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NEW EXAMPLES OF CYLINDRICAL FANO FOURFOLDS

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Abstract. We produce new families of smooth Fano fourfolds with Picard rank 1, which contain cylinders, i.e., Zariski open subsets of form $Z \times \mathbb{A}^1$, where Z is a quasiprojective variety. The affine cones over such a fourfold admit effective $\mathbb{G}_{\mathrm{a}}$-actions. Similar constructions of cylindrical Fano threefolds and fourfolds were done previously in [KPZ11, KPZ14, PZ15].

1. Introduction

All varieties in this paper are algebraic and are defined over $\mathbb{C}$. A smooth projective variety V is called *cylindrical* if it contains a cylinder, i.e., a principal Zariski open subset U isomorphic to a product $Z \times \mathbb{A}^1$, where Z is a variety and $\mathbb{A}^1$ stands for the affine line ([KPZ11, KPZ13]).

Provided that $\operatorname{rk}\operatorname{Pic}(V) = 1$, the affine cone over V admits an effective action of the additive group $\mathbb{G}_{\mathrm{a}}$ if and only if V is cylindrical; see [KPZ13, Cor. 3.2]. Furthermore, the existence of a $\mathbb{G}_{\mathrm{a}}$-action on the affine cone over V implies that V is uniruled. Since $\operatorname{rk}\operatorname{Pic}(V) = 1$, V is a Fano variety.

In [KPZ11, KPZ14, PZ15] several families of smooth cylindrical Fano threefolds and fourfolds with Picard number 1 were constructed. Here we provide further examples of such fourfolds. Let us recall the standard terminology and notation.

1.1. **Notation.** Given a smooth Fano fourfolds V with Picard rank 1, the *index* of V is the integer r such that $-K_V = rH$, where H is the ample divisor generating the Picard group: $\operatorname{Pic}(V) = \mathbb{Z} \cdot H$ (by abuse of notation, we denote by the same letter a divisor and its class in the Picard group). The *degree* $d = \deg V$ is the degree with respect

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to H. It is known that $1 \leq r \leq 5$. Moreover, if $r = 5$ then $V \cong \mathbb{P}^4$, and if $r = 4$ then V is a quadric in $\mathbb{P}^5$. Smooth Fano fourfolds of index $r = 3$ are called *del Pezzo fourfolds*; their degrees vary in the range $1 \leq d \leq 5$ ([Fuj80]-[Fuj81]). Smooth Fano fourfolds of index $r = 2$ are called *Mukai fourfolds*; their degrees are even and can be written as $d = 2g - 2$, where g is called the *genus* of V. The genera of Mukai fourfolds satisfy $2 \leq g \leq 10$ ([Muk89]). Up to now, there is no classification of Fano fourfolds of index $r = 1$.

According to [PZ15, Thm. 0.1] a smooth intersection of two quadrics in $\mathbb{P}^6$ is a cylindrical del Pezzo fourfold of degree 4. A smooth del Pezzo fourfold $W = W_5 \subset \mathbb{P}^7$ of degree 5 is also cylindrical (*ibid.*).

1.2. **On the content.** Starting with the del Pezzo quintic fourfold W and performing suitable Sarkisov links we constructed in [PZ15] two families of cylindrical Mukai fourfolds V_{12} of genus 7 and V_{14} of genus 8. Proceeding in a similar fashion, in the present paper we produce two more families of cylindrical Mukai fourfolds V_{16} of genus 9 and V_{18} of genus 10, see Theorem 2.1 and Corollary 2.4. These are the main results of the paper.

The paper is divided into 6 sections. After formulating in Section 2 our principal results, we give in Section 3 necessary preliminaries. In particular, we recall some useful facts from [PZ15]. In Section 5 we prove Theorem 2.1 about the existence of suitable Sarkisov links. This theorem depends on the existence of certain specific surfaces in the quintic fourfold W. Section 4 is devoted to constructions of such surfaces, see Proposition 4.1. The resulting Mukai fourfolds V_{16} and V_{18} occur to be cylindrical, with a cylinder coming from a one on W via the corresponding Sarkisov link, see Corollary 2.4. Section 6 contains concluding remarks and some open problems.

2. Main results

The following theorem describes the Sarkisov links used in our constructions.

Theorem 2.1. *Let $W = W_5 \subset \mathbb{P}^7$ be a del Pezzo fourfold of degree 5, and let $F \subset W \cap \mathbb{P}^6$ be a smooth surface of one of the following types:*

a) *$F \subset \mathbb{P}^6$ is a rational normal quintic scroll, $F \cong \mathbb{F}_1$, with $c_2(W) \cdot F = 22$, and*
b) *$F \subset \mathbb{P}^6$ is an anticanonically embedded sextic del Pezzo surface with $c_2(W) \cdot F = 26$ (see Lemma 3.5).*

Suppose that F does not intersect any plane in W along a (possibly, degenerate) conic. Then there is a commutative diagram

$$\begin{tikzcd}
 & D \arrow[rr, hook] \arrow[dl] & & \widetilde{W} \arrow[dl, "\rho"'] \arrow[dr, "\varphi"] & & \tilde{E} \arrow[ll, hook'] \arrow[dr] & \\
F \arrow[rr, hook] & & W \arrow[rr, dashed, "\phi"] & & V & & S \arrow[ll, hook']
\end{tikzcd} \tag{2.1}$$

where

- $V = V_{2g-2} \subset \mathbb{P}^{g+2}$ *is a Mukai fourfold of genus $g = 10$ in case* a) *and $g = 9$ in case* b)*;*
- *the map $\phi : W \dashrightarrow V \subset \mathbb{P}^{g+2}$ is given by the linear system of quadrics passing through F, while $\phi^{-1} : V \dashrightarrow W$ is the projection from the linear span $\langle S \rangle$ of S.*

Furthermore,

(i) *$\rho : \widetilde{W} \longrightarrow W$ is the blowup of F with exceptional divisor D, and $\varphi : \widetilde{W} \longrightarrow V$ is the blowup of a smooth surface $S \subset V$ with exceptional divisor $\tilde{E}$, where*
   - *in case* a) *$S \subset \mathbb{P}^4 \subset \mathbb{P}^{12}$ is a normal cubic scroll with $c_2(V) \cdot S = 7$, and*
   - *in case* b) *$S \subset \mathbb{P}^3 \subset \mathbb{P}^{11}$ is a quadric with $c_2(V) \cdot S = 5$;*

(ii) *if H is an ample generator of* $\mathrm{Pic}(W)$ *and L is an ample generator of* $\mathrm{Pic}(V)$*, then on $\widetilde{W}$ we have*

$$\begin{aligned} &\rho^* H \equiv \varphi^* L - \tilde{E}, \quad D \equiv \varphi^* L - 2\tilde{E}, \\ &\varphi^* L \equiv 2\rho^* H - D, \quad \tilde{E} \equiv \rho^* H - D. \end{aligned} \tag{2.2}$$

The proof is done in Section 5. In Section 4 we establish the existence of surfaces F as in Theorem 2.1.

Using this theorem, we deduce our main results.

Theorem 2.2. *Under the assumptions of Theorem 2.1, there is an isomorphism*

$$V \setminus \varphi(D) \cong W \setminus \rho(\tilde{E}),$$

where $\varphi(D)$ is a hyperplane section of $V = V_{2g-2} \subset \mathbb{P}^{g+2}$ singular along $S = \varphi(\tilde{E})$, and $\rho(\tilde{E}) = W \cap \langle F \rangle$ is a singular hyperplane section of $W = W_5 \subset \mathbb{P}^7$ by the linear span of F.

