The monodromies of knots, hypersurface singularities and polynomials

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Intoduction

There are similar situations in three fields in which monodromy appears: in classical knot theory, in the theory of singularities and in algebraic geometry. For example, the Alexander polynomial of a knot corresponds to the characteristic polynomial in cohomology of the Milnor fibre of a singularity. We want to give a survey of some results concerning this subject concluding by the results of our recent paper [KK] (and some other results which are not contained in [KK]).

Now we give a more detailed description of this paper. In section 1 we define following Milnor the Alexander invariants and in particular the Alexander polynomials in a general situation of a CW-complex X and its infinite cyclic covering X_∞. Besides, we recall the Milnor exact sequence which connects homology of X and X_∞.

In section 2 we recall some facts of classical knot theory. In particular, we recall Stallings'es theorem, characterizing the fibred knots as knots whose groups possess a finitely generated commutator subgroups, to be compared with corresponding result on the fundamental group of the complement of a plane algebraic curve [K1]. In section 3 we consider algebraic knots throwing a bridge between knot theory and singularity theory.

In section 4 we consider the Milnor fibration $f' : X' \to S'$ of a germ of a hypersurface singularity. If $X_\infty = X' \times_S U$ is the canonical Milnor fibre, where $U \to S'$ is the unramified covering, then the monodromy transformation h of the Milnor fibre can be considered as a generator of the group of covering transformations of X_∞ / X'. We pay especial attention to the Monodrony theorem and the limit mixed Hodge structure on cohomology of X_∞ to be compared with results [KK] in global situation (see sections 9 and 10).

In section5 we consider the monodromy of a quasihomogeneous singularity to connect results on local and global situations. Besides, we recall a construction representing the Milnor fibre X_1 as a cyclic covering of the complement $U = \mathbb{P} \setminus V$ of a hypersurface in a weighted projective space.

In section 6 we investigate the global case, that is, the complement $X' = \mathbb{C}^2 \setminus D$ and $\mathbb{P}^2 \setminus \bar{D}$ of a plane algebraic curve. The definitions of the monodromy h and the (first) Alexander

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polynomial $\Delta(t) = \det(tid - h_*)$ are general. We recall the divisibility theorems of Libgober [L1] and Vik. Kulikov [K1].

In section 7 we review methords of calculations of $\Delta(t)$ in case of irreducible curve D intersecting the line at infinity transversally ([E], [LV]). They are based on Randell's theorem [R] reducing the calculation of $\Delta(t)$ to the calculation of the characteristic polynomial of the monodromy of the Milnor fibre X_1 of a homogeneous function $F(x_0, x_1, x_2)$. As we mentioned above X_1 is an unramified cyclic covering of $U = \mathbb{P}^2 \setminus \bar{D}$. Esnault imbeds it to a ramified coverinh of a "blown-up" plane $Y = \bar{\mathbb{P}^2}$, and expresses $\Delta(t)$ in terms of cohomology of invertible sheaves on Y. Loeser and Vaquie express $\Delta(t)$ in terms of cohomology of some sheaves on $\mathbb{P}^2$.

In sections 8-10 we review the results of our paper [KK]. In section 8 under weak conditions of connectivity and irreducibility, we obtain a relation between homology of X'_n, the unramified n-sheeted covering of $X' = \mathbb{C}^2 \setminus D$, and homology of its nonsingular projective model $\bar{X}_n$, and also with homology of X_∞, the infinite cyclic covering of X'. This generalizes Libgober's result [L1-L3]. Besides, we generalize the result of Kohno [Ko] on calculation of $\Delta(t)$ in terms of cohomology of rational differentioal forms. In section 9 we sketch the proof of our theorem [KK] on the semisimplicity of the monodromy h on $H_1(X_\infty)_{\neq 1}$. Besides, we five a sketch of the proof of a new result of the semisimplicity of h on $H_1(X_\infty)$ under some condition of transversality. At last in section 10, as a consequence of the semisimplicity theorem, we show how to introduce a natural mixed Hodge structure on $H_1(X_\infty)$.

1 Alexander invariants of infinite cyclic coverings

For the first time the notion of the Alexander invariants appeared in classical knot theory. Then it was transfered to other geometric contents. We begin with a general geometric situation (Milnor [$M2$]).

1.1. Let X be a finite connected complex or CW-complex and $\varphi : \widetilde{X} \to X$ be the infinite cyclic covering of X , determined by some epimorphism $p : \pi_1(X) \to \mathbb{F}_1$ onto the free group $\mathbb{F}_1 = \mathbb{Z}$. Then $\mathbb{F}_1$ acts freely on $\widetilde{X}$ as the group of covering transformations $Deck(\widetilde{X}/X)$ and $\widetilde{X}/\mathbb{F}_1 = X$.

Let k be a commutative ring and $\Lambda = \Lambda(k)$ be group ring $k[\mathbb{F}_1]$ of the group $\mathbb{F}_1$. If t is one of two generators of $\mathbb{F}_1$, then $\Lambda = k[t, t^{-1}]$ is the ring of Laurent polynomials in t with coefficients in k. The homology group $H_i(\widetilde{X}, k)$ has a natural structure of a Λ-module, where $t \cdot c = H_i(t)(c)$ for $c \in H_i(\widetilde{X})$, $H_i(t)$ is the automorphism corresponding to the covering transformation $t : \widetilde{X} \to \widetilde{X}$.

Definition 1 *The Λ-module $A_i = A_i(k) = H_i(\widetilde{X}, k)$ is called the i-th Alexander invariant (module) of a space X (more exactly of the pair (X, p)).*

1.2. If k is a field, then Λ is a principal ideal domain. The homology group $A = H_i(\widetilde{X}, k)$ is finitely generated over Λ. Hence by a general theorem of algebra A is isomorphic to a direct sum of cyclic modules

$$A \simeq \oplus_{j=1}^{l} \Lambda/(p_j) = \Lambda^b \oplus (\oplus_{i=1}^{k} \Lambda/(p_i)),$$

where (p_j) is a principal ideal generated by a polynomial $p_j(t)$. Here $\Lambda^b = \Lambda_{\text{free}}$ is a free part corresponding to $p_j(t) \equiv 0$, and $\oplus_{i=1}^k \Lambda/(p_i) = A_{\text{tors}}$ is a torsion submodule of the Λ-module A.

The product ideal $(p_1 \cdot \ldots \cdot p_l)$ is called the *order* of A. Obviously the order of A equals to 0 if and only if A has a free part, $b \neq 0$. In other words, the order $A \neq 0 \Leftrightarrow A = A_{\text{tors}}$ is a torsion module over Λ, i.e. A is a vector space of finite dimension over k.

Definition 2 *If $A = H_i(\widetilde{X}, k)$, then the polynomial $\Delta_i(t) = p_1(t) \cdot \ldots \cdot p_k(t)$ is called the i-th Alexander polynomial of a space X.*

Obviously we have

Proposition 1 *If $A = H_i(\tilde{X}, k)$ is finite dimensional over k, then $\Delta_i(t)$ coincides (to within a unit of Λ) with the characteristic polynomial of linear transformation $H_i(t) : H_i(\widetilde{X}, k) \to H_i(\widetilde{X}, k)$.*

1.3. In general, if k is not a field, then the ring $\Lambda = k[t, t^{-1}]$ is not a principal ideal domain. The most important example is $k = \mathbb{Z}$. In any case, with any finitely presented module A over a commutative ring Λ we can associate so called *Fitting ideals* $F_k(A)$. They are defined invariantly and are calculated in such a way. Let $\Lambda^t \xrightarrow{p} \Lambda^s \to A \to 0$ be a presentation of a Λ-module A and let P be the matrix of the linear map p. Then the k-th Fitting ideal $F_k(A) \subset \Lambda$ is generated by all minors of order $s - k$ of the matrix P.

Definition 3 *The ideals $F_{k-1}(A)$ are called the k-th Alexander ideals for Λ-module A.*

For any ideal $I \subset \Lambda$ denote by $\bar{I}$ the minimal principal ideal containing I.

Definition 4 *Any generator $\Delta_k(t)$ of the ideal $\overline{F_{k-1}(A)}$ is called the k-th Alexander polynomial for the Λ-module A.*

Note that in the knot theory only the first Alexander polynomial of the space $X = S^3 \setminus K$ is nontrivial for a knot K. So in this case the k-th Alexander polynomial of the module $H_1(\widetilde{X})$ is called the k-th Alexander polynomial of the knot K.

1.4. *The Milnor exact sequence.* There is an exact sequence which is very usefull for applications of $H_i(\widetilde{X})$ for study of $H_i(X)$. It is analogous to Wang exact sequence for a locally trivial fibration over a circle. Consider a short exact sequence

$$0 \to C_*(\widetilde{X}) \xrightarrow{t-1} C_*(\widetilde{X}) \to C_*(X) \to 0$$

of chain complexes. Then *homological (cohomological) Milnor exact sequence* is the corresponding homology (cohomology) exact sequence

$$\ldots \xrightarrow{\partial} H_i(\widetilde{X}) \xrightarrow{t-1} H_i(\widetilde{X}) \to H_i(X) \xrightarrow{\partial} H_{i-1}(\tilde{X}) \to \ldots \to H_0(X) \to 0.$$

2 Classical Knot Theory

2.1. Let $K \subset S^3$ be a knot, i.e. a connected submanifold in S^3 diffeomorphic to a circle S^1. The knot group is the most important invariant of a knot. The *group of the knot* K is the fundamental group $\pi = \pi_1(S^3 \setminus K)$ of the complement $S^3 \setminus K = X$, which is a $K(\pi, 1)$ Eilenberg-Maclane space. It is easy to see that $H_i(S^3 \setminus K) = \mathbb{Z}$ for $i = 0$, 1 and $H_i(S^3 \setminus K) = 0$ for $i \geq 2$. Since $H_1(S^3 \setminus K)$ is the abelianization of the group $\pi_1(S^3 \setminus K)$, we have an exact sequence

$$0 \to N \to \pi_1(X) \to H_1(X) \to 0,$$

where $N = [G, G] = G'$ is the commutator subgroup of G. Consider an infinite cyclic covering $\varphi : \widetilde{X} \to X$ determined by the epimorphism $p : \pi_1(X) \to H_1(X) = \mathbb{F}_1$.

Let $\Lambda = k[\mathbb{F}_1]$ be the group ring of the group $\mathbb{F}_1$.

Definition 5 *The first Alexander invariant* $A = H_1(\widetilde{X})$ *of the space* $X = S^3 \setminus K$ *is called the Alexander invariant of the knot* K.

Any presentation matrix for the Λ-module $H_1(\widetilde{X})$ is called an *Alexander matrix.* The $(k-1)$-th Fitting ideal $F_{k-1}(H_1(\widetilde{X}))$, correspondingly, the k-th Alexander polynomial $\Delta_k(t)$ for the Λ-module $H_1(\tilde{X})$ is called the *k-th Alexander ideal,* correspondingly, the *k-th Alexander polynomial for the knot* K.

Theorem 1 *The 1-st Alexander ideal* $F_0(H_1(\widetilde{X}))$ *of a knot* K *is principal.*

The first Alexander ideal $F_0(H_1(\widetilde{X}))$ is called simply an *Alexander ideal* and its generater $\Delta(t) = \Delta_1(t)$ is called an *Alexander polynomial for the knot* K.

One can show that $\Delta(t) = \det(V - tV^{\tau})$, where V is the so-called Seifert matrix for a knot K. It is a $2g \times 2g$ matrix defined by means of notions of a Seifert surface and a linking number of cycles.

The following result shows that there is great freedom for the Alexander polynomials.

Theorem 2 (Seifert). *If a polynomial* $\Delta \in A$ *satisfies the conditions* $\Delta(t) = \pm 1$ *and* $\Delta(t) = t^d \cdot \Delta(t^{-1})$, *where* $d = \deg \Delta$, *then there is a knot* $K \subset S^3$ *such that its Alexander polynomial is* Δ.

2.2. Now we define a more restrictive class of knots which contains all algebraic knots (the knots of singularities).

Definition 6 *A knot* $K \subset S^3$ *is fibered if there is a fibration map* $p : S^3 \setminus K \to S^1$ *such that* K *has a tubular neighborhood* $T \simeq S^1 \times D^2$ *in* S^3 *for which the diagram*

$$\begin{array}{ccc} T \setminus K & \xrightarrow{\sim} & S^1 \times (D^2 \setminus \{0\}) \\ & {\scriptstyle p|_{T\setminus K}} \searrow \quad \swarrow {\scriptstyle p_0} & \\ & S^1 & \end{array}$$

is commutative, where $D^2 = \{y \in \mathbb{C} \mid \mid y \mid < 1\}$ *is the unit disk and* $p_0(x, y) = \frac{y}{|y|}$.

Fibres $F_t = p^{-1}(t)$, $t \in S^1$, are the interiors of compact surfaces $\bar{F}_t \subset S^3$ with common boundary $\partial \bar{F}_t = K$. ($\bar{F}_t$ is obtained from a compact surface without boundary $\tilde{F}_t$ by means of removing an open disk and is called a Seifert surface for the knot K).

Let $F = F_1 = p^{-1}(1)$. The exact homotopy sequence of the fibration p gives that the commutator subgroup of $\pi_1(S^3 \setminus K)$ coinsides with $\pi_1(F)$ and, consequently, is a finitely generated free group. The fibered knots are charecterized by this property.

Theorem 3 (Stallings). *A knot K is fibered if and only if the commutator subgroup $[\pi_1(S^3 \setminus K), \pi_1(S^3 \setminus K)]$ is finitely generated (and free).*

The fibration p is locally trivial over a circle S^1 and so defines a *monodromy homeomorphism* $h : F \to F$ and a *momodromy operator* $H_1(h) : H_1(F) \to H_1(F)$, which we'll denote often simply by h. Let $[h]$ be a matrix of h. One can prove

Proposition 2 *The matrix $t \cdot Id - [h]$ is an Alexander matrix for the knot K. In particular, the Alexander polynomial $\Delta(t)$ for the fibered knot coincides with the characteristic polynomial of the monodromy operator*

$$\Delta(t) = \det(t \cdot Id \; - [h]).$$

3 Algebraic Knots

3.1. Let $(C, 0) \subset (\mathbb{C}^2, 0)$ be an isolated singularity of a plane curve (i.e. the curve is reduced). Then we can associate with $(C, 0)$ a link $L \subset S^3$, where $L = C \cap S^3_\varepsilon$ and S^3_ε is a sphere of sufficienly small radius ε. Such links are called *algebraic links*. If $(C, 0)$ has k components, then the link $L = L(C, 0)$ has also k components. In particular, if $(C, 0)$ is irreducible, $k = 1$, then we obtain an *algebraic knot* $L = K \subset S^3$.

Let $f(x, y) = 0$ be an equation of $(C, 0)$, i.e. $(C, 0)$ is the zero fibre of a morphism of germs $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$. Consider the map

$$p : S^3_\varepsilon \setminus L \to S^1, \qquad p(x, y) = \frac{f(x, y)}{| f(x, y) |}.$$

By a general result of Milnor $[M1]$ it follows that p is a locally trivial fibration, i.e. we have the following proposition.

Proposition 3 *All algebraic knots (and links) are fibred.*

3.2. An algebraic knot $K = K(C, 0)$ is entirely defined by the type of the singularity $(C, 0)$.

Theorem 4 (K.Brauner). *If*

$$P(C, 0) = \{(m_1, n_1), (m_2, n_2), \dots, (m_s, n_s)\}$$

is the sequence of Puiseux pairs of a plane curve singularity $(C, 0)$, then the knot $K(C, 0)$ is equivalent to the iterated torus knot associated to the sequence of pairs $P(C, 0)$.

The Alexander polynomial $\Delta(t)$ of a knot $K(C,0)$ also can be expressed in terms of Puiseux pairs.

Theorem 5 (Lê Dũng Tráng [Le]).

$$\Delta(t) = \Delta_1(t) = P_{\lambda_1,n_1}(t^{\nu_2}) \cdot \ldots P_{\lambda_s,n_s}(t^{\nu_{s+1}}),$$

where

$$P_{\lambda,n}(t) = \frac{(t^{\lambda n}-1)(t-1)}{(t^\lambda - 1)(t^n-1)},$$

and $\nu_i = n_i \cdot \ldots \cdot n_s$ *for* $i = 1, \ldots, s$, $\nu_{s+1} = 1$, *and* $\lambda_1 = m_1$, $\lambda_i = m_i - m_{i-1}n_i + \lambda_{i-1}n_i n_{i-1}$ *for* $i = 2, \ldots, s$.

For example, if $(C,0)$ is a cusp, $x^3 + y^2 = 0$, then $K(C,0)$ is the trefoil knot and

$$\Delta(t) = \frac{(t^6-1)(t-1)}{(t^3-1)(t^2-1)} = t^2 - t + 1.$$

Besides, Lê Dũng Tráng proved the following theorem.

Theorem 6 . *The quotient Δ_1/Δ_2 of the first two Alexander polynomial of the knot $K(C,0)$ is the minimal polynomial of the monodromy operator $M = h_* : H_1(F) \to H_1(F)$ of the singularrity $(C,0)$. Moreover, this polynomial has distinct roots and hence the monodromy M has finite order and, in particular, M is semisimple.*

Remark 1 If the singularity $(C,0)$ is not irreducible, then the monodromy operator M can be not semisimple. For example [AC1], if $f = (x^2+y^3)(x^3+y^2)$, then the minimal polynomial of M is equal to $(t^5+1)(t^2-1)$ and has a double root $t=-1$. Hence the monodromy operator M has infinite order.

4 Milnor fibrations of germs of analytic functions

4.1. The Milnor fibration of an algebraic knot is a particular case of the Milnor fibration of an analytic function germ $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$. Let $(Y,0) \subset (\mathbb{C}^{n+1}, 0)$ be a germ of a hypersurface with the equation $f(x_0, x_1, \ldots, x_n) = 0$. Let the germ f be defined in a neighborhood of the closed ball $\bar{B}_\varepsilon$ of sufficienly small radius $\varepsilon > 0$. Let $S_\varepsilon^{2n+1} = \partial \bar{B}_\varepsilon$ be the boundary sphere.

Definition 7 $K = K(Y,0) = Y \cap S_\varepsilon^{2n+1} \subset S_\varepsilon^{2n+1}$ *is called the knot of a singularity f or $(Y,0)$.*

Theorem 7 $[M1]$. *If $\varepsilon > 0$ is sufficienly small, then the map*

$$\varphi : S_\varepsilon^{2n+1} \setminus K \to S^1, \qquad \varphi(x) = \frac{f(x)}{| f(x) |},$$

is a smooth locally trivial fibration.

If the singularity f is isolated, then the knot K is a smooth $(2n-1)$ -manifold. Any fiber $F_t = \varphi^{-1}(t)$ is a smooth open manifold whose closure $\bar{F}_t = F_t \cup K$. $\bar{F}_t$ is a manifold with boundary $\partial \bar{F}_t = K$.

The fibration φ is called the *Milnor fibration of a singularity* f.

From the historical point of view the fibration φ is a natural generalization of fibred knots (and then of algebraic knots). But there is another equivalent fibration associated to f which is also called the Milnor fibration.

4.2. Let $S = S_\delta = \{t \in \mathbb{C} \mid |t| < \delta\}$, $S' = S \setminus \{0\}$, $X = X_{\varepsilon,\delta} = B_\varepsilon \cap f^{-1}(S_\delta)$, $\bar{X} = \bar{B}_\varepsilon \cap f^{-1}(S_\delta)$, $X' = X \setminus f^{-1}(0)$, and $f : X \to S$, $\bar{f} : \bar{X} \to S$ denote the restrictions of f to X and $\bar{X}$.

Theorem 8 ([$M1$]; Lê Dũng Tráng, 1977). *If $\varepsilon \gg \delta > 0$ are sufficiently small, then the map $\bar{f} : \bar{X}' \to S'$ is topological locally trivial fibration, and $f' : X' \to S'$ is a smooth locally trivial fibration.*

Definition 8 *The fibration $f' : X' \to S'$, and also $\bar{f}' : \bar{X}' \to S'$, is called the Milnor fibration of a singularity f.*

Sometimes, one uses the terms open Milnor fibration and closed Milnor fibration to distinguish between f' and $\bar{f}'$.

The fibre $X_t = f^{-1}(t)$, $t \in S'$, is a Stein complex manifold, $\dim X_t = n$. The fibre $\bar{X}_t = \bar{f}^{-1}(t)$ is a manifold with boundary. X_t and $\bar{X}_t$ have the homotopy type of a CW-complex of real dimension n. The fibre X_t (and also $\bar{X}_t$) is called the *Milnor fibre of a singularity* f.

Obviously the fibrations f and $\bar{f}$ are homotopically equivalent to their restrictions over the circle $S^1_{\delta/2} \subset S_\delta$ of radius $\delta/2$. Identifying $S^1_{\delta/2}$ and S^1, we can assume that the radius of the circle equals to 1. Denote the restrictions of f and $\bar{f}$ over S^1 by ψ and $\bar{\psi}$ correspondingly.

Theorem 9 *(i) The fibrations φ and ψ are fibre diffeomorphic equivalent.*
(ii) The fibrations ψ and $\bar{\psi}$ are fibre homotopy equivalent.

Thus introduced definitions of the notion of Milnor fibration are equivalent.

Remark 2 The Milnor fibration associated to a hypersurface singularity $(Y, 0)$ does not depend on the choice of an equation $f = 0$ for $(Y, 0)$. This comes from the fact that K-arbits are connected. Moreover, the equivalence class of the Milnor fibration for an isolated singularity does not change under μ-const deformations.

4.3. A locally trivial fibration over a circle S^1 determine (and is determined by) a *monodromy transformation*

$$h : X_t \to X_t.$$

The monodromy transformation determines homology and cohomology operators

$$M = h_* : H_*(X_t) \to H_*(X_t), \quad T = (h^*)^{-1} : H^*(X_t) \to H^*(X_t).$$

The basic property of monodromy operators is given by

Monodromy Theorem. *Let* $T : H^p(X_t, \mathbb{C}) \to H^p(X_t, \mathbb{C})$ *be the monodromy operator in p-dimensional cohomology of the Milnor fibration of a hypersurface singularity* $(Y, 0) \subset (\mathbb{C}^{n+1}, 0)$ *or of a germ of a function* $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$. *Then*

(i) All the eigenvalues of T are roots of unity, or in other words, the operator T is quasiunipotent, i.e there exist positive intergers l and q such that

$$(T^l - id)^q = 0.$$

(ii) T^l *has index of unipotency at most p, i.e. we can take* $q = p + 1$, *i.e. the dimensions of Jordan blocks of T are less or equal to* $p + 1$.

