb{K}^*_{\infty,+}$-action

$$\theta_\lambda|_{\mathbb{Z}_p}(f) := \int_{H=G\times\mathbb{Z}_p} \ell(x, y)\langle\lambda\rangle^y U_{(x,y)} d\mu(x, y), \quad \forall \lambda \in \mathbb{K}^*_{\infty,+} \tag{5.18}$$

while the restriction to $G \subset H$ of θ is the $\mathbb{K}^*_{\infty,+}$-action

$$\theta_\lambda|_G(f) := \int_{H=G\times\mathbb{Z}_p} \ell(x, y)x^{\deg(\lambda)} U_{(x,y)} d\mu(x, y) \quad \forall \lambda \in \mathbb{K}^*_{\infty,+}. \tag{5.19}$$

Proposition 5.8. *The restriction to $\mathbb{Z}_p \subset H$ of the scaling action θ on the dual system $\hat{\mathcal{A}}_H(\mathcal{L}_{\mathbb{K},1})$ corresponds, under the homomorphism* (5.17)*, to the action on $\tilde{\mathcal{L}}_{\mathbb{K},1}$ of the subgroup of $\mathbb{K}^*_{\infty,+}$ that maps to* $\mathrm{Gal}(\mathbb{K}^{ab}_\infty/\mathbb{K}^{un}_\infty)$ *under the local class field homomorphism. The restriction of the scaling action θ to the subgroup $\{s = (x, 0)|x \in G\} \subset H \subset S_\infty$ corresponds in the same way to the group of integer powers of the Frobenius.*

Proof. The result follows by combining Lemma 5.5 with Lemma 5.7. By Lemma 5.7 we see that we can write the restriction (5.18) as an action of U_1 of the form

$$\theta_\gamma(f) = \int_{H=G\times\mathbb{Z}_p} \ell(x, y)\gamma^y U_{(x,y)} d\mu(x, y), \quad \forall f \in \hat{\mathcal{A}}_H(\mathcal{L}_{\mathbb{K},1}), \quad \forall \gamma \in U_1. \tag{5.20}$$

Similarly, we write the restriction (5.19) as an action of $\mathbb{Z}$ of the form

$$\theta_m(f) = \int_{H=G\times\mathbb{Z}_p} \ell(x, y)x^{-d_\infty m} U_{(x,y)} d\mu(x, y), \quad \forall f \in \hat{\mathcal{A}}_H(\mathcal{L}_{\mathbb{K},1}), \quad \forall m \in \mathbb{Z}. \tag{5.21}$$

Lemma 5.5 shows that the scaling action (5.13) on the dual system $\hat{\mathcal{A}}_H(\mathcal{L}_{\mathbb{K},1})$ corresponds, under the homomorphism (5.17), to the action of $\mathbb{K}^*_{\infty,+}$ by scaling $L \mapsto \lambda L$ on commensurability classes of $\mathbb{K}$-lattices $L \in \tilde{\mathcal{L}}_{\mathbb{K},1}$. In particular, this implies that the action (5.20) corresponds to the action $L \mapsto \gamma L$, with $\Theta(\gamma) \in \mathrm{Gal}(\mathbb{K}^{ab}_\infty/\mathbb{K}^{un}_\infty)$. Similarly, the action (5.21) corresponds under (5.17) to the action $L \mapsto u_\infty^m L$ on $\tilde{\mathcal{L}}_{\mathbb{K},1}$, where $u_\infty^m = \Theta^{-1}(Fr^m)$. □

It is interesting to notice here that the Frobenius is recovered from the part of the scaling action that corresponds to the time evolution σ_x in "imaginary time" (here the subgroup of the $s = (x, 0)$ in S_∞ plays the role of the imaginary time it, $t \in \mathbb{R}$, in the complex case). The scaling action associated to the usual time evolution σ_y in the complementary direction $s = (1, y)$ gives the inertia, while in the archimedean case, the scaling associated to the real time evolution σ_t has instead properties comparable to a Frobenius (*cf.* [8]). We like to interpret this phenomenon as another instance of the presence of a "Wick rotation" in passing from an archimedean to a non-archimedean place. Here this is seen in the time evolution, while other such instances occur in the behavior of L-functions. This phenomenon in the case of L-functions was already observed by Manin in [26] (p. 135) and we also encountered it in the context of Mumford curves in [14] (end of §5.5).