Recall the following fact ([PZ15, Thm. 4.1]).

Theorem 2.3. *For any hyperplane section M of W, the complement $W \setminus M$ contains a cylinder.*

Corollary 2.4. *Any Fano fourfold V as in Theorem 2.1 is cylindrical.*

Proof. Since $M = \rho(\tilde{E})$ is a hyperplane section of W, the complement $W \setminus \rho(\tilde{E})$ contains a cylinder. Hence also $V \setminus \varphi(D) \cong W \setminus \rho(\tilde{E})$ does, and so, V is cylindrical. □

3. Preliminaries

3.1. Recall the following notation, see e.g. [PZ15, §3]. There are two types of planes in the Grassmannian $\mathrm{Gr}(2,5)$, namely, the Schubert varieties $\sigma_{3,1}$ and $\sigma_{2,2}$ ([GH78, Ch. 1, §5]), where

- $\sigma_{3,1} = \{l \in \mathrm{Gr}(2,5) \mid p \in l \subset h\}$ with $h \subset \mathbb{P}^4$ a fixed hyperplane and $p \in h$ a fixed point;
- $\sigma_{2,2} = \{l \in \mathrm{Gr}(2,5) \mid l \subset e\}$ with $e \subset \mathbb{P}^4$ a fixed plane.

In the terminology of [Fuj81, §10], the $\sigma_{3,1}$-planes (the $\sigma_{2,2}$-planes, respectively) are called planes of *vertex type* (of *non-vertex type*, respectively).

3.2. Let $W = W_5 \subset \mathbb{P}^7$ be a del Pezzo fourfold of index 3 and degree 5. Due to [Fuj81] such a variety is unique up to isomorphism and can be realized as a section of $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$ by two general hyperplanes. By the Lefschetz hyperplane section theorem $\mathrm{Pic}(W) \cong \mathbb{Z}$. We have $-K_W = 3H$, where H is the ample generator of $\mathrm{Pic}(W)$. The variety W is an intersection of quadrics (see [GH78, Ch. 1, §5]).

The following proposition proven in [Tod30] (see also [Fuj86, Sect. 2]) deals with the planes in the fourfold $W = W_5$.

Proposition 3.3. *Let $W = W_5 \subset \mathbb{P}^7$ be a Fano fourfold of index 3 and degree 5. Then the following hold.*

(i) *W contains a unique $\sigma_{2,2}$-plane Ξ, a one-parameter family (Π_t) of $\sigma_{3,1}$-planes, and no further plane.*
(ii) *Any $\sigma_{3,1}$-plane Π meets Ξ along a tangent line to a fixed conic $\delta \subset \Xi$.*
(iii) *Any two $\sigma_{3,1}$-planes Π' and Π'' meet at a point $p \subset \Xi \setminus \delta$.*
(iv) *Let R be the union of all $\sigma_{3,1}$-planes on W. Then R is a hyperplane section of W and $\mathrm{Sing}\, R = \Xi$.*
(v) *There is a 1-parameter family of lines in W through each point in W. A line $l \subset W$ meets the plane Ξ if and only if $l \subset R$, and then l is contained in a plane in R.*

By abuse of notation, the cohomology class associated with an algebraic subvariety will be denoted by the same letter as the subvariety itself. By the Lefschetz hyperplane section theorem, the group $H^4(W, \mathbb{Z})$ is torsion free, since the group $H^4(\mathrm{Gr}(2,5), \mathbb{Z})$ is. In the next lemma we describe a natural basis in $H^4(W, \mathbb{Z})$, see [PZ15, Cor. 4.2 and 4.7].

Lemma 3.4. *The group $H^4(W,\mathbb{Z})$ is freely generated by the classes of the planes Ξ and Π, where*

$$\Pi^2 = 1, \quad \Xi^2 = 2, \quad \Pi\cdot\Xi = -1, \quad \text{and} \quad c_2(W) = 9\Xi + 13\Pi\,. \tag{3.1}$$

Lemma 3.5. a) *Let $F \subset W \cap \mathbb{P}^6$ be a smooth rational quintic scroll. Then*

$$F \equiv 2\Xi + 3\Pi \quad \text{and} \quad F\cdot\Xi = 1,\ F\cdot\Pi = 1,\ c_2(W)\cdot F = 22\,.$$

b) *Let $F \subset W \cap \mathbb{P}^6$ be a smooth anticanonically embedded sextic del Pezzo surface. Then either*

b1) *$F \equiv 2\Xi + 4\Pi$ and $F\cdot\Xi = 0$, $F\cdot\Pi = 2$, $c_2(W)\cdot F = 26$, or*

b2) *$F \equiv 3\Xi + 3\Pi$ and $F\cdot\Xi = 3$, $F\cdot\Pi = 0$, $c_2(W)\cdot F = 27$.*

Proof. By Lemma 3.4 one can write $F \equiv a\Xi + b\Pi$, where

$$a + b = \deg F, \qquad c_2(W)\cdot F = 5a + 4b. \tag{3.2}$$

From the exact sequence

$$0 \longrightarrow \mathscr{T}_F \longrightarrow \mathscr{T}_W \longrightarrow \mathscr{N}_{F/W} \longrightarrow 0$$

we deduce

$$c_1(W)|_F = c_1(F) + c_1(\mathscr{N}_{F/W}) \tag{3.3}$$

and

$$\begin{aligned} c_2(W)\cdot F &= c_2(F) + c_1(F)\cdot c_1(\mathscr{N}_{F/W}) + c_2(\mathscr{N}_{F/W}) \\ &= c_2(F) - c_1(F)^2 + c_1(F)\cdot c_1(W)|_F + c_2(\mathscr{N}_{F/W})\,. \end{aligned} \tag{3.4}$$

The Noether formula for the rational surface F can be written as follows:

$$c_2(F) - c_1(F)^2 = 2c_2(F) - 12\,.$$

Note that

$$c_2(\mathscr{N}_{F/W}) = F^2 = 2a^2 + b^2 - 2ab\,.$$

Since $c_1(W) = \mathcal{O}_W(3)$, from (3.2) and (3.4) we obtain

$$c_2(W)\cdot F = 5a + 4b = 2c_2(F) - 12 + c_1(F)\cdot\mathcal{O}_W(3)|_F + 2a^2 + b^2 - 2ab\,.$$

In case a) using (3.2) and the latter equality we get $b = 5 - a$ and

$$c_2(W)\cdot F \;=\; 20 + a = 5a^2 - 20a + 42\,, \quad \text{hence} \quad a = 2\,.$$

Similarly, in case b) we have $b = 6 - a$ and

$$c_2(W)\cdot F = 24 + a = 5a^2 - 24a + 54\,, \quad \text{hence} \quad a \in \{2, 3\}\,.$$

Now the assertions follow. $\square$

Remark 3.6. For a surface F as in Lemma 3.5 we have $\dim\langle F\rangle = 6$. Hence F is contained in a unique hyperplane section $\langle F\rangle \cap W \subset \mathbb{P}^7$.

4. Construction of quintic and sextic surfaces $F \subset W$

In this section we prove the existence of surfaces F satisfying the assumptions of Theorem 2.1. Our main results can be stated as follows.

Proposition 4.1. *The quintic fourfold $W \subset \mathbb{P}^7$ admits hyperplane sections which contain*

a) *a rational quintic scroll $F = F_5 \subset \mathbb{P}^6$,*

and other ones which contain

b) *an anticanonically embedded sextic del Pezzo surface $F = F_6 \subset \mathbb{P}^6$ of type* b)-b1).