There are several different proofs of the Monodromy Theorem (Grothendieck A., Landman A., Clemens C.H., Katz N.M., Borel A., Brieskorn E.,..., see, for example, [G-S]).

The Monodromy Theorem is closely connected with the theory of mixed Hodge structures (MHS). One can find an introduction to this theory in [G-S].

4.4. Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be an isolated singularity and let X_t be its Milnor fibre. We consider the *canonical Milnor fibre* X_∞, i.e. the total space of the pullback of the Milnor fibration $f' : X' \to S'$ to the universal cover $U \to S'$ of S', $X_\infty = X' \times_S U$. As each X_t is homotopy equivalent to X_∞, there is a canonical isomorphism between $H^n(X_t)$ and $H^n(X_\infty)$. The vanishing cohomology groups carry a mixed Hodge structure first defined by Steenbrink [*S*], who used an embedding of $f : X \to S$ to a family of projective hypersurfaces and resolution of singularities of this family. Then Varchenko A.N. and Sherk-Steenbrink [S-S] gave another description of the Hodge filtration on $H^n(X_\infty)$ which does not use resolution of singularities. The weight filtration $W.$ on $H^n(X_\infty)$ is connected with the monodromy operator T. It is the weight filtration of the nilpotent operator $N = -\frac{1}{2\pi i} \log T_u$, where T_u is the unipotent part of the monodromy.

We can make more precise formulation of part (ii) of the Monodromy Theorem for $H^n(X_t)$:

ii') The Jordan blocks of T are of size at most $n + 1$. *The Jordan blocks for eigenvalue 1 of T are of size at most n.*

Van Doorn M.G.M. and Steenbrink J.H.M. [DS] gave the following supplement to the Monodromy Theorem :

Theorem 10 *If the monodromy operator T on* $H^n(X_t)$ *has a Jordan block of size* $n + 1$ *(necessarily for an eigenvalue* $\neq 1$*), then T also has a Jordan block of size n for the eigenvalue 1.*

This theorem is an analogue in higher dimensions of the following result of Lê D.T. (1972): *The monodromy of an irreducible plane curve singularity is of finite order.*

4.5. At last we want to mention about the MHS on cohomology of the knot (link) of a singularity and the Wang sequence (cf. [Ka]) to be compared with the Milnor exact sequence and our result in the last section of this article.

Let $K = X_0 \cap S^{2n+1}$, $X_0 = f^{-1}(0)$, be the knot of an isolated singularity f. It is known that the pair (X, X_0) is homeomorphic to the cone on (S^{2n+1}, K) with the vertex $x_0 = \operatorname{Sing} f$. So K is homotopy equivalent to $X_0 \setminus \{x_0\}$ and hence $H_i(X_0 \setminus \{x_0\}) \simeq H_i(K)$. By the

Poincaré duality we have an isomorphism $H_i(K) \simeq H^{2n-1-i}(K)$, and by the Alexander duality $H^{2n-1-i}(K) \simeq H_{i+1}(S^{2n+1} \setminus K)$. Again $H_{i+1}(S^{2n+1} \setminus K) \simeq H_{i+1}(X')$, $X' = X \setminus X_0$, since $S^{2n+1} \setminus K$ is homotopy equivalent to X'. For the cohomology with coefficients in a field we obtain dual isomorphisms

$$H^i(K) \simeq H^i(X_0 \setminus \{x_0\}) \simeq H^{i+1}(X').$$

So we can think about each of these cohomology as cohomology of the knot of a singularity.

Using the above isomorphism we can introduce a MHS on $H^i(K)$. Indeed, because X_0 is contractible, the long exact cohomology sequence of the couple $(X_0, X_0 \setminus \{x_o\})$ implies isomorphism $H^i(X_0 \setminus \{x_0\}) \simeq H^{i+1}(X_0, X_0 \setminus \{x_0\}) \stackrel{\text{def}}{=} H^{i+1}_{\{x_0\}}(X_0)$. But X_0 can be extended to a complete variety X_0, and by excision $H^{\cdot}_{\{x_0\}}(X_0) \simeq H^{\cdot}_{\{x_0\}}(\bar{X}_0)$. On $H^{\cdot}_{\{x_0\}}(\bar{X}_0)$ there is a canonical and functorial MHS.

The monodromy operator T appears in *Wang exact sequence*

$$\ldots \to H^i(X') \to H^i(X_t) \xrightarrow{T-\text{id}} H^i(X_t) \to H^{i+1}(X') \to \ldots .$$

For an isolated singularity f the Milnor fibre X_t is homotopy equivalent to a buquet of n-spheres and hence $H^i(X_t) \neq 0$ only for $i = 0$ and $i = n$. Consequently the only interesting cohomology groups of the knot are $H^n(X')$ and $H^{n+1}(X')$ which are the kernel and the cokernel, respectively, of the map $T - Id : H^n(X_t) \to H^n(X_t)$.

The terms of the Wang sequence carry the MHS. The map $T - Id$ need not be a morphism of Hodge structures. But if we change $T - Id$ by $N = -\frac{1}{2\pi i} \log T_u$, then the sequence remains exact and becomes a MHS exact sequence.

5 Monodromy of a quasihomogeneous singularity

5.1. Let $f \in \mathbb{C}[x_0, \ldots, x_n]$ be a quasihomogeneous (= weighted homogeneous) polynomial of degree $\deg f = N$ with respect to the weights $wt\, x_i = w_i$, i.e. f satisfies the Euler relation

$$f(r^{w_0} x_0, \ldots, r^{w_n} x_n) = r^N f(x_0, \ldots, x_n), \;\; \forall r \in \mathbb{C}^*, \quad x \in \mathbb{C}^{n+1}.$$

Consider the singularity $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$. Then the local Milnor fibration $f' : X' \to S'$ defined in the previous section is equivalent to the global *affine Milnor fibration* f', where we denote by $f : X \to S$ a morphism defined by the polynomial f,

$$\begin{array}{ccccc} \mathbb{C}^{n+1} = X & \supset & X' = X \setminus X_0\,, & & X_0 = f^{-1}(0) \\ \Big\downarrow f & & \Big\downarrow f' & & \\ \mathbb{C} = S & \supset & S' = \mathbb{C} \setminus \{0\}\,. & & \end{array}$$

Indeed, we can consider the $\mathbb{C}^*$-action on $\mathbb{C}^{n+1}$ associated to weights $w = (w_0, \ldots, w_n)$

$$r \circ x = (r^{w_0} x_0, \ldots, r^{w_n} x_n), \quad r \in \mathbb{C}^*, \quad x \in \mathbb{C}^{n+1}.$$

The fibre $X_t = f^{-1}(t)$ has the equation $f(x_0,\ldots,x_n) = t$. The Euler relation $f(r \circ x) = r^N f(x) = r^N t$ shows that $r \in \mathbb{C}^*$ translates the fibre X_t to the fibre $X_{r^N t}$, and $\mathbb{C}^*$ acts on the set of fibres X_t, $t \in \mathbb{C}^*$, transitively. If we define the action of $\mathbb{C}^*$ on $S = \mathbb{C}$, $r \circ t = r^N t$, then we see that the morphism $f : \mathbb{C}^{n+1} \to \mathbb{C}$ is equivariant.

5.2. Let us calculate the monodromy $h : X_t \to X_t$. Consider the fibre $F = X_1$ over the point $t = 1 \in S^1 = \{| t |= 1\}$. If we construct a family of diffeomorphisms $h_\varphi : X_1 \to X_t$, $t = e^{2\pi i\varphi}$, then $h = h_1 : X_1 \to X_1$ is the monodromy. Obviously, we can take $h_\varphi =: e^{\frac{2\pi i\varphi}{N}} \circ (\cdot)$ and then we have $r \circ x \in X_{r^N} = X_{e^{2\pi i\varphi}} = X_t$ for $x \in X_1$, $f(x) = 1$, and $r = e^{\frac{2\pi i\varphi}{N}}$. Thus we obtain a concrete description of the geometric monodromy $h : X_1 \to X_1$,

$$h(x) = (e^{\frac{2\pi i w_0}{N}} x_0, \ldots, e^{\frac{2\pi i w_n}{N}} x_n).$$

In particular, $h^N = 1$ and the monodromy $T = h^* : H^\cdot(X_t, \mathbb{C}) \to H^\cdot(X_t, \mathbb{C})$ has a finite order, $T^N = id$. We obtain

Corollary 1 *The monodromy of a quasihomogeneous singularity is semisimple, $T = T_s$, and all eigenvalues λ are roots of unity, $\lambda^N = 1$.*

5.3. We can use the weighted projective space

$$\mathbb{P} = \mathbb{P}(w) = \mathbb{C}^{n+1} \setminus \{0\}/\mathbb{C}^*,$$

where the action of $\mathbb{C}^*$ on $\mathbb{C}^{n+1}$ is defined above, to calculate the eigensubspaces of $H^k(X_t, \mathbb{C})$. The space $\mathbb{P}(w)$ generalizes the usual projective space $\mathbb{P}^n$ corresponding to weights $w = (1,\ldots,1)$ (in the case of homogeneous polynomial f). The equation $f(x) = 0$ defines a hypersurface $V \subset \mathbb{P}$ and let $U = \mathbb{P} \setminus V$ be the complement to V. We can consider $Y = X_0 \subset \mathbb{C}^{n+1}$ as a quasicone $Y = C_V$ over V. We obtain a diagram

$$\begin{array}{ccccc}
Y \setminus \{0\} & & \longrightarrow & & V \\
\cap & & & & \cap \\
\mathbb{C}^{n+1} \setminus \{0\} & & \longrightarrow & & \mathbb{P} \\
\cup & & & & \cup \\
X = \mathbb{C}^{n+1} & \supset & X' & \longrightarrow & U = \mathbb{P} \setminus V \\
f \downarrow & & f' \downarrow \hookleftarrow & F = X_1 \nearrow p & \\
S = \mathbb{C} & \supset & S' \ni 1 & &
\end{array}$$

Thus, $\mathbb{C}^{n+1} \setminus \{0\}$ is partitioned into $\mathbb{C}^*$-orbits and the set of orbits is $\mathbb{P}$. The fibre $Y = f^{-1}(0)$ consists of orbits, the generators of the quasicone C_V, $Y \setminus \{0\}/\mathbb{C}^* = V$. And any fibre X_t, $t \neq 0$, is mapped onto U. Moreover, the subgroup $\mu_N = \{r \in \mathbb{C}^* \mid r^N = 1\}$ is the stationary subgroup of the fibre X_t ($r^N t = t \Rightarrow r^N = 1$). The group μ_N acts on X_t and the generator $e^{2\pi i/N}$ of this group acts as the monodromy h, and $X_t/\mu_N = U$.

Thus, $F/_{\mu_N} = U$ and we can use the differential forms on the principal open subset $U = \mathbb{P} \setminus V$ to calculate summands of $H^{\cdot}(F, \mathbb{C})$, $H^{\cdot}(F, \mathcal{O}_F)$, $H^{\cdot}(F, \Omega_F^p)$ in their decomposition into the sum corresponding to the characters $\chi \in \mathrm{Hom}(\mu_N, \mathbb{C}^*)$.

6 Alexander polynomial of a plane curve

6.1. Let $D \subset \mathbb{C}^2$ be a plane affine curve of degree d defined by an equation $f(x,y) = 0$. Let

$$f(x,y) = \prod_{i=1}^{k} f_i^{m_i}(x,y) \tag{1}$$

be the decomposition into irreducible factors. Denote by $D_i \subset \mathbb{C}^2$ an irreducible component of D defined by the equation $f_i(x,y) = 0,\ i = 1, \ldots, k$. The function $t = f(x,y)$ defines a morphism $f : X \to S$, where $X = \mathbb{C}^2,\ S = \mathbb{C}^1$. Then $D = f^{-1}(0)$. Denote $S' = \mathbb{C} \setminus \{0\},\ X' = X \setminus D$. Consider the infinite cyclic covering $\varphi = \varphi_\infty : X_\infty \to X'$ corresponding to the universal covering $e : U \to S'$, where $U = \mathbb{C},\ t = e(u) = e^{2\pi i u}$,

$$\begin{array}{ccc} X_\infty & \xrightarrow{\varphi} & X' \\ {\scriptstyle f_\infty}\downarrow & & \downarrow{\scriptstyle f'} \\ U & \xrightarrow[e]{} & S' \end{array}$$

It is well known that $H_1(X', \mathbb{Z}) = \mathbb{Z}^k$ is generated by the loops γ_i "surrounding" the components D_i. The homomorphism $f'_* : \pi_1(X') \to \pi_1(S') = \mathbb{Z}$ factors through the Hurewitz homomorphism H

$$f'_* : \pi_1(X') \xrightarrow{H} H_1(X', \mathbb{Z}) = \mathbb{Z}^k \xrightarrow{\delta} \mathbb{Z},$$

where $\delta(s_1[\gamma_1] + \ldots + s_k[\gamma_k]) = \sum_{i=1}^k s_i m_i$.

Definition 9 *We say that f is primitive if the generic fibre of $f : \mathbb{C}^2 \to \mathbb{C}$ is irreducible.*

It is well known that if f is not primitive, then f factors through a covering $p : \mathbb{C} \to \mathbb{C},\ f(x,y) = p(g(x,y))$ such that $g(x,y)$ is primitive.

Proposition 4 ($[K1], [Sa]$). *The vector space $H_1(X_\infty, \mathbb{C})$ is finite-dimensional if and only if f is a power of a primitive polynomial.*

In the sequel we'll assume that f is primitive, in particular,

$$G.C.D.(m_1, \ldots, m_k) = 1.$$

Then $f'_* : \pi_1(X') \to \mathbb{Z}$ is an epimorphism.

The following theorem is an analog of Theorem 3 (Stallings) in the case of algebraic curves.

Theorem 11 ([K1], [K2]) *If f is primitive, then $Ker\, f'_*$ is finitely generated. In particular, if $D \subset \mathbb{C}^2$ is an irreducible curve, then $[\pi_1(\mathbb{C}^2 \setminus D), \pi_1(\mathbb{C}^2 \setminus D)]$ is finitely generated.*

Definition 10 *The i-th Alexander polynomial $\Delta_{D,i}$ of a curve $D \subset \mathbb{C}^2$ is the i-th Alexander polynomial of the space $X' = \mathbb{C}^2 \setminus D$ associated with the epimorphism f'_*. The first Alexander polynomial $\Delta_D(t) = \Delta_{D,1}(t)$ is called simply the Alexander polynomial of D.*

Definition 11 *The Alexander polynomial $\Delta_D(t)$ of a projective curve $D \subset \mathbb{P}^2$ is the Alexander polynomial of an affine curve $D \subset \mathbb{C}^2$, where $\mathbb{C}^2 = \mathbb{P}^2 \setminus L$, $D = D \cap \mathbb{C}^2$, and $L \subset \mathbb{P}^2$ is a generic line.*

We can begin with an affine curve $D \subset \mathbb{C}^2$ and consider its projective closure $\bar{D} \subset \mathbb{P}^2$. Then for the equality $\Delta_{\bar{D}}(t) = \Delta_D(t)$ it is necessary for L to intersect D trasversally.

Theorem 12 (Randell [*R*]). *If $\bar{D} \subset \mathbb{P}^2$ is a reduced curve defined by the equation $F(x_0, x_1, x_2) = 0$, then $\Delta_{\bar{D}}(t)$ is equal to the characteristic polynomial of the monodromy h on first homology of the Milnor fibre of the homogeneous singularity $F : \mathbb{C}^3 \to \mathbb{C}$,*

$$\Delta_{\bar{D}}(t) = \det(t \cdot id - h).$$

In particular, this yields that the monodromy of a reduced curve transversally intersecting the line at infinity is semisimple because it is so for the monodromy of a quasihomogeneous singularity.

6.2. *The divisibility by the Alexander polynomial* ([L1], [K1]). Let $p_i \in D$ be a singular point. Denote by $\Delta_{p_i,D}(t)$ the characteristic polynomial of the monodromy T on cohomology $H^1(X_t)$ of the Milnor fibre of the singularity (D, p_i). Equivalently, $\Delta_{p_i,D}(t)$ is the Alexander polynomial of the algebraic link $K(D, p_i)$.

Let $S^3 = \partial T(L)$ be the boundary of the tubular neighbouhood of the line $L = L_\infty \subset \mathbb{P}^2$ at infinity. Denote by $\Delta_{\infty,\bar{D}}(t)$ the Alexander polynomial of the link $\bar{D} \cap \partial T(L) \subset \partial T(L)$. If L is in general position relative to $\bar{D}$, then

$$\Delta_{\infty,\bar{D}}(t) = (t-1)(t^d-1)^{d-2}.$$

Theorem 13 (Libgober [L1]). *If $D \subset \mathbb{C}^2$ is an irreducible curve, then*

(i) $\Delta_{\bar{D}}(t)$ divides the product $\prod \Delta_{p_i,D}(t)$ of the local Alexander polynomials of all singularities $p_i \in \bar{D}$.

(ii) $\Delta_{\bar{D}}(t)$ divides $\Delta_{\infty,\bar{D}}(t)$.

This theorem provides some information about the fundamental group $\pi_1(\mathbb{C}^2 \setminus D)$. The application of it to curves with only cusps and nodes one can find in [*L*3].

Let $X_{t_1}, \ldots, X_{t_q}$ be the degenerate fibres of $f' : X' \to S'$ such that $X' \setminus (\cup X_{t_j}) \to S' \setminus (\cup t_j)$ is a C^∞ locally trivial fibration. Let X_t be a generic fibre. Let γ_0 and γ_∞ be circles with centers at 0 and of radius $r_0 \ll 1$ and $r_\infty \gg 1$. Denote by h_0 and h_∞ the monodromy operators on $H^1(X_t)$ corresponding to γ_0 and γ_∞ (and defined modulo an inner automorphism). We call the characteristic polynomial $\Delta_{in}(t) = \det(h_0 - t \cdot Id)$ (correspondingly, $\Delta_{ex}(t) = \det(h_\infty - t \cdot Id)$) the *internal* (and (correspondingly, *external*) *Alexander polynomial.*

Theorem 14 (Kulikov [K1]). *If the polynomial $f(x, y)$ is primitive, then $\Delta_D(t)$ divides $\Delta_{in}(t)$ and it divides $\Delta_{ex}(t)$.*

Let $\sigma : \bar{\mathbb{P}}^2 \to \mathbb{P}^2$ be a composition of σ -processes resolving the points of indeterminacy of the rational map

$$\begin{array}{ccc} f : \mathbb{P}^2 & - - \to & \mathbb{P}^1 \\ \cup & & \cup \\ \mathbb{C}^2 & \longrightarrow & \mathbb{C}^1 \end{array} ,$$

Put $\tilde{f} = \sigma \circ f : \bar{\mathbb{P}}^2 \to \mathbb{P}^1$. We can assume that the fibres

$$\tilde{f}^{-1}(0) = \sum_{i=1}^{N_0} m_i \bar{D}_i, \qquad \tilde{f}^{-1}(\infty) = \sum_{i=1}^{N_\infty} r_i R_i$$

are divisors with normal crossings. Put

$$D_i^0 = \bar{D}_i \setminus (\bigcup_{i \neq j} (\bar{D}_i \cap \bar{D}_j)), \qquad R_i^0 = R_i \setminus (\bigcup_{i \neq j} (R_i \cap R_j)).$$

In this case the internal and external Alexander polynomials can be calculated in terms of Euler characteristics $\chi(D_i^0)$ (respectively $\chi(R_i^0)$) and multiplicities m_i (respectively r_i) [AC2] :

$$\begin{aligned} \Delta_{in}(t) &= (t-1) \prod_{i=1}^{N_0} (t^{m_i} - 1)^{-\chi(D_i^0)}, \\ \Delta_{ex}(t) &= (t-1) \prod_{i=1}^{N_\infty} (t^{r_i} - 1)^{-\chi(R_i^0)}. \end{aligned}$$

In particular, if the curve $\bar{D} = \bar{D}_1 \cup \ldots \cup \bar{D}_k \subset \mathbb{P}^2$ intersects L_∞ transversally, then

$$\Delta_{ex}(t) = (t-1)(t^{\sum d_i m_i} - 1)^{\sum d_i - 2},$$

where $d_i = \deg D_i$.

7 Calculations of Alexander polynomials of reduced curves

We review the results of Esnault [E], Loeser-Vacuié [LV], Kohno [Ko].

7.1. *The Alexander polynomial coincides with the characteristic polynomial of the monodromy for a homogeneous singularity.* Let $D = D_1 + \ldots + D_k \subset \mathbb{P}^2$ be a reduced curve of degree d with the equation $F(x_0, x_1, x_2) = 0$. In virtue of Randell's Theorem the question is reduced to the calculation of the characteristic polynomial $\Delta(t) = \Delta_D(t)$ for the monodromy $h^* : H^1(X_1, \mathbb{C}) \to H^1(X_1, \mathbb{C})$ of the Milnor fibre $X_1 = F^{-1}(1)$ of the homogeneous singularity $F : \mathbb{C}^3 \to \mathbb{C}$.

7.2. *The monodromy transformation is a generator of the group of automorphisms of a cyclic unramified covering.* Consider the diagram from section 5

$$
\begin{array}{ccccc}
X_0 \setminus \{0\} & & \longrightarrow & & D \\
\cap & & & & \cap \\
\mathbb{C}^3 \setminus \{0\} & & \longrightarrow & & \mathbb{P}^2 \\
\cup & & & & \cup \\
X = \mathbb{C}^3 & \supset & X' & \longrightarrow & U = \mathbb{P}^2 \setminus D \\
F \big\downarrow & & \big\downarrow \quad \hookrightarrow & X_1 & \nearrow p \\
S = \mathbb{C} & \supset & S' & &
\end{array}
$$

The calculation of $\Delta(t)$ is based on the fact that $p : X_1 \to U$ is an unramified cyclic covering of degree d, and the monodromy $h : X_1 \to X_1$ acting as $h(x_0, x_1, x_2) = (\zeta x_0, \zeta x_1, \zeta x_2)$, where $\zeta = e^{2\pi i/d}$ is the primitive root of unity of degree d, coincides with the generator of the group $\mathrm{Aut}_U X_1 = \mathbb{Z}/d\mathbb{Z}$.

