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by

Michael E. Hoffman
Kentaro Ihara

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Michael E. Hoffman
Kentaro Ihara

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Department of Mathematics
U.S. Naval Academy
Annapolis, MD 21402
USA

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Michael E. Hoffman and Kentaro Ihara
U. S. Naval Academy
Annapolis, MD 21402 USA
and
Max-Planck Institut für Mathematik
Vivatsgasse 7, Bonn, D-53111 Germany
meh@usna.edu
k.ihara72@mpim-bonn.mpg.de

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Abstract

Quasi-shuffle products, introduced by the first author, have been useful in studying multiple zeta values and some of their analogues and generalizations. Recently the second author, together with Kajikawa, Ohno, and Okuda, significantly extended the definition of quasi-shuffle algebras so it could be applied to multiple q-zeta values. This article extends some of the algebraic machinery of the first author's original paper to the more general definition, and demonstrates how various algebraic formulas in the quasi-shuffle algebra can be obtained in a transparent way.

1 Introduction

The point of this article is, as the title indicates, to revisit the construction of quasi-shuffle products in [5]. In [7] the construction of [5] was put in a more general setting that had two chief advantages: (i) it simultaneously applied to "multiple zeta" and "star-multiple zeta" values and their extensions; and (ii) it could be applied to the q-series version of multiple zeta values studied in [3]. Here we show that some of the machinery developed in [5], particularly the coalgebra structure (not considered in [7]), can be carried over to the more general setting and used to make many of the calculations of [7] more transparent. We also describe some applications of quasi-shuffle algebras not considered in [7].

The original quasi-shuffle product was inspired by the multiplication of multiple zeta values, i.e.,

$$\sum_{n_1>\cdots>n_k\geq 1} \frac{1}{n_1^{i_1}\cdots n_k^{i_k}}, \tag{1}$$

with $i_1 > 1$ to insure convergence. One can associate to the series (1) the monomial $z_{i_1}\cdots z_{i_k}$ in the noncommuting variables $z_1, z_2, \ldots$; then we write the value (1) as $\zeta(z_{i_1}\cdots z_{i_k})$. For any monomials $w = z_i w'$ and $v = z_j v'$, define the product $w * v$ recursively by

$$w * v = z_i(w' * v) + z_j(w * v') + z_{i+j}(w' * v'). \tag{2}$$

Then $\zeta(w)\zeta(v) = \zeta(w * v)$, where we think of ζ as a linear function on monomials. As we shall see in the next section, the recursive rule (2) is a quasi-shuffle product on monomials in the z_i derived from the product $\diamond$ on the vector space of z_i's given by $z_i \diamond z_j = z_{i+j}$.

In [3] the multiple q-zeta values were defined as

$$\sum_{n_1>\cdots>n_k\geq 1} \frac{q^{(i_1-1)n_1}\cdots q^{(i_k-1)n_k}}{[n_1]_q^{i_1}\cdots [n_k]_q^{i_k}}, \tag{3}$$

where $[n]_q = 1 + q + \cdots + q^{n-1} = (1-q^n)/(1-q)$. If we denote (3) by $\zeta_q(z_{i_1}\cdots z_{i_k})$, then to have $\zeta_q(w)\zeta_q(v) = \zeta_q(w * v)$ the recursion (2) must be significantly modified: in place of $z_i \diamond z_j = z_{i+j}$ we must have

$$z_i \diamond z_j = z_{i+j} + (1-q)z_{i+j-1}.$$

This means that to have a theory of quasi-shuffle algebras that applies to multiple q-zeta values, two restrictions in the original construction of [5] must be removed: that the product $a \diamond b$ of two letters be a letter, and that the operation $\diamond$ preserve a grading. This was done in [7]. The same paper also addressed the relation between multiple zeta values (1) and the "star-multiple zeta values"

$$\zeta^\star(z_{i_1}\cdots z_{i_k}) = \sum_{n_1\ge\cdots\ge n_k\ge 1} \frac{1}{n_1^{i_1}\cdots n_k^{i_k}}. \tag{4}$$

This relation can be expressed in terms of a linear isomorphism (here denoted Σ) from the vector space of monomials in the z_i's to itself. The function Σ acts on monomials as, e.g.,

$$\Sigma(z_iz_jz_k) = z_iz_jz_k + (z_i \diamond z_j)z_k + z_i(z_j \diamond z_k) + z_i \diamond z_j \diamond z_k$$

and then $\zeta^\star(w) = \zeta(\Sigma(w))$. If we define analogously "star-multiple q-zeta values" $\zeta_q^\star(w)$, then $\zeta_q^\star(w) = \zeta_q(\Sigma(w))$.