In both cases, the surface F can be chosen so that none of the planes in W meets F along a (possibly, degenerate) conic.

Proof of Proposition 4.1 b). We start with a smooth sextic del Pezzo threefold $X = X_6 \subset \mathbb{P}^7$. Up to isomorphism, there is a unique such threefold X with $\mathrm{rk}\,\mathrm{Pic}\,X = 2$ ([Fuj80], [IP99]). In fact, the latter is the threefold which parametrizes the complete flags in $\mathbb{P}^2$. Consider the following diagram ([Pro13, §8]):

$$\begin{tikzcd} & \tilde{X} \arrow[dl] \arrow[dr] & \\ X \arrow[rr, dashed] & & U \subset \mathbb{P}^6 \end{tikzcd}$$

where $U = U_5 \subset \mathbb{P}^6$ is a quintic del Pezzo threefold with two nodes (ordinary double points), $X \dashrightarrow U = U_5 \subset \mathbb{P}^6$ is the projection from a general point $P \in X$, and $\tilde{X} \to X$ is the blowup of P. Recall that X can be realized as a smooth divisor of bidegree $(1,1)$ in $\mathbb{P}^2 \times \mathbb{P}^2$ (see, e.g., [Fuj80], [IP99]). The natural projections $\mathrm{pr}_1, \mathrm{pr}_2 : X \to \mathbb{P}^2$ define $\mathbb{P}^1$-bundles with total space X. Let l_i, $i = 1, 2$, be the corresponding fibers passing through P. Then l_1, l_2 are contracted to the nodes $P_1, P_2 \in U$. The threefold U contains a unique plane $\mathcal{P}$, and this plane is the image of the exceptional divisor of $\tilde{X} \to X$ ([Pro13, §8]).

The intersection Z of X with a general divisor of bidegree $(1,1)$ in $\mathbb{P}^2 \times \mathbb{P}^2$ is a smooth sextic del Pezzo surface $Z \cong Z_6 \subset \mathbb{P}^6$. We can choose Z so that $P \notin Z$. Let $F \subset U$ be the image of Z. Then $F = F_6 \subset W \cap \mathbb{P}^6$ is an anticanonically embedded smooth sextic del Pezzo surface, and $F \cap \mathcal{P} = \{P_1, P_2\}$.

Note that the del Pezzo quintic threefold $U = U_5 \subset \mathbb{P}^6$ with two nodes as above is unique up to isomorphism (see [Pro13, Corollary 8.7, Corollary 8.2]). On the other hand, such a variety can be obtained as a section of $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$ by a general hyperplane Λ and two general Schubert subvarieties Σ_1, Σ_2 of codimension one in $\mathrm{Gr}(2,5)$ (see [Tod30, 3.18, 3.3]). Letting Σ' be a general linear combination of Σ_1 and Σ_2, the section of $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$ by Λ and Σ' is smooth. Therefore, this section is a del Pezzo fourfold $W = W_5 \subset \mathbb{P}^7$. By construction, W contains F and $\mathcal{P}$. Since $F \cdot \mathcal{P} = 2$ in W, it follows that F is of type b)-b1), see (2) in Lemma 3.4 and Lemma 3.5.

Since U contains a unique plane $\mathcal{P}$, and F meets $\mathcal{P}$ just in two points and not along a conic, F satisfies the last condition of Proposition 4.1. Indeed, it is easily seen that $U = W \cap \langle F \rangle$. If $\mathcal{T}$ is a plane, which meets F along a conic, then $\mathcal{T}$ is contained in U. So, $\mathcal{T} = \mathcal{P}$ due to the uniqueness of $\mathcal{P}$. The latter equality leads to a contradiction, since $\mathcal{P} \cap F$ is not a conic. $\square$

To show Proposition 4.1 a) we need to recall Proposition 4.11 in [PZ15]. It describes a construction (borrowed in [Fuj81, Sect. 10] and [Pro94]), which allows to recover the fourfold W via a Sarkisov link starting with a certain 2-dimensional cubic scroll S in $\mathbb{P}^5$ contained in a smooth quadric Q^4.

Proposition 4.2. *Let as before $W = W_5 \subset \mathbb{P}^7$ be a del Pezzo quintic fourfold, and let $l \subset W$ be a line, which is not contained in any plane in W, that is, $l \not\subset R$. Then there is a commutative diagram*

$$(4.1)\qquad \begin{tikzcd}
& \hat{D} \arrow[dl] \arrow[rr, hook] & & \widehat{W} \arrow[dl, "\hat\rho"'] \arrow[dr, "\hat\varphi"] & & \hat{E} \arrow[ll, hook'] \arrow[dr] & \\
l \arrow[rr, hook] & & W \arrow[rr, dashed, "\hat\phi"] & & Q^4 & & S \arrow[ll, hook']
\end{tikzcd}$$

where

(i) *$\hat\rho : \widehat{W} \longrightarrow W$ is the blowup of l, $\hat\phi : W \dashrightarrow \mathbb{P}^5$ is the projection from l, $Q^4 = \hat\phi(\widehat{W}) \subset \mathbb{P}^5$ is a smooth quadric, and $\hat\varphi : \widehat{W} \longrightarrow Q^4$ is the blowup of a cubic scroll $S \subset Q^4 \subset \mathbb{P}^5$ with exceptional divisor $\hat{E}$;*

(ii) *the morphism $\hat\rho : \widehat{W} \longrightarrow W \subset \mathbb{P}^7$ is defined by the linear system $|\hat\rho^* H - \hat{D}|$, where $H \subset W$ is a hyperplane section and $\hat{D} = \hat\rho^{-1}(l)$ is the exceptional divisor of $\hat\rho$;*

(iii) *$\hat\varphi(\hat{D}) = Q^4 \cap \langle S \rangle$ is a quadric cone, where $\langle S \rangle \cong \mathbb{P}^4$ is the linear span of S in $\mathbb{P}^5$;*

(iv) *the image $\hat\rho(\hat{E}) \subset W$ is a hyperplane section of W singular along l and swept out by lines in W meeting l;*

(v) *for a hyperplane section $\mathcal{L}$ of Q^4 we have on $\widehat{W}$*

$$\hat\varphi^* \mathcal{L} \sim \hat{D} - \hat{E} \quad \text{and} \quad \hat\rho^* H \sim \hat{D} - 2\hat{E} \sim \hat\varphi^* \mathcal{L} - \hat{E} \sim 2\hat\varphi^* \mathcal{L} - \hat{D}\,.$$

Conversely, given a pair (Q^4, S), where $Q^4 \subset \mathbb{P}^5$ is a smooth quadric fourfold and $S \subset Q^4$is a cubic scroll in $\mathbb{P}^5$ such that the hyperplane section $Q^4 \cap \langle S \rangle$ is a quadric cone, one can recover the quintic fourfold W together with diagram (4.1)*) satisfying* (i)-(v)*.*

To construct surfaces $F \subset W$ as in Proposition 4.1 a) we use the following Lemmas 4.3-4.4.