7.3. *The imbedding of an unramified covering to a ramified one.* Let $\bar{X}_1$ be a projective closure of the surface X_1 in $\mathbb{P}^3$, defined by the equation $F(x_0, x_1, x_2) = x_3^d$. The projection $\mathbb{P}^3 \to \mathbb{P}^2$, $(x_0 : x_1 : x_2 : x_3) \mapsto (x_0 : x_1 : x_2)$ from the point $(0 : 0 : 0 : 1) \notin \bar{X}_1$ determines the covering $\bar{p} : \bar{X}_1 \to \mathbb{P}^2$ extending p and ramified over the curve $D \subset \mathbb{P}^2$. Resolve the singularities of $\bar{X}_1$. First take the imbedded resolution $\sigma : Y \to \mathbb{P}^2$ of the curve D to obtain a divisor with normal crossings $\bar{C} = \sigma^{-1}(D)$. Let $C \subset Y$ be the proper preimage of D and E_j be the exeptional curves. Then take the pullback image $Y' = \bar{X}_1 \times_{\mathbb{P}^2} Y$ of the surface $\bar{X}_1$ over Y and take its normalization $\bar{Y} \to Y'$. At last, consider a good resolution $Z \xrightarrow{\pi} \bar{Y}$ of the singularities of $\bar{Y}$. The surface $\bar{Y}$ has only rational (in fact, quotient) singularities, because $\bar{C}$ is a divisor with normal crossings, and after minimal resolution of $\bar{Y}$ chains of rational curves are "glued". We obtain a commutative diagram

$$
\begin{array}{ccc}
X_1 \subset Z & & \\
\pi \big\downarrow & \searrow \bar{\pi} & \\
\bar{Y} & \longrightarrow & \bar{X}_1 \supset X_1 \\
q \big\downarrow & & \big\downarrow \bar{p} \\
U \subset Y & \xrightarrow[\sigma]{} & \mathbb{P}^2 \supset U = \mathbb{P}^2 \setminus D .
\end{array}
$$

Let $\varphi = q \circ \pi$, $\bar{\varphi} = \bar{p} \circ \bar{\pi}$; $\bar{\varphi}^{-1}(D) = \varphi^{-1}(\bar{C}) = \Delta$ is a divisor with normal crossings. Then $Z \setminus \Delta \simeq X_1$, $Y \setminus \bar{C} \simeq U$.

7.4. *The description of ramified cyclic coverings in terms of invertible sheaves* ([E]). Let (temporarily) Y be a nonsingular algebraic variety (of arbitrary dimension m). In a local situation, if for example $Y = \mathbb{C}^2$, a cyclic covering $q' : Y' \to Y$ of degree n ramified over a divisor $D \subset Y$ defined by an equation $f(x,y) = 0$ is a projection of the subvariety $Y' \subset \mathbb{C}^2 \times \mathbb{C}$ defined by the equation $z^n = f(x,y)$ in the trivial fibration. We can write $Y'=$ Spec $\mathcal{O}_{Y'}$, where $\mathcal{O}_{Y'} = \mathcal{O}_{\mathbb{C}^2} \oplus \mathcal{O}_{\mathbb{C}^2} z \oplus \ldots \oplus \mathcal{O}_{\mathbb{C}^2} z^{n-1}$, and the structure of algebra on $\mathcal{O}_{Y'}$ is given by the rule $z^n = f(x,y)$.

In a global situation instead of trivial fibration we must take a locally trivial fibration or an invertible sheaf. The construction of a cyclic covering runs as follows. Let $\mathcal{L}$ be an invertible sheaf on Y such that the sheaf $\mathcal{L}^n$ has a section $s : \mathcal{O}_Y \to \mathcal{L}^n$ and its zeroes determine a divisor $D \subset Y$. Then $\mathcal{L}^{-n} \subset \mathcal{O}_Y$ is the sheaf of ideals of the divisor D. Set

$$Y' = \operatorname{Spec}_Y(\bigoplus_{j=0}^{n-1} \mathcal{L}^{-j}),$$

where the $\mathcal{O}_Y$-algebra structure is defined by inclusion $\mathcal{L}^{-n} \subset \mathcal{O}_Y$. Then $q' : Y' \to Y$ is a cyclic covering ramified over D.

Now let $D = \sum \nu_l E_l$ be a divisor with normal crossings. Then the normalization $\nu : \bar{Y} \to Y'$ can be described concretely in terms of the following sheaves $\mathcal{L}^{(j)}$. Put $q = \nu \circ q' : \bar{Y} \xrightarrow{\nu} Y' \xrightarrow{q'} Y$. Then the direct image of the structure sheaf

$$q_* \mathcal{O}_{\bar{Y}} = \bigoplus_{j=0}^{n-1} (\mathcal{L}^{(j)})^{-1},$$

where

$$\mathcal{L}^{(j)} = \mathcal{L}^j \otimes \mathcal{O}_Y(-\sum_l [\frac{j}{n} \nu_l] E_l),$$

and $[\cdot]$ denote the entire part of a number.

The group $\operatorname{Aut}_Y Y' = \operatorname{Aut}_Y \bar{Y} = \mathbb{Z}/n\mathbb{Z}$ acts semisimply on $q_* \mathcal{L}_{\bar{Y}}$. Let $\zeta = e^{2\pi i/n}$ be a root of unity, and

$$q_* \mathcal{O}_{\bar{Y}} = \bigoplus_{j=0}^{n-1} F_j$$

be the decomposition into the sum of eigensubsheaves, where F_j corresponds to the eigenvalue ζ^j of the generator h of $\mathbb{Z}/n\mathbb{Z}$. Actually the decomposition described above coincides with the decomposition according to eigenvalues of h,

$$F_j = \mathcal{L}^{(j)}, \quad j = 0, \ldots, n-1.$$

Return to our situation. The covering $\bar{p} : \bar{X}_1 \to \mathbb{P}^2$ is ramified over a divisor D and is determined by the sheaf $\mathcal{L} = \mathcal{O}_{\mathbb{P}^2}(1)$ and inclusion $F : \mathcal{L}^{-d} = \mathcal{O}_{\mathbb{P}^2}(-d) \to \mathcal{O}_{\mathbb{P}^2}$, $\bar{X}_1 = \operatorname{Spec}_{\mathbb{P}^2}(\oplus_{j=0}^{d-1} \mathcal{O}_{\mathbb{P}^2}(-j))$, and the covering $q' : Y' \to Y$ is determined by $\mathcal{L} = \sigma^*(\mathcal{O}_{\mathbb{P}^2}(1))$ and inclusion $\mathcal{L}^{-d} = \mathcal{O}_Y(-\bar{C}) \subset \mathcal{O}_Y$, where $\bar{C} = C + \sum \nu_j E_j$.

7.5. *The decomposition of cohomology of the Milnor fibre.* The fibre X_1 is a nonsingular algebraic variety and so there is a MHS on $H^{\cdot}(X_1, \mathbb{Q})$ (Deligne). We need a good compactification of X_1 to introduce the MHS. We can take a good resolution $Z \supset X_1$, because $\Delta = Z \setminus X_1$ is a divisor with normal crossings. We have a spectral sequence

$$E_1^{pq} = H^q(Z, \Omega_Z^p(\log \Delta)) \Rightarrow H^{p+q}(X_1, \mathbb{C}),$$

degenerating in the term E_1. In particular, we obtain that

$$H^1(X_1,\mathbb{C}) = H^1(Z,\mathcal{O}_Z) \oplus H^0(Z,\Omega^1_Z(\log\Delta)).$$

7.6. *The descent to Y.* The singularities of the surface $\bar{Y}$ are rational (being quotient singularities) and the morphism q is finite. This involves

Corollary 2 *We have*

$$\begin{aligned} \varphi_*\Omega^p_Z(\log\Delta) &= \Omega^p_Y(\log C)\otimes q_*\mathcal{O}_{\bar{Y}}, \\ R^i\varphi_*\Omega^p_Z(\log\Delta) &= 0 \qquad \text{for } i>0. \end{aligned}$$

Consequently,

$$\begin{aligned} H^q(Z,\Omega^p_Z(\log\Delta)) &= H^q(Y,\Omega^p_Y(\log\bar{C})\otimes q_*\mathcal{O}_{\bar{Y}}) \\ &= \oplus_{j=0}^{d-1} H^q(Y,\Omega^p_Y(\log\bar{C})\otimes(\mathcal{L}^{(j)})^{-1} \end{aligned}$$

and the last equality is the decomposition into the sum of eigensubspaces for the operator h with eigenvalues ζ^j.

In particular, we obtain

Corollary 3 *We have*

$$H^1(X_1,\mathbb{C})_{\zeta^j} = H^1(Y,(\mathcal{L}^{(j)})^{-1}) \oplus H^0(Y,\Omega^1_Y(\log\bar{C})\otimes(\mathcal{L}^{(j)})^{-1})$$

for $j=0,\ldots,d-1$.

We have $\mathcal{L}^{(j)} = \mathcal{O}_Y$ for $j=0$, $H^1(Y,\mathcal{O}_Y)=0$ since Y is a rational surface, and

$$h^0(Y,\Omega^1_Y(\log\bar{C})) = k-1.$$

Therefore, $\dim H^1(X_1,\mathbb{C})_1 = k-1$.

We obtain the following expression for the Alexander polynomial

Theorem 15 ([*E*]). *If a curve $D\subset\mathbb{P}^2$ is reduced, then*

$$\Delta_D(t) = \prod_{j=0}^{d-1}(t-\zeta^j)^{h_j},$$

where $\zeta = e^{2\pi i/d}$, and

$$h_j = \dim H^1(Y,(\mathcal{L}^{(j)})^{-1}) + \dim H^0(Y,\Omega^1_Y(\log\bar{C})\otimes(\mathcal{L}^{(j)})^{-1}).$$

We have $h_0 = k-1$ for $j=0$. Besides,

$$\dim H^0(Y,\Omega^1_Y(\log\bar{C})\otimes(\mathcal{L}^{(j)})^{-1}) = \dim H^1(Y,(\mathcal{L}^{(d-j)})^{-1})$$

for $j=1,\ldots,d-1$.

The last equality is proven in $[LV]$. Strictly saying, this theorem is contained in the Esnault's paper $[E]$ implicitly. She calculates $b_1(X_1)$, $b_2(X_1)$ and also the rank and the signature of the intersection quadratic form on $H^2_c(X_1, \mathbb{C})$. This theorem is contained in $[LV]$, where the further descent to $\mathbb{P}^2$ is realized with the help of Vanishing theorem and the theory of MHS in vanishing cohomology of an isolated hypersurface singularity.

7.7. *The descent to* $\mathbb{P}^2$ ([LV]). Let $\sigma : Y \to \mathbb{P}^2$ be an embedded resolution of singularities of a curve $D \subset \mathbb{P}^2$. In the paper $[LV]$ the sheaves $\sigma_*(\mathcal{L}^{(j)} \otimes \omega_Y)$, where ω_Y is the canonical sheaf, are calculated. They prove that

$$\sigma_*(\mathcal{L}^{(j)} \otimes \omega_Y) \simeq \mathcal{A}_\alpha(j-3),$$

where $\alpha = j/d - 1$, $1 \le j \le d-1$, and the subsheaf $\mathcal{A}_\alpha \subset \mathcal{O}_{\mathbb{P}^2}$, for $\alpha \in \mathbb{Q}$, $-1 < \alpha < 0$, coincides with $\mathcal{O}_{\mathbb{P}^2}$ outside Sing$D$ and for a singular point $x \in$ Sing$D$ is defined by the condition

$$(\mathcal{A}_\alpha)_x = \{g \in \mathcal{O}_{\mathbb{P}^2,x} \mid \alpha_f(g\omega_0) > \alpha\},$$

where $f = 0$ is a local equation of the curve D at x, ω_0 is a 2-form regular and not vanishing at x, and $\alpha_f(\omega)$ is the order of a form ω. [If $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ is an isolated singularity and ω is a $(n+1)$-form, then the order $\alpha_f(\omega)$ is the minimal exponent of t in the asymptotic development of integrals $\int \frac{\omega}{df}$ over manyvalued horisontal sections of homological Milnor fibration.] A.N.Varchenko expressed $\alpha_f(\omega)$ (in the case $\alpha_f(\omega) \le 0$) in terms of embedded resolution $\pi : X \to \mathbb{C}^{n+1}$ of the singularity f :

$$\alpha_f(\omega) = \inf\left(\frac{1 + v_j(\omega)}{m_j} - 1\right),$$

where $\pi^{-1}(f^{-1}(0)) = \sum m_j E_j$ and $v_j(\omega)$ is the order of the form $\pi^*(\omega)$ along the component E_j. Moreover, Loeser and Vacuié prove in $[LV]$ with the help of the vanishing theorem (E.Vieweg) that

$$R^i\sigma_*(\mathcal{L}^{(j)} \otimes \omega_Y) = 0 \quad \text{for } i > 0 \text{ and } \quad j = 1, \dots, d-1.$$

This involves that

$$\dim H^1(Y, (\mathcal{L}^{(j)})^{-1}) = \dim H^1(\mathbb{P}^2, \mathcal{A}_\alpha(j-3))$$

for $j = 1, \dots, d-1$ and $\alpha = j/d - 1$.

It is proven in $[LV]$ that

$$H^1(\mathbb{P}^2, \mathcal{A}_\alpha(j-3)) = 0,$$

if $\alpha = j/d - 1$ does not belong to one of the spectra of $x \in \operatorname{Sing} D$. Summarizing we obtain the Loeser-Vaquie's result

Theorem 16 ($[LV]$). *If $D = D_1 + \ldots + D_k \subset \mathbb{P}^2$ is a reduced curve, then*

$$\Delta_D(t) = (t-1)^{k-1} \prod_{\alpha \in A_D} (\Delta_\alpha(t))^{l_\alpha},$$

where A_D is the set of $\alpha \in \mathbb{Q}$ for which $-1 < \alpha < 0$, $d \cdot \alpha \in \mathbb{Z}$ and α belongs to the spectrum of one of the singularities $x \in SingD$, and

$$\begin{aligned} \Delta_\alpha(t) &= (t - \exp(2\pi i\alpha))(t - \exp(-2\pi i\alpha)), \\ l_\alpha &= \dim H^1(\mathbb{P}^2, \mathcal{A}_\alpha(d(\alpha+1) - 3)). \end{aligned}$$

7.8. *Calculation of the Alexander polinomial in terms of cohomology of rational differential forms* ([Ko]). T.Kohno calculates $\Delta_D(t)$ for a reduced irreducible curve $D \subset \mathbb{C}^2$ transversally intersecting the line L_∞ at infinity. To calculate $\Delta_D(t)$ in terms of differential forms one needs some preliminary results about the connection between cohomology of infinite and d-fold coverings of $\mathbb{C}^2 \backslash D$. We'll obtain these results in general setting in the next section, and then we'll calculate $\Delta_D(t)$ in section 8.6.

8 The homology of cyclic coverings of the complement to a plane curve

We pass to the exposition of the main results of the paper $[KK]$. The problem is to consider as general curve as possible, not reduced and without conditions at infinity as it was assumed in the previous section. We return to the notation of section 6 : $D = m_1 D_1 + \ldots + m_k D_k \subset \mathbb{C}^2$ is a curve of degree d defined by the equation $f(x,y) = 0$ and so on.

8.1. Let $\varphi_n : X_n \to X$ be n-fold cyclic covering of $X = \mathbb{C}^2$, where X_n is a normalization of the surface defined by the equation $z^n = f(x,y)$ in $\mathbb{C}^3$. Denote $X' = X \setminus D,\ X'_n = X_n \setminus B,\ B = \varphi_n^{-1}(D)$. Then the infinite cyclic covering $\varphi = \varphi_\infty : X_\infty \to X'$ factors through the unramified covering $\varphi_n : X'_n \to X'$ and we have a commutative diagram

$$\begin{array}{ccccc} \phi : X_\infty & \xrightarrow{\phi_{\infty,n}} & X'_n & \xrightarrow{\phi_n} & X' \\ \downarrow f_\infty & & \downarrow f'_n & & \downarrow f' \\ e : U & \longrightarrow & S'_n & \xrightarrow{e_n} & S' , \end{array} \qquad (2)$$

where $e(u) = t = e^{2\pi i u}$ for $u \in U = \mathbb{C}$, $e_n(z) = t = z^n$ and $z = e^{2\pi i u/n}$ for $z \in S'_n = \mathbb{C} \setminus \{0\}$.

We'll be interested in the connection between the homology of the affine variety X'_n and its projective completion $\bar{X}_n$.

Remark 3 The investigation of cyclic coverings $\bar{X}_n$ of the plane $\mathbb{P}^2$ (theory of algebraic surfaces) was the main reason for O.Zariski ([Z1], [Z2]) to study $\pi_1(\mathbb{C}^2 \setminus D)$. The computation of the irregularity $q(\bar{X}_n)$ and other invariants of $\bar{X}_n$ is one of the directions of the subject which we don't touch (see the Sakai's survey [Sa]).

Let $f : \mathbb{P}^2 \to \bar{S} = \mathbb{P}^1$ be a rational map corresponding to the morphism $f : X \to S$, not defined only at some points of the infinite line $L_\infty = \mathbb{P}^2 \setminus \mathbb{C}^2 = f^{-1}(\infty)$. Resolving the points of indeterminacy by means of σ-processes $\sigma : \bar{X} \to \mathbb{P}^2$ we get a morphism $\bar{f} = f \cdot \sigma : \bar{X} \to \bar{S}$. We can imagine $X = \mathbb{C}^2$ to be obtained from $\bar{X}$ by means of throwing out a curve $\sigma^{-1}(L_\infty)$ which consists of some quasisections and some components of fibres of the morphism $\bar{f}$. Analogously we can construct a completion $\bar{X}_n$. We begin with a hypersurface in $\mathbb{P}^3$ defined by the equation $x_3^n = x_0^m \bar{f}(x_0, x_1, x_2)$, where $\bar{f}(x_0, x_1, x_2)$ is a homogeneous polynomial of degree $\deg f$ associated with f, $m = n - \deg f$. Let $\widetilde{X}_n$ be a normalization of this surface and $\tilde{\varphi}_n : \widetilde{X}_n \to \mathbb{P}^2$ be induced by the morphism $\varphi_n : X_n \to X$. Resolving the singularities and the points of indeterminacy of morphisms we get a smooth surface $\bar{X}_n \supset X_n^0$, where $X_n^0 = X_n \setminus \mathrm{Sing} X_n$, and a commutative diagram

$$\begin{array}{ccccccccc} X'_n & \subset & X_n^0 & \subset & \bar{X}_n & \xrightarrow{\phi_n} & \bar{X} & \supset & X \\ \downarrow{\scriptstyle f'_n} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \bar{f}_n} & & \downarrow{\scriptstyle \bar{f}} & & \downarrow{\scriptstyle f} \\ S'_n & \subset & S_n & \subset & \bar{S}_n & \xrightarrow{\bar{e}_n} & \bar{S} & \supset & S \end{array} \qquad (3)$$

8.2. Let us formulate the conditions which we impose on the curve D. We say that the *condition* (Irr_n) holds for a curve D if all the curves $B_i = \varphi_n^{-1}(D_i)$, $i = 1, \dots, k$, are irreducible.

Definition 12 *A curve D is connected modulo n, if the support of the divisor*

$$D_{mod\, n} = \sum_{m_i \not\equiv 0 \, mod\, n} m_i D_i$$

is connected. D is absolutely connected modulo n , or shorter, satisfies the condition (C_n), if

$$D \text{ is connected modulo } n_1 \text{ for each } n_1,\ n_1 \mid n. \qquad (C_n)$$

We need the condition (C_n) to get the following

Theorem 17 ([KK]). *If D satisfies the condition (C_n) , then*

$$H^0(X_n, \mathcal{O}^*_{X_n}) = \mathbb{C}.$$

This theorem affims that the regular and regular invertible functions on the affine variety X_n are only constants, i.e. the matter is the same as on $\mathbb{C}^2$. From this theorem follows that from the point of view of one-dimensional homology only the components $B_i = \varphi_n^{-1}(D_i)$, which lie on the normalization X_n^0, are essential for a compactification $X'_n \subset \bar{X}_n$.

Corollary 4 . *If a curve D satisfies the condition (C_n), then the inclusion $i : X_n^0 \subset \bar{X}_n$ induces an isomorphism*

$$i_* : H_1(X_n^0) \xrightarrow{\sim} H_1(\bar{X}_n).$$

8.3. *The relation between homology of X'_n and $\bar{X}_n$* is given by

Theorem 18 ([KK]).

(i) If D satisfies the conditions (C_n) and (Irr_n) , then there is an exact sequence

$$0 \to \bigoplus_{i=1}^{k} \mathbb{C}\bar{\gamma}_i \to H_1(X'_n; \mathbb{C}) \xrightarrow{i_*} H_1(\bar{X}_n, \mathbb{C}) \to 0, \qquad (4_n)$$

where $i = i_n : X'_n \subset \bar{X}_n$ is an imbedding, and the cycle $\bar{\gamma}_i \in Ker\, i_*$ corresponds to "going arroud the component B_i" .

(ii) Moreover, if D satisfies the conditions ($C_{n(D)}$) and ($Irr_{n(D)}$) , then the sequence (4_n) is exact for all n.

The condition of Theorem 17 (i) holds if the support of the curve D is connected in $\mathbb{C}^2$ and $(m_i, n) = 1$ for each multiplicity m_i (in particular, if D is a reduced curve). If D is an irreducible curve, we get a generalization of the Libgober's result: $\dim Ker\, i_* = 1$ (the conditions at infinity are superfluous).

8.4. *The relation of $H_1(X_\infty)$ and $H_1(X'_n)$.* The proof of Theorem 17 (ii) is based on the application of the Milnor exact sequence (see section 1) to analyse the relation of $H_1(X_\infty)$ and $H_1(X'_n)$ for different n.