Important properties of Σ were established in [7], though some of the inductive proofs are tedious. Here we make use of two aspects of the theory developed in [5] not used in [7]. First, for any formal power series

$$f = c_1t + c_2t^2 + \cdots$$

with $c_1 \ne 0$, it is possible to define a linear isomorphism (but not necessarily an algebra homomorphism) Ψ_f from (the vector space underlying) the quasi-shuffle algebra to itself. This process respects composition (i.e., $\Psi_{f\circ g} = \Psi_f\Psi_g$), and many important isomorphisms can be represented this way, e.g., $\Sigma = \Psi_{\frac{t}{1-t}}$. Second, the quasi-shuffle algebra together with the "deconcatenation" coproduct is a Hopf algebra: in fact, it turns out that its antipode is closely related to Σ.

This paper is organized as follows. In §2 we define the quasi-shuffle products $*$ and $\star$ on the vector space $k\langle A\rangle$, where A is a set of noncommuting letters equipped with a commutative product $\diamond$. Then in §3 we explain how to obtain linear isomorphisms from $k\langle A\rangle$ to itself from formal power series: as noted above, this gives a useful representation of Σ. In §4 we describe three Hopf algebras: the ordinary Hopf algebras $(k\langle A\rangle, *, \Delta)$ and $(k\langle A\rangle, \star, \Delta)$, and the infinitesimal Hopf algebra $(k\langle A\rangle, \diamond, \tilde\Delta)$, where Δ is deconcatenation, $\tilde\Delta(w) = \Delta(w) - w\otimes 1 - 1\otimes w$, and $\diamond$ is an extension of the original operation on A to a (noncommutative) product on $k\langle A\rangle$. Each of these Hopf algebras is associated with a represention of Σ via the antipode. In §5 we apply the machinery of the preceding two sections to obtain many of the algebraic formulas of [7] (and generalizations thereof) in a transparent way. Finally, in §6 we illustrate one of these algebraic formulas (specifically Theorem 5.2 below) for four different homomorphic images of quasi-shuffle algebras.

2 The quasi-shuffle products

We start with a field k containing $\mathbf{Q}$, and a countable set A of "letters". We let kA be the vector space with A as basis, and suppose there is an associative and commutative product $\diamond$ on kA.

Now let $k\langle A\rangle$ be the noncommutative polynomial algebra over A. So $k\langle A\rangle$ is the vector space over k generated by "words" (monomials) $a_1a_2\cdots a_n$, with $a_i \in A$: a word $w = a_1\cdots a_n$ has length $\ell(w) = n$. (We think of 1 as the empty word, and set $\ell(1) = 0$.) Following [7], we define two k-bilinear products $*$ and $\star$ on $k\langle A\rangle$ by making $1 \in k\langle A\rangle$ the identity element for each product, and requiring that $*$ and $\star$ satisfy the relations

$$aw * bv = a(w * bv) + b(aw * v) + (a \diamond b)(w * v) \tag{5}$$
$$aw \star bv = a(w \star bv) + b(aw \star v) - (a \diamond b)(w \star v) \tag{6}$$

for all $a, b \in A$ and all monomials w, v in $k\langle A\rangle$. As in [5] we have the following result.

Theorem 2.1. *If equipped with either the product $*$ or the product $\star$, the vector space $k\langle A\rangle$ becomes a commutative algebra.*

Proof. We prove the result for $*$, as the proof for $\star$ is almost identical. It suffices to show that $*$ is commutative and associative. For commutativity, it is enough to show that $u_1 * u_2 = u_2 * u_1$ for words u_1, u_2: we proceed by induction on $\ell(u_1) + \ell(u_2)$. This is trivial if either u_1 or u_2 is empty, so write $u_1 = aw$ and $u_2 = bv$ for $a, b \in A$ and words w, v. Then by equation (5),

$$u_1 * u_2 - u_2 * u_1 = (a \diamond b)(w * v) - (b \diamond a)(v * w),$$

and the right-hand side is zero by the induction hypothesis and the commutativity of $\diamond$.