Lemma 4.3. *Consider a quadric cone threefold $Q^3 \subset \mathbb{P}^4$ with a zero-dimensional vertex P, a smooth hyperplane section $Q^2 = Q^3 \cap \mathcal{H}$, where $Q^2 \cong \mathbb{P}^1 \times \mathbb{P}^1$, and a smooth conic $C \subset Q^2$. Consider also a plane $\mathcal{T} \subset Q^3$, $\mathcal{T} \cong \mathbb{P}^2$, and a general quadric $Q^{\bullet 3} \subset \mathbb{P}^4$ which contains $\mathcal{T} \cup C$. Then $Q^3 \cap Q^{\bullet 3} = \mathcal{T} \cup S$, where $S \cong \mathbb{F}_1$ is a smooth rational normal cubic scroll in $\mathbb{P}^4$ passing through P and C.*

Proof. The exact sequence

$$0 \longrightarrow \mathcal{O}_{Q^3}(1) \longrightarrow \mathcal{O}_{Q^3}(2) \longrightarrow \mathcal{O}_{Q^2}(2) \longrightarrow 0$$

yields the exact cohomology sequence

$$0 \to H^0(\mathcal{O}_{Q^3}(1)) \longrightarrow H^0(\mathcal{O}_{Q^3}(2)) \xrightarrow{\psi} H^0(\mathcal{O}_{Q^2}(2)) \to 0\,. \tag{4.2}$$

Let l_1 and l_2 be general horizontal and vertical generators of the quadric Q^2, and let $s \in H^0(\mathcal{O}_{Q^2}(2))$ be a section vanishing along the $(2,2)$-divisor $C + l_1 + l_2$. By virtue of (4.2) the affine subspace $\psi^{-1}(s) \subset H^0(\mathcal{O}_{Q^3}(2))$ has dimension 5. It projects into a 5-dimensional family of divisors $D \in |\mathcal{O}_{Q^3}(2)|$ such that $D \cap Q^2 = C + l_1 + l_2$. The plane $\mathcal{T} \subset Q^3$ is spanned by l_1 and P. It defines a 2-dimensional subfamily $\mathcal{Q}$ of divisors D containing $\mathcal{T}$ and such that $D \cap Q^2 = C + l_1 + l_2$.

Write $D = \mathcal{T} \cup S$, where S is the residual cubic surface. Then $S \cap Q^2 = C + l_2$. Suppose that S is reducible: $S = \mathcal{T}_2 \cup S'$, where $\mathcal{T}_2 \cap Q^2 = l_2$ and $S' \cap Q^2 = C$. Then $D = \mathcal{T}_1 \cup \mathcal{T}_2 \cup S'$, where $\mathcal{T}_1 = \mathcal{T}$, $\mathcal{T}_2 = \mathrm{span}(l_2, P)$ is a plane, and S' is a hyperplane section of Q^3. Here $\mathcal{T}_1 \cup \mathcal{T}_2$ is uniquely determined by $l_1 \cup l_2$, and S' runs over a 1-parameter family.

Since $\dim \mathcal{Q} = 2$, one can conclude that a general divisor $D \in \mathcal{Q}$ has the form $D = \mathcal{T} \cup S$, where $S \subset \mathbb{P}^4$ is an irreducible cubic surface.

The cubic surface S is linearly nondegenerate, because a hyperplane section of Q^3 is a quadric surface. Thus, S is a linearly nondegenerate surface of minimal degree 3 in $\mathbb{P}^4$. Such a surface is either a cone over a twisted cubic $\Gamma \subset \mathbb{P}^3$, or a rational normal scroll $S = S_{2,1} \cong \mathbb{F}_1$ (see [GH78, Ch. 4, Prop. on p. 525]).

If S were a cone over $\Gamma \subset \mathbb{P}^3$ with vertex P', then the twisted cubic Γ would be dominated by the conic C under the projection from P', which is impossible. Thus $F \cong \mathbb{F}_1$ is smooth.

Finally, $P \in S$ since otherwise S would be a Cartier divisor on Q^3 linearly proportional to a hyperplane section. $\square$

Lemma 4.4. *Let $Q^4 \subset \mathbb{P}^5$ be a smooth quadric. There exist two smooth cubic scrolls S and S' in $Q^4 \subset \mathbb{P}^5$ such that*

- *$S \cong \mathbb{F}_1 \cong S'$;*

- *S and S' span hyperplanes L and L' in $\mathbb{P}^5$, respectively, where $L \neq L'$;*
- *$L \cap Q^4 = Q^3$ and $L' \cap Q^4 = Q'^3$ are quadric cones with zero-dimensional vertices P and P', respectively, where $P \neq P'$;*
- *the scheme theoretical intersection $C = S \cdot S'$ is a smooth conic.*

Proof. A general pencil (Q^3_λ) of hyperplane sections of Q^4 contains exactly 6 degenerate members. Consider two of them, say, $Q^3 = Q \cap T_P Q$ and $Q'^3 = Q \cap T_{P'} Q$, where $P, P' \in Q$. Then Q^3 and Q'^3 are quadric cones with zero-dimensional vertices P and P', respectively. The base locus of the pencil (Q^3_λ) is the smooth quadric $Q^2 = Q^3 \cap Q'^3 \cong \mathbb{P}^1 \times \mathbb{P}^1$. Applying Lemma 4.3 to Q^3 and Q'^3, the assertions follow. $\square$

Using Lemma 4.4 and Proposition 4.2 we proceed now with construction of surfaces F as in Proposition 4.1 a).

Construction 4.5. Consider the smooth cubic scrolls S and S' in $\mathbb{P}^5$ as in Lemma 4.4. The embedding $\mathbb{F}_1 \xrightarrow{\cong} S' \hookrightarrow \mathbb{P}^4$ is given by the linear system $|\sigma + 2f|$ on $\mathbb{F}_1$, where σ is the exceptional section of $\mathbb{F}_1 \to \mathbb{P}^1$ and f is a fiber. On S' we have $C = S \cdot S' \sim \sigma + f$, where the images of σ and f on S' are denoted by the same letters.

In what follows we employ the notation of Proposition 4.2. Let $\hat{S}'$ be the proper transform of S' in $\widehat{W}$ (see diagram (4.1)). Then, clearly, $\hat{S}' \cong S' \cong \mathbb{F}_1$. By Proposition 4.2(v), the morphism $\hat{\rho} : \widehat{W} \to W \subset \mathbb{P}^7$ is defined by the linear system $|\hat{\rho}^* H| = |2\hat{\varphi}^* \mathcal{L} - \hat{D}|$, where $\mathcal{L}$ is a hyperplane section of $Q^4 \subset \mathbb{P}^5$ and $\hat{D} = \hat{\varphi}^*(S)$ is the exceptional divisor of $\hat{\varphi}$. Identifying S' with $\tilde{S}'$ one can write

$$(4.3) \qquad (2\varphi^* \mathcal{L} - \hat{D})|_{\hat{S}'} = 2\mathcal{L}|_{S'} - S|_{S'} \sim 2(\sigma + 2f) - C \sim \sigma + 3f\,.$$

We let $F = \hat{\rho}(\hat{S}') \subset W$. Since $\hat{S}' \not\subset \hat{D}$, the map $\hat{\rho}|_{\hat{S}'} : \hat{S}' \to F$ is a birational morphism, and the surface F is a quintic scroll.

Remark 4.6. Since $S' \cap \langle S \rangle = C + f_0$, where f_0 is a fiber of S', we have $\hat{S}' \cap \hat{D} \supset \hat{f}_0$. Therefore, $\hat{\rho}(\hat{f}_0) = l \subset F$ (because $l = \hat{\rho}(\hat{D})$ and $F = \hat{\rho}(\hat{S}')$). Moreover, l is a ruling of F.