Consider the Milnor exact sequence for the infinite cyclic covering $\varphi_{\infty,n} : X_\infty \to X'_n$. If $G_n \subset \mathbb{F}_1$ is the infinite cyclic group generated by h^n, then $X'_n = X_\infty/G_n$ and $X' = X'_n/\mu_n$, where $\mu_n = \mathbb{F}_1/G_n$ is the cyclic group of order n. Denote by h_n the automorphism of X'_n induced by the monodromy h. Then h_n is the generator of the group μ_n which corresponds also to the generator of the Galois group $\mathrm{Gal}(k(X'_n)/k(X'))$.

Put together the Milnor exact sequence and the exact sequence (4_n)

$$
\begin{array}{ccccccc}
 & & \vdots & & & & \\
 & & \downarrow & & & & \\
 & & H_1(X_\infty) & & & & \\
 & & \downarrow{\scriptstyle h^n - id} & & & & \\
 & & H_1(X_\infty) & & & & \\
 & & \downarrow{\scriptstyle (\phi_{\infty,n})_*} & & & & \\
0 \longrightarrow & \oplus_{i=1}^k \mathbb{C}\bar{\gamma}_i \longrightarrow & H_1(X'_n) & \xrightarrow{(i_n)_*} & H_1(\bar{X}_n) \longrightarrow 0 & & (5_n) \\
 & & \downarrow & & & & \\
 & & H_0(X_\infty) & & & & \\
 & & \downarrow & & & & \\
 & & 0 & & & &
\end{array}
$$

The group $\mathbb{F}_1 = \mathbb{Z}$ with the generator $h = h_*$ acts on the spaces $H_1(X_\infty)$, $H_1(X'_n)$ and $H_1(\bar{X}_n)$ (on $H_1(X'_n)$ and $H_1(\bar{X}_n)$ the action is reduced to the group $\mu_n = \mathbb{F}_1/G_n$ with the generator h_n). Clearly, the homomorphisms $(\varphi_{\infty,n})_*$ and $(i_n)_*$ are equivariant.

We denote

$$H_1(X_\infty) = \oplus_i H_1(X_\infty)_{\lambda_i} = H_1(X_\infty)_{\lambda=1} \oplus H_1(X_\infty)_{\neq 1}$$

the root decomposition of the automorphism h, where $H_1(X_\infty)_{\neq 1} = \oplus_{\lambda\neq 1} H_1(X_\infty)_\lambda$. We shall apply the analogous notation for the root decomposition of the spaces $H_1(X'_n)$ and $H_1(\bar{X}_n)$ corresponding to the automorphism $h_n = (h_n)_*$.

The colomn in diagram (5_n) gives that $H_1(X'_n)$ is almost $Im(\varphi_{\infty,n})_* \subset H_1(X'_n)$, $Coker(\varphi_{\infty,n})_* \simeq H_0(X_\infty)$, $H_0(X_\infty) = \mathbb{C}$, and h acts trivially on $H_0(X_\infty)$. On the other hand, $Im(\varphi_{\infty,n})_* = Coker(h^n - id)$. We need an easy exercise in the linear algebra.

Lemma 1 . *Let h be an automorphism of a vector space $L/\mathbb{C}$. Then* *(i) If the Jordan decomposition of h has only one Jordan block with a eigenvalue $\lambda \neq 0$, then for $\forall n \in \mathbb{N}$ the automorphism h^n also has one Jordan block with the eigenvalue λ^n. This involves*

(ii) If h has no eigenvalues $\lambda = 0$, then the number and the dimensions of Jordan blocks for h^n are the same as for h.

(iii) If h has only one Jordan block with $\lambda \neq 1$, then $h - id : L \to L$ is an isomorphism, and if $\lambda = 1$, then $\dim Ker(h - id) = \dim L/(h\text{-}id)L = 1$.

Therefore, applying this lemma to h^n in the diagram (5_n), we get that, moving from $H_1(X'_n)$ to $H_1(X_\infty)$, the Jordan blocks of $H_1(X_\infty)$ with $\lambda^n \neq 1$ disappear, and every Jordan block with $\lambda^n = 1$ gives one eigenvector in the space $H_1(X'_n)$ with the same eigenvalue λ. Denote by $J = J(h)$ the number of Jordan blocks of the automorphism h, and by $J_1 = J_{\lambda=1}$, $J_n = J_{\lambda^n=1}$, $J_{\neq 1}$ the number of Jordan blocks with eigenvalues $\lambda = 1$, with eigenvalues λ such that $\lambda^n = 1$, with eigenvalues not equal to 1 correspondingly. Thus lemma involves that

$$\dim H_1(X'_n) = J_1(h^n) + 1 = J_n(h) + 1.$$

8.5. *The decomposition of $H_1(X'_n)$ into eigensubspaces.* Now consider the line in the diagram (5_n). On one hand, the fact that X_n/μ_n is a rational surface (and hence there are no invariant holomorphic forms on $\bar{X}_n$) involves that the operator h_n on $H_1(\bar{X}_n, \mathbb{C})$ has no eigenvalues $\lambda = 1$, $H_1(\bar{X}_n, \mathbb{C}) = H_1(\bar{X}_n, \mathbb{C})_{\neq 1}$. On the other hand, under the condition (Irr_n), i.e. if the curves B_i are irreducible, the cycles $\bar{\gamma}_i$, $i = 1, \dots, k$, are invariant relative to h_n. This involves the proof of the Theorem 17, and we obtain that

$$H_1(X'_n)_1 = \oplus_{i=1}^k \mathbb{C} \cdot \bar{\gamma}_i, \qquad H_1(X'_n)_{\neq 1} \simeq H_1(\bar{X}_n).$$

This involves

$$\dim H_1(\bar{X}_n) = J_n - J_1 = J_{\neq 1}(D, n),$$

where $J_{\neq 1}(D, n)$ is the number of Jordan blocks of the monodromy h on $H_1(X_\infty, \mathbb{C})$ with eigenvalues $\lambda \neq 1$ for which $\lambda^n = 1$.

8.6. *Calculation of the Alexander polinomial of an irreducible curve in terms of cohomology of rational differential forms* ([Ko]). Let D be an irreducible curve ($k = 1$) of degree $deg\, D = d$. If D intersects the line at infinity transversally, then according to the Randell's theorem in section 6 the monodromy h is semisimple, and $\lambda^d = 1$. In the next section we obtain the theorem on the semisimplicity of h without the assumption of the transversality of intersection at infinity. Thus (8.4) and (8.5) involve that

$$H^1(X_\infty) \simeq H^1(X'_d)_{\neq 1}.$$

The surface $X_d \subset \mathbb{C}^3$ defined by the equation $z^n = f(x, y)$ is normal, because the curve D is reduced. Recall that $X' = \mathbb{C}^2 \setminus D$, $X'_d = X_d \setminus B$, where $B \subset X_d$ is the curve with the equation $z = 0$ and $\varphi_d : X'_d \to X'$ is an unramified covering of degree d.

We apply the Grothendieck's theorem to calculate the cohomology $H^1(X'_d, \mathbb{C})$. If X is a complex variety, $D \subset X$ is a hypersurface, $j : X' = X \setminus D \to X$ is the imbedding, then

$$H^{\cdot}(X', \mathbb{C}) = \mathbb{H}^{\cdot}(\Omega^{\cdot}_{X'}) = \mathbb{H}^{\cdot}(j_* \Omega^{\cdot}_{X'}) = \mathbb{H}^{\cdot}(\Omega^{\cdot}_X(*D)),$$

where $\Omega^{\cdot}_X(*D)$ is de Rham's complex of meromorphic forms with poles along D. If X is an affine variety, then

$$H^{\cdot}(X', \mathbb{C}) = H^{\cdot}(\Gamma(\Omega^{\cdot}_X(*D)).$$

Apply this to $X = \mathbb{C}^2$ and $X = X_d$. We obtain

$$H^{\cdot}(X'_d, \mathbb{C}) = H^{\cdot}(\Gamma(\Omega^{\cdot}_{X_d}(*B)).$$

Let $h = h_d$ be a generator of the group $\mathrm{Aut}_{X'} X'_d$, $h^d = 1$. Let $\zeta = e^{2\pi i/d}$ be a root of unity of degree d and

$$H^{\cdot}(X'_d, \mathbb{C}) = \oplus_{j=0}^{d-1} H^1(X'_d, \mathbb{C})_{\zeta^j}$$

be the decomposition into eigensubspaces. The monodromy h acts semisimply also on the differential forms on X'_d. The part corresponding to the eigenvalue ζ^j is equal to

$$\Omega^{\cdot}_{X_d}(*B)_j = \Omega^{\cdot}_X(*D) z^j.$$

Since $\frac{df}{f} = \frac{d(z^d)}{z^d} = d\frac{dz}{z}$, we have

$$d(z^j \omega) = j z^{j-1} dz \wedge \omega + z^j d\omega = (\frac{j}{d}\frac{df}{f} \wedge \omega + d\omega) z^j.$$

So the multiplication by $z^j = f^{j/d}$ defines an isomorphism of complexes

$$\begin{array}{ccccccc} \ldots \to & \Omega^p_{X_d}(*B)_j & \xrightarrow{d} & \Omega^{p+1}_{X_d}(*B)_j & \to \ldots \\ & \uparrow z^j & & \uparrow z^j & \\ \ldots \to & \Omega^p_X(*D) & \xrightarrow{\nabla_j} & \Omega^{p+1}_X(*D) & \to \ldots \end{array}$$

where ∇_j is a regular connection in $\Omega^{\cdot}_X(*D)$ defined by the formula

$$\nabla_j(\omega) = d\omega + \frac{j}{d}\frac{df}{f} \wedge \omega.$$

Therefore,

$$H^1(X'_d, \mathbb{C})_{\zeta^j} = H^1(\Gamma(\Omega^{\cdot}_{X_d}(*B)_j, d) = H^1(\Gamma(\Omega^{\cdot}_X(*D), \nabla_j)$$

and we obtain Kohno's theorem.

Theorem 19 ([*Ko*]). *If $D \subset \mathbb{C}^2$ is an irreducible curve of degree d transversally intersecting the line at infinity, then the Alexander polynomial is*

$$\Delta_D(t) = \prod_{1 \le j \le d-1} (t - \zeta^j)^{h_j},$$

where $h_j = \dim_{\mathbb{C}} H^1(\Gamma(\Omega^{\cdot}_{\mathbb{C}^2}(*D), \nabla_j)$.

In virtue of the theorem on the simplicity of monodromy in the next section we obtain the generalization of Kohno's theorem:

Theorem 18'

i). The Kohno's theorem is true without the condition of transversality at infinity;

ii). if D is a connected reduced curve, then

$$\Delta_D(t) = (t-1)^l \prod_{1\le j\le d-1} (t-\zeta^j)^{h_j}.$$

9 The semisimplicity of the monodromy

9.1. If a curve $D \subset \mathbb{C}^2$ is reduced and $\bar{D}$ transversally intersects the line at infinity, then the monodromy h on $H_1(X_\infty)$ is semisimple, because it coincides with the monodromy of a quasihomogeneous singularity (see section 6). We generalize it to the case of nonreduced curves without any conditions at infinity, and use quite different ideas based on the Milnor exact sequence and the theory of mixed Hodge structures.

Theorem 20 ([KK]). *If a curve D satisfies the conditions $(C_{n(D)})$ and $(Irr_{n(D)})$, then the monodromy h on $H_1(X_\infty)_{\neq 1}$ is semisimple.*

We sketch the proof of this Theorem. We have to prove that the Jordan blocks of the automorphism h on $H_1(X_\infty)$ with eigenvalues $\lambda \neq 1$ are one-dimensional. We compare the monodromies on homology of X_∞ and on homology of a nonsingular fibre Y. Let $Y = X_t = f^{-1}(t)$, correspondingly $\bar{Y} = \bar{X}_t = \bar{f}^{-1}(t)$, be a nonsingular fibre (close to X_0) of the morphism f, correspondingly $\bar{f}$, in the diagram (3). The morphisms φ_n and φ_∞ are unramified coverings and the fibers $\varphi_n^{-1}(Y)$ and $\varphi_\infty^{-1}(Y)$ break up into components isomorphic to Y. Choosing points $u \in e_\infty^{-1}(t)$ and $\tilde{t} \in e_n^{-1}(t)$ we can assume that Y is embedded into X_∞ and X'_n as fibres $f_\infty^{-1}(u)$ and $f_n^{-1}(\tilde{t})$ such that the diagram

$$\begin{array}{ccccc} Y & = & Y & \hookrightarrow & \bar{Y} \\ \Big\downarrow j_\infty & & \Big\downarrow j & & \Big\downarrow \bar{j} \\ X_\infty & \xrightarrow[\phi_{\infty,n}]{} & X'_n & \underset{i}{\hookrightarrow} & \bar{X}_n. \end{array}$$

is commutative. We get a commutative diagram for homology

$$\begin{array}{ccccc} H_1(Y) & = & H_1(Y) & \longrightarrow & H_1(\bar{Y}) \\ \Big\downarrow (j_\infty)_* & & \Big\downarrow j_* & & \Big\downarrow \bar{j}_* \\ H_1(X_\infty) & \longrightarrow & H_1(X'_n) & \underset{i_*}{\longrightarrow} & H_1(\bar{X}_n). \end{array}$$

The monodromy operator acts on the spaces $H_1(X_\infty)$, $H_1(X'_n)$, $H_1(Y)$ and $H_1(\bar{Y})$ and the homomorphisms in the above diagram are equivariant, i.e. commute with the action of the monodromy operators. The part of the diagram corresponding to the eigenvalues $\lambda \neq 1$ is the following

$$
\begin{array}{ccccc}
 & & H_1(Y)_{\neq 1} & \longrightarrow & H_1(\bar{Y}) \\
 & \swarrow^{(j_\infty)_*} & \downarrow & & \downarrow \bar{j}_* \\
H_1(X_\infty)_{\neq 1} & \xrightarrow[(\phi_{\infty,n})_*]{} & H_1(X'_n)_{\neq 1} & \xrightarrow[i_*]{\sim} & H_1(\bar{X}_n).
\end{array}
$$

The key place of the proof is the following. On one hand, in virtue of (8.4) every Jordan block of the automorphism h on $H_1(X_\infty)_{\neq 1}$ (i.e. the invariant subspace L on which h consists of one Jordan block) gives in $H_1(\bar{X}_n)$ one non zero vector, $\dim i_* \cdot (\varphi_{\infty,n})_* L = 1$. On the other hand, a two-dimensional block L can be obtained from a two-dimensional block L in $H_1(Y)$. At last the theory of mixed Hodge structures yieds that we can choose L in such a way that $j_*(L) = 0$ in $H_1(X_n)$. This shows that the blocks in $H_1(X_\infty)_{\neq 1}$ can be only one-dimensional.

9.2. If D is an irreducible curve, then $\Delta_D(1) = \pm 1$ [K1]. Hence from Theorem 20 we obtain the following theorem.

Theorem 21 *The monodromy h on $H_1(X_\infty)$ is semisimple for an irreducible curve D.*

Remark 4 The analogous statement is not true for knots. For example, for the knot 8_{10} the monodromy h on $H_1(X_\infty)$ isn't semisimple.

Proposition 5 *Let $D = m_1 D_1 + D'$ be a curve satisfying the following conditions:*

i) D_1 is irreducible and $D_1 \not\subset supp(D')$,

ii) there exists a point $x \in D_1 \cap supp(D')$ at which the divisor $D_1 + D'_{red}$ is locally a divisor with normal crossings,

iii) for the curve D' the monodromy h on $H_1(X_\infty)_1$ is semisimple.

Then for the curve D the monodromy h on $H_1(X_\infty)_1$ is semisimple.

We sketch the proof of this proposition. From the homological Milnor exact sequence it follows that one needs to show that the Alexander polynomial $\Delta_D(t)$ of the curve D satisfies the condition: $\Delta_D(t) = (t-1)^{k-1} \cdot \Delta'(t)$, where Δ' is a polynomial such that $\Delta'(1) \neq 0$, and k is the number of irreducible components of D. The straightforward calculations of the Alexander polynomial (as in [K1] and [K2]), using Fox's free calculus, show that the polynomial $\Delta_D(t)$ possesses the required property under the conditions of the proposition.

As a consequence of this proposition and Theorem 20 we obtain the following theorem.

Theorem 22 *Let $D = m_1 D_1 + \dots + m_k D_k$ be a curve satisfying the conditions of Theorem 20 and such that for $i = 1, \dots, k-1$, there exists a point $x_i \in D_{i+1} \cap (\cup_{j=1}^{i} D_j)$ such that the curve $D^{(i+1)} = D_1 + \dots + D_{i+1}$ is locally a divisor with normal crossings at x_i. Then for the curve D the monodromy h on $H_1(X_\infty)$ is semisimple.*

Conjecture. *If a curve D satisfies the conditions $(C_{n(D)})$ and $(Irr_{n(D)})$, then the monodromy h on $H_1(X_\infty)$ is semisimple.*

10 On the mixed Hodge structure on $H^1(X_\infty)$

The construction of X_∞ for $X' = \mathbb{C}^2 \setminus D$ is analogous to the construction of a canonical fibre for a family of nonsingular projective varieties over the punctured disk S' or for the Milnor fibration of a hypersurface singularity, but in our case the fibration $f' : X' \to S'$ is not locally trivial. In these cases there is a limit MHS (W.Schmid, J.Steenbrink). We want to introduce a MHS on $H^1(X_\infty, \mathbb{Q})$ such that the homomorphisms

$$(\varphi_{\infty,n})^* : H^1(X'_n) \to H^1(X_\infty)$$

are MHS morphisms.

The surface X'_n is a nonsingular algebraic variety and so there is the MHS on $H^1(X'_n)$ introduced by Deligne [G-S]. Remind that we must take a nonsingular projective variety $\bar{X}_n \supset X'_n$ such that $\bar{X}_n \setminus X'_n = \bar{D}$ is a divisor with normal crossings. The weight filtration $W.$ defines spectral sequence which involves an exact MHS sequence

$$0 \to H^1(\bar{X}_n) \to H^1(X'_n) \to H^0(\bar{D}^{(1)}) \to H^2(\bar{X}_n),$$

where there is a pure Hodge structure of weight 1 on $H^1(\bar{X}_n)$, $\bar{D}^{(1)}$ is a disjoint union of the components of $\bar{D}$ and there is a pure Hodge structure of weight 2 and type (1,1) on $H^0(\bar{D}^{(1)})$. In our case by the Theorem 17 this exact sequence is the exact sequence

$$0 \to H^1(\bar{X}_n) \to H^1(X'_n) \to \oplus_{i=1}^k \mathbb{C} \cdot \bar{\gamma}_i^* \to 0 \qquad (\tilde{4}_n)$$

dual to the sequence (4_n). So if $W.$ is the weight filtration on $H^1(X'_n)$, $H^1(X'_n) = W_2 \supset W_1 \supset 0$, then $W_1 = H^1(\bar{X}_n)$ and $Gr_2^W = W_2/W_1 = \oplus \mathbb{C} \cdot \bar{\gamma}_i^*$ and the pure Hodge structure on Gr_2^W is of type (1,1).

In our case the cyclic group $\mu_n = \mathbb{Z}/n\mathbb{Z}$, generated by h_n^*, acts on $H^1(X'_n)$. The monodromy h_n^* is a MHS isomorphism since h_n is an isomorphism of the algebraic variety X'_n. From section (8.5) we have

$$H^1(\bar{X}_n) = H^1(X'_n)_{\neq 1}, \quad H^1(X'_n)_1 \simeq \oplus_{i=1}^k \mathbb{C} \cdot \bar{\gamma}_i^*.$$

Hence the MHS on $H^1(X'_n)$ splits and is the direct sum of pure Hodge structures of weight 1 on $H^1(\bar{X}_n)$ and of weight 2 and type (1,1) on $\oplus_{i=1}^k \mathbb{C} \cdot \bar{\gamma}_i^*$.

Consider the diagram dual to the diagram (5_n)

$$
\begin{array}{ccccccccc}
 & & & & \vdots & & & & \\
 & & & & \uparrow & & & & \\
 & & & & H^1(X_\infty) & & & & \\
 & & & & \uparrow{\scriptstyle (h^*)^n - id} & & & & \\
 & & & & H^1(X_\infty) & & & & \\
 & & & & \uparrow{\scriptstyle (\phi_{\infty,n})^*} & & & & \\
0 & \longrightarrow & H^1(\bar{X}_n) & \longrightarrow & H^1(X'_n) & \longrightarrow & \oplus_{i=1}^{k}\mathbb{C}\cdot\bar{\gamma}_i^* & \longrightarrow & 0 \\
 & & & & \uparrow & & & & \\
 & & & & H^0(X_\infty) & & & & \\
 & & & & \uparrow & & & & \\
 & & & & 0 & & & &
\end{array}
\qquad (\tilde{5}_n)
$$

By the Theorem 20 if $n(D) \mid n$, then

$$H^1(X'_n)_{\neq 1} \overset{(\varphi_{\infty,n})^*}{\longrightarrow} H^1(X_\infty)_{\neq 1}.$$

is an isomorphism. So if we introduce the MHS on $H^1(X_\infty)$ as a direct sum of the pure Hodge structure of weight 1 on $H^1(X_\infty)_{\neq 1}$, obtained by the isomorphism $(\varphi_{\infty,n(D)})^*$, and the pure Hodge structure of weight 2 and type (1,1) on $H^1(X_\infty)_1$, then we obtain the desired MHS.

Theorem 23 ([KK]). *If a curve D satisfies the conditions $(C_{n(D)})$ and $(Irr_{n(D)})$, then there exists a natural mixed Hodge structure on $H^1(X_\infty, \mathbb{Q})$ such that the homomorphisms $\varphi^*_{\infty,n}$: $H^1(X'_n, \mathbb{Q}) \to H^1(X_\infty, \mathbb{Q})$ are MHS morphisms. If a curve D is irreducible, then the MHS on $H^1(X_\infty, \mathbb{Q})$ is pure.*

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The monodromies of knots, hypersurface singularities and polynomials

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The monodromies of knots, hypersurface singularities and polynomials

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Intoduction

There are similar situations in three fields in which monodromy appears: in classical knot theory, in the theory of singularities and in algebraic geometry. For example, the Alexander polynomial of a knot corresponds to the characteristic polynomial in cohomology of the Milnor fibre of a singularity. We want to give a survey of some results concerning this subject concluding by the results of our recent paper $[KK]$ (and some other results which are not contained in $[KK]$).