5.4. **Non-archimedean measures on $\mathbb{Z}_p$ and the dual system.**

We now describe the dual system in the non-archimedean case more concretely, for time evolutions of the form $\sigma : \mathbb{Z}_p \to \mathrm{Aut}(\mathcal{A})$. We consider here the case where $H = \mathbb{Z}_p$ and we use the description of non-archimedean measures on $\mathbb{Z}_p$ as in [18] §8.4 and [20] §2.3.
Suppose given a measure μ on $\mathbb{Z}_p$ determined by the momenta

$$\int_{\mathbb{Z}_p} \binom{y}{k} d\mu(y) = X^{-k} \tag{5.22}$$

Then one can define the transform (*cf.* [20])

$$\hat{f}(X) = \sum_{k=0}^{\infty} f_k X^{-k} = \int_{\mathbb{Z}_p} f(y)\, d\mu(y). \tag{5.23}$$

We begin by a reformulation of the time evolution in the following way.

Lemma 5.9. *Suppose given a quantum statistical mechanical system $(\mathcal{A}, \sigma)$. We can write the time evolution $\sigma_y(a)$ in the form*

$$\sigma_y(a) = \sum_{k=0}^{\infty} \sigma_k(a) \binom{y}{k}, \tag{5.24}$$

with $\sigma_k(a) \in \mathcal{A}$ for $k \in \mathbb{Z}_{\geq 0}$. The coefficients $\sigma_k(a)$ satisfy

$$\sigma_{k+m}(a) = \sigma_k(\sigma_m(a)), \quad \forall k, m \in \mathbb{Z}_{\geq 0}, \quad \forall a \in \mathcal{A}, \tag{5.25}$$

$$\sigma_k(ab) = \sum_{j=0}^{k} \sigma_j(a)\sigma_{k-j}(b), \quad \forall k \in \mathbb{Z}_{\geq 0}, \quad \forall a, b \in \mathcal{A}. \tag{5.26}$$

Proof. We write continuous functions $f : \mathbb{Z}_p \to \mathcal{A}$ in the form

$$f(y) = \sum_{k=0}^{\infty} f_k \binom{y}{k}, \tag{5.27}$$

with coefficients $f_k \in \mathcal{A}$. For a given $a \in \mathcal{A}$ the properties of the time evolution σ ensure that the family $y \mapsto \sigma_y(a)$ defines a continuous function $\mathbb{Z}_p \to \mathcal{A}$, which we can then write in the form (5.24). We then use the fact that σ satisfies $\sigma_{y+x}(a) = \sigma_y(\sigma_x(a))$ and we identify the expressions

$$\sigma_{y+x}(a) = \sum_{k=0}^{\infty} \sigma_k(a) \binom{y+x}{k}$$

and

$$\sigma_y(\sigma_x(a)) = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \sigma_{k-j}(\sigma_j(a)) \binom{y}{k-j} \binom{x}{j},$$

The identity

$$\binom{y+x}{k} = \sum_{j=0}^{k} \binom{y}{k-j} \binom{x}{j}$$

then leads to the identifications $\sigma_k(a) = \sigma_{k-j}(\sigma_j(a))$.
We then set

$$\Sigma_a(X) = \widehat{\sigma_\cdot(a)}(X) = \int_{\mathbb{Z}_p} \sigma_y(a) d\mu(y). \tag{5.28}$$

Then we have

$$\Sigma_{ab}(X) = \sum_{k=0}^{\infty} \sigma_k(ab) X^{-k}$$

while the transform of $\sigma_y(a)\sigma_y(b)$ is given by the ordinary product of formal series

$$\Sigma_a(X)\Sigma_b(X) = \sum_{k=0}^{\infty}\sum_{j=0}^{k} \sigma_j(a)\sigma_{k-j}(b) X^{-k}.$$

The equality $\sigma_y(ab) = \sigma_y(a)\sigma_y(b)$ then gives the identity (5.26). $\square$

In order to pass to the dual system, we now consider continuous functions $\ell(y)$ of the form

$$\ell(y) = \sum_{k=0}^{\infty} \ell_k \binom{y}{k}, \tag{5.29}$$

with coefficients $\ell_k \in \mathcal{A}$, and the corresponding transforms

$$\hat{\ell}(X) = \sum_{k=0}^{\infty} \ell_k X^{-k}. \tag{5.30}$$

Lemma 5.10. *Consider the vector space of functions $\hat{\ell}$ of the form* (5.30)*. The product* (5.10) *induces on this space an algebra structure with product*

$$(\hat{\ell}_1 *_\sigma \hat{\ell}_2)(X) := \sum_{r=0}^{\infty}\sum_{k=0}^{r}\sum_{j=0}^{r-k} a_k \sigma_{r-k-j}(b_j)\, X^{-r}, \tag{5.31}$$

where $\ell_1(y) = \sum_k a_k \binom{y}{k}$ and $\ell_2(y) = \sum_k b_k \binom{y}{k}$, with $a_k, b_k \in \mathcal{A}$.