Lemma 4.7. *The morphism $\hat{\rho}|_{\hat{S}'} : \hat{S}' \to F$ is an isomorphism onto a smooth rational normal quintic scroll $F \supset l$ contained in a hyperplane in $\mathbb{P}^7$.*

Proof. It suffices to show that the morphism $\hat{\rho}|_{\hat{S}'} : \hat{S}' \to \mathbb{P}^6 \subset \mathbb{P}^7$ is given by the (very ample) *complete* linear system $|\sigma + 3f|$ on $\hat{S}' \cong \mathbb{F}_1$ (cf. (4.3)), or, in other words, that the induced morphism $\mathbb{F}_1 \to F$ is an isomorphism, see [GH78, Ch. 4, p. 523] or [Har92].

Suppose to the contrary that $\langle F\rangle \cong \mathbb{P}^5$, that is, F is cut out in W by two hyperplanes. Then the quintic scroll F cannot be normal. Indeed, for a general hyperplane section γ on F we have by adjunction $\omega_\gamma = (K_W+3H)|_\gamma \sim 0$. Hence the arithmetic genus of γ equals 1. The genus of the proper transform of γ on the normalization of F equals 0, hence γ is a rational curve with one double point. Such double points of hyperplane sections of F fill in a line in F, and F is singular along this line. In particular, F is not normal. This leads to the following claim.

Claim 4.8. *If* $\langle F\rangle \cong \mathbb{P}^5$ *then* $\operatorname{Sing} F = l$ *is a ruling of* F.

Proof. We know that $l \subset F$ is a ruling, see Remark 4.6. Since $\hat{W} \to W$ is an isomorphism over $W \setminus l$, its restriction $\hat{S}' \to F$ is an isomorphism over $F \setminus l$. Since F is not normal, the claim follows. $\square$

On the other hand, we have

Claim 4.9. *Let as before* $\langle F\rangle \cong \mathbb{P}^5$, *and let* $\nu : \mathbb{F}_1 \to F$ *be the normalization. Then on* $\mathbb{F}_1$ *we have* $K_{\mathbb{F}_1} \sim \nu^*\omega_F - B$, *where* $B \sim \sigma$ *is an effective divisor supported by the proper transform in* $\mathbb{F}_1$ *of the non-normal locus of* F.

Proof. Under our assumption, F is a complete intersection in a smooth variety W. Hence F is Cohen-Macaulay, and so, the standard formula $K_{\mathbb{F}_1} \sim \nu^*\omega_F - B$ holds with B supported by the proper transform in $\mathbb{F}_1$ of the non-normal locus of F. Using this formula and adjunction one gets on $\mathbb{F}_1$:

$$\begin{aligned} B \sim \nu^*\omega_F - K_{\mathbb{F}_1} &\sim (K_W + 2H)|_F + (2\sigma + 3f) \sim -H|_F + (2\sigma + 3f) \\ &\sim -(\sigma + 3f) + (2\sigma + 3f) \quad \sim \sigma\,, \end{aligned}$$

as stated. $\square$

Due to Claim 4.8 we have $\operatorname{supp}(B) = f$, and so, $B \cdot f = 0$. This yields a contradiction, since by Claim 4.9, $B \cdot f = \sigma \cdot f = 1$ on $\mathbb{F}_1$. $\square$

Lemma 4.10. *None of the planes in* W *meets the quintic scroll* $F \subset W$ *along a* (*possibly, degenerate*) *conic.*

Proof. Recall that R stands for the hyperplane section of W swept out by the 1-parameter family of planes (Π_t) in W. It is singular along the plane Ξ, see Proposition 3.3(iv). Since $l \subset F$ and $l \not\subset R$, we have $F \not\subset R$ and $l \cap \Xi = \emptyset$, see Proposition 3.3(v).

Suppose to the contrary that F meets a plane $\mathcal{P} \subset W$ along a conic, say, η.

Claim. *The conic* η *coincides with the exceptional section* σ_F *of the scroll* $F \cong \mathbb{F}_1$.

Proof. Suppose that the conic η is degenerate. Since any two lines on F are disjoint, $\eta \subset P$ cannot be a bouquet of two distinct lines. Hence η is a double line $2f$.

For any line $f' \neq f$ in F there exists an automorphism $\alpha \in \operatorname{Aut} F \cong \operatorname{Aut} \mathbb{F}_1$ such that $\alpha(f) = f'$. Since the embedding

$$\mathbb{F}_1 \xrightarrow{\cong} F \hookrightarrow \mathbb{P}^6 \subset \mathbb{P}^7$$

is given by an $(\operatorname{Aut} F)$-invariant linear system $|\sigma + 3f|$, α can be extended to an automorphism $\bar{\alpha} \in \operatorname{Aut} \mathbb{P}^7$, which leaves $\langle F \rangle \cong \mathbb{P}^6$ invariant. Hence there exists a second plane $\mathcal{P}' = \bar{\alpha}(\mathcal{P})$, which meets F along a double line $2f'$ (this plane $\mathcal{P}'$ does not need to be contained in W). [1]

The planes $\mathcal{P}$ and $\mathcal{P}'$ span a subspace $\mathcal{N} \subset \mathbb{P}^7$ with $\dim \mathcal{N} \leq 5$. Thus, there exists a hyperplane $\mathcal{M} \supset \mathcal{N}$ in $\mathbb{P}^7$ different from $\langle F \rangle$. We have $\mathcal{M} \cdot F = 2f + 2f' + f''$, where $f'' \subset F$ is an extra line. However, this divisor $\mathcal{M} \cdot F$ on F is not ample, which is a contradiction.

Thus, the conic $\eta = F \cap \mathcal{P}$ is smooth. Since the image σ of the exceptional section $\sigma_{\hat{S}'} \subset \hat{S}'$ is a unique smooth conic in the quintic scroll $F \cong \mathbb{F}_1$, we obtain that $\eta = \sigma_F$. $\square$

The line $l \subset F$ meets the section $\sigma = \sigma_F$ in a point $p \in \sigma$. Hence it meets also the plane $\mathcal{P}$ in p. The projection $\hat{\phi} : W \dashrightarrow \mathbb{P}^5$ with center l sends σ_F to the exceptional section $\sigma_{S'} \subset S'$, and $\mathcal{P}$ to a line on $S' \cong \mathbb{F}_1$, which should coincide with $\sigma_{S'}$. Recall that by our construction $S \cap S' = C \sim \sigma_{S'} + f_{S'}$ is a smooth conic on S'. Since $\sigma_{S'} \cap C = \emptyset$, the exceptional divisor $\hat{E} \subset \widehat{W}$ does not meet the section $\sigma_{\hat{S}'}$ of the scroll $\hat{S}' \subset \widehat{W}$. Thus $\hat{\varphi} : \widehat{W} \to Q^4$ is an isomorphism near $\sigma_{\hat{S}'}$.

On the other hand, let $\hat{\mathcal{P}}$ be the proper transform of $\mathcal{P}$ in $\widehat{W}$. Then $\hat{\mathcal{P}} \to \mathcal{P}$ is the blowup of the point $p = \mathcal{P} \cap l$, and $\hat{\mathcal{P}} \cap \hat{S}' \supset \sigma_{\hat{S}'}$. Thus the image $\hat{\varphi}(\hat{\mathcal{P}}) \subset Q^4$ should be a surface, and not a line. This yields as well a contradiction. $\square$

Examples show that the last assumption in Theorem 2.1 cannot be omitted. Without this assumption one arrives at a singular fourfold V in diagram (2.1), or else φ is the blowup of a singular surface. According to Proposition 4.1, this does not happen for our choice of F.