Now we give a more detailed description of this paper. In section 1 we define following Milnor the Alexander invariants and in particular the Alexander polynomials in a general situation of a CW-complex X and its infinite cyclic covering X_∞. Besides, we recall the Milnor exact sequence which connects homology of X and X_∞.

In section 2 we recall some facts of classical knot theory. In particular, we recall Stallings'es theorem, characterizing the fibred knots as knots whose groups possess a finitely generated commutator subgroups, to be compared with corresponding result on the fundamental group of the complement of a plane algebraic curve [K1]. In section 3 we consider algebraic knots throwing a bridge between knot theory and singularity theory.

In section 4 we consider the Milnor fibration $f' : X' \to S'$ of a germ of a hypersurface singularity. If $X_\infty = X' \times_S U$ is the canonical Milnor fibre, where $U \to S'$ is the unramified covering, then the monodromy transformation h of the Milnor fibre can be considered as a generator of the group of covering transformations of X_∞/X'. We pay especial attention to the Monodrony theorem and the limit mixed Hodge structure on cohomology of X_∞ to be compared with results [KK] in global situation (see sections 9 and 10).

In section5 we consider the monodromy of a quasihomogeneous singularity to connect results on local and global situations. Besides, we recall a construction representing the Milnor fibre X_1 as a cyclic covering of the complement $U = \mathbb{P} \setminus V$ of a hypersurface in a weighted projective space.

In section 6 we investigate the global case, that is, the complement $X' = \mathbb{C}^2 \setminus D$ and $\mathbb{P}^2 \setminus \bar{D}$ of a plane algebraic curve. The definitions of the monodromy h and the (first) Alexander

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polynomial $\Delta(t) = \det(tid - h_*)$ are general. We recall the divisibility theorems of Libgober [L1] and Vik. Kulikov [K1].

In section 7 we review methords of calculations of $\Delta(t)$ in case of irreducible curve D intersecting the line at infinity transversally ([E], [LV]). They are based on Randell's theorem [R] reducing the calculation of $\Delta(t)$ to the calculation of the characteristic polynomial of the monodromy of the Milnor fibre X_1 of a homogeneous function $F(x_0, x_1, x_2)$. As we mentioned above X_1 is an unramified cyclic covering of $U = \mathbb{P}^2 \setminus \bar{D}$. Esnault imbeds it to a ramified coverinh of a "blown-up" plane $Y = \bar{\mathbb{P}}^2$, and expresses $\Delta(t)$ in terms of cohomology of invertible sheaves on Y. Loeser and Vaquie express $\Delta(t)$ in terms of cohomology of some sheaves on $\mathbb{P}^2$.

In sections 8-10 we review the results of our paper [KK]. In section 8 under weak conditions of connectivity and irreducibility, we obtain a relation between homology of X'_n, the unramified n-sheeted covering of $X' = \mathbb{C}^2 \setminus D$, and homology of its nonsingular projective model $\bar{X}_n$, and also with homology of X_∞, the infinite cyclic covering of X'. This generalizes Libgober's result [L1-L3]. Besides, we generalize the result of Kohno [Ko] on calculation of $\Delta(t)$ in terms of cohomology of rational differentioal forms. In section 9 we sketch the proof of our theorem [KK] on the semisimplicity of the monodromy h on $H_1(X_\infty)_{\neq 1}$. Besides, we five a sketch of the proof of a new result of the semisimplicity of h on $H_1(X_\infty)$ under some condition of transversality. At last in section 10, as a consequence of the semisimplicity theorem, we show how to introduce a natural mixed Hodge structure on $H_1(X_\infty)$.

1 Alexander invariants of infinite cyclic coverings

For the first time the notion of the Alexander invariants appeared in classical knot theory. Then it was transfered to other geometric contents. We begin with a general geometric situation (Milnor [$M2$]).

1.1. Let X be a finite connected complex or CW-complex and $\varphi : \widetilde{X} \to X$ be the infinite cyclic covering of X , determined by some epimorphism $p : \pi_1(X) \to \mathbb{F}_1$ onto the free group $\mathbb{F}_1 = \mathbb{Z}$. Then $\mathbb{F}_1$ acts freely on $\widetilde{X}$ as the group of covering transformations $Deck(\widetilde{X}/X)$ and $\widetilde{X}/\mathbb{F}_1 = X$.

Let k be a commutative ring and $\Lambda = \Lambda(k)$ be group ring $k[\mathbb{F}_1]$ of the group $\mathbb{F}_1$. If t is one of two generators of $\mathbb{F}_1$, then $\Lambda = k[t, t^{-1}]$ is the ring of Laurent polynomials in t with coefficients in k. The homology group $H_i(\widetilde{X}, k)$ has a natural structure of a Λ-module, where $t \cdot c = H_i(t)(c)$ for $c \in H_i(\widetilde{X})$, $H_i(t)$ is the automorphism corresponding to the covering transformation $t : \widetilde{X} \to \widetilde{X}$.

Definition 1 *The Λ-module $A_i = A_i(k) = H_i(\widetilde{X}, k)$ is called the i-th Alexander invariant (module) of a space X (more exactly of the pair (X, p)).*

1.2. If k is a field, then Λ is a principal ideal domain. The homology group $A = H_i(\widetilde{X}, k)$ is finitely generated over Λ. Hence by a general theorem of algebra A is isomorphic to a direct sum of cyclic modules

$$A \simeq \oplus_{j=1}^{l} \Lambda/(p_j) = \Lambda^b \oplus (\oplus_{i=1}^{k} \Lambda/(p_i)),$$

where (p_j) is a principal ideal generated by a polynomial $p_j(t)$. Here $\Lambda^b = \Lambda_{\text{free}}$ is a free part corresponding to $p_j(t) \equiv 0$, and $\oplus_{i=1}^k \Lambda/(p_i) = A_{\text{tors}}$ is a torsion submodule of the Λ-module A.

The product ideal $(p_1 \cdot \ldots \cdot p_l)$ is called the *order* of A. Obviously the order of A equals to 0 if and only if A has a free part, $b \neq 0$. In other words, the order $A \neq 0 \Leftrightarrow A = A_{\text{tors}}$ is a torsion module over Λ, i.e. A is a vector space of finite dimension over k.

Definition 2 *If $A = H_i(\widetilde{X}, k)$, then the polynomial $\Delta_i(t) = p_1(t) \cdot \ldots \cdot p_k(t)$ is called the i-th Alexander polynomial of a space X.*

Obviously we have

Proposition 1 *If $A = H_i(\tilde{X}, k)$ is finite dimensional over k, then $\Delta_i(t)$ coincides (to within a unit of Λ) with the characteristic polynomial of linear transformation $H_i(t) : H_i(\widetilde{X}, k) \to H_i(\widetilde{X}, k)$.*

1.3. In general, if k is not a field, then the ring $\Lambda = k[t, t^{-1}]$ is not a principal ideal domain. The most important example is $k = \mathbb{Z}$. In any case, with any finitely presented module A over a commutative ring Λ we can associate so called *Fitting ideals* $F_k(A)$. They are defined invariantly and are calculated in such a way. Let $\Lambda^t \xrightarrow{p} \Lambda^s \to A \to 0$ be a presentation of a Λ-module A and let P be the matrix of the linear map p. Then the k-th Fitting ideal $F_k(A) \subset \Lambda$ is generated by all minors of order $s - k$ of the matrix P.

Definition 3 *The ideals $F_{k-1}(A)$ are called the k-th Alexander ideals for Λ-module A.*

For any ideal $I \subset \Lambda$ denote by $\bar{I}$ the minimal principal ideal containing I.

Definition 4 *Any generator $\Delta_k(t)$ of the ideal $\overline{F_{k-1}(A)}$ is called the k-th Alexander polynomial for the Λ-module A.*

Note that in the knot theory only the first Alexander polynomial of the space $X = S^3 \setminus K$ is nontrivial for a knot K. So in this case the k-th Alexander polynomial of the module $H_1(\widetilde{X})$ is called the k-th Alexander polynomial of the knot K.

1.4. *The Milnor exact sequence.* There is an exact sequence which is very usefull for applications of $H_i(\widetilde{X})$ for study of $H_i(X)$. It is analogous to Wang exact sequence for a locally trivial fibration over a circle. Consider a short exact sequence

$$0 \to C_*(\widetilde{X}) \xrightarrow{t-1} C_*(\widetilde{X}) \to C_*(X) \to 0$$

of chain complexes. Then *homological (cohomological) Milnor exact sequence* is the corresponding homology (cohomology) exact sequence

$$\ldots \xrightarrow{\partial} H_i(\widetilde{X}) \xrightarrow{t-1} H_i(\widetilde{X}) \to H_i(X) \xrightarrow{\partial} H_{i-1}(\tilde{X}) \to \ldots \to H_0(X) \to 0.$$

2 Classical Knot Theory

2.1. Let $K \subset S^3$ be a knot, i.e. a connected submanifold in S^3 diffeomorphic to a circle S^1. The knot group is the most important invariant of a knot. The *group of the knot* K is the fundamental group $\pi = \pi_1(S^3 \setminus K)$ of the complement $S^3 \setminus K = X$, which is a $K(\pi, 1)$ Eilenberg-Maclane space. It is easy to see that $H_i(S^3 \setminus K) = \mathbb{Z}$ for $i = 0$, 1 and $H_i(S^3 \setminus K) = 0$ for $i \geq 2$. Since $H_1(S^3 \setminus K)$ is the abelianization of the group $\pi_1(S^3 \setminus K)$, we have an exact sequence

$$0 \to N \to \pi_1(X) \to H_1(X) \to 0,$$

where $N = [G, G] = G'$ is the commutator subgroup of G. Consider an infinite cyclic covering $\varphi : \widetilde{X} \to X$ determined by the epimorphism $p : \pi_1(X) \to H_1(X) = \mathbb{F}_1$.

Let $\Lambda = k[\mathbb{F}_1]$ be the group ring of the group $\mathbb{F}_1$.

Definition 5 *The first Alexander invariant* $A = H_1(\widetilde{X})$ *of the space* $X = S^3 \setminus K$ *is called the Alexander invariant of the knot* K.

Any presentation matrix for the Λ-module $H_1(\widetilde{X})$ is called an *Alexander matrix*. The $(k-1)$-th Fitting ideal $F_{k-1}(H_1(\widetilde{X}))$, correspondingly, the k-th Alexander polynomial $\Delta_k(t)$ for the Λ-module $H_1(\tilde{X})$ is called the *k-th Alexander ideal,* correspondingly, the *k-th Alexander polynomial for the knot* K.

Theorem 1 *The 1-st Alexander ideal* $F_0(H_1(\widetilde{X}))$ *of a knot* K *is principal.*

The first Alexander ideal $F_0(H_1(\widetilde{X}))$ is called simply an *Alexander ideal* and its generater $\Delta(t) = \Delta_1(t)$ is called an *Alexander polynomial for the knot* K.

One can show that $\Delta(t) = \det(V - tV^T)$, where V is the so-called Seifert matrix for a knot K. It is a $2g \times 2g$ matrix defined by means of notions of a Seifert surface and a linking number of cycles.

The following result shows that there is great freedom for the Alexander polynomials.

Theorem 2 (Seifert). *If a polynomial* $\Delta \in A$ *satisfies the conditions* $\Delta(t) = \pm 1$ *and* $\Delta(t) = t^d \cdot \Delta(t^{-1})$*, where* $d = \deg \Delta$*, then there is a knot* $K \subset S^3$ *such that its Alexander polynomial is* Δ.

2.2. Now we define a more restrictive class of knots which contains all algebraic knots (the knots of singularities).

Definition 6 *A knot* $K \subset S^3$ *is fibered if there is a fibration map* $p : S^3 \setminus K \to S^1$ *such that* K *has a tubular neighborhood* $T \simeq S^1 \times D^2$ *in* S^3 *for which the diagram*

$$\begin{array}{ccc} T \setminus K & \xrightarrow{\ \sim\ } & S^1 \times (D^2 \setminus \{0\}) \\ & {\scriptstyle p|_{T\setminus K}} \searrow \quad \swarrow {\scriptstyle p_0} & \\ & S^1 & \end{array}$$

is commutative, where $D^2 = \{y \in \mathbb{C} \mid \, \mid y \mid < 1\}$ *is the unit disk and* $p_0(x, y) = \frac{y}{|y|}$.

Fibres $F_t = p^{-1}(t)$, $t \in S^1$, are the interiors of compact surfaces $\bar{F}_t \subset S^3$ with common boundary $\partial \bar{F}_t = K$. ($\bar{F}_t$ is obtained from a compact surface without boundary $\tilde{F}_t$ by means of removing an open disk and is called a Seifert surface for the knot K).

Let $F = F_1 = p^{-1}(1)$. The exact homotopy sequence of the fibration p gives that the commutator subgroup of $\pi_1(S^3 \setminus K)$ coinsides with $\pi_1(F)$ and, consequently, is a finitely generated free group. The fibered knots are charecterized by this property.

Theorem 3 (Stallings). *A knot K is fibered if and only if the commutator subgroup $[\pi_1(S^3 \setminus K), \pi_1(S^3 \setminus K)]$ is finitely generated (and free).*

The fibration p is locally trivial over a circle S^1 and so defines a *monodromy homeomorphism* $h : F \to F$ and a *momodromy operator* $H_1(h) : H_1(F) \to H_1(F)$, which we'll denote often simply by h. Let $[h]$ be a matrix of h. One can prove

Proposition 2 *The matrix $t \cdot Id - [h]$ is an Alexander matrix for the knot K. In particular, the Alexander polynomial $\Delta(t)$ for the fibered knot coincides with the characteristic polynomial of the monodromy operator*

$$\Delta(t) = \det(t \cdot Id - [h]).$$

3 Algebraic Knots

3.1. Let $(C, 0) \subset (\mathbb{C}^2, 0)$ be an isolated singularity of a plane curve (i.e. the curve is reduced). Then we can associate with $(C, 0)$ a link $L \subset S^3$, where $L = C \cap S^3_\varepsilon$ and S^3_ε is a sphere of sufficienly small radius ε. Such links are called *algebraic links*. If $(C, 0)$ has k components, then the link $L = L(C, 0)$ has also k components. In particular, if $(C, 0)$ is irreducible, $k = 1$, then we obtain an *algebraic knot* $L = K \subset S^3$.

Let $f(x, y) = 0$ be an equation of $(C, 0)$, i.e. $(C, 0)$ is the zero fibre of a morphism of germs $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$. Consider the map

$$p : S^3_\varepsilon \setminus L \to S^1, \qquad p(x, y) = \frac{f(x, y)}{| f(x, y) |}.$$

By a general result of Milnor $[M1]$ it follows that p is a locally trivial fibration, i.e. we have the following proposition.

Proposition 3 *All algebraic knots (and links) are fibred.*

3.2. An algebraic knot $K = K(C, 0)$ is entirely defined by the type of the singularity $(C, 0)$.

Theorem 4 (K.Brauner). *If*

$$P(C, 0) = \{(m_1, n_1), (m_2, n_2), \ldots, (m_s, n_s)\}$$

is the sequence of Puiseux pairs of a plane curve singularity $(C, 0)$, then the knot $K(C, 0)$ is equivalent to the iterated torus knot associated to the sequence of pairs $P(C, 0)$.

The Alexander polynomial $\Delta(t)$ of a knot $K(C,0)$ also can be expressed in terms of Puiseux pairs.

Theorem 5 (Lê Dũng Tráng [Le]).

$$\Delta(t) = \Delta_1(t) = P_{\lambda_1,n_1}(t^{\nu_2}) \cdot \ldots P_{\lambda_s,n_s}(t^{\nu_{s+1}}),$$

where

$$P_{\lambda,n}(t) = \frac{(t^{\lambda n}-1)(t-1)}{(t^{\lambda}-1)(t^n-1)},$$

and $\nu_i = n_i \cdot \ldots \cdot n_s$ *for* $i = 1,\ldots,s$, $\nu_{s+1} = 1$, *and* $\lambda_1 = m_1$, $\lambda_i = m_i - m_{i-1}n_i + \lambda_{i-1}n_in_{i-1}$ *for* $i = 2,\ldots,s$.

For example, if $(C,0)$ is a cusp, $x^3+y^2=0$, then $K(C,0)$ is the trefoil knot and

$$\Delta(t) = \frac{(t^6-1)(t-1)}{(t^3-1)(t^2-1)} = t^2 - t + 1.$$

Besides, Lê Dũng Tráng proved the following theorem.

Theorem 6 . *The quotient Δ_1/Δ_2 of the first two Alexander polynomial of the knot $K(C,0)$ is the minimal polynomial of the monodromy operator $M = h_* : H_1(F) \to H_1(F)$ of the singularrity $(C,0)$. Moreover, this polynomial has distinct roots and hence the monodromy M has finite order and, in particular, M is semisimple.*

Remark 1 If the singularity $(C,0)$ is not irreducible, then the monodromy operator M can be not semisimple. For example [AC1], if $f = (x^2+y^3)(x^3+y^2)$, then the minimal polynomial of M is equal to $(t^5+1)(t^2-1)$ and has a double root $t=-1$. Hence the monodromy operator M has infinite order.

4 Milnor fibrations of germs of analytic functions

4.1. The Milnor fibration of an algebraic knot is a particular case of the Milnor fibration of an analytic function germ $f : (\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$. Let $(Y,0) \subset (\mathbb{C}^{n+1},0)$ be a germ of a hypersurface with the equation $f(x_0,x_1,\ldots,x_n) = 0$. Let the germ f be defined in a neighborhood of the closed ball $\bar{B}_\varepsilon$ of sufficiently small radius $\varepsilon > 0$. Let $S^{2n+1}_\varepsilon = \partial\bar{B}_\varepsilon$ be the boundary sphere.

Definition 7 $K = K(Y,0) = Y \cap S^{2n+1}_\varepsilon \subset S^{2n+1}_\varepsilon$ *is called the knot of a singularity f or $(Y,0)$.*

Theorem 7 [M1]. *If $\varepsilon > 0$ is sufficienly small, then the map*

$$\varphi : S^{2n+1}_\varepsilon \setminus K \to S^1, \qquad \varphi(x) = \frac{f(x)}{|\,f(x)\,|},$$

is a smooth locally trivial fibration.

If the singularity f is isolated, then the knot K is a smooth $(2n-1)$ -manifold. Any fiber $F_t = \varphi^{-1}(t)$ is a smooth open manifold whose closure $\bar{F}_t = F_t \cup K$. $\bar{F}_t$ is a manifold with boundary $\partial \bar{F}_t = K$.

The fibration φ is called the *Milnor fibration of a singularity f*.

From the historical point of view the fibration φ is a natural generalization of fibred knots (and then of algebraic knots). But there is another equivalent fibration associated to f which is also called the Milnor fibration.

4.2. Let $S = S_\delta = \{t \in \mathbb{C} \mid |t| < \delta\}$, $S' = S \setminus \{0\}$, $X = X_{\varepsilon,\delta} = B_\varepsilon \cap f^{-1}(S_\delta)$, $\bar{X} = \bar{B}_\varepsilon \cap f^{-1}(S_\delta)$, $X' = X \setminus f^{-1}(0)$, and $f : X \to S$, $\bar{f} : \bar{X} \to S$ denote the restrictions of f to X and $\bar{X}$.

Theorem 8 ([$M1$]; Lê Dũng Tráng, 1977). *If $\varepsilon \gg \delta > 0$ are sufficiently small, then the map $\bar{f} : \bar{X}' \to S'$ is topological locally trivial fibration, and $f' : X' \to S'$ is a smooth locally trivial fibration.*

Definition 8 *The fibration $f' : X' \to S'$, and also $\bar{f}' : \bar{X}' \to S'$, is called the Milnor fibration of a singularity f.*

Sometimes, one uses the terms open Milnor fibration and closed Milnor fibration to distinguish between f' and $\bar{f}'$.

The fibre $X_t = f^{-1}(t)$, $t \in S'$, is a Stein complex manifold, $\dim X_t = n$. The fibre $\bar{X}_t = \bar{f}^{-1}(t)$ is a manifold with boundary. X_t and $\bar{X}_t$ have the homotopy type of a CW-complex of real dimension n. The fibre X_t (and also $\bar{X}_t$) is called the *Milnor fibre of a singularity f*.

Obviously the fibrations f and $\bar{f}$ are homotopically equivalent to their restrictions over the circle $S^1_{\delta/2} \subset S_\delta$ of radius $\delta/2$. Identifying $S^1_{\delta/2}$ and S^1, we can assume that the radius of the circle equals to 1. Denote the restrictions of f and $\bar{f}$ over S^1 by ψ and $\bar{\psi}$ correspondingly.

Theorem 9 *(i) The fibrations φ and ψ are fibre diffeomorphic equivalent.*

(ii) The fibrations ψ and $\bar{\psi}$ are fibre homotopy equivalent.

Thus introduced definitions of the notion of Milnor fibration are equivalent.

Remark 2 The Milnor fibration associated to a hypersurface singularity $(Y, 0)$ does not depend on the choice of an equation $f = 0$ for $(Y, 0)$. This comes from the fact that K-arbits are connected. Moreover, the equivalence class of the Milnor fibration for an isolated singularity does not change under μ-const deformations.

4.3. A locally trivial fibration over a circle S^1 determine (and is determined by) a *monodromy transformation*

$$h : X_t \to X_t.$$

The monodromy transformation determines homology and cohomology operators

$$M = h_* : H_*(X_t) \to H_*(X_t), \quad T = (h^*)^{-1} : H^*(X_t) \to H^*(X_t).$$

The basic property of monodromy operators is given by

Monodromy Theorem. *Let* $T : H^p(X_t, \mathbb{C}) \to H^p(X_t, \mathbb{C})$ *be the monodromy operator in p-dimensional cohomology of the Milnor fibration of a hypersurface singularity* $(Y, 0) \subset (\mathbb{C}^{n+1}, 0)$ *or of a germ of a function* $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$. *Then*

(i) All the eigenvalues of T are roots of unity, or in other words, the operator T is quasiunipotent, i.e there exist positive intergers l and q such that

$$(T^l - id)^q = 0.$$

(ii) T^l *has index of unipotency at most p, i.e. we can take* $q = p + 1$, *i.e. the dimensions of Jordan blocks of T are less or equal to* $p + 1$.