Similarly, to prove associativity it is enough to show that $u_1 * (u_2 * u_3) = (u_1 * u_2) * u_3$ for words u_1, u_2, u_3, and this can be done by induction on $\ell(u_1) + \ell(u_2) + \ell(u_3)$. The required identity is trivial if any of u_1, u_2, u_3 is 1, so we can write $u_1 = aw$, $u_2 = bv$, and $u_3 = cy$ for $a, b, c \in A$ and words w, v, y. Then

$$\begin{aligned}
u_1*(u_2*u_3)-(u_1*u_2)*u_3 &= a(w*b(v*cy)+b(aw*(v*cy))+(a\diamond b)(w*(v*cy)) \\
&+ a(w*c(bv*y))+c(aw*(bv*y))+(a\diamond c)(w*(bv*y)) \\
&+ a(w*(b\diamond c)(v*y))+(b\diamond c)(aw*(v*y))+(a\diamond(b\diamond c))(w*(v*y)) \\
&- a((w*bv)*cy)-c(a(w*bv)*y)-(a\diamond c)((w*bv)*y) \\
&- b((aw*v)*cy)-c(b(aw*v)*y)-(b\diamond c)((aw*v)*y) \\
&- (a\diamond b)((w*v)*cy)-c((a\diamond b)(w*v)*y)-((a\diamond b)\diamond c)((w*v)*y) = \\
a(w*(bv*cy))+c(aw*(bv*y))&-a((w*bv)*cy)-c((aw*bv)*y)=0,
\end{aligned}$$

by the induction hypothesis and the associativity of $\diamond$. □

If the product $\diamond$ is identically zero, then $*$ and $\star$ coincide with the usual shuffle product $⧢$ on $k\langle A\rangle$. We call both $*$ and $\star$ quasi-shuffle products.

We note that $\diamond$ can be extended to a product of on all of $k\langle A\rangle$ by defining $1\diamond w = w\diamond 1$ for all words w, and $w\diamond v = w'(a\diamond b)v'$ for nonempty words $w = w'a$ and $v = bv'$ (where a, b are letters). Then $(k\langle A\rangle, \diamond)$ is a noncommutative algebra that contains the commutative subalgebra $k1 + kA$.

3 Linear maps induced by power series

Let $a_1, a_2, \ldots a_n \in A$. If $w = a_1a_2\cdots a_n$, and $I = (i_1, \ldots, i_m)$ is a composition of n (i.e., a sequence of positive integers whose sum is n), define (as in [5])

$$I[w] = (a_1\diamond\cdots\diamond a_{i_1})(a_{i_1+1}\diamond\cdots\diamond a_{i_1+i_2})\cdots(a_{i_1+\cdots+i_{m-1}+1}\diamond\cdots\diamond a_n). \quad (7)$$

We call $n = |I|$ the weight of the composition of I, and $m = \ell(I)$ its length. Note that the parentheses in equation (7) are not really necessary: the right-hand side is simultaneously an m-fold product in the concatenation algebra $k\langle A\rangle$ and a product of length

$$1 + (i_1 - 1) + (i_2 - 1) + \cdots + (i_m - 1) = n + 1 - m$$

in the algebra $(k\langle A\rangle, \diamond)$. If we set

$$I\langle w\rangle = a_1\cdots a_{i_1}\diamond a_{i_1+1}\cdots a_{i_1+i_2}\diamond\cdots\diamond a_{i_1+\cdots+i_{m-1}+1}\cdots a_n,$$

so, e.g.,

$$(2,1,2)[a_1a_2a_3a_4a_5] = a_1 \diamond a_2a_3a_4 \diamond a_5 = (1,3,1)\langle a_1a_2a_3a_4a_5\rangle,$$

then $I[w] = I^*\langle w\rangle$ defines an involution $*$ on compositions such that $|I^*| = |I|$ and $\ell(I^*) = |I| + 1 - \ell(I)$.

Let $\mathcal{P} \subset k[[t]]$ be the set of formal power series

$$f = c_1t + c_2t^2 + c_3t^3 + \cdots$$

with $c_1 \neq 0$. For $f \in \mathcal{P}$ we define the k-linear map $\Psi_f : k\langle A\rangle \to k\langle A\rangle$ by

$$\Psi_f(w) = \sum_{I=(i_1,\ldots,i_m)\in\mathcal{C}(\ell(w))} c_{i_1}\cdots c_{i_m} I[w], \tag{8}$$

where $\mathcal{C}(n)$ is the set of compositions of n.