[1] Alternatively, the further proof can proceed as follows. The plane $\mathcal{P}'$ is tangent to F along the ruling f'. Thus, the Gauss map of F is degenerate, and so, F is a developable surface. Such a surface, which is not a plane, is a cone or the tangential developable of a curve, see, e.g., [Ca64] or more general results in [GH78, (2.29)], [Zak87, Cor. 5], or [FP01, §2.3.3]. Hence F cannot be smooth, a contradiction. Our argument in the text is more elementary.

5. Proof of Theorem 2.1.

Let us start with the following well known lemmas.

Lemma 5.1. *Any surface F as in Theorem* 2.1 *is a scheme theoretical intersection of quadrics.*

Proof. In case a) the assertion follows from [Dol12, Thm. 8.4.1], and in case b) from [Har92, Lect. 9, Exs. 9.10–9.11]. $\square$

The next well known lemma is immediate.

Lemma 5.2. *Let a smooth surface $F \subset \mathbb{P}^n$, $n \geq 4$, be a scheme theoretical intersection of quadrics. Let $\tilde{\mathbb{P}}^n \to \mathbb{P}^n$ be the blowup of F with exceptional divisor T. Then the linear system $|2H^* - T|$ defines a morphism $\tilde{\mathbb{P}}^n \to \mathbb{P}^N$, which contracts the proper transform of any 2-secant line of F.*

5.3. In what follows we keep the notation as in Theorem 2.1. In particular, we let $g = 10$ in case a) and $g = 9$ in case b).

A surface $F \subset W$ as in Theorem 2.1 is contained in a unique hyperplane section $E = \langle F \rangle \cap W$ of W, see Remark 3.6. We let

- $\rho : \widetilde{W} \longrightarrow W$ be the blowup of F with exceptional divisor D,
- $\tilde{E} \subset \widetilde{W}$ be the proper transform of E,
- $H \subset W$ be a general hyperplane section, and
- $H^* = \rho^* H \in \operatorname{Div} W$.

Clearly, one has $\operatorname{rk} \operatorname{Pic} \widetilde{W} = 2$ and $\tilde{E} \sim H^* - D$ on $\widetilde{W}$.

Lemma 5.4. *The variety $\widetilde{W}$ is a smooth Fano fourfold.*

Proof. We have

$$-K_{\widetilde{W}} = 3H^* - D = 2H^* - D + H^*,$$

where both $2H^* - D$ and H^* are nef, because the linear systems $|2H^* - D|$ and $|H^*|$ are free. Since $\operatorname{rk} \operatorname{Pic} \widetilde{W} = 2$ and the nef divisors $2H^* - D$ and H^* are not proportional, their sum is an ample divisor by the Kleiman ampleness criterion. $\square$

The nef and non-ample linear systems $|H^*|$ and $|2H^* - D|$ on $\widetilde{W}$ define the two extremal Mori contractions on $\widetilde{W}$. The first one is $\rho : \widetilde{W} \to W$; the second one $\varphi : \widetilde{W} \to V$ makes the subject of our following studies. We need the next lemma.

Lemma 5.5. *On $\widetilde{W}$ one has* $(H^*)^4 = 5, \quad (H^*)^3 \cdot D = 0,$

$$(H^*)^2 \cdot D^2 = \begin{cases} -5 \\ -6 \end{cases}, \quad H^* \cdot D^3 = \begin{cases} -8 \\ -12 \end{cases}, \quad D^4 = \begin{cases} -6 & \textit{in case a)} \\ -16 & \textit{in case b)}. \end{cases}$$

Proof. The lemma follows easily from the equalities (see [PZ15, Lem. 1.4])

$$(H^*)^2 \cdot D^2 = -F \cdot H^2\,,$$
$$H^* \cdot D^3 = -H|_F \cdot K_F - 3H \cdot H \cdot F\,,$$

and

$$D^4 = c_2(W) \cdot F + K_W|_F \cdot K_F - c_2(F) - K_W^2 \cdot F\,.$$

□

Lemma 5.6. *Let U be a Mukai fourfold of genus $g(U) \geq 4$ with at worst terminal Gorenstein singularities and with* $\operatorname{rk}\operatorname{Pic} U = 1$. *Assume that the linear system $|-\frac{1}{2}K_U|$ is base point free. Then the divisor $-\frac{1}{2}K_U$ is very ample and defines an embedding $U \hookrightarrow \mathbb{P}^{g+2}$.*

Proof. This follows from the corresponding result in the three-dimensional case, see [Muk89, Prop. 1], [IP99], and [PCS05], by recursion on the dimension, likewise this is done in [Isk77, Lem. (2.8)]. □

5.7. Using Lemma 5.5 we obtain

$$\deg V = (2H^* - D)^4 = 2g - 2 = \begin{cases} 18 & \text{in case a)} \\ 16 & \text{in case b)} \end{cases} \tag{5.1}$$

and

$$\tilde{E} \cdot (2H^* - D)^3 = (H^* - D) \cdot (2H^* - D)^3 = 0\,. \tag{5.2}$$

Therefore, the linear system $|2H^* - D|$ defines a generically finite morphism

$$\Phi_{|2H^*-D|} : \widetilde{W} \to V \subset \mathbb{P}^{g+2}$$

onto a fourfold V, where $\Phi_{|2H^*-D|}$ contracts the divisor $\tilde{E} \sim H^* - D$. Consider the Stein factorization

$$\Phi_{|2H^*-D|} : \widetilde{W} \xrightarrow{\varphi} U \to V \subset \mathbb{P}^{g+2}\,.$$

Here φ is a divisorial Mori contraction, and $\operatorname{Pic} U = \mathbb{Z} \cdot L$, where L is an ample Cartier divisor with $\varphi^*L = 2H^* - D$. Once again, the exceptional divisor of φ is $\tilde{E} \sim H^* - D$. Hence $D \sim \varphi^*L - 2E$.

Lemma 5.8. *The variety U as in* 5.7 *is a Mukai fourfold with at worst terminal Gorenstein singularities and* $\operatorname{rk}\operatorname{Pic} U = 1$.

Proof. Since φ is a divisorial Mori contraction, U has at worst terminal singularities. We have $\operatorname{rk}\operatorname{Pic} U = 1$ because $\operatorname{rk}\operatorname{Pic} \widetilde{W} = 2$. Since

$$-K_{\tilde{W}} = 3H^* - D = 2(2H^* - D) - \tilde{E}$$

we also have $-K_U = 2L$. Hence $-K_U$ is an ample Cartier divisor divisible by 2 in $\operatorname{Pic} U$. So U is a Mukai fourfold. □

Convention 5.9. The morphism $U \to V \subset \mathbb{P}^{g+2}$ is given by the linear system $|L| = |-\frac{1}{2}K_U|$. As follows from Lemma 5.6, this is an isomorphism. In the sequel we identify V with U and $\Phi_{|2H^*-D|}$ with φ.

Lemma 5.10. *For the image $V = \varphi(\widetilde{W}) \subset \mathbb{P}^{g+2}$ the following hold.*

(i) *The morphism $\varphi : \widetilde{W} \to V$ is birational and $\deg V = 2g-2$;*
(ii) *the morphism φ contracts the divisor $\tilde{E}$ to an irreducible surface $S \subset V$;*
(iii) $\deg S = g - 7 = \begin{cases} 3 & \textit{in case} \quad \text{a)} \\ 2 & \textit{in case} \quad \text{b)}\,; \end{cases}$
(iv) *S can have only isolated singularities.*

Proof. Upon convention 5.9, φ is birational. By (5.1) we have $\deg V = 2g-2$. By virtue of (5.2), $\tilde{E}$ is the exceptional divisor of φ. Using Lemma 5.5 we deduce the equalities

$$(2H^* - D)^2 \cdot \tilde{E}^2 = (2H^* - D)^2 (H^* - D)^2 = \begin{cases} -3 & \text{in case} \quad \text{a)} \\ -2 & \text{in case} \quad \text{b)}\,. \end{cases}$$

Since the latter number is nonzero, S is a surface of degree

$$\deg S = -(2H^* - D)^2 \cdot \tilde{E}^2$$

satisfying (iii).

