On Injectivity Of The Satoh Lifting Of Modular Forms And The Taylor Coefficients Of Jacobi Forms

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Abstract

In this paper many interesting properties of the Satoh map from elliptic cusp forms to Jacobi forms are proven. Moreover an important link to the Gross-Prasad conjecture is established.

Introduction

The Gross-Prasad conjecture [G-P92] proclaims fundamental insights into the pairing of automorphic forms restricted to subdomains with automorphic forms living there. The non-vanishing of this pairing is deeply related to the behaviour of certain automorphic L-functions at the center of symmetry and properties of special values. Recently huge progress was performed regarding to this conjecture by S. Böcherer, M. Furusawa, R. Schulze-Pillot and A. Ichino (e.g. [B-F-S05], [Ich05]).

One of the most interesting research areas towards proving this conjecture at this time is the setting of automorphic forms with respect to the symplectic group of degree 2. More precisely, let F be a Siegel modular form of degree 2. Let us consider these forms being evaluated on the subdomain $\mathbb{H} \times \mathbb{H}$, the cartesian product of two copies of the Siegel upper half-space. The resulting function we denote by φ_F. Let further f, g be elliptic cusp forms. Then the conjecture relates to the multi-Petersson scalar product

$$\langle\langle \varphi_F\,,\, f \otimes g \rangle\rangle \tag{1}$$

with a certain special value of a L-function attached to f, g and F and a fudge factor. Recently a beautiful formula for this expression has been found by A. Ichino, for F is considered as a Maass lift.

It is remarkable, that with a different viewpoint in mind, the author who did not know about the Gross-Prasad conjecture, also worked on the same topic [Hei98]. He searched for integral representations of automorphic L-functions using the Rankin-Selberg method. He found out that by generalizing the work of P. Garrett [Ga84] naturally expressions of the form (1) came up. In addition several interesting results had been proven. For example, let f and g be Hecke eigenforms linear independent, then

$$\langle\langle \varphi_F \, , \, f \otimes g \rangle\rangle = 0. \tag{2}$$

Encouraged by this we proceed further with our origin ideas and provide different kind of generalizations. Let F be a Siegel cusp form of degree 2, which is a Maass lift attached to an elliptic cusp form, which is an Hecke eigenform. It is well known that the first Fourier-Jacobi coefficient Φ_1^F of F is a Jacobi cusp form on $\mathbb{H} \times \mathbb{C}$. Then (1) can be reduced to the task of calculating the scalar product

$$\langle \mathcal{D}_0(\Phi_1^F) \, , \, g \rangle, \tag{3}$$

here $\mathcal{D}_0(\Phi_1^F)$ is the 0-th Taylor coefficient of Φ_1^F with respect to the Jacobi variable. We can generalize this to higher Taylor coefficients:

$$\langle \mathcal{D}_\nu(\Phi_1^F) \, , \, g \rangle, \tag{4}$$

where $\mathcal{D}_\nu(\Phi_1^F)$ are modular forms attached to the higher Taylor coefficients up to the degree ν ([E-Z85], §3). We believe that the given formulas are deeply related to the special values near the center of symmetry of the involved L-function ([Ich05]). We found out [Hei98], and prove now in the most general context that the operators $\mathcal{D}_\nu$ are the adjoint operators to the mapping introduced by T. Satoh [Sa89]. Next, and this is much more relevant, we answer several important questions raised by Satoh. For example Satoh asked in which cases his construction does not vanish. Even in the case of Jacobi forms of index 1 there had been no answer given until now. More precisely we prove that for example in this context Satohs approach never vanishes. It turns out that the questions of Satoh are not only deeply related to the Gross-Prasad conjecture, they also lead for example to the unsolved questions of the following type:

> Let $\tilde{g}$ be a modular form of half-integral weight, let us assume that all Fourier coefficients parametrized by fundamental discriminants vanish. Does then $\tilde{g}$ has to be already identically zero?

If $\tilde{g}$ is a cusp form which is an Hecke eigenform, then the answer is yes. Another question is the following:

> Is it possible, that the first Fourier-Jacobi coefficient of a Klingen Eisenstein series and the Siegel Eisenstein series are equal at all Fourier coefficients parametrized by fundamental discriminants?