Proof. In (5.31) the coefficients $\sigma_{r-k-j}(b_j)$ are defined as in (5.24). We write

$$\sigma_y(\ell(x)) = \sum_{k=0}^{\infty}\sum_{j=0}^{k} \sigma_{k-j}(\ell_j) \binom{y}{k-j}\binom{x}{j}. \tag{5.32}$$

This then gives, for $\ell_1(y) = \sum_k a_k \binom{y}{k}$ and $\ell_1(x) = \sum_k b_k \binom{x}{k}$, the expression

$$\ell_1(y)\sigma_y(\ell_2(x)) = \sum_{r=0}^{\infty}\sum_{k=0}^{k}\sum_{j=0}^{r-k} a_k \sigma_{u-k-j}(b_j) \binom{y}{u-k-j}\binom{x}{j}, \tag{5.33}$$

from which we see that the convolution product (5.31) corresponds to (5.10). $\square$

We refer to the algebra of Lemma 5.10 with the product $*_\sigma$ as $\hat{\mathcal{A}}_X$.

We now describe the effect of the scaling action θ_λ on the dual system $\hat{\mathcal{A}}_X$. Recall that, given the decomposition $\lambda = u_\infty^m \langle\lambda\rangle$ in $\mathbb{K}^*_{\infty,+} = u_\infty^{\mathbb{Z}} \times U_1$, one has

$$\langle\lambda\rangle^y = \sum_{j=0}^{\infty} \alpha_\lambda^j \binom{y}{j}, \tag{5.34}$$

for $y \in \mathbb{Z}_p$, where $\langle\lambda\rangle = 1 + \alpha_\lambda$, with $v_\infty(\alpha_\lambda) > 0$.

Lemma 5.11. *For $\lambda \in \mathbb{K}^*_{\infty,+}$ and $y \in \mathbb{Z}_p$, the scaling action θ_λ of* (5.13) *is given in the form*

$$\theta_\lambda(\hat{\ell}) = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \ell_j \alpha_\lambda^{k-j} X^{-k}, \tag{5.35}$$

where $\hat{\ell}$ is as in (5.30) *and α_λ as in* (5.34).

Proof. The scaling action is given by the expression

$$\int \ell(y) \langle\lambda\rangle^y U_y d\mu(y).$$

The transform (5.23) of $\langle\lambda\rangle^y$ is given by the function

$$\hat{\alpha}_\lambda(X) = \sum_{j=0}^{\infty} \alpha_\lambda^j X^{-j}. \tag{5.36}$$

Thus, we see that the transform of $\ell(y)\langle\lambda\rangle^y$ yields the expression (5.35). $\square$

In particular, in the case of $\mathbb{K}$-lattices, the time evolution can be written in the form of Lemma 5.9 with

$$\sigma_y(f)(L, L') = \sum_{k=0}^{\infty} \alpha_{IJ^{-1}}^k \binom{y}{k},$$

where $\alpha_{IJ^{-1}}$ is defined as in (5.34), for $\langle I\rangle/\langle J\rangle$ as in (4.18).

6. Questions and directions

We want to outline briefly some natural questions posed by the setting for quantum statistical mechanics over function fields that we introduced in this paper.

The construction of noncommutative spaces of n-pointed Drinfeld modules works for arbitrary rank. Although in this paper we concentrate mostly on the rank 1 case, it would be interesting to study quantum statistical mechanical systems associated to higher rank. In particular we have seen that in the rank 1 case the partition function is a Goss zeta function and it would be interesting to see what arithmetic information the partition functions of higher rank cases give. Along these lines one may ask if the quantum statistical mechanical methods can give any information about special values of these functions.

We showed that the points of the moduli scheme of Drinfeld modules define "low temperature" KMS_x states of the system. A natural question is to study the "zero temperature" limits as $|x| \to \infty$. It is especially interesting to know if in this case the Artin homomorphism of local class field theory intertwines the action of symmetries of the quantum statistical mechanical system with the Galois action on values of ground states on a suitable subalgebra. A related question is how much one can parallel the cooling and distillation procedure described in the theory of endomotives to the positive characteristic setting.

Finally, there are other classes of objects, besides Drinfeld modules, that may have noncommutative counterparts (like our n-pointed Drinfeld modules) and associated quantum statistical mechanical systems. For instance, one could investigate similar constructions for Anderson's t-motives or Drinfeld's shtukas.

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