Since $\operatorname{rk}\operatorname{Pic}\widetilde{W} = 2$, the exceptional locus of φ coincides with $\tilde{E}$, and $\tilde{E}$ is a prime divisor. Therefore, φ has at most a finite number of 2-dimensional fibers. By the Andreatta–Wisniewski Theorem ([AW98]) S has at most isolated singularities. □

Corollary 5.11. *The surface S is normal.*

Proof. The assertion is certainly true if $\deg S = 2$. If $\deg S = 3$ and the cubic surface $S \subset \mathbb{P}^4$ is not normal, then S is contained in a 3-dimensional subspace and the singular locus of S is 1-dimensional, which contradicts (iv). □

Lemma 5.12. *In the notation of* 5.7 *the morphism $\varphi : \widetilde{W} \to V$ is the blowup of the surface S, where both S and V are smooth.*

Proof. If to the contrary S or V were singular, then by [And85, Thm. 2.3] the extremal $K_{\widetilde{W}}$-negative contraction $\varphi : \widetilde{W} \to V$ would have a 2-dimensional fiber, say, $\widetilde{Y} \subset \widetilde{W}$. Since S is normal (see Corollary 5.11), by the main theorem and Prop. 4.11 in [AW98] one has $\widetilde{Y} \cong \mathbb{P}^2$ and

$$(3H^* - D)|_{\widetilde{Y}} = -K_{\widetilde{W}}|_{\tilde{Y}} = \mathcal{O}_{\mathbb{P}^2}(1)\,.$$

Since $\tilde{Y}$ is contracted to a point under φ, we have $(2H^* - D)|_{\widetilde{Y}} \sim 0$. Thus $H^*|_{\widetilde{Y}} = \mathcal{O}_{\mathbb{P}^2}(1)$ and $D|_{\widetilde{Y}} = \mathcal{O}_{\mathbb{P}^2}(2)$.

It follows that the image $Y = \rho(\widetilde{Y}) \subset W$, where $\rho = \Phi_{|H^*|}$, is a plane, $Y \neq F$, and $Y \cap F \cong \widetilde{Y} \cap D$ is a conic in $Y \cong \mathbb{P}^2$. However, the latter contradicts our assumption in Theorem 2.1 that F does not meet any plane in W along a conic.

Therefore, φ has no 2-dimensional fiber. Hence the surface S and the fourfold V are smooth, and φ is the blowup of S by [And85, Thm. 2.3]. $\square$

Corollary 5.13. *The surface $S \subset V \subset \mathbb{P}^{g+2}$ is a smooth normal cubic scroll in case* a) *and a smooth quadric in case* b).

Proof. By Lemmas 5.10(iii) and 5.12, S is a smooth surface of degree 3 in case a) and of degree 2 in case b). It remains to show that in case a), S is a normal scroll in $\mathbb{P}^4$ and not a smooth cubic surface in $\mathbb{P}^3$. Using (2.2) and Lemma 5.5 one can compute

$$L^* \cdot \tilde{E}^3 = (2H^* - D) \cdot (H^* - D)^3 = -1.$$

On the other hand,

$$L^* \cdot \tilde{E}^3 = -L|_S \cdot K_S + K_V \cdot L \cdot S$$

(see e.g. [PZ15, Lem. 1.4]), and so, due to 5.9,

$$L|_S \cdot K_S = -L^* \cdot \tilde{E}^3 - 2L^2 \cdot S = 1 - 6 = -5.$$

If $\dim\langle S\rangle < 4$, then S is a cubic surface in $\mathbb{P}^3$ and we have $L|_S \cdot K_S = -K_S^2 = -3$, a contradiction. Therefore, $\dim\langle S\rangle = 4$, and so, $S \subset \mathbb{P}^4$ is a linearly nondegenerate surface of degree 3, i.e., a normal cubic scroll. $\square$

Lemma 5.14. *Under the setting as before, the following hold.*

- *$\varphi(D)$ is a hyperplane section of V singular along $S = \varphi(\tilde{E})$,*
- *there is an isomorphism $V \setminus \varphi(D) \cong W \setminus \rho(\tilde{E})$.*

Proof. We have $D \sim \varphi^* L - 2\tilde{E}$ in $\widetilde{W}$ and $S \subset \varphi(D)$, because any fiber $\varphi^{-1}(s)$, $s \in S$, meets D. Thus, $\varphi(D) \sim L$ is a hyperplane section of $V \subset \mathbb{P}^{g+2}$ singular along $S = \varphi(\tilde{E})$.

Finally, since $F \subset E = \rho(\tilde{E})$ we have isomorphisms

$$W \setminus \rho(\tilde{E}) \cong \widetilde{W} \setminus (\tilde{E} \cup D) \cong V \setminus (S \cup \varphi(D)) = V \setminus \varphi(D)\,.$$

$\square$

The following corollary is immediate from (5.1) and Lemma 5.8. It ends the proof of Theorem 2.1.

Corollary 5.15. *Under the assumptions of Theorem* 2.1*,*

- *in case* a) $V \cong V_{18} \subset \mathbb{P}^{12}$ *is a smooth Mukai fourfold of genus* $g = 10$*, and*

- *in case* b) $V \cong V_{16} \subset \mathbb{P}^{11}$ *is a smooth Mukai fourfold of genus* $g = 9$.

6. Concluding remarks.

6.1. **Cylindricity in families.** Our Theorem 2.2 and the results in [PZ15] show that for any $g \geq 7$, in the family of all Mukai fourfolds of genus g there exist subfamilies of cylindrical such fourfolds. The question about cylindricity of all the Mukai fourfolds of genus $g \geq 7$ remains open, and as well the question about cylindricity of Mukai fourfolds of lower genera is. We expect that the answers to both questions are negative in general. However, at the moment we do not dispose suitable tools to prove this.

6.2. **Rationality questions.** The question about cylindricity is ultimately related to the rationality problem. For instance, in dimension 3 cylindricity of a Fano variety implies its rationality. Note that for any $g = 5, \ldots, 8$ there exist rational Mukai fourfolds $V = V_{2g-2} \subset \mathbb{P}^{g+2}$ of genus g. We also have the following fact.

Proposition 6.1. *Any Mukai fourfold* $V = V_{2g-2} \subset \mathbb{P}^{g+2}$ *of genus* $g \in \{7, 9, 10\}$ *is rational.*

Proof. By Shokurov's theorem ([Sho79]) applied to a hyperplane section, there exists a line λ on V. By an easy parameter count (see [PZ15, Lem. 2.4]) a general hyperplane section of V passing through λ is smooth. Hence one can take a pencil $\mathcal{H}$ of hyperplane sections of V passing through λ whose general member $U = H_{2g-2} \in \mathcal{H}$ is a smooth anticanonically embedded Fano threefold of genus g with $\operatorname{Pic} U = \mathbb{Z} \cdot K_U$. Blowing up the base locus of $\mathcal{H}$ yields a family $\mathfrak{V} \to \mathbb{P}^1$, whose fibers are the members of $\mathcal{H}$ and the total space $\mathfrak{V}$ is birational to V.