This shows directly that the Gross-Prasad conjecture, the Satoh mapping and Taylor coefficients of Jacobi forms are worth to study and lead to a much better understanding of the field of automorphic forms and hence modern number theory.

Acknowledgement I would like to say thank you to Prof. Zagier for several motivating and inspireing discussions.

Notation
The transpose, determinant and trace of a matrix A will be denoted by $A^t, det(A)$ and $tr(A)$. For matrices A and B of appropriate size we set $A[B] = B^t AB$. For $z \in \mathbb{C}^{n,n}$ we set $e\{z\} = e^{2\pi i \mathrm{tr}(z)}$. Let $\mathbb{A}_2^m$ be the set of all half-integral positive definite matrices of size 2 and right lower entry m. By abuse of notation we put $[n, r, m] = \begin{pmatrix} n & \frac{r}{2} \\ \frac{r}{2} & m \end{pmatrix}$. Let $\lfloor x \rfloor = \max\{n \in \mathbb{Z} \mid n \leqslant x\}$.

1 Main results

1.1 Modular and Jacobi forms

In this paper we are mainly interested in properties of the space of Siegel modular forms M_k of weight k on the Siegel upper half space $\mathbb{H}$ of degree one with respect to $\Gamma = SL_2(\mathbb{Z})$ and Jacobi forms $J_{k,m}$ of weight k and index m on $\mathbb{H} \times \mathbb{C}$ with respect to the Jacobi group $\Gamma^J = \Gamma \ltimes H(\mathbb{Z})$. Here $H(\mathbb{Z})$ denotes the group of integral points of the Heisenberg group and S_k and $J_{k,m}^{\mathrm{cusp}}$ the subspaces of cusp forms. Further let $|_k$, $|_{k,m}$ and $|_m$ denote the corresponding slash operators and $\langle \cdot\,,\cdot \rangle$ and $\langle \cdot\;,\;\cdot \rangle_J$ the Petersson scalar products. Let $(\tau, z) \in \mathbb{H} \times \mathbb{C}$. Then we put $q = e\{\tau\}$ and $\xi = e\{z\}$. For all $g \in M_k$ we have Fourier coefficients $a_g(n)$. For $\Phi \in J_{k,m}$ we have the Fourier expansion

$$\Phi(\tau, z) = \sum_{\substack{n \in \mathbb{N}_0,\, r \in \mathbb{Z} \\ 4nm - r^2 \geq 0}} c(n, r)\, q^n \xi^r \tag{5}$$

and we have a Taylor expansion around $z = 0$:

$$\Phi(\tau, z) = \sum_{\nu=0}^{\infty} \chi_\nu(\tau)\, z^\nu. \tag{6}$$

In general the functions $\chi_\nu(\tau)$ are only modular forms in the case $\nu = 0$. Nevertheless there exists a map

$$\mathcal{D}_\nu^m : J_{k,m} \longrightarrow M_{k+\nu} \tag{7}$$

given by

$$\left(\mathcal{D}_\nu^m \Phi\right)(\tau) = \frac{(2\pi i)^{-\nu}\,\nu!}{(k+\frac{\nu}{2}-2)!} \sum_{0 \le \mu \le \frac{\nu}{2}} \frac{(-2\pi i m)^\mu (k+\nu-\mu-2)!}{\mu!}\;\chi_{\nu-2\mu}^{(\mu)}(\tau). \tag{8}$$

Here $\chi_n^{(\mu)}(\tau)$ denotes the μ-derivative of χ_n ([E-Z85], $\S 3$).