Any two "functions" $f, g \in \mathcal{P}$, say

$$f = \sum_{i\geq 1} c_it^i \in k[[t]] \quad \text{and} \quad g = \sum_{i\geq 1} d_it^i \in k[[t]], \quad c_1d_1 \neq 0,$$

have a "functional composition"

$$\begin{aligned} f\circ g = \sum_{i\geq 1} c_ig^i &= c_1(d_1t + d_2t^2 + \cdots) + c_2(d_1t + d_2t^2 + \cdots)^2 + \cdots \\ &= c_1d_1t + (c_1d_2 + c_2d_1^2)t^2 + \cdots \in \mathcal{P}. \end{aligned}$$

Writing $[t^i]f$ for the coefficient of t^i in $f \in k[[t]]$, it is not hard to see that

$$[t^k]f\circ g = \sum_{j=1}^{k} [t^j]f[t^k]g^j. \tag{9}$$

The following result generalizes Lemma 2.4 of [5].

Theorem 3.1. *For $f, g \in \mathcal{P}$, $\Psi_f\Psi_g = \Psi_{f\circ g}$.*

Proof. Since

$$\Psi_g(w) = \sum_{I=(i_1,\ldots,i_m)\in\mathbf{C}(\ell(w))} [t^{i_1}]g\cdots[t^{i_m}]gI[w]$$

we have

$$\Psi_f\Psi_g(w) = \sum_{m=1}^{\ell(w)} \sum_{J=(j_1,\ldots,j_l)\in\mathbf{C}(m)} \sum_{I=(i_1,\ldots,i_m)\in\mathbf{C}(\ell(w))} [t^{j_1}]f\cdots[t^{j_l}]f[t^{i_1}]g\cdots[t^{i_m}]gJ[I[w]].$$

On the other hand,

$$\Psi_{f\circ g}(w) = \sum_{K=(k_1,\ldots,k_l)\in\ell(w)} [t^{k_1}]f\circ g\cdots[t^{k_l}]f\circ gK[w],$$

so we need to show that, for all compositions $K=(k_1,\ldots,k_l)\in\mathbf{C}(n)$,

$$[t^{k_1}]f\circ g\cdots[t^{k_l}]f\circ g = \sum_{m=1}^{n} \sum_{\substack{J=(j_1,\ldots,j_l)\in\mathbf{C}(m)\\ JI=K}} \sum_{I=(i_1,\ldots,i_m)\in\mathbf{C}(n)} [t^{j_1}]f\cdots[t^{j_l}]f[t^{i_1}]g\cdots[t^{i_m}]g \quad (10)$$

where

$$JI = (i_1+\cdots+i_{j_1}, i_{j_1+1}+\cdots+i_{j_1+j_2},\ldots,i_{j_1+\cdots+j_{l-1}+1}+\cdots+i_m)$$

is the obvious "composition" of the compositions $I=(i_1,\ldots,i_m)$ and $J=(j_1,\ldots,j_l)$, with $J\in\mathbf{C}(m)$. Now the right-hand side of equation (10) can be rewritten

$$\sum_{\substack{J=(j_1,\ldots,j_l)\\ JI=K}} \sum_{m=1}^{n} \sum_{I=(i_1,\ldots,i_m)\in\mathbf{C}(n)} \prod_{s=1}^{l}[t^{j_s}]f[t^{i_{j_1+\cdots+j_{s-1}+1}}]g\cdots[t^{i_{j_1+\cdots+j_s}}]g$$

$$= \sum_{\substack{J=(j_1,\ldots,j_l)\\ JI=K}} \prod_{s=1}^{l}[t^{j_s}]f[t^{k_s}]g^{j_s},$$

from which equation (10) follows by use of (9). □

3.1 The isomorphisms T and Σ

We now consider some particular linear isomorphisms from $k\langle A\rangle$ to itself. First, it is immediate from equation (8) that Ψ_t is the identity homomorphism of $k\langle A\rangle$. Now, following [7], consider

$$T = \Psi_{-t} \quad \text{and} \quad \Sigma = \Psi_{\frac{t}{1-t}}.$$

(The function we call Σ is written S in [7], but as in [5] we wish to reserve S for a Hopf algebra antipode.) For words w of $k\langle A\rangle$, $T(w) = (-1)^{\ell(w)}w$ and