Consider the generic fiber $X = \mathfrak{V} \times \operatorname{Spec} \mathbb{C}(\mathbb{P}^1)$, where $\mathbb{P}^1$ is the parameter space of the pencil $\mathcal{H}$. As before, X is a Fano threefold of genus g over the non-closed field $\mathbb{C}(\mathbb{P}^1)$ with $\operatorname{Pic} X = \mathbb{Z} \cdot K_X$. It suffices to show the $\mathbb{C}(\mathbb{P}^1)$-rationality of X.

By construction, the line $\lambda \subset V$ gives a line $\Lambda \subset X$ defined over $\mathbb{C}(\mathbb{P}^1)$. Then we can apply the Fano-Iskovskikh double projection $\Psi : X \dashrightarrow Y$ from Λ, see [IP99]. For $g = 9$ ($g = 10$, respectively) the map Ψ is birational and Y is a form of $\mathbb{P}^3$, i.e., a Brauer-Severi scheme, (a smooth quadric $Q \subset \mathbb{P}^4$, respectively). Since $\mathbb{C}(\mathbb{P}^1)$ is a c_1-field, by Tsen's theorem, Y is $\mathbb{C}(\mathbb{P}^1)$-rational, and so, X is as well. In the case $g = 7$ we have $Y \cong \mathbb{P}^1$ and Ψ is a birational map to a del Pezzo fibration of degree 5. Thus, the original variety V has a birational structure of a del Pezzo fibration of degree 5 over a surface. Then V is rational by the Enriques–Manin–Swinnerton-Dyer theorem (see, e.g., [ShB92]). □

We do not know whether the rationality as in Proposition 6.1 holds also for the Mukai fourfolds V_{2g-2} of genera $g = 5, 6, 8$.

6.3. **Compactifications of $\mathbb{C}^4$.** The Hirzebruch problem about compactifications of the affine space $\mathbb{A}^n$ ([Hir54]) is also very close in spirit to our cylindricity problem. One can ask the following natural question:

> Which Mukai fourfolds can serve as compactifications of $\mathbb{A}^4$?

We hope that the corresponding examples can be constructed via Sarkisov links, likewise this is done in the present paper for cylindricity. For the del Pezzo fourfolds, a similar problem was completely solved in [Pro94].

References

[And85] T. Ando. On extremal rays of the higher-dimensional varieties. *Invent. Math.*, 81(2):347–357, 1985.

[AW98] M. Andreatta and J. A. Wiśniewski. On contractions of smooth varieties. *J. Algebraic Geom.*, 7(2):253–312, 1998.

[Ca64] A. Cayley. On certain developable surfaces. *Quart. J. Pure Appl. Math.*, VI:108–126, 1864.

[Dol12] I. V. Dolgachev. *Classical algebraic geometry.* Cambridge University Press, Cambridge, 2012.

[FP01] G. Fischer, J. Piontkowski. *Ruled varieties. Advanced Lectures in Mathematics.* Fridrich Vieweg & son, Braunschweig, 2001.

[Fuj80] T. Fujita. On the structure of polarized manifolds with total deficiency one. I. *J. Math. Soc. Japan*, 32(4):709–725, 1980.

[Fuj81] T. Fujita. On the structure of polarized manifolds with total deficiency one. II. *J. Math. Soc. Japan*, 33(3):415–434, 1981.

[Fuj86] T. Fujita. Projective varieties of Δ-genus one. In *Algebraic and topological theories (Kinosaki, 1984)*, 149–175. Kinokuniya, Tokyo, 1986.

[GH78] Ph. Griffiths and J. Harris. *Principles of algebraic geometry.* Wiley-Interscience [John Wiley & Sons], New York, 1978. Pure and Applied Mathematics.

[GH79] Ph. Griffiths and J. Harris. Algebraic geometry and local differential geometry. *Ann. Sci. Ecole Norm. Sup.* (4) 12(3):355–452, 1979.

[Har92] J. Harris. *Algebraic geometry*, volume 133 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1992. A first course.

[Hir54] F. Hirzebruch. Some problems on differentiable and complex manifolds. *Ann. Math.* 60(2):213–236, 1954.

[Isk77] V. A. Iskovskikh, Fano threefolds. I. *Izv. Akad. Nauk SSSR Ser. Mat.* 41:516–562, 1977.

[IP99] V. A. Iskovskikh and Yu. Prokhorov. *Fano varieties. Algebraic geometry V.*, volume 47 of *Encyclopaedia Math. Sci.* Springer, Berlin, 1999.

[KPZ11] T. Kishimoto, Yu. Prokhorov, and M. Zaidenberg. Group actions on affine cones. In: *Affine Algebraic Geometry. CRM Proceedings and Lecture Notes*, 54:123–163. American Mathematical Society, Providence, 2011.

[KPZ13] T. Kishimoto, Yu. Prokhorov, and M. Zaidenberg. $\mathbf{G}_a$-actions on affine cones. *Transformation Groups*, 18(4):1137–1153, 2013.

[KPZ14] T. Kishimoto, Yu. Prokhorov, and M. Zaidenberg. Affine cones over Fano threefolds and additive group actions. *Osaka J. Math.*, 51(4):1093–1113, 2014.

[Mor82] S. Mori. Threefolds whose canonical bundles are not numerically effective. *Ann. Math.*, 115:133–176, 1982.

[Muk89] Sh. Mukai. Biregular classification of Fano 3-folds and Fano manifolds of coindex 3. *Proc. Nat. Acad. Sci. U.S.A.*, 86(9):3000–3002, 1989.

[Pro94] Yu. Prokhorov. Compactifications of $\mathbf{C}^4$ of index 3. In *Algebraic geometry and its applications (Yaroslavl′, 1992)*, Aspects Math., E25, 159–169. Vieweg, Braunschweig, 1994.

[Pro13] Yu. Prokhorov. G-Fano threefolds, I. *Adv. Geom.*, 13(3):389–418, 2013.

[PZ15] Yu. Prokhorov and M. Zaidenberg. Examples of cylindrical Fano fourfolds. *European J. Mathem.* 1–21, 2015, DOI 10.1007/s40879-015-0051-7.

[PCS05] V. V. Przhiyalkovski, I. A. Cheltsov, and K. A. Shramov. Hyperelliptic and trigonal Fano threefolds. (Russian). *Izv. Ross. Akad. Nauk Ser. Mat.* 69:145–204, 2005. English translation in: *Izv. Math.* 69:365421, 2005.

[ShB92] N. I. Shepherd-Barron. The rationality of quintic Del Pezzo surfaces—a short proof. *Bull. London Math. Soc.* 24:249–250, 1992.

[Sho79] V. V. Shokurov. The existence of a straight line on Fano 3-folds. (Russian) *Izv. Akad. Nauk SSSR Ser. Mat.* 43:922–964, 1979. English translation in: *Math. USSR Izvestija* 15:173–209, 1980.

[Tod30] J. A. Todd. The locus representing the lines of four-dimensional space and its application to linear complexes in four dimensions. *Proc. London Math. Soc.*, 30:513–550, 1930. Ser. 2.

[Zak87] F. L. Zak. The structure of Gauss mappings. (Russian) *Funktsional. Anal. i Prilozhen.* 21(1):39–50, 1987. English translation in: *Funct. Anal. Appl.* 21:32–41, 1987.

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