Remark 1.1 *Notice that $M_{2k+1} = 0$. We have $\mathcal{D}_0^m(\Phi) = \chi_0 \in M_k$. It is a cusp form if and only if Φ is a Jacobi cusp form. For convenience we put $\mathcal{D}_\nu = \mathcal{D}_\nu^1$.*

Special examples of cusp forms and Jacobi cusp forms are given by Poincarè series. Let $t \in \mathbb{N}$ and $T = [n, r, m] \in \mathbb{A}_2^m$ then

$$\begin{aligned} P_{k,t}(\tau) &:= \sum_{\Gamma_\infty \backslash \Gamma} q^t \,|_k\, \gamma \in S_k & (9)\\ P_{k,T}^J(\tau, z) &:= \sum_{\Gamma_\infty^J \backslash \Gamma^J} q^n \xi^r \,|_{k,m}\, \gamma \in J_{k,m}^{\mathrm{cusp}}. & (10) \end{aligned}$$

Here Γ_∞ is the subgroup of translations of Γ. And further we have the group $\Gamma_\infty^J = \Gamma_\infty \ltimes \{(0, \mu, \kappa) |\ \mu, \kappa \in \mathbb{Z}\}$.

1.2 The lifting of Satoh

Definition 1.2 *For non-negative integers λ and j the Gegenbauer polynomial is defined by*

$$\mathcal{C}_{2j}^\lambda(X) := \sum_{l=0}^{j} \frac{(-1)^j \lambda^{[2j-l]}}{l!(2j-2l)!} (2X)^{(2j-2l)}. \tag{11}$$

Here $\lambda^{[n]} = \lambda \cdot (\lambda+1) \cdot \ldots \cdot (\lambda+n-1)$ if $n > 0$ and $\lambda^{(0)} = 1$ else (see also [Sa89], 2.6).

Definition 1.3 *Let $T = \begin{pmatrix} n & \frac{r}{2} \\ \frac{r}{2} & m \end{pmatrix} \in \mathbb{A}_2^m$ and $k \geqslant 4$ be an even integer and $j \in \mathbb{N}_0$ and $s \in \mathbb{C}$ sufficiently large. Then we attach to $g \in S_{k+2j}$ the Dirichlet series:*

$$\tilde{L}_{k,j}(T, g; s) := \sum_{t \in \mathbb{Z}} a(T[\tbinom{1}{t}]) \left(T[\tbinom{1}{t}]\right)^{-s}. \tag{12}$$

Definition 1.4 *Let $k, m \in \mathbb{N}$ and k be an even. Let $\nu \in \mathbb{N}_0$. Then we denote by*

$$\{\}_{k,m}^{\nu} : \begin{cases} S_{k+2\nu} & \longrightarrow J_{k,m}^{\text{cusp}} \\ g(\tau) = \sum a(n) q^n & \longmapsto \sum_{T=[n,r,m]\in\mathbb{A}_2^m} (4\det T)^{k-\frac{3}{2}} \\ & \qquad \tilde{L}_{k,\nu}(T, g; k-1+\nu)\, q^n \xi^r \end{cases} \tag{13}$$

the **Satoh map**, *which is well defined. To simplify we put $\{\ \}_{k,m} = \{\ \}_{k,m}^{0}$.*

Remark 1.5 *(i) T. Satoh has first introduced and studied the map $\{\}_{k,m}^{\nu}$. He has shown by an explicit construction, using a famous formula from Cohen and Kuznetsov (see [E-Z85], Theorem 3.3 for details) and a certain average operator, that the infinite sum is well defined and has the expected properties.*

(ii) In this paper we give a new and independent proof of Satohs results for the most general operator $\{\}_{k,m}^{\nu}$.

Now we are ready to state the first main result of this paper. The proof will be given in section 2.

Theorem 1.6 *Let $k, m, \lambda \in \mathbb{N}_0$ and k be even and $k, m > 0$. Let $\Phi \in J_{k,m}^{\text{cusp}}$ and $g \in S_{k+2\nu}$. Then*

$$\langle \mathcal{D}_{\nu}^{m} \Phi, g \rangle = \alpha_{\nu,m,k} \langle \Phi, \{g\}_{k,m}^{2\nu} \rangle_J, \tag{14}$$

where

$$\alpha_{\nu,m,k} = \frac{(2\nu)!(k-2)!}{(k+\nu-2)!} \frac{\Gamma(k+2\nu-1}{\Gamma(k-\frac{3}{2})} m^{\nu-k+2}\, \pi^{(-2\nu-\frac{1}{2})}\, 2^{4-2k-4\nu} \tag{15}$$

Hence the two operators $\mathcal{D}_{\nu}^{m}$ and $\{g\}_{k,m}^{2\nu}$ are adjoint with respect to Petersson scalar products.

