# Knots and Galois actions.

Norbert A'Campo

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**Introduction.** Let $f : \mathbf{C}^{n+1} \to \mathbf{C}$ be a map defined by a polynomial. We assume that $f(0) = 0$ and that $0 \in \mathbf{C}^{n+1}$ is an isolated critical point of $f$. For $p \in \mathbf{C}^{n+1}$ let $||p||$ denote the square root of $|z_0(p)|^2 + |z_1(p)|^2 + \cdots + |z_n(p)|^2$. Let $B_\epsilon := \{p \in \mathbf{C}^{n+1} \mid ||p|| \leq \epsilon\}, 0 < \epsilon$, be a Milnor ball for the singularity of $f$ and let $\mathrm{Tube}_{\epsilon,\delta} := \{p \in B_\epsilon \mid |f(p)| \leq \delta\}, 0 < \delta << \epsilon$, be a regular tubular neighborhood of $\{p \in B_\epsilon \mid f(p) = 0\}$ in $B_\epsilon$. A monodromy vector field $X$ for the singularity is a smooth vector field $p \in \mathrm{Tube}_{\epsilon,\delta} \mapsto X_p \in \mathbf{C}^{n+1}$ such that we have the following properties for $p \in \mathrm{Tube}_{\epsilon,\delta}$ (remember $i = \sqrt{-1}$):

- $(df)_p(X_p) = 2\pi i f(p)$,

- $X_p$ is tangent to $\partial B_\epsilon$ if $p \in \partial B_\epsilon$,

- trajectories of $X$ starting at $p \in \partial B_\epsilon$ are periodic with period 1.

Using partition of unity, one can construct monodromy vector fields. The flow at time 1 of a monodromy vector field $X$ defines a monodromy diffeomorphism $T = T_X \colon F \to F$, where the manifold with boundary $(F, \partial(F)) := \{p \in B_\epsilon \mid f(p) = \delta\}$ is the Milnor fiber of the singularity. The relative isotopy class of the diffeomorphism $T$ is independent from the chosen monodromy vector field and is the geometric monodromy of the singularity. The geometric monodromy is a topological invariant of the singularity.

From now on we will assume in addition, that the polynomial $f$ is real meaning that its coefficients are real numbers. Let $c : \mathbf{C}^{n+1} \to \mathbf{C}^{n+1}$ denote the involution on complex space given by the complex conjugation of coordinate values. Hence with the above notations, we have $c(\mathrm{Tube}_{\epsilon,\delta}) = \mathrm{Tube}_{\epsilon,\delta}$

and $c(F) = F$. We denote by $c_F : F \to F$ the restriction of the involution $c$ to $F$.

Let $X\colon \mathrm{Tube}_{\epsilon,\delta} \to \mathbf{C}^{n+1}$ be a monodromy vector field for the isolated singularity of $f$. We may assume that we have constructed the vector field $X$ with more care near the boundary of the Milnor ball in order to achieve that for some $\epsilon' < \epsilon$ we have the symmetry $c(X_p) = -c(X_{c(p)}), p \in \mathrm{Tube}_{\epsilon,\delta}, ||p|| > \epsilon'$.

Since $f$ is real, we have

$$c((df)_p(X_p)) = (df)_{c(p)}(c(X_p)) = -2\pi i f(c(p))$$

hence, we see (by substituting $q$ for $c(p)$ and accordingly $c(q)$ for $p$) that the vector field $X^c$ defined by:

$$q \in \mathrm{Tube}_{\epsilon,\delta} \mapsto X^c_q := -c(X_{c(q)}) \in \mathbf{C}^{n+1},$$

is a monodromy vector field too. Let $Y : \mathrm{Tube}_{\epsilon,\delta} \to \mathbf{C}^{n+1}$ be the vector field $Y := \frac{X+X^c}{2}$, which due to the extra care is also a monodromy vector field. We have $Y^c = Y$. The following is an important symmetry of the geometric monodromy:

**Lemma 1** *Let $T_Y$ be a monodromy diffeomorphism, which has been computed with a monodromy vector field $Y$ satisfiyng $Y^c = Y$. We have the symmetry*

$$c_F \circ T_Y \circ c_F = T_Y^{-1}.$$

*The geometric monodromy $T$ satisfies (up to relative isotopy) the symmetry*

$$c_F \circ T \circ c_F = T^{-1}.$$

**Proof:** The restriction of complex conjugation $c_{\mathrm{Tube}_{\epsilon,\delta}}\colon \mathrm{Tube}_{\epsilon,\delta} \to \mathrm{Tube}_{\epsilon,\delta}$ maps the monodromy vector field $Y$ to $-Y$ and $F$ to $F$. Hence, since $c_{\mathrm{Tube}_{\epsilon,\delta}}$ reverses the orientations of the trajectories, we have

$$T_Y^{-1} = (c_F)^{-1} \circ T_Y \circ c_F = c_F \circ T_Y \circ c_F.$$

Since the geometric monodromy $T$ is in the relative mapping class group of the Milnor fiber represented by $T_Y$, for $T$ in the relative mapping class group we have the symmetry $c_F \circ T \circ c_F = T^{-1}$. ■

Symmetries of monodromies as in the lemma can occur in the more general context of so-called *strongly invertible knots*, see for instance [H-To],[T],[K].

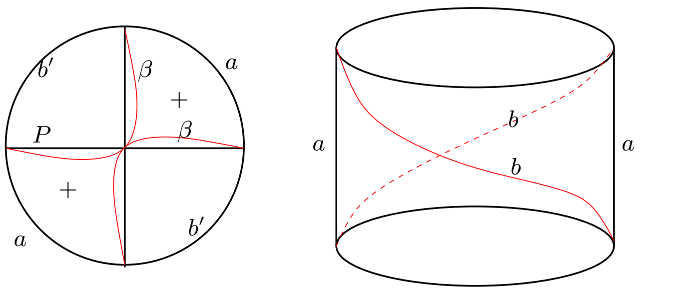
The symmetry property $c_F \circ T \circ c_F = T^{-1}$ expresses that the geometric monodromy $T$ of a complex hypersurface with real defining equation is conjugate in the mapping class group by an element of order 2 to its inverse $T^{-1}$. This statement does not refer to any complex conjugation, so it can be stated for any complex hypersurface singularity with complex defining equation. We say that the singularity is *strongly invertible* if its geometric monodromy diffeomorphism $T$ is conjugate by an element of order 2 in the relative mapping class group of the Milnor fiber to its inverse $T^{-1}$. The property of strong invertibility is a topological property for hypersurface singularities.

We can rewrite the symmetry property as follows: $T_Y \circ c_F \circ T_Y \circ c_F = \mathrm{Id}_F$. We see that $T_Y \circ c_F : F \to F$ is an involution of $F$. It follows, see also [G-Z], the

**Corollary 1** *The geometric monodromy $T$ of an isolated complex hypersurface singularity, which is defined by a real equation, is the composition of two involutions of the fiber $F_\delta, \delta \in \mathbf{R}$, namely: $T = (T \circ c_F) \circ c_F$, where $c_F$ is the restriction of the complex conjugation.*

The above observations can be applied to plane curve singularities in general, since it follows from the Theory of Puiseux Pairs that every plane curve singularity is topologically equivalent to a singularity given by a rational equation and hence plane curve singularities are strongly invertible.

In the next picture we decompose a Dehn twist as composition of two involutions $I_a \circ I_b$, where $I_a, I_b$ are the orientation reversing diffeomorphism of the cylinder, that fixe pointwise the arc systems $a$ or $b$.




Divide for $xy = 0$ Fiber surface with arc systems $a, b$

Fig. 5. Dehn twist as composition of two involutions.

For complex hypersurface singularities of higher dimension the situation seems to be opposite. In $\mathbf{C}^{n+1}$, $n > 1$, there exist isolated hypersurface singularities which are not topologically equivalent to a singularity with a real defining equation [T]. We expect that in general the geometric monodromy $T$ of a complex hypersurface singularity fails to be strongly invertible.

For complex hypersurface singularities the eigenvalues of the monodromy $T_*$ acting on the homology $E := \bigoplus_k H_k(F, \mathbf{Q})$ are roots of unity by the monodromy theorem and the characteristic polynomial of $T_*$ is a product of cyclotomic polynomials. It follows, that the complex homological monodromy ist strongly invertible. Mathias Schulze has proved strong invertibility of the homological monodromy with real coefficients and the following is a strengthening of his result.