There are several different proofs of the Monodromy Theorem (Grothendieck A., Landman A., Clemens C.H., Katz N.M., Borel A., Brieskorn E.,..., see, for example, [G-S]).

The Monodromy Theorem is closely connected with the theory of mixed Hodge structures (MHS). One can find an introduction to this theory in [G-S].

4.4. Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be an isolated singularity and let X_t be its Milnor fibre. We consider the *canonical Milnor fibre* X_∞, i.e. the total space of the pullback of the Milnor fibration $f' : X' \to S'$ to the universal cover $U \to S'$ of S', $X_\infty = X' \times_S U$. As each X_t is homotopy equivalent to X_∞, there is a canonical isomorphism between $H^n(X_t)$ and $H^n(X_\infty)$. The vanishing cohomology groups carry a mixed Hodge structure first defined by Steenbrink [*S*], who used an embedding of $f : X \to S$ to a family of projective hypersurfaces and resolution of singularities of this family. Then Varchenko A.N. and Sherk-Steenbrink [S-S] gave another description of the Hodge filtration on $H^n(X_\infty)$ which does not use resolution of singularities. The weight filtration $W.$ on $H^n(X_\infty)$ is connected with the monodromy operator T. It is the weight filtration of the nilpotent operator $N = -\frac{1}{2\pi i} \log T_u$, where T_u is the unipotent part of the monodromy.

We can make more precise formulation of part (ii) of the Monodromy Theorem for $H^n(X_t)$:

ii') The Jordan blocks of T are of size at most $n + 1$. *The Jordan blocks for eigenvalue 1 of T are of size at most n.*

Van Doorn M.G.M. and Steenbrink J.H.M. [DS] gave the following supplement to the Monodromy Theorem :

Theorem 10 *If the monodromy operator T on* $H^n(X_t)$ *has a Jordan block of size* $n + 1$ *(necessarily for an eigenvalue* $\neq 1$*), then T also has a Jordan block of size n for the eigenvalue 1.*

This theorem is an analogue in higher dimensions of the following result of Lê D.T. (1972): *The monodromy of an irreducible plane curve singularity is of finite order.*

4.5. At last we want to mention about the MHS on cohomology of the knot (link) of a singularity and the Wang sequence (cf. [Ka]) to be compared with the Milnor exact sequence and our result in the last section of this article.

Let $K = X_0 \cap S^{2n+1}$, $X_0 = f^{-1}(0)$, be the knot of an isolated singularity f. It is known that the pair (X, X_0) is homeomorphic to the cone on (S^{2n+1}, K) with the vertex $x_0 = \mathrm{Sing}\, f$. So K is homotopy equivalent to $X_0 \setminus \{x_0\}$ and hence $H_i(X_0 \setminus \{x_0\}) \simeq H_i(K)$. By the

Poincaré duality we have an isomorphism $H_i(K) \simeq H^{2n-1-i}(K)$, and by the Alexander duality $H^{2n-1-i}(K) \simeq H_{i+1}(S^{2n+1} \setminus K)$. Again $H_{i+1}(S^{2n+1} \setminus K) \simeq H_{i+1}(X')$, $X' = X \setminus X_0$, since $S^{2n+1} \setminus K$ is homotopy equivalent to X'. For the cohomology with coefficients in a field we obtain dual isomorphisms

$$H^i(K) \simeq H^i(X_0 \setminus \{x_0\}) \simeq H^{i+1}(X').$$

So we can think about each of these cohomology as cohomology of the knot of a singularity.

Using the above isomorphism we can introduce a MHS on $H^i(K)$. Indeed, because X_0 is contractible, the long exact cohomology sequence of the couple $(X_0, X_0 \setminus \{x_o\})$ implies isomorphism $H^i(X_0 \setminus \{x_0\}) \simeq H^{i+1}(X_0, X_0 \setminus \{x_0\}) \stackrel{\text{def}}{=} H^{i+1}_{\{x_0\}}(X_0)$. But X_0 can be extended to a complete variety $\bar{X}_0$, and by excision $H^{\cdot}_{\{x_0\}}(X_0) \simeq H^{\cdot}_{\{x_0\}}(\bar{X}_0)$. On $H^{\cdot}_{\{x_0\}}(\bar{X}_0)$ there is a canonical and functorial MHS.

The monodromy operator T appears in *Wang exact sequence*

$$\ldots \to H^i(X') \to H^i(X_t) \xrightarrow{T-\text{id}} H^i(X_t) \to H^{i+1}(X') \to \ldots.$$

For an isolated singularity f the Milnor fibre X_t is homotopy equivalent to a buquet of n-spheres and hence $H^i(X_t) \neq 0$ only for $i = 0$ and $i = n$. Consequently the only interesting cohomology groups of the knot are $H^n(X')$ and $H^{n+1}(X')$ which are the kernel and the cokernel, respectively, of the map $T - Id : H^n(X_t) \to H^n(X_t)$.

The terms of the Wang sequence carry the MHS. The map $T - Id$ need not be a morphism of Hodge structures. But if we change $T - Id$ by $N = -\frac{1}{2\pi i}\log T_u$, then the sequence remains exact and becomes a MHS exact sequence.

5 Monodromy of a quasihomogeneous singularity

5.1. Let $f \in \mathbb{C}[x_0, \ldots, x_n]$ be a quasihomogeneous (= weighted homogeneous) polynomial of degree $\deg f = N$ with respect to the weights $wt\, x_i = w_i$, i.e. f satisfies the Euler relation

$$f(r^{w_0}x_0, \ldots, r^{w_n}x_n) = r^N f(x_0, \ldots, x_n), \ \forall r \in \mathbb{C}^*, \ \ x \in \mathbb{C}^{n+1}.$$

Consider the singularity $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$. Then the local Milnor fibration $f' : X' \to S'$ defined in the previous section is equivalent to the global *affine Milnor fibration* f', where we denote by $f : X \to S$ a morphism defined by the polynomial f,

$$\begin{array}{ccccc} \mathbb{C}^{n+1} = X & \supset & X' = X \setminus X_0\,, & & X_0 = f^{-1}(0) \\ \downarrow f & & \downarrow f' & & \\ \mathbb{C} = S & \supset & S' = \mathbb{C} \setminus \{0\}\,. & & \end{array}$$

Indeed, we can consider the $\mathbb{C}^*$-action on $\mathbb{C}^{n+1}$ associated to weights $w = (w_0, \ldots, w_n)$

$$r \circ x = (r^{w_0}x_0, \ldots, r^{w_n}x_n), \quad r \in \mathbb{C}^*, \ \ x \in \mathbb{C}^{n+1}.$$

The fibre $X_t = f^{-1}(t)$ has the equation $f(x_0, \dots, x_n) = t$. The Euler relation $f(r \circ x) = r^N f(x) = r^N t$ shows that $r \in \mathbb{C}^*$ translates the fibre X_t to the fibre $X_{r^N t}$, and $\mathbb{C}^*$ acts on the set of fibres X_t, $t \in \mathbb{C}^*$, transitively. If we define the action of $\mathbb{C}^*$ on $S = \mathbb{C}$, $r \circ t = r^N t$, then we see that the morphism $f : \mathbb{C}^{n+1} \to \mathbb{C}$ is equivariant.

5.2. Let us calculate the monodromy $h : X_t \to X_t$. Consider the fibre $F = X_1$ over the point $t = 1 \in S^1 = \{| t |= 1\}$. If we construct a family of diffeomorphisms $h_\varphi : X_1 \to X_t$, $t = e^{2\pi i \varphi}$, then $h = h_1 : X_1 \to X_1$ is the monodromy. Obviously, we can take $h_\varphi =: e^{\frac{2\pi i \varphi}{N}} \circ (\cdot)$ and then we have $r \circ x \in X_{r^N} = X_{e^{2\pi i \varphi}} = X_t$ for $x \in X_1$, $f(x) = 1$, and $r = e^{\frac{2\pi i \varphi}{N}}$. Thus we obtain a concrete description of the geometric monodromy $h : X_1 \to X_1$,

$$h(x) = (e^{\frac{2\pi i w_0}{N}} x_0, \dots, e^{\frac{2\pi i w_n}{N}} x_n).$$

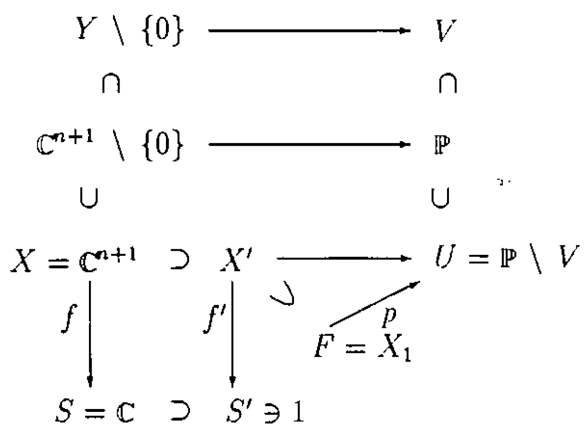
In particular, $h^N = 1$ and the monodromy $T = h^* : H^\cdot(X_t, \mathbb{C}) \to H^\cdot(X_t, \mathbb{C})$ has a finite order, $T^N = id$. We obtain

Corollary 1 *The monodromy of a quasihomogeneous singularity is semisimple, $T = T_s$, and all eigenvalues λ are roots of unity, $\lambda^N = 1$.*

5.3. We can use the weighted projective space

$$\mathbb{P} = \mathbb{P}(w) = \mathbb{C}^{n+1} \setminus \{0\}/\mathbb{C}^*,$$

where the action of $\mathbb{C}^*$ on $\mathbb{C}^{n+1}$ is defined above, to calculate the eigensubspaces of $H^k(X_t, \mathbb{C})$. The space $\mathbb{P}(w)$ generalizes the usual projective space $\mathbb{P}^n$ corresponding to weights $w = (1, \dots, 1)$ (in the case of homogeneous polynomial f). The equation $f(x) = 0$ defines a hypersurface $V \subset \mathbb{P}$ and let $U = \mathbb{P} \setminus V$ be the complement to V. We can consider $Y = X_0 \subset \mathbb{C}^{n+1}$ as a quasicone $Y = C_V$ over V. We obtain a diagram



Thus, $\mathbb{C}^{n+1} \setminus \{0\}$ is partitioned into $\mathbb{C}^*$-orbits and the set of orbits is $\mathbb{P}$. The fibre $Y = f^{-1}(0)$ consists of orbits, the generators of the quasicone C_V, $Y \setminus \{0\}/\mathbb{C}^* = V$. And any fibre X_t, $t \neq 0$, is mapped onto U. Moreover, the subgroup $\mu_N = \{r \in \mathbb{C}^* \mid r^N = 1\}$ is the stationary subgroup of the fibre X_t ($r^N t = t \Rightarrow r^N = 1$). The group μ_N acts on X_t and the generator $e^{2\pi i/N}$ of this group acts as the monodromy h, and $X_t/\mu_N = U$.

Thus, $F/_{\mu_N} = U$ and we can use the differential forms on the principal open subset $U = \mathbb{P} \setminus V$ to calculate summands of $H^{\cdot}(F, \mathbb{C})$, $H^{\cdot}(F, \mathcal{O}_F)$, $H^{\cdot}(F, \Omega_F^p)$ in their decomposition into the sum corresponding to the characters $\chi \in \mathrm{Hom}(\mu_N, \mathbb{C}^*)$.

6 Alexander polynomial of a plane curve

6.1. Let $D \subset \mathbb{C}^2$ be a plane affine curve of degree d defined by an equation $f(x,y) = 0$. Let

$$f(x,y) = \prod_{i=1}^{k} f_i^{m_i}(x,y) \tag{1}$$

be the decomposition into irreducible factors. Denote by $D_i \subset \mathbb{C}^2$ an irreducible component of D defined by the equation $f_i(x,y) = 0$, $i = 1, \ldots, k$. The function $t = f(x,y)$ defines a morphism $f : X \to S$, where $X = \mathbb{C}^2$, $S = \mathbb{C}^1$. Then $D = f^{-1}(0)$. Denote $S' = \mathbb{C} \setminus \{0\}$, $X' = X \setminus D$. Consider the infinite cyclic covering $\varphi = \varphi_\infty : X_\infty \to X'$ corresponding to the universal covering $e : U \to S'$, where $U = \mathbb{C}$, $t = e(u) = e^{2\pi i u}$,

$$\begin{array}{ccc} X_\infty & \xrightarrow{\varphi} & X' \\ {\scriptstyle f_\infty}\downarrow & & \downarrow{\scriptstyle f'} \\ U & \xrightarrow[e]{} & S' \end{array}$$

It is well known that $H_1(X', \mathbb{Z}) = \mathbb{Z}^k$ is generated by the loops γ_i "surrounding" the components D_i. The homomorphism $f'_* : \pi_1(X') \to \pi_1(S') = \mathbb{Z}$ factors through the Hurewitz homomorphism H

$$f'_* : \pi_1(X') \xrightarrow{H} H_1(X', \mathbb{Z}) = \mathbb{Z}^k \xrightarrow{\delta} \mathbb{Z},$$

where $\delta(s_1[\gamma_1] + \ldots + s_k[\gamma_k]) = \sum_{i=1}^{k} s_i m_i$.

Definition 9 *We say that f is primitive if the generic fibre of $f : \mathbb{C}^2 \to \mathbb{C}$ is irreducible.*

It is well known that if f is not primitive, then f factors through a covering $p : \mathbb{C} \to \mathbb{C}$, $f(x,y) = p(g(x,y))$ such that $g(x,y)$ is primitive.

Proposition 4 ([K1], [Sa]). *The vector space $H_1(X_\infty, \mathbb{C})$ is finite-dimensional if and only if f is a power of a primitive polynomial.*

In the sequel we'll assume that f is primitive, in particular,

$$G.C.D.(m_1, \ldots, m_k) = 1.$$

Then $f'_* : \pi_1(X') \to \mathbb{Z}$ is an epimorphism.

The following theorem is an analog of Theorem 3 (Stallings) in the case of algebraic curves.

Theorem 11 ([K1], [K2]) *If f is primitive, then $Ker\, f'_*$ is finitely generated. In particular, if $D \subset \mathbb{C}^2$ is an irreducible curve, then $[\pi_1(\mathbb{C}^2 \setminus D), \pi_1(\mathbb{C}^2 \setminus D)]$ is finitely generated.*

Definition 10 *The i-th Alexander polynomial $\Delta_{D,i}$ of a curve $D \subset \mathbb{C}^2$ is the i-th Alexander polynomial of the space $X' = \mathbb{C}^2 \setminus D$ associated with the epimorphism f'_*. The first Alexander polynomial $\Delta_D(t) = \Delta_{D,1}(t)$ is called simply the Alexander polynomial of D.*

Definition 11 *The Alexander polynomial $\Delta_D(t)$ of a projective curve $D \subset \mathbb{P}^2$ is the Alexander polynomial of an affine curve $D \subset \mathbb{C}^2$, where $\mathbb{C}^2 = \mathbb{P}^2 \setminus L$, $D = D \cap \mathbb{C}^2$, and $L \subset \mathbb{P}^2$ is a generic line.*

We can begin with an affine curve $D \subset \mathbb{C}^2$ and consider its projective closure $\bar{D} \subset \mathbb{P}^2$. Then for the equality $\Delta_{\bar{D}}(t) = \Delta_D(t)$ it is necessary for L to intersect D trasversally.

Theorem 12 (Randell [*R*]). *If $\bar{D} \subset \mathbb{P}^2$ is a reduced curve defined by the equation $F(x_0, x_1, x_2) = 0$, then $\Delta_{\bar{D}}(t)$ is equal to the characteristic polynomial of the monodromy h on first homology of the Milnor fibre of the homogeneous singularity $F : \mathbb{C}^3 \to \mathbb{C}$,*

$$\Delta_{\bar{D}}(t) = \det(t \cdot id - h).$$

In particular, this yields that the monodromy of a reduced curve transversally intersecting the line at infinity is semisimple because it is so for the monodromy of a quasihomogeneous singularity.

6.2. *The divisibility by the Alexander polynomial* ([L1], [K1]). Let $p_i \in D$ be a singular point. Denote by $\Delta_{p_i,D}(t)$ the characteristic polynomial of the monodromy T on cohomology $H^1(X_t)$ of the Milnor fibre of the singularity (D, p_i). Equivalently, $\Delta_{p_i,D}(t)$ is the Alexander polynomial of the algebraic link $K(D, p_i)$.

Let $S^3 = \partial T(L)$ be the boundary of the tubular neighbouhood of the line $L = L_\infty \subset \mathbb{P}^2$ at infinity. Denote by $\Delta_{\infty,\bar{D}}(t)$ the Alexander polynomial of the link $\bar{D} \cap \partial T(L) \subset \partial T(L)$. If L is in general position relative to $\bar{D}$, then

$$\Delta_{\infty,\bar{D}}(t) = (t-1)(t^d - 1)^{d-2}.$$

Theorem 13 (Libgober [L1]). *If $D \subset \mathbb{C}^2$ is an irreducible curve, then*

(i) $\Delta_{\bar{D}}(t)$ divides the product $\prod \Delta_{p_i,D}(t)$ of the local Alexander polynomials of all singularities $p_i \in \bar{D}$.

(ii) $\Delta_{\bar{D}}(t)$ divides $\Delta_{\infty,\bar{D}}(t)$.

This theorem provides some information about the fundamental group $\pi_1(\mathbb{C}^2 \setminus D)$. The application of it to curves with only cusps and nodes one can find in [*L*3].

Let $X_{t_1}, \dots, X_{t_q}$ be the degenerate fibres of $f' : X' \to S'$ such that $X' \setminus (\cup X_{t_j}) \to S' \setminus (\cup t_j)$ is a C^∞ locally trivial fibration. Let X_t be a generic fibre. Let γ_0 and γ_∞ be circles with centers at 0 and of radius $r_0 \ll 1$ and $r_\infty \gg 1$. Denote by h_0 and h_∞ the monodromy operators on $H^1(X_t)$ corresponding to γ_0 and γ_∞ (and defined modulo an inner automorphism). We call the characteristic polynomial $\Delta_{in}(t) = \det(h_0 - t \cdot Id)$ (correspondingly, $\Delta_{ex}(t) = \det(h_\infty - t \cdot Id)$) the *internal* (and (correspondingly, *external*) *Alexander polynomial*.

Theorem 14 (Kulikov [K1]). *If the polynomial $f(x,y)$ is primitive, then $\Delta_D(t)$ divides $\Delta_{in}(t)$ and it divides $\Delta_{ex}(t)$.*

Let $\sigma : \bar{\mathbb{P}}^2 \to \mathbb{P}^2$ be a composition of σ -processes resolving the points of indeterminacy of the rational map

$$\begin{array}{ccc} f : \mathbb{P}^2 & \dashrightarrow & \mathbb{P}^1 \\ \cup & & \cup \\ \mathbb{C}^2 & \longrightarrow & \mathbb{C}^1 \end{array}$$

Put $\tilde{f} = \sigma \circ f : \bar{\mathbb{P}}^2 \to \mathbb{P}^1$. We can assume that the fibres

$$\tilde{f}^{-1}(0) = \sum_{i=1}^{N_0} m_i \bar{D}_i, \qquad \tilde{f}^{-1}(\infty) = \sum_{i=1}^{N_\infty} r_i R_i$$

are divisors with normal crossings. Put

$$D_i^0 = \bar{D}_i \setminus (\bigcup_{i \neq j} (\bar{D}_i \cap \bar{D}_j)), \qquad R_i^0 = R_i \setminus (\bigcup_{i \neq j} (R_i \cap R_j)).$$

In this case the internal and external Alexander polynomials can be calculated in terms of Euler characteristics $\chi(D_i^0)$ (respectively $\chi(R_i^0)$) and multiplicities m_i (respectively r_i) [AC2] :

$$\begin{aligned} \Delta_{in}(t) &= (t-1) \prod_{i=1}^{N_0} (t^{m_i} - 1)^{-\chi(D_i^0)}, \\ \Delta_{ex}(t) &= (t-1) \prod_{i=1}^{N_\infty} (t^{r_i} - 1)^{-\chi(R_i^0)}. \end{aligned}$$

In particular, if the curve $\bar{D} = \bar{D}_1 \cup \ldots \cup \bar{D}_k \subset \mathbb{P}^2$ intersects L_∞ transversally, then

$$\Delta_{ex}(t) = (t-1)(t^{\sum d_i m_i} - 1)^{\sum d_i - 2},$$

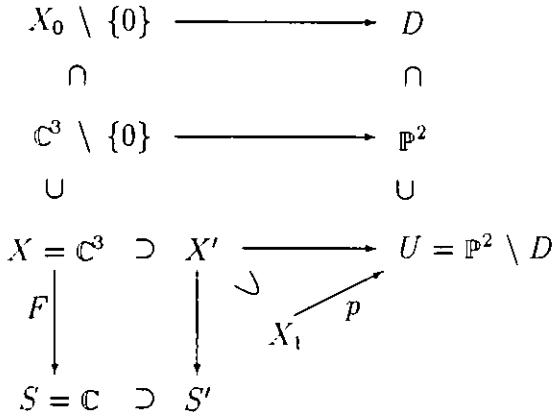
where $d_i = \deg D_i$.

7 Calculations of Alexander polynomials of reduced curves

We review the results of Esnault [E], Loeser-Vacuié [LV], Kohno [Ko].

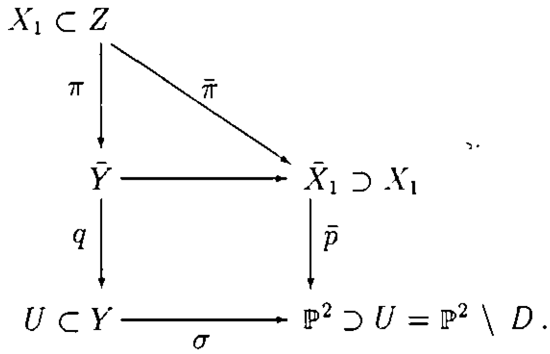
7.1. *The Alexander polynomial coincides with the characteristic polynomial of the monodromy for a homogeneous singularity.* Let $D = D_1 + \ldots + D_k \subset \mathbb{P}^2$ be a reduced curve of degree d with the equation $F(x_0, x_1, x_2) = 0$. In virtue of Randell's Theorem the question is reduced to the calculation of the characteristic polynomial $\Delta(t) = \Delta_D(t)$ for the monodromy $h^* : H^1(X_1, \mathbb{C}) \to H^1(X_1, \mathbb{C})$ of the Milnor fibre $X_1 = F^{-1}(1)$ of the homogeneous singularity $F : \mathbb{C}^3 \to \mathbb{C}$.