1.3 On the injectivity of the Satoh map

Let us consider now the map

$$\mathcal{D} := \bigoplus_{\nu=0}^{m} \mathcal{D}_{2\nu}^{m} : J_{k,m} \longmapsto M_k \oplus \bigoplus_{\nu=1}^{m} S_{k+2\nu} \tag{16}$$

It is known ([E-Z85], Theorem 8.5) that every $\Phi(\tau, z) \in J_{k,m}$, for fixed $\tau \in \mathbb{H}$ if not identically zero, has exactly $2m$ zeros in every fundamental domain. Hence the map $\mathcal{D}$ is injectiv and

$$\dim J_{k,m} \leqslant \dim M_k + \sum_{\nu=0}^{m} \dim S_{k+2\nu}. \tag{17}$$

We are also interested in the restriction of $\mathcal{D}$ on the space of cuspidal forms:

$$\mathcal{D}^{\,\mathrm{cusp}} : J_{k,m}^{\mathrm{cusp}} \longmapsto \bigoplus_{\nu=0}^{m} S_{k+2\nu}. \tag{18}$$

Before the most complicated general case is considered, we start with the results in the important case $m = 1$, which are very nice. Gathering and reorganizing insights from the standard work on Jacobi forms [E-Z85], we state

Proposition 1.7 *Let $\Delta(\tau) = q \prod_{n=1}^{\infty}(1-q^n)^{24}$ and $E_k = \sum_{\Gamma_\infty \backslash \Gamma} 1 \,|_k\, \gamma$. Then we have in the case of index one:*

a) the maps $\mathcal{D}$ and $\mathcal{D}^{\,\mathrm{cusp}}$ are isomorphisms,

b) the map

$$\begin{aligned} M_{k-10} \oplus M_{k-12} &\longrightarrow \mathcal{J}_{*,1}^{\mathrm{cusp}} && (19)\\ (f, g) &\longmapsto f(\tau)\,\Phi_{10,1}(\tau, z) + g(\tau)\,\Phi_{12,1}(\tau, z) && (20) \end{aligned}$$

is an isomorphism. Here $\Phi_{10,1}$ and $\Phi_{12,1}$ denote the wellknown Jacobi cusp forms of weight 10 and 12 (see [E-Z85], §3),

c) the Jacobi forms

$$E_4(\tau)^a\, E_6(\tau)^b\, \Phi_{j,1}(\tau, z); \qquad a, b \geqslant 0,\, j \in \{10, 12\},\, 4a+6b+j = k \tag{21}$$

form an additive basis of the space of Jacobi cusp forms of weight k and index 1

d) and we have the identities

$$\begin{aligned} \mathcal{D}_0\Phi_{10,1} &= \mathcal{D}_2\Phi_{12,1} &&= 0\\ \mathcal{D}_0\Phi_{12,1} &= \frac{12}{20}\,\mathcal{D}_2\Phi_{10,1} &&= 12\Delta. \end{aligned}$$

In his fundamental work T. Satoh ([Sa89],second part of $\S 3$) raised the question about non-vanishing of $\{\}_{k,m}^{\nu}$ and stated in a honest way, that for $\nu > 0$ he has no idea how to proceed and gave two important obstacles. The new approach in this article is able to bring some light to this topic. Even in the "easiest case" of index 1, the question has been open until now. Nevertheless we have:

Corollary 1.8 *The Satoh maps $\{\quad\}_{k,1}^{0}$ and $\{\quad\}_{k,1}^{2}$ are injectiv whereas $\{\quad\}_{12,1}^{12}$ is not.*

Proof:
We have to consider $\mathcal{D}_{\nu}^{m}$ for $m = 1$ and examine in which cases the operator is surjectiv. This gives a transparent way, how the general case can be solved and reduces a problem of infinite analysis to finite algebra, which is a enormous insight. Now since we know that $\mathcal{D}_{12}(\mathcal{J}_{12,1}^{\text{cusp}}) \subseteq S_{24}$ the corollar is proven.