**Theorem 1** *The rational homological monodromy of a complex hypersurface singularity is strongly invertible.*

**Proof:** Let $E = \bigoplus_i E_i$ a be finest possible direct sum decomposition in $T_*$-invariant $\mathbf{Q}$-subspaces of $E$. The characteristic polynomial of the restriction $A_i : E_i \to E_i$ of $T_*$ to a summand $E_i$ is a power $\psi(t) = \phi(t)^L$ of a cyclotomic polynomial $\phi(t)$. We have $\psi(A_i) = 0$ by the Hamilton-Cayley theorem and we have $\psi(A_i^{-1}) = 0$ since a power of cyclotomic polynomial satisfies $\psi(1/t) = \pm t^{\text{degree}(\psi)}\psi(t)$.

Since the decomposition has no refinement, we may choose a vector $e_1 \in E_i$, that is cyclic for $A_i$ and for $A_i^{-1}$. The systems

$$e_1, e_2 := A_i(e_1), \cdots, e_{\dim E_i} := A^{\dim E_i - 1}(e_1),$$

$$f_1 := e_1, f_2 := A_i^{-1}(e_1), \cdots, f_{\dim E_i} := A^{-\dim E_i + 1}(e_1)$$

are bases for the space $E_i$. Let $b_i : E_i \to E_i$ be the linear map defined by $b_i(e_j) = f_j, 1 \leq j \leq \dim E_i$.

We have $A_i^{-1} b_i(e_j) = b_i A_i(e_j), 1 \leq j < \dim E_i$. The polynomial $\psi_1(t) := -\psi(t) + t^{\dim E_i}$ satisfies $\psi_1(A_i^{\pm 1}) = A_i^{\pm \dim E_i}$. Since the degree of the polynomial $\psi_1(t)$ is less than $\dim E_i$, we have

$$A_i^{-1} b_i(e_{\dim E_i}) = A^{-\dim E_i} b_i(e_1) = \psi_1(A_i^{-1}) b_i(e_1) =$$

$$b_i \psi_1(A_i)(e_1) = b_i A_i^{\dim E_i}(e_1) = b_i A_i(e_{\dim E_i}).$$

We conclude that $A_i^{-1} b_i = b_i A_i$ and $A_i^{-1} = b_i A_i b_i^{-1}$ hold.

The polynomial $\psi_0(t) := \frac{\psi(t) - \psi(0)}{-t\psi(0)}$ satisfies $A_i^{-1} = \psi_0(A_i)$ and $A_i = \psi_0(A_i^{-1})$. We deduce $b_i A_i^{-1} = b_i \psi_0(A_i) = \psi_0(A_i^{-1}) b_i = A_i b_i$ and conclude $A_i^{-1} = b_i^{-1} A_i b_i$. We observe at this point that both the conjugates of $A_i$ by $b_i$ and by $b_i^{-1}$ are equal to the inverse $A_i^{-1}$.

For $0 \leq j < \dim E_i$ we have (remember $e_1 = f_1 = b_i(e_1) = b_i(f_1)$)

$$b_i(f_{j+1}) = b_i A_i^{-j}(f_1) = A_i^{j} b_i(f_1) = A_i^{j}(e_1) = e_{j+1}.$$

Hence $b_i$ is of order two, which shows that $A_i$ is strongly invertible over $\mathbf{Q}$. The sum $b := \bigoplus b_i$ is a rational strong inversion for $T_*$. ∎

**Question:** Is the integral homological monodromy of a complex hypersurface singularity strongly invertible? The answer is yes for plane curve singularities. We would like to know the answer for the isolated surface singularity $f(x, y, z) = L_1 L_2^2 L_3^3 L_4^4 L_5^5 L_6^6 L_7^7 L_8^8 L_9^9 + x^{46} + y^{46} + z^{46}$ where $L_j, 1 \leq j \leq 9$ are linear forms on $\mathbf{C}^3$, such that the 9 lines $\{L_j = 0\}$ in the complex projective plane span the $9_3$ configuration of flex tangents to a nonsingular cubic. No real equation for this singularity can exist, since the configuration $9_3$ cannot be realized in the real projective plane.