7.2. *The monodromy transformation is a generator of the group of automorphisms of a cyclic unramified covering.* Consider the diagram from section 5



The calculation of $\Delta(t)$ is based on the fact that $p : X_1 \to U$ is an unramified cyclic covering of degree d, and the monodromy $h : X_1 \to X_1$ acting as $h(x_0, x_1, x_2) = (\zeta x_0, \zeta x_1, \zeta x_2)$, where $\zeta = e^{2\pi i/d}$ is the primitive root of unity of degree d, coincides with the generator of the group $\mathrm{Aut}_U X_1 = \mathbb{Z}/d\mathbb{Z}$.

7.3. *The imbedding of an unramified covering to a ramified one.* Let $\bar{X}_1$ be a projective closure of the surface X_1 in $\mathbb{P}^3$, defined by the equation $F(x_0, x_1, x_2) = x_3^d$. The projection $\mathbb{P}^3 \to \mathbb{P}^2$, $(x_0 : x_1 : x_2 : x_3) \mapsto (x_0 : x_1 : x_2)$ from the point $(0:0:0:1) \notin \bar{X}_1$ determines the covering $\bar{p} : \bar{X}_1 \to \mathbb{P}^2$ extending p and ramified over the curve $D \subset \mathbb{P}^2$. Resolve the singularities of $\bar{X}_1$. First take the imbedded resolution $\sigma : Y \to \mathbb{P}^2$ of the curve D to obtain a divisor with normal crossings $\bar{C} = \sigma^{-1}(D)$. Let $C \subset Y$ be the proper preimage of D and E_j be the exeptional curves. Then take the pullback image $Y' = \bar{X}_1 \times_{\mathbb{P}^2} Y$ of the surface $\bar{X}_1$ over Y and take its normalization $\bar{Y} \to Y'$. At last, consider a good resolution $Z \xrightarrow{\pi} \bar{Y}$ of the singularities of $\bar{Y}$. The surface $\bar{Y}$ has only rational (in fact, quotient) singularities, because $\bar{C}$ is a divisor with normal crossings, and after minimal resolution of $\bar{Y}$ chains of rational curves are "glued". We obtain a commutative diagram



Let $\varphi = q \circ \pi$, $\bar{\varphi} = \bar{p} \circ \bar{\pi}$; $\bar{\varphi}^{-1}(D) = \varphi^{-1}(\bar{C}) = \Delta$ is a divisor with normal crossings. Then $Z \setminus \Delta \simeq X_1$, $Y \setminus \bar{C} \simeq U$.

7.4. *The description of ramified cyclic coverings in terms of invertible sheaves* ([E]). Let (temporarily) Y be a nonsingular algebraic variety (of arbitrary dimension m). In a local situation, if for example $Y = \mathbb{C}^2$, a cyclic covering $q' : Y' \to Y$ of degree n ramified over a divisor $D \subset Y$ defined by an equation $f(x,y)=0$ is a projection of the subvariety $Y' \subset \mathbb{C}^2 \times \mathbb{C}$ defined by the equation $z^n = f(x,y)$ in the trivial fibration. We can write $Y'=$ Spec $\mathcal{O}_{Y'}$, where $\mathcal{O}_{Y'} = \mathcal{O}_{\mathbb{C}^2} \oplus \mathcal{O}_{\mathbb{C}^2} z \oplus \ldots \oplus \mathcal{O}_{\mathbb{C}^2} z^{n-1}$, and the structure of algebra on $\mathcal{O}_{Y'}$ is given by the rule $z^n = f(x,y)$.

In a global situation instead of trivial fibration we must take a locally trivial fibration or an invertible sheaf. The construction of a cyclic covering runs as follows. Let $\mathcal{L}$ be an invertible sheaf on Y such that the sheaf $\mathcal{L}^n$ has a section $s : \mathcal{O}_Y \to \mathcal{L}^n$ and its zeroes determine a divisor $D \subset Y$. Then $\mathcal{L}^{-n} \subset \mathcal{O}_Y$ is the sheaf of ideals of the divisor D. Set

$$Y' = \operatorname{Spec}_Y(\bigoplus_{j=0}^{n-1} \mathcal{L}^{-j}),$$

where the $\mathcal{O}_Y$-algebra structure is defined by inclusion $\mathcal{L}^{-n} \subset \mathcal{O}_Y$. Then $q' : Y' \to Y$ is a cyclic covering ramified over D.

Now let $D = \sum \nu_l E_l$ be a divisor with normal crossings. Then the normalization $\nu : \bar{Y} \to Y'$ can be described concretely in terms of the following sheaves $\mathcal{L}^{(j)}$. Put $q = \nu \circ q' : \bar{Y} \xrightarrow{\nu} Y' \xrightarrow{q'} Y$. Then the direct image of the structure sheaf

$$q_* \mathcal{O}_{\bar{Y}} = \bigoplus_{j=0}^{n-1} (\mathcal{L}^{(j)})^{-1},$$

where

$$\mathcal{L}^{(j)} = \mathcal{L}^j \otimes \mathcal{O}_Y(-\sum_l [\frac{j}{n} \nu_l] E_l),$$

and $[\cdot]$ denote the entire part of a number.

The group $\operatorname{Aut}_Y Y' = \operatorname{Aut}_Y \bar{Y} = \mathbb{Z}/n\mathbb{Z}$ acts semisimply on $q_* \mathcal{L}_{\bar{Y}}$. Let $\zeta = e^{2\pi i/n}$ be a root of unity, and

$$q_* \mathcal{O}_{\bar{Y}} = \bigoplus_{j=0}^{n-1} F_j$$

be the decomposition into the sum of eigensubsheaves, where F_j corresponds to the eigenvalue ζ^j of the generator h of $\mathbb{Z}/n\mathbb{Z}$. Actually the decomposition described above coincides with the decomposition according to eigenvalues of h,

$$F_j = \mathcal{L}^{(j)}, \quad j = 0, \ldots, n-1.$$

Return to our situation. The covering $\bar{p} : \bar{X}_1 \to \mathbb{P}^2$ is ramified over a divisor D and is determined by the sheaf $\mathcal{L} = \mathcal{O}_{\mathbb{P}^2}(1)$ and inclusion $F : \mathcal{L}^{-d} = \mathcal{O}_{\mathbb{P}^2}(-d) \to \mathcal{O}_{\mathbb{P}^2}$, $\bar{X}_1 = \operatorname{Spec}_{\mathbb{P}^2}(\oplus_{j=0}^{d-1} \mathcal{O}_{\mathbb{P}^2}(-j))$, and the covering $q' : Y' \to Y$ is determined by $\mathcal{L} = \sigma^*(\mathcal{O}_{\mathbb{P}^2}(1))$ and inclusion $\mathcal{L}^{-d} = \mathcal{O}_Y(-\bar{C}) \subset \mathcal{O}_Y$, where $\bar{C} = C + \sum \nu_j E_j$.

7.5. *The decomposition of cohomology of the Milnor fibre.* The fibre X_1 is a nonsingular algebraic variety and so there is a MHS on $H^\cdot(X_1, \mathbb{Q})$ (Deligne). We need a good compactification of X_1 to introduce the MHS. We can take a good resolution $Z \supset X_1$, because $\Delta = Z \setminus X_1$ is a divisor with normal crossings. We have a spectral sequence

$$E_1^{pq} = H^q(Z, \Omega_Z^p(\log \Delta)) \Rightarrow H^{p+q}(X_1, \mathbb{C}),$$

degenerating in the term E_1. In particular, we obtain that

$$H^1(X_1, \mathbb{C}) = H^1(Z, \mathcal{O}_Z) \oplus H^0(Z, \Omega^1_Z(\log \Delta)).$$

7.6. *The descent to Y.* The singularities of the surface $\bar{Y}$ are rational (being quotient singularities) and the morphism q is finite. This involves

Corollary 2 *We have*

$$\begin{aligned} \varphi_* \Omega^p_Z(\log \Delta) &= \Omega^p_Y(\log C) \otimes q_* \mathcal{O}_{\bar{Y}}, \\ R^i \varphi_* \Omega^p_Z(\log \Delta) &= 0 \qquad \text{for } i > 0. \end{aligned}$$

Consequently,

$$\begin{aligned} H^q(Z, \Omega^p_Z(\log \Delta)) &= H^q(Y, \Omega^p_Y(\log \bar{C}) \otimes q_* \mathcal{O}_{\bar{Y}}) \\ &= \oplus_{j=0}^{d-1} H^q(Y, \Omega^p_Y(\log \bar{C}) \otimes (\mathcal{L}^{(j)})^{-1} \end{aligned}$$

and the last equality is the decomposition into the sum of eigensubspaces for the operator h with eigenvalues ζ^j.

In particular, we obtain

Corollary 3 *We have*

$$H^1(X_1, \mathbb{C})_{\zeta^j} = H^1(Y, (\mathcal{L}^{(j)})^{-1}) \oplus H^0(Y, \Omega^1_Y(\log \bar{C}) \otimes (\mathcal{L}^{(j)})^{-1})$$

for $j = 0, \ldots, d-1$.

We have $\mathcal{L}^{(j)} = \mathcal{O}_Y$ for $j = 0$, $H^1(Y, \mathcal{O}_Y) = 0$ since Y is a rational surface, and

$$h^0(Y, \Omega^1_Y(\log \bar{C})) = k - 1.$$

Therefore, $\dim H^1(X_1, \mathbb{C})_1 = k - 1$.

We obtain the following expression for the Alexander polynomial

Theorem 15 ([E]). *If a curve $D \subset \mathbb{P}^2$ is reduced, then*

$$\Delta_D(t) = \prod_{j=0}^{d-1} (t - \zeta^j)^{h_j},$$

where $\zeta = e^{2\pi i/d}$, and

$$h_j = \dim H^1(Y, (\mathcal{L}^{(j)})^{-1}) + \dim H^0(Y, \Omega^1_Y(\log \bar{C}) \otimes (\mathcal{L}^{(j)})^{-1}).$$

We have $h_0 = k - 1$ for $j = 0$. Besides,

$$\dim H^0(Y, \Omega^1_Y(\log \bar{C}) \otimes (\mathcal{L}^{(j)})^{-1}) = \dim H^1(Y, (\mathcal{L}^{(d-j)})^{-1})$$

for $j = 1, \ldots, d-1$.

The last equality is proven in $[LV]$. Strictly saying, this theorem is contained in the Esnault's paper $[E]$ implicitly. She calculates $b_1(X_1)$, $b_2(X_1)$ and also the rank and the signature of the intersection quadratic form on $H_c^2(X_1, \mathbb{C})$. This theorem is contained in $[LV]$, where the further descent to $\mathbb{P}^2$ is realized with the help of Vanishing theorem and the theory of MHS in vanishing cohomology of an isolated hypersurface singularity.

7.7. *The descent to* $\mathbb{P}^2$ ([LV]). Let $\sigma : Y \to \mathbb{P}^2$ be an embedded resolution of singularities of a curve $D \subset \mathbb{P}^2$. In the paper $[LV]$ the sheaves $\sigma_*(\mathcal{L}^{(j)} \otimes \omega_Y)$, where ω_Y is the canonical sheaf, are calculated. They prove that

$$\sigma_*(\mathcal{L}^{(j)} \otimes \omega_Y) \simeq \mathcal{A}_\alpha(j-3),$$

where $\alpha = j/d - 1$, $1 \leq j \leq d-1$, and the subsheaf $\mathcal{A}_\alpha \subset \mathcal{O}_{\mathbb{P}^2}$, for $\alpha \in \mathbb{Q}$, $-1 < \alpha < 0$, coincides with $\mathcal{O}_{\mathbb{P}^2}$ outside $\mathrm{Sing} D$ and for a singular point $x \in \mathrm{Sing} D$ is defined by the condition

$$(\mathcal{A}_\alpha)_x = \{g \in \mathcal{O}_{\mathbb{P}^2,x} \mid \alpha_f(g\omega_0) > \alpha\},$$

where $f = 0$ is a local equation of the curve D at x, ω_0 is a 2-form regular and not vanishing at x, and $\alpha_f(\omega)$ is the order of a form ω. [If $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ is an isolated singularity and ω is a $(n+1)$-form, then the order $\alpha_f(\omega)$ is the minimal exponent of t in the asymptotic development of integrals $\int \frac{\omega}{df}$ over manyvalued horisontal sections of homological Milnor fibration.] A.N.Varchenko expressed $\alpha_f(\omega)$ (in the case $\alpha_f(\omega) \leq 0$) in terms of embedded resolution $\pi : X \to \mathbb{C}^{n+1}$ of the singularity f :

$$\alpha_f(\omega) = \inf\left(\frac{1 + v_j(\omega)}{m_j} - 1\right),$$

where $\pi^{-1}(f^{-1}(0)) = \sum m_j E_j$ and $v_j(\omega)$ is the order of the form $\pi^*(\omega)$ along the component E_j. Moreover, Loeser and Vacuié prove in $[LV]$ with the help of the vanishing theorem (E.Vieweg) that

$$R^i\sigma_*(\mathcal{L}^{(j)} \otimes \omega_Y) = 0 \quad \text{for } i > 0 \text{ and } \quad j = 1, \ldots, d-1.$$

This involves that

$$\dim H^1(Y, (\mathcal{L}^{(j)})^{-1}) = \dim H^1(\mathbb{P}^2, \mathcal{A}_\alpha(j-3))$$

for $j = 1, \ldots, d-1$ and $\alpha = j/d - 1$.

It is proven in $[LV]$ that

$$H^1(\mathbb{P}^2, \mathcal{A}_\alpha(j-3)) = 0,$$

if $\alpha = j/d - 1$ does not belong to one of the spectra of $x \in \mathrm{Sing}\, D$. Summarizing we obtain the Loeser-Vaquie's result

Theorem 16 ($[LV]$). *If $D = D_1 + \ldots + D_k \subset \mathbb{P}^2$ is a reduced curve, then*

$$\Delta_D(t) = (t-1)^{k-1} \prod_{\alpha \in A_D} (\Delta_\alpha(t))^{l_\alpha},$$

where A_D is the set of $\alpha \in \mathbb{Q}$ for which $-1 < \alpha < 0$, $d \cdot \alpha \in \mathbb{Z}$ and α belongs to the spectrum of one of the singularities $x \in SingD$, and

$$\begin{aligned} \Delta_\alpha(t) &= (t - \exp(2\pi i\alpha))(t - \exp(-2\pi i\alpha)), \\ l_\alpha &= \dim H^1(\mathbb{P}^2, \mathcal{A}_\alpha(d(\alpha+1) - 3)). \end{aligned}$$

7.8. *Calculation of the Alexander polinomial in terms of cohomology of rational differential forms* ([Ko]). T.Kohno calculates $\Delta_D(t)$ for a reduced irreducible curve $D \subset \mathbb{C}^2$ transversally intersecting the line L_∞ at infinity. To calculate $\Delta_D(t)$ in terms of differential forms one needs some preliminary results about the connection between cohomology of infinite and d-fold coverings of $\mathbb{C}^2 \setminus D$. We'll obtain these results in general setting in the next section, and then we'll calculate $\Delta_D(t)$ in section 8.6.

8 The homology of cyclic coverings of the complement to a plane curve

We pass to the exposition of the main results of the paper $[KK]$. The problem is to consider as general curve as possible, not reduced and without conditions at infinity as it was assumed in the previous section. We return to the notation of section 6 : $D = m_1 D_1 + \ldots + m_k D_k \subset \mathbb{C}^2$ is a curve of degree d defined by the equation $f(x,y) = 0$ and so on.

8.1. Let $\varphi_n : X_n \to X$ be n-fold cyclic covering of $X = \mathbb{C}^2$, where X_n is a normalization of the surface defined by the equation $z^n = f(x,y)$ in $\mathbb{C}^3$. Denote $X' = X \setminus D$, $X'_n = X_n \setminus B$, $B = \varphi_n^{-1}(D)$. Then the infinite cyclic covering $\varphi = \varphi_\infty : X_\infty \to X'$ factors through the unramified covering $\varphi_n : X'_n \to X'$ and we have a commutative diagram

$$\begin{array}{ccccc} \phi : X_\infty & \xrightarrow{\phi_{\infty,n}} & X'_n & \xrightarrow{\phi_n} & X' \\ \downarrow f_\infty & & \downarrow f'_n & & \downarrow f' \\ e : U & \longrightarrow & S'_n & \xrightarrow{e_n} & S', \end{array} \qquad (2)$$

where $e(u) = t = e^{2\pi i u}$ for $u \in U = \mathbb{C}$, $e_n(z) = t = z^n$ and $z = e^{2\pi i u/n}$ for $z \in S'_n = \mathbb{C} \setminus \{0\}$.

We'll be interested in the connection between the homology of the affine variety X'_n and its projective completion $\bar{X}_n$.

Remark 3 The investigation of cyclic coverings $\bar{X}_n$ of the plane $\mathbb{P}^2$ (theory of algebraic surfaces) was the main reason for O.Zariski ([Z1], [Z2]) to study $\pi_1(\mathbb{C}^2 \setminus D)$. The computation of the irregularity $q(\bar{X}_n)$ and other invariants of $\bar{X}_n$ is one of the directions of the subject which we don't touch (see the Sakai's survey [Sa]).

Let $f : \mathbb{P}^2 \to \bar{S} = \mathbb{P}^1$ be a rational map corresponding to the morphism $f : X \to S$, not defined only at some points of the infinite line $L_\infty = \mathbb{P}^2 \setminus \mathbb{C}^2 = f^{-1}(\infty)$. Resolving the points of indeterminacy by means of σ-processes $\sigma : \bar{X} \to \mathbb{P}^2$ we get a morphism $\bar{f} = f \cdot \sigma : \bar{X} \to \bar{S}$. We can imagine $X = \mathbb{C}^2$ to be obtained from $\bar{X}$ by means of throwing out a curve $\sigma^{-1}(L_\infty)$ which consists of some quasisections and some components of fibres of the morphism $\bar{f}$. Analogously we can construct a completion $\bar{X}_n$. We begin with a hypersurface in $\mathbb{P}^3$ defined by the equation $x_3^n = x_0^m \bar{f}(x_0, x_1, x_2)$, where $\bar{f}(x_0, x_1, x_2)$ is a homogeneous polynomial of degree $\deg f$ associated with f, $m = n - \deg f$. Let $\widetilde{X}_n$ be a normalization of this surface and $\tilde{\varphi}_n : \widetilde{X}_n \to \mathbb{P}^2$ be induced by the morphism $\varphi_n : X_n \to X$. Resolving the singularities and the points of indeterminacy of morphisms we get a smooth surface $\bar{X}_n \supset X_n^0$, where $X_n^0 = X_n \setminus \mathrm{Sing} X_n$, and a commutative diagram

$$\begin{array}{ccccccccc} X'_n & \subset & X_n^0 & \subset & \bar{X}_n & \xrightarrow{\phi_n} & \bar{X} & \supset & X \\ \downarrow f'_n & & \downarrow f & & \downarrow \bar{f}_n & & \downarrow \bar{f} & & \downarrow f \\ S'_n & \subset & S_n & \subset & \bar{S}_n & \xrightarrow{\bar{\varepsilon}_n} & \bar{S} & \supset & S \end{array} \qquad (3)$$

8.2. Let us formulate the conditions which we impose on the curve D. We say that the *condition* (Irr_n) holds for a curve D if all the curves $B_i = \varphi_n^{-1}(D_i)$, $i = 1, \dots, k$, are irreducible.

Definition 12 *A curve D is connected modulo n, if the support of the divisor*

$$D_{mod\,n} = \sum_{m_i \not\equiv 0\, mod\, n} m_i D_i$$

is connected. D is absolutely connected modulo n , or shorter, satisfies the condition (C_n), if

$$D \text{ is connected modulo } n_1 \text{ for each } n_1,\ n_1 \mid n. \qquad (\dot{C}_n)$$

We need the condition (C_n) to get the following

Theorem 17 ([KK]). *If D satisfies the condition (C_n) , then*

$$H^0(X_n, \mathcal{O}^*_{X_n}) = \mathbb{C}.$$

This theorem affims that the regular and regular invertible functions on the affine variety X_n are only constants, i.e. the matter is the same as on $\mathbb{C}^2$. From this theorem follows that from the point of view of one-dimensional homology only the components $B_i = \varphi_n^{-1}(D_i)$, which lie on the normalization X_n^0, are essential for a compactification $X'_n \subset \bar{X}_n$.

Corollary 4 . *If a curve D satisfies the condition (C_n), then the inclusion $i : X_n^0 \subset \bar{X}_n$ induces an isomorphism*

$$i_* : H_1(X_n^0) \longrightarrow H_1(\bar{X}_n).$$

8.3. *The relation between homology of X'_n and $\bar{X}_n$* is given by

Theorem 18 ([KK]).

(i) If D satisfies the conditions (C_n) and (Irr_n) , then there is an exact sequence

$$0 \to \bigoplus_{i=1}^{k} \mathbb{C}\bar{\gamma}_i \to H_1(X'_n, \mathbb{C}) \xrightarrow{i_*} H_1(\bar{X}_n, \mathbb{C}) \to 0, \qquad (4_n)$$

where $i = i_n : X'_n \subset \bar{X}_n$ is an imbedding, and the cycle $\bar{\gamma}_i \in Ker\, i_*$ corresponds to "going arroud the component B_i" .

(ii) Moreover, if D satisfies the conditions ($C_{n(D)}$) and ($Irr_{n(D)}$) , then the sequence (4_n) is exact for all n.

The condition of Theorem 17 (i) holds if the support of the curve D is connected in $\mathbb{C}^2$ and $(m_i, n) = 1$ for each multiplicity m_i (in particular, if D is a reduced curve). If D is an irreducible curve, we get a generalization of the Libgober's result: $\dim Ker\, i_* = 1$ (the conditions at infinity are superfluous).