$\square$

We are ready now to state our main results

Theorem 1.9 *Let k and m be natural numbers and $k > 2$ be even. Then the Sato map*

$$\{\ \}_{k,m} : S_k \longrightarrow J_{k,m}^{\text{cusp}} \tag{22}$$

is injectiv for all weights k which are larger or equal to the index m.

Theorem 1.10 *Let k and m be natural numbers and $k > 2$ be even. Let us assume that $k \geqslant m$. Then we have: The Satoh map $\{\}_{k,m}^{\nu}$ is injectiv for all $0 \leqslant \nu \leqslant m$ if and only if $m \in \{1, 2, 3\}$.*

2 Proofs

To complete this article we have to give the remaining proofs. Let us start with Theorem 1.6

Proof:
First we prove a pullback formula for $\mathcal{D}_{2\nu}^{m} P_{k,T}^{J}$ for $T \in \mathbb{A}_2^m$ with respect to the embedding

$$\mathbb{H} \hookrightarrow \mathbb{H} \times \mathbb{C}.$$

We utilize certain invariant properties of the differential operator. Then we calculate all the Fourier coefficients of $(\mathcal{D}_{2\nu}^m)^{\#}(g)$.
Let $\Phi : \mathbb{H} \times \mathbb{C} \longrightarrow \mathbb{H}$ and $X \in \mathbb{Z} \times \mathbb{Z} \times \{0\}$ and $\gamma \in SL_2(\mathbb{R})$. Then we have

$$\mathcal{D}_{2\nu}^m (\Phi \mid_m X \mid_{k,m} \gamma) = \mathcal{D}_{2\nu}^m (\Phi \mid_m X) \mid_{k+2\nu,m} \gamma. \tag{23}$$

Hence the action of the SL_2-part of the Jacobi group intertwines with the Operator $\mathcal{D}_{2\nu}^m$ by raising the weight by 2ν (see [E-Z85], §3 (11)).
The point is now, that we can find a representative sytem of $\Gamma_\infty^J \backslash \Gamma^J$ of the form

$$(\text{Heisenberg-part}) \times (SL_2 - \text{part}).$$

We have

$$P_{k,T}^J(\tau, z) = \sum_{\lambda \in \mathbb{Z}} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} (q^n \xi^r) \mid_m (\lambda, 0, 0) \mid_{k,m} \gamma$$

and the inner operation turns out to be easy calculable

$$\begin{aligned} \varphi_{T,\lambda}(\tau, z) &= (q^n \xi^r) \mid_m (\lambda, 0, 0) && (24)\\ &= e\{T[\tbinom{1}{\lambda}]\tau + (r + 2m\lambda)z\} && (25) \end{aligned}$$

We are going to use the following result. The proof is straight forward. Here I would like to thank Prof. Böcherer for a clarifying discussion.

Lemma 2.1 *The action of the differential operator on the test function $\varphi_{T,\lambda}$ is given by*

$$\begin{aligned} \left(\mathcal{D}_{2\nu}^{(m)}(\varphi_{T,\lambda})\right)(\tau) &= \frac{(2\nu)!(mT[\tbinom{1}{\lambda}]^\nu (k-2)!}{(k+\nu-2)!} && (26)\\ &\quad \times \mathcal{C}_{\mathbf{2\nu}}^{\mathbf{k-1}} \left(\frac{2\, m\lambda + r}{2(mT[\tbinom{1}{\lambda})]^{\frac{1}{2}}} \right) e\{T[\tbinom{1}{\lambda}])\tau\}. && (27) \end{aligned}$$

Now we are ready to obtain the desired pullback formula for Jacobi Poincarè series:

$$\left(\mathcal{D}_{2\nu}^m(P_{k,T}^J)\right)(\tau) = \frac{(2\nu)!(k-2)!}{(k+\nu-2)!}\, m^\nu \sum_{\lambda \in \mathbb{Z}} \mathcal{C}_{\mathbf{2\nu}}^{\mathbf{k-1}} \left(\frac{2\, m\lambda + r}{2(mT[\tbinom{1}{\lambda}])^{\frac{1}{2}}} \right) P_{k+2\nu, T[\binom{1}{\lambda}]}(\tau). \tag{28}$$