The autor does not know any general feasible algorithmic method for computing the integral homological monodromy of isolated surface singularities. For instance, compute for the surface singularity $f(x,y,z) = x^3 + x^2y^2z^2 + yz^{10} + zy^{11}$ of very low multiplicity 3 the integral monodromy. With SINGULAR you can discover a Jordan block of length 3 for the eigenvalue $-1$ in the monodromy of $f$..

Let $f(z_0, z_1, \cdots, z_n) \in K[z_0, z_1, \cdots, z_n]$ be a polynomial with coefficients in a number field $K$. We assume that $f$ has an isolated singularity at $0 \in \mathrm{K}^{n+1}$. How does the local topology at $0 \in \mathbf{C}^{n=1}$ of the mapping $f^\sigma : \mathbf{C}^{n+1} \to \mathbf{C}$ change depending upon Galois twists $\sigma$? For $n = 1$ the local topology depends upon the Puiseux pairs of $f^\sigma$, hence does not change. For $n = 2$ many interesting changes occur in the local monodromy diffeomorphism. The rational homological monodromy can be computed using the Gauss-Manin connection and the algorithm of Brieskorn, which is implemented in SINGULAR. The interesting changes stay hidden to us, since monodromy computation beyond the rational homological monodromy are not known yet.

So we try global monodromy of mappings $f^\sigma : \mathbf{C}^2 \to \mathbf{C}$ for $f(z_0, z_1) \in K[z_0, z_1]$.

We recall from at1999:

A divide $P$ is the image of a relative generic immersion of a 1-manifold $(M, \partial M)$ in the unit 2-disk $(D, \partial D)$. To a divide $P \subset D$ is associated a classical link in the 3-sphere, as follows. Consider the tangent space $T\mathbf{R}^2$ as $\mathbf{R}^2 \times \mathbf{R}^2 = \mathbf{R}^2 + i\mathbf{R}^2 = \mathbf{C}^2$. The link $L(P)$ of the divide $P$ is the following subset in the unit sphere $S^3$ of $\mathbf{C}^2$:

$$L(P) := \{(x,u) \in T\mathbf{R}^2 = \mathbf{R}^4 \mid x \in P, u \in T_xP, ||x||^2 + ||u||^2 = 1\}$$

The involution $c : (x,u) \mapsto (x,-u)$ induces a rigid symmetry of the link $L(P)$, which induces for each component a strong inversion.

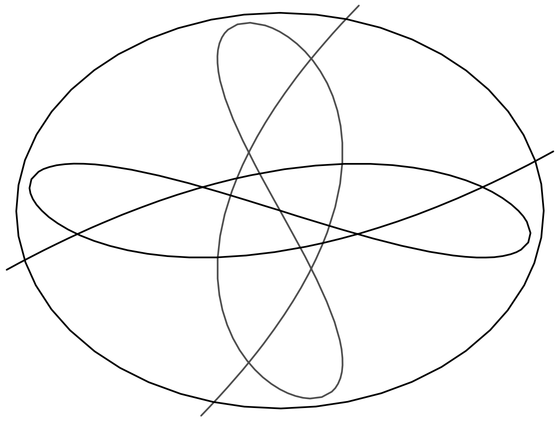


Figure 1: A divide with 2 branches

Links of divides tend to be fibered:

**Theorem 2** *The link of a connected divide is a fibered link. The fibration can be choosen to be invariant by the involution $c : (x, u) \mapsto (x, -u)$ with $c(F) = F$ for the fiber $F$ over $1$. Moreover, the monodromy diffeomorphism $T$ can be choosen, such we have $c_F \circ T \circ c_F = T^{-1}$.*

The fibration of the link of a divide is very similar to the Milnor fibration of links of plane curve singularities. This holds too for the construction of the fibration, that we explain now.