8.4. *The relation of $H_1(X_\infty)$ and $H_1(X'_n)$.* The proof of Theorem 17 (ii) is based on the application of the Milnor exact sequence (see section 1) to analyse the relation of $H_1(X_\infty)$ and $H_1(X'_n)$ for different n.

Consider the Milnor exact sequence for the infinite cyclic covering $\varphi_{\infty,n} : X_\infty \to X'_n$. If $G_n \subset \mathbb{F}_1$ is the infinite cyclic group generated by h^n, then $X'_n = X_\infty/G_n$ and $X' = X'_n/\mu_n$, where $\mu_n = \mathbb{F}_1/G_n$ is the cyclic group of order n. Denote by h_n the automorphism of X'_n induced by the monodromy h. Then h_n is the generator of the group μ_n which corresponds also to the generator of the Galois group $\mathrm{Gal}(k(X'_n)/k(X'))$.

Put together the Milnor exact sequence and the exact sequence (4_n)

$$\begin{array}{ccccccccc}
 & & & & \vdots & & & & \\
 & & & & \downarrow & & & & \\
 & & & & H_1(X_\infty) & & & & \\
 & & & & \downarrow{\scriptstyle h^n - id} & & & & \\
 & & & & H_1(X_\infty) & & & & \\
 & & & & \downarrow{\scriptstyle (\varphi_{\infty,n})_*} & & & & \\
0 & \longrightarrow & \oplus_{i=1}^{k}\mathbb{C}\bar{\gamma}_i & \longrightarrow & H_1(X'_n) & \xrightarrow{(i_n)_*} & H_1(\bar{X}_n) & \longrightarrow & 0 \\
 & & & & \downarrow & & & & \\
 & & & & H_0(X_\infty) & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array} \qquad (5_n)$$

The group $\mathbb{F}_1 = \mathbb{Z}$ with the generator $h = h_*$ acts on the spaces $H_1(X_\infty)$, $H_1(X'_n)$ and $H_1(\bar{X}_n)$ (on $H_1(X'_n)$ and $H_1(\bar{X}_n)$ the action is reduced to the group $\mu_n = \mathbb{F}_1/G_n$ with the generator h_n). Clearly, the homomorphisms $(\varphi_{\infty,n})_*$ and $(i_n)_*$ are equivariant.

We denote

$$H_1(X_\infty) = \oplus_i H_1(X_\infty)_{\lambda_i} = H_1(X_\infty)_{\lambda=1} \oplus H_1(X_\infty)_{\neq 1}$$

the root decomposition of the automorphism h, where $H_1(X_\infty)_{\neq 1} = \oplus_{\lambda\neq 1} H_1(X_\infty)_\lambda$. We shall apply the analogous notation for the root decomposition of the spaces $H_1(X'_n)$ and $H_1(\bar{X}_n)$ corresponding to the automorphism $h_n = (h_n)_*$.

The colomn in diagram (5_n) gives that $H_1(X'_n)$ is almost $Im(\varphi_{\infty,n})_* \subset H_1(X'_n)$, $Coker(\varphi_{\infty,n})_* \simeq H_0(X_\infty)$, $H_0(X_\infty) = \mathbb{C}$, and h acts trivially on $H_0(X_\infty)$. On the other hand, $Im(\varphi_{\infty,n})_* = Coker(h^n - id)$. We need an easy exercise in the linear algebra.

Lemma 1 . *Let h be an automorphism of a vector space $L/\mathbb{C}$. Then* *(i) If the Jordan decomposition of h has only one Jordan block with a eigenvalue $\lambda \neq 0$, then for $\forall n \in \mathbb{N}$ the automorphism h^n also has one Jordan block with the eigenvalue λ^n. This involves*

(ii) If h has no eigenvalues $\lambda = 0$, then the number and the dimensions of Jordan blocks for h^n are the same as for h.

(iii) If h has only one Jordan block with $\lambda \neq 1$, then $h - id : L \to L$ is an isomorphism, and if $\lambda = 1$, then $\dim Ker(h - id) = \dim L/(h\text{-}id)L = 1$.

Therefore, applying this lemma to h^n in the diagram (5_n), we get that, moving from $H_1(X'_n)$ to $H_1(X_\infty)$, the Jordan blocks of $H_1(X_\infty)$ with $\lambda^n \neq 1$ disappear, and every Jordan block with $\lambda^n = 1$ gives one eigenvector in the space $H_1(X'_n)$ with the same eigenvalue λ. Denote by $J = J(h)$ the number of Jordan blocks of the automorphism h, and by $J_1 = J_{\lambda=1}$, $J_n = J_{\lambda^n=1}$, $J_{\neq 1}$ the number of Jordan blocks with eigenvalues $\lambda = 1$, with eigenvalues λ such that $\lambda^n = 1$, with eigenvalues not equal to 1 correspondingly. Thus lemma involves that

$$\dim H_1(X'_n) = J_1(h^n) + 1 = J_n(h) + 1.$$

8.5. *The decomposition of $H_1(X'_n)$ into eigensubspaces.* Now consider the line in the diagram (5_n). On one hand, the fact that X_n/μ_n is a rational surface (and hence there are no invariant holomorphic forms on $\bar{X}_n$) involves that the operator h_n on $H_1(\bar{X}_n, \mathbb{C})$ has no eigenvalues $\lambda = 1$, $H_1(\bar{X}_n, \mathbb{C}) = H_1(\bar{X}_n, \mathbb{C})_{\neq 1}$. On the other hand, under the condition (Irr_n), i.e. if the curves B_i are irreducible, the cycles $\bar{\gamma}_i$, $i = 1, \ldots, k$, are invariant relative to h_n. This involves the proof of the Theorem 17, and we obtain that

$$H_1(X'_n)_1 = \oplus_{i=1}^{k} \mathbb{C} \cdot \bar{\gamma}_i, \qquad H_1(X'_n)_{\neq 1} \simeq H_1(\bar{X}_n).$$

This involves

$$\dim H_1(\bar{X}_n) = J_n - J_1 = J_{\neq 1}(D, n),$$

where $J_{\neq 1}(D,n)$ is the number of Jordan blocks of the monodromy h on $H_1(X_\infty, \mathbb{C})$ with eigenvalues $\lambda \neq 1$ for which $\lambda^n = 1$.

8.6. *Calculation of the Alexander polinomial of an irreducible curve in terms of cohomology of rational differential forms* ([Ko]). Let D be an irreducible curve ($k = 1$) of degree $deg\ D = d$. If D intersects the line at infinity transversally, then according to the Randell's theorem in section 6 the monodromy h is semisimple, and $\lambda^d = 1$. In the next section we obtain the theorem on the semisimplicity of h without the assumption of the transversality of intersection at infinity. Thus (8.4) and (8.5) involve that

$$H^1(X_\infty) \simeq H^1(X'_d)_{\neq 1}.$$

The surface $X_d \subset \mathbb{C}^3$ defined by the equation $z^n = f(x, y)$ is normal, because the curve D is reduced. Recall that $X' = \mathbb{C}^2 \setminus D$, $X'_d = X_d \setminus B$, where $B \subset X_d$ is the curve with the equation $z = 0$ and $\varphi_d : X'_d \to X'$ is an unramified covering of degree d.

We apply the Grothendiek's theorem to calculate the cohomology $H^1(X'_d, \mathbb{C})$. If X is a complex variety, $D \subset X$ is a hypersurface, $j : X' = X \setminus D \to X$ is the imbedding, then

$$H^{\cdot}(X', \mathbb{C}) = \mathbb{H}^{\cdot}(\Omega^{\cdot}_{X'}) = \mathbb{H}^{\cdot}(j_* \Omega^{\cdot}_{X'}) = \mathbb{H}^{\cdot}(\Omega^{\cdot}_X(*D)),$$

where $\Omega^{\cdot}_X(*D)$ is de Rham's complex of meromorphic forms with poles along D. If X is an affine variety, then

$$H^{\cdot}(X', \mathbb{C}) = H^{\cdot}(\Gamma(\Omega^{\cdot}_X(*D)).$$

Apply this to $X = \mathbb{C}^2$ and $X = X_d$. We obtain

$$H^{\cdot}(X'_d, \mathbb{C}) = H^{\cdot}(\Gamma(\Omega^{\cdot}_{X_d}(*B)).$$

Let $h = h_d$ be a generator of the group $\mathrm{Aut}_{X'}X'_d$, $h^d = 1$. Let $\zeta = e^{2\pi i/d}$ be a root of unity of degree d and

$$H^{\cdot}(X'_d, \mathbb{C}) = \oplus_{j=0}^{d-1} H^1(X'_d, \mathbb{C})_{\zeta^j}$$

be the decomposition into eigensubspaces. The monodromy h acts semisimply also on the differential forms on X'_d. The part corresponding to the eigenvalue ζ^j is equal to

$$\Omega^{\cdot}_{X_d}(*B)_j = \Omega^{\cdot}_X(*D)z^j.$$

Since $\frac{df}{f} = \frac{d(z^d)}{z^d} = d\frac{dz}{z}$, we have

$$d(z^j\omega) = jz^{j-1}dz \wedge \omega + z^j d\omega = (\frac{j}{d}\frac{df}{f} \wedge \omega + d\omega)z^j.$$

So the multiplication by $z^j = f^{j/d}$ defines an isomorphism of complexes

$$\begin{array}{ccc} \ldots \to \Omega^p_{X_d}(*B)_j & \xrightarrow{d} & \Omega^{p+1}_{X_d}(*B)_j \to \ldots \\ \uparrow z^j & & \uparrow z^j \\ \ldots \to \Omega^p_X(*D) & \xrightarrow{\nabla_j} & \Omega^{p+1}_X(*D) \to \ldots \end{array}$$

where ∇_j is a regular connection in $\Omega^{\cdot}_X(*D)$ defined by the formula

$$\nabla_j(\omega) = d\omega + \frac{j}{d}\frac{df}{f} \wedge \omega.$$

Therefore,

$$H^1(X'_d, \mathbb{C})_{\zeta^j} = H^1(\Gamma(\Omega^{\cdot}_{X_d}(*B)_j, d) = H^1(\Gamma(\Omega^{\cdot}_X(*D), \nabla_j)$$

and we obtain Kohno's theorem.

Theorem 19 ([*Ko*]). *If $D \subset \mathbb{C}^2$ is an irreducible curve of degree d transversally intersecting the line at infinity, then the Alexander polynomial is*

$$\Delta_D(t) = \prod_{1 \le j \le d-1} (t - \zeta^j)^{h_j},$$

where $h_j = \dim_{\mathbb{C}} H^1(\Gamma(\Omega^{\cdot}_{\mathbb{C}^2}(*D), \nabla_j)$.

In virtue of the theorem on the simplicity of monodromy in the next section we obtain the generalization of Kohno's theorem:

Theorem 18'

i). The Kohno's theorem is true without the condition of transversality at infinity;

ii). if D is a connected reduced curve, then

$$\Delta_D(t) = (t-1)^l \prod_{1\leq j\leq d-1} (t-\zeta^j)^{h_j}.$$

9 The semisimplicity of the monodromy

9.1. If a curve $D \subset \mathbb{C}^2$ is reduced and $\bar{D}$ transversally intersects the line at infinity, then the monodromy h on $H_1(X_\infty)$ is semisimple, because it coincides with the monodromy of a quasihomogeneous singularity (see section 6). We generalize it to the case of nonreduced curves without any conditions at infinity, and use quite different ideas based on the Milnor exact sequence and the theory of mixed Hodge structures.

Theorem 20 ([KK]). *If a curve D satisfies the conditions $(C_{n(D)})$ and $(Irr_{n(D)})$, then the monodromy h on $H_1(X_\infty)_{\neq 1}$ is semisimple.*

We sketch the proof of this Theorem. We have to prove that the Jordan blocks of the automorphism h on $H_1(X_\infty)$ with eigenvalues $\lambda \neq 1$ are one-dimensional. We compare the monodromies on homology of X_∞ and on homology of a nonsingular fibre Y. Let $Y = X_t = f^{-1}(t)$, correspondingly $\bar{Y} = \bar{X}_t = \bar{f}^{-1}(t)$, be a nonsingular fibre (close to X_0) of the morphism f, correspondingly $\bar{f}$, in the diagram (3). The morphisms φ_n and φ_∞ are unramified coverings and the fibers $\varphi_n^{-1}(Y)$ and $\varphi_\infty^{-1}(Y)$ break up into components isomorphic to Y. Choosing points $u \in e_\infty^{-1}(t)$ and $\tilde{t} \in e_n^{-1}(t)$ we can assume that Y is embedded into X_∞ and X_n' as fibres $f_\infty^{-1}(u)$ and $f_n^{-1}(\tilde{t})$ such that the diagram

$$\begin{array}{ccccc} Y & = & Y & \hookrightarrow & \bar{Y} \\ \downarrow{\scriptstyle j_\infty} & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle \bar{j}} \\ X_\infty & \xrightarrow[\phi_{\infty,n}]{} & X_n' & \underset{i}{\hookrightarrow} & \bar{X}_n. \end{array}$$

is commutative. We get a commutative diagram for homology

$$\begin{array}{ccccc} H_1(Y) & = & H_1(Y) & \longrightarrow & H_1(\bar{Y}) \\ \downarrow{\scriptstyle (j_\infty)_*} & & \downarrow{\scriptstyle j_*} & & \downarrow{\scriptstyle \bar{j}_*} \\ H_1(X_\infty) & \longrightarrow & H_1(X_n') & \underset{i_*}{\longrightarrow} & H_1(\bar{X}_n). \end{array}$$

The monodromy operator acts on the spaces $H_1(X_\infty)$, $H_1(X_n')$, $H_1(Y)$ and $H_1(\bar{Y})$ and the homomorphisms in the above diagram are equivariant, i.e. commute with the action of the monodromy operators. The part of the diagram corresponding to the eigenvalues $\lambda \neq 1$ is the following

$$
\begin{array}{ccccc}
 & & H_1(Y)_{\neq 1} & \longrightarrow & H_1(\bar{Y}) \\
 & \swarrow^{(j_\infty)_*} & \downarrow & & \downarrow \bar{j}_* \\
H_1(X_\infty)_{\neq 1} & \xrightarrow[(\phi_{\infty,n})_*]{} & H_1(X'_n)_{\neq 1} & \xrightarrow[i_*]{\sim} & H_1(\bar{X}_n).
\end{array}
$$

The key place of the proof is the following. On one hand, in virtue of (8.4) every Jordan block of the automorphism h on $H_1(X_\infty)_{\neq 1}$ (i.e. the invariant subspace L on which h consists of one Jordan block) gives in $H_1(\bar{X}_n)$ one non zero vector, $\dim i_* \cdot (\varphi_{\infty,n})_* L = 1$. On the other hand, a two-dimensional block L can be obtained from a two-dimensional block L in $H_1(Y)$. At last the theory of mixed Hodge structures yieds that we can choose L in such a way that $j_*(L) = 0$ in $H_1(X_n)$. This shows that the blocks in $H_1(X_\infty)_{\neq 1}$ can be only one-dimensional.

9.2. If D is an irreducible curve, then $\Delta_D(1) = \pm 1$ [K1]. Hence from Theorem 20 we obtain the following theorem.

Theorem 21 *The monodromy h on $H_1(X_\infty)$ is semisimple for an irreducible curve D.*

Remark 4 The analogous statement is not true for knots. For example, for the knot 8_{10} the monodromy h on $H_1(X_\infty)$ isn't semisimple.

Proposition 5 *Let $D = m_1 D_1 + D'$ be a curve satisfying the following conditions:*

i) D_1 is irreducible and $D_1 \not\subset supp(D')$,

ii) there exists a point $x \in D_1 \cap supp(D')$ at which the divisor $D_1 + D'_{red}$ is locally a divisor with normal crossings,

iii) for the curve D' the monodromy h on $H_1(X_\infty)_1$ is semisimple.

Then for the curve D the monodromy h on $H_1(X_\infty)_1$ is semisimple.

We sketch the proof of this proposition. From the homological Milnor exact sequence it follows that one needs to show that the Alexander polynomial $\Delta_D(t)$ of the curve D satisfies the condition: $\Delta_D(t) = (t-1)^{k-1} \cdot \Delta'(t)$, where Δ' is a polynomial such that $\Delta'(1) \neq 0$, and k is the number of irreducible components of D. The straightforward calculations of the Alexander polynomial (as in [K1] and [K2]), using Fox's free calculus, show that the polynomial $\Delta_D(t)$ possesses the required property under the conditions of the proposition.

As a consequence of this proposition and Theorem 20 we obtain the following theorem.

Theorem 22 *Let $D = m_1 D_1 + ... + m_k D_k$ be a curve satisfying the conditions of Theorem 20 and such that for $i = 1, ..., k-1$, there exists a point $x_i \in D_{i+1} \cap (\cup_{j=1}^{i} D_j)$ such that the curve $D^{(i+1)} = D_1 + ... + D_{i+1}$ is locally a divisor with normal crossings at x_i. Then for the curve D the monodromy h on $H_1(X_\infty)$ is semisimple.*

Conjecture. *If a curve D satisfies the conditions $(C_{n(D)})$ and $(Irr_{n(D)})$, then the monodromy h on $H_1(X_\infty)$ is semisimple.*

10 On the mixed Hodge structure on $H^1(X_\infty)$

The construction of X_∞ for $X' = \mathbb{C}^2 \setminus D$ is analogous to the construction of a canonical fibre for a family of nonsingular projective varieties over the punctured disk S' or for the Milnor fibration of a hypersurface singularity, but in our case the fibration $f' : X' \to S'$ is not locally trivial. In these cases there is a limit MHS (W.Schmid, J.Steenbrink). We want to introduce a MHS on $H^1(X_\infty, \mathbb{Q})$ such that the homomorphisms

$$(\varphi_{\infty,n})^* : H^1(X'_n) \to H^1(X_\infty)$$

are MHS morphisms.

The surface X'_n is a nonsingular algebraic variety and so there is the MHS on $H^1(X'_n)$ introduced by Deligne [G-S]. Remind that we must take a nonsingular projective variety $\bar{X}_n \supset X'_n$ such that $\bar{X}_n \setminus X'_n = \bar{D}$ is a divisor with normal crossings. The weight filtration $W.$ defines spectral sequence which involves an exact MHS sequence

$$0 \to H^1(\bar{X}_n) \to H^1(X'_n) \to H^0(\bar{D}^{(1)}) \to H^2(\bar{X}_n),$$

where there is a pure Hodge structure of weight 1 on $H^1(\bar{X}_n)$, $\bar{D}^{(1)}$ is a disjoint union of the components of $\bar{D}$ and there is a pure Hodge structure of weight 2 and type (1,1) on $H^0(\bar{D}^{(1)})$. In our case by the Theorem 17 this exact sequence is the exact sequence

$$0 \to H^1(\bar{X}_n) \to H^1(X'_n) \to \oplus_{i=1}^{k} \mathbb{C} \cdot \bar{\gamma}_i^* \to 0 \qquad (\tilde{4}_n)$$

dual to the sequence (4_n). So if $W.$ is the weight filtration on $H^1(X'_n)$, $H^1(X'_n) = W_2 \supset W_1 \supset 0$, then $W_1 = H^1(\bar{X}_n)$ and $Gr_2^W = W_2/W_1 = \oplus \mathbb{C} \cdot \bar{\gamma}_i^*$ and the pure Hodge structure on Gr_2^W is of type (1,1).

In our case the cyclic group $\mu_n = \mathbb{Z}/n\mathbb{Z}$, generated by h_n^*, acts on $H^1(X'_n)$. The monodromy h_n^* is a MHS isomorphism since h_n is an isomorphism of the algebraic variety X'_n. From section (8.5) we have

$$H^1(\bar{X}_n) = H^1(X'_n)_{\neq 1}, \quad H^1(X'_n)_1 \simeq \oplus_{i=1}^{k} \mathbb{C} \cdot \bar{\gamma}_i^*.$$

Hence the MHS on $H^1(X'_n)$ splits and is the direct sum of pure Hodge structures of weight 1 on $H^1(\bar{X}_n)$ and of weight 2 and type (1,1) on $\oplus_{i=1}^{k} \mathbb{C} \cdot \bar{\gamma}_i^*$.

Consider the diagram dual to the diagram (5_n)

$$
\begin{array}{ccccccc}
 & & \vdots & & & & \\
 & & \uparrow & & & & \\
 & & H^1(X_\infty) & & & & \\
 & & \uparrow (h^*)^n - id & & & & \\
 & & H^1(X_\infty) & & & & \\
 & & \uparrow (\phi_{\infty,n})^* & & & & \\
0 \longrightarrow & H^1(\bar{X}_n) \longrightarrow & H^1(X'_n) & \longrightarrow & \oplus_{i=1}^{k} \mathbb{C} \cdot \bar{\gamma}_i^* \longrightarrow 0 & & (\tilde{5}_n) \\
 & & \uparrow & & & & \\
 & & H^0(X_\infty) & & & & \\
 & & \uparrow & & & & \\
 & & 0 & & & &
\end{array}
$$

By the Theorem 20 if $n(D) \mid n$, then

$$H^1(X'_n)_{\neq 1} \stackrel{(\varphi_{\infty,n})^*}{\longrightarrow} H^1(X_\infty)_{\neq 1}.$$

is an isomorphism. So if we introduce the MHS on $H^1(X_\infty)$ as a direct sum of the pure Hodge structure of weight 1 on $H^1(X_\infty)_{\neq 1}$, obtained by the isomorphism $(\varphi_{\infty,n(D)})^*$, and the pure Hodge structure of weight 2 and type (1,1) on $H^1(X_\infty)_1$, then we obtain the desired MHS.

Theorem 23 ([KK]). *If a curve D satisfies the conditions $(C_{n(D)})$ and $(Irr_{n(D)})$, then there exists a natural mixed Hodge structure on $H^1(X_\infty, \mathbb{Q})$ such that the homomorphisms $\varphi^*_{\infty,n}$: $H^1(X'_n, \mathbb{Q}) \to H^1(X_\infty, \mathbb{Q})$ are MHS morphisms. If a curve D is irreducible, then the MHS on $H^1(X_\infty, \mathbb{Q})$ is pure.*

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