By definition we have:

$$\langle \mathcal{D}_{2\nu}^m P_{k,T}^J, g\rangle = \langle P_{k,T}^J, (\mathcal{D}_{2\nu}^m)^\sharp (g)\rangle_J$$

Using well known properties of Poincarè series (e.g. [Hei98] (4)(5)) we have for the left side

$$\frac{(2\nu)!(k-2)!}{(k+\nu-2)!}\, m^\nu\, 2\, c_{1,k+2\nu} \sum_{\lambda\in\mathbb{Z}} [\tbinom{1}{\lambda}]^{(k+2\nu-1)}\, a_g\left([\tbinom{1}{\lambda}]\right) \mathcal{C}_{\mathbf{2\nu}}^{\mathbf{k-1}} \left(\frac{2\, m\lambda + r}{2(mT[\binom{1}{\lambda}]^{\frac{1}{2}}}\right)$$
$$= \frac{(2\nu)!(k-2)!}{(k+\nu-2)!}\, m^\nu\, 2\, c_{1,k+2\nu} \tilde{L}_{k,\nu}(T, g; k+2\nu-1) \quad (29)$$

and on the right side we have:

$$2\, c_{2,k}^J\, m^{k-2} \mathsf{det}(T)^{-(k-\frac{3}{2})}\;\; a_{\mathcal{D}_{2\nu}^m{}^\sharp(g)}(T). \quad (30)$$

So we have for all $T \in \mathbb{A}_2^m$ and $g \in S_{k+2\nu}$:

$$\mathcal{D}_{2\nu}^{m\,\sharp}(g)(T) = \frac{(2\nu)!(k-2)!}{(k+\nu-2)!} \frac{c_{1,k+2\nu}}{c_{2,k}^J} m^{\nu-k+2}\, 2^{3-2k}\;\; \{g\}_{k,m}^{2\nu} \quad (31)$$

This leads tp the desired result.

□

It remains to finish the proof of (Theorem 1.9). We would like to mention, that the proof is worthwile to consider in the future also for other questions.

Proof:
We have seen that $\mathcal{D}_{2\nu}^m$ is the adjoint operator of the Satoh map $\{\}_{k,m}^\nu$. Hence we can proof the theorem by examine several surjectiv properties of the operator $\mathcal{D}^{\,\mathrm{cusp}}$. More general we consider k, m arbitrary and the vector space $J_{k,m}$. Then the subspaces

$$J_{k,m}^{(\nu)} = \{\Phi \in J_{k,m} \,|\, \Phi(\tau, z) = O(z^\nu) \text{ for } z \to 0\} \quad (32)$$

lead to a filtration, with $J_{k,m}^{(0)} = J_{k,m}$ and $J_{k,m}^{(2m+2)} = \{0\}$. Let us denote $M_k^{\nu,m} = \{f \in M_{k+2\nu} \mid \sum_{n \geqslant \frac{\nu^2}{4m}} a_f(n) q^n\}$. Then we have an exact sequenze

$$0 \longrightarrow J_{k,m}^{(\nu+1)} \longrightarrow J_{k,m}^{(\nu)} \xrightarrow{p_\nu} M_k^{\nu,m} \quad (33)$$

where $p_\nu(f(\tau)z^\nu + O(z^{\nu+1}) = f(\tau)$ ([E-Z85], §9). Now it follows from Theorem 9.2 [E-Z85] that for all p_ν and $k \geqslant m$ even the sequenze

$$0 \longrightarrow J_{k,m}^{(\nu+1)} \longrightarrow J_{k,m}^{(\nu)} \xrightarrow{p_\nu} M_k^{\nu,m} \longrightarrow 0 \tag{34}$$

is exact. Since $p_0 = \mathcal{D}_0^m$ we have

$$\mathcal{D}_0^m\left(J_{k,m}\right) = M_k \qquad \text{for } k \geqslant m.$$

Mainly we are interested in the surjectivity of $\mathcal{D}_0^m\mid_{J_{k,m}^{\text{cusp}}}$. It is known that

$$J_{k,m} = J_{k,m}^{\text{Eis}} \oplus J_{k,m}^{\text{cusp}}, \tag{35}$$

the direct sum of the subspaces of Eisenstein series and cusp forms. Moreover let $m = ab^2$, with a square-free, then since $k > 2$ we have

$$\dim J_{k,m}^{\text{Eis}} = \left\lfloor \frac{b}{2} \right\rfloor + 1. \tag{36}$$