$$\Sigma(w) = \sum_{I\in\mathbf{C}(\ell(w))} I[w].$$

Evidently T is an involution, and $\Sigma^{-1} = \Psi_{\frac{t}{1+t}}$ is given by

$$\Sigma^{-1}(w) = \sum_{I\in\mathbf{C}(\ell(w))} (-1)^{\ell(w)-\ell(I)} I[w],$$

where $\ell(I)$ is the number of parts of the composition I. We note that, for letters a and words w,

$$T(aw) = -aT(w) \tag{11}$$

$$\Sigma(aw) = a\Sigma(w) + a \diamond \Sigma(w) \tag{12}$$

$$\Sigma^{-1}(aw) = a\Sigma^{-1}(w) - a \diamond \Sigma^{-1}(w) \tag{13}$$

and (as in [7]) the property (12) can be used to define Σ. The functions Σ and T are not inverses, but we have the following result.

Corollary 3.1. *The functions Σ and T satisfy $T\Sigma T = \Sigma^{-1}$, and generate the infinite dihedral group.*

Proof. From Theorem 3.1 we have $\Sigma^n = \Psi_{\frac{t}{1-nt}}$, so all powers of Σ are distinct. We have also $T\Sigma T = \Psi_{\frac{t}{1+t}} = \Sigma^{-1}$. □

It follows immediately that $T\Sigma$ and ΣT are involutions (cf. [7, Prop. 2]). For future reference we note that the equation $\Sigma^p = \Psi_{\frac{t}{1-pt}}$ defines Σ^p for any $p \in k$: from Theorem 3.1 we have $\Sigma^p\Sigma^q = \Sigma^{p+q}$, and Σ^p is the pth iterate of Σ when p is an integer.

From [5] we have the (inverse) functions $\exp = \Psi_{e^t-1}$ and $\log = \Psi_{\log(1+t)}$. As shown in [5, Theorem 2.5], exp is an algebra isomorphism from $(k\langle A\rangle, ⊔⊔)$ to $(k\langle A\rangle, *)$. The functions exp and log are related to Σ and T as follows.

Corollary 3.2. $\Sigma = \exp T \log T$.

Proof. This is immediate from Theorem 3.1, since $\exp T = \Psi_{e^{-t}-1}$, $\log T = \Psi_{\log(1-t)}$, and $\log(1-t)$ composed with $e^{-t}-1$ gives

$$\frac{1}{1-t} - 1 = \frac{1-(1-t)}{1-t} = \frac{t}{1-t}.$$

□

We now turn to the algebraic properties of T and Σ.

Proposition 3.1. *$T : (k\langle A\rangle, *) \to (k\langle A\rangle, \star)$ and $T : (k\langle A\rangle, \star) \to (k\langle A\rangle, *)$ are homomorphisms.*

Proof. We prove the first statement; the second then follows because T is an involution. We shall show that $T(u_1 * u_2) = T(u_1) \star T(u_2)$ for any words u_1, u_2 by induction on $\ell(u_1) + \ell(u_2)$. The result is immediate if u_1 or u_2 is 1, so write $u_1 = aw$ and $u_2 = bv$ for letters a, b and words w, v. Then

$$\begin{aligned} T(u_1 * u_2) &= T(a(w * bv) + b(aw * v) + (a \diamond b)(w * v)) \\ &= -a(T(w) * T(bv)) - b(T(aw) * T(v)) - (a \diamond b)(T(w) * T(v)) \\ &= a(T(w) * bT(v)) + b(aT(w) * T(v)) - (a \diamond b)(T(w) * T(v)) \\ &= aT(w) \star bT(v) = T(u_1) \star T(u_2), \end{aligned}$$

where we have used the induction hypothesis and equation (11). □

The following result was proved as Theorem 1 in [7] in a much less direct way.

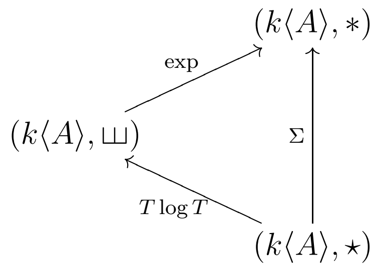
Corollary 3.3. *The linear isomorphism $\Sigma : (k\langle A\rangle, \star) \to (k\langle A\rangle, *)$ is an algebra isomorphism.*

Proof. This follows from Corollary 3.2, since Σ is the composition

$$(k\langle A\rangle, \star) \xrightarrow{T} (k\langle A\rangle, *) \xrightarrow{\