First we make by a small isotopy of $P$ the divide to be linear with right angles near its crossing points. For a connected divide $P \subset D(0, \rho)$, let $f_P : D(0, \rho) \to \mathbf{R}$ be a generic $C^\infty$ function, such that $P$ is its 0-level and that each region has exactly one local maximum or minimum. Such a function exist for a connected divide and is well defined up to sign and isotopy. In particular, there are no critical points of saddle type other then the crossing points of the divide. We assume moreover that the function $f_P$ is quadratic and euclidean in a neighborhood of its critical points, i.e. for euclidean coordinates $(X, Y)$ with center at a critical point $c$ of $f_P$ we have in a neighborhood of $c$ the expression $f_P(X, Y) = f_P(c) + XY$ or $f_P(X, Y) = f_P(c) + X^2 + Y^2$. Let $\chi : D(0, \rho) \to [0, 1]$ be a $C^\infty$, positive function, which evaluates to zero outside of the neighborhoods where $f_P$ is quadratic and to 1 in some smaller neighborhood of the critical points of $f_P$. Let $\theta_P : \partial B(0,\rho) \to \mathbf{C}$ be given by: $\theta_{P,\eta}(J(v)) := f_P(x) + i\eta df_P(x)(u) - \frac{1}{2}\eta^2\chi(x)H_{f_P}(v)$ for $J(v) = (x,u) \in TD(0,\rho) = D \times \mathbf{R}^2$ and $\eta \in \mathbf{R}, \eta > 0$. Observe that the Hessian $H_{f_P}$ is locally constant in a the neighborhood of the critical points of $f_P$. The function $\theta_{P,\eta}$ is $C^\infty$. Let $\pi_{P,\eta} : \partial B(0,\rho) \setminus L(P) \to S^1$ be defined by:$\pi_{P,\eta}(J(v)) := \theta_{P,\eta}(J(v))/|\theta_{P,\eta}(J(v))|$.

**Theorem 3** *Let $P \subset D(0,\rho)$ be a divide, such that the system of immersed curves is connected. The link $L(P)$ in $\partial B(0,\rho) = \{J(v) \mid v \in T(D(0,\rho))$ and $\|J(v)\| = \rho\}$ is a fibered link. The map $\pi_P := \pi_{P,\eta}$ is for $\eta$ sufficiently small, a fibration of the complement of $L(P)$ over $S^1$. The geometric monodromy is the product of $\mu$ positive Dehn twist, where $\mu$ is the first Betti number of the fiber.*

Connected divides $P$ such that the auxilary Morse function $f_P$ has only two critical values are in the world of divides a natural generalization of simple singularities, see [A'C3].

A natural problem is the classification of the knots, that come from simple divides. Since simple divides are "slalom" curves around trees we call the corresponding knots also slalom knots or divides.

Since the Alexander polynomial suffices for the classification of the knots of irreducible plane curve singularities, I had as first naive guess that the Alexander polynomial suffices too for the classification of slalom knots. This turned out to be very naive.

The following two slalom divides $P_1, P_2$ have equal abstract underlying trees. The knots $K(P_1)$ and $K(P_2)$ are not equivalent, but related by a mutation. So, many invariants coincide. They have equal Khovanov homology as computed by Alexander Shumakovitch, although Khovanov homology is not invariant under mutation, as shown by Stefan Wehrli [W]. The knots $K(P_1)$ and $K(P_2)$ are hyperbolic and can be distinguised by the isometry groups of the their complements with the Mostow metrics. Yvonne Gerber has constructed the Thurston foliation on the fibers of hyperbolic slalom links [S]. Her method allows to distinguish the knots $K(P_1)$ and $K(P_2)$ too. Soo the classification of slalom knots is not typically easy.

Instead of formulating properties, that the knots $K(P_1)$ and $K(P_2)$ do not have in common, I ask the following question: Do the fundamental groups of the complements of the knots $K(P_1)$ and $K(P_2)$ have isomorphic profinite completions?

A positive answer would explain why the two knots $K(P_1)$ and $K(P_2)$ are not easy to separate by knot invariants.

If the answer is yes, a reason could be that the monodromies of the knots $K(P_1)$ and $K(P_2)$ have the same orbit under the action of $G_{\mathbf{Q}} := \mathbf{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ on the profinite completion of the mapping class group $\mathbf{Mod}_{5,1}$.

The second part of the talk is devoted to construct by using the Theorem of Belyi a polynomial function $f_\Gamma : \mathbf{C}^2 \to \mathbf{C}$ for an finite planar graph. The coefficients of $f_\Gamma$ belong to a number field. We are especially interested to detect the monodromy of the slalom link of the given graph as a global monodromy of the mapping $f_\Gamma : \mathbf{C}^2 \to \mathbf{C}$.

As work in progress we propose a construction, such that a Galois transformation of the number field acts. The aim is to relate by a Galois action the monodromies of the slalom knots of trees, that have Galois related Belyi functions.

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