Since $\dim J_{k,m} = \sum_{\nu=0}^m M_k^{\nu,m}$ and $\mathcal{D}_0^m(J_{k,m}^{\text{cusp}}) \subseteq S_k$, we can deduce for m square-free directly the theorem. For m arbitrary we have to proceed in a more refined way.

Let us consider the vector spaces $J_{k,m}^{\text{cusp},(\nu)} = J_{k,m}^{\text{cusp}} \cap J_{k,m}^{(\nu)}$ and
$M_k^{\text{cusp},\nu,m} = \{f \in M_k^{\nu,m} \mid a_f(\lfloor \frac{\nu^2}{4m} \rfloor) = 0\}$. Then we obtain the exact sequence

$$0 \longrightarrow J_{k,m}^{\text{cusp},(\nu+1)} \longrightarrow J_{k,m}^{\text{cusp},(\nu)} \xrightarrow{p_\nu} M_k^{\text{cusp},\nu,m}. \tag{37}$$

This leads to

$$\dim J_{k,m}^{\text{cusp}} \leqslant \sum_{\nu=0}^{m} \dim M_k^{\text{cusp},\nu,m}. \tag{38}$$

A straightforward calculation shows

$$\sum_{\nu=0}^{m} \dim M_k^{\nu,m} - \sum_{\nu=0}^{m} \dim M_k^{\text{cusp},\nu,m} = J_{k,m}^{\text{Eis}}. \tag{39}$$

Putting everything carefully together we obtain

$$J_{k,m}^{\text{cusp}} = \sum_{\nu=0}^{m} \dim M_k^{\text{cusp},\nu,m}. \tag{40}$$

In general this is false for $2 < k < m$. Observing that p_ν is surjectiv and $S_k = M_k^{\text{cusp},0,m}$ gives the desired result.

$\square$

To proof Theorem 1.10 we have to examine the following question: When is the following expression valid:

$$\dim S_{k+2\nu} = \dim M_k^{\text{cusp},\nu,m}? \tag{41}$$

A short calculation leads to the result.

We would like to end this paper we an interesting non-trivial example. Let $k = m = 4$ then we have $\{\Delta\}_{4,4}^4 = 0$.

References

[B-F-S05] **S. Böcherer, M. Furusawa and R. Schulze-Pillot**, On the global Gross-Prasad conjecture for Yoshida liftings, *to appear*

[Co75] **H. Cohen**, Sums involving the values at negative integers of L-functions of quadratic characters, *Math. Ann.217* (1975), 271-285

[E-Z85] **M. Eichler and D. Zagier**, The theory of Jacobi forms, *Progress in Mathematics Vol. 55, Birkhäuser Verlag* (1985)

[Ga84] **P. Garrett**, Pullbacks of Eisenstein series; Applications. In: Automorphic forms of several variables. *Taniguchi symposium 1983, Birkhäuser* (1984)

[G-P92] **Gross and Prasad**, On the decomposition of a representation of $S0_n$ when restricted to $S0_{n-1}$, *Canad. J. Math. 46* (1992), 974-1002

[Hei98] **B. Heim**, Über Poincare Reihen und Restriktionsabbildungen, *Abh. Math. Sem. Univ. Hamburg 68* (1998), 79-89

[Ich05] **A. Ichino**, Pullbacks of Saito-Kurokawa Lifts, *Preprint, April, 2005*

[Ku75] **N.V. Kuznetsov**, A new class of identities for the Fourier coefficients of modular forms, *Acta. Arith. 27* (1975), 505-519

[Sa89] **T. Satoh**, Jacobi forms and certain special values of Dirichlet series asscociated to modular forms, *Math. Ann. 285* (1989), 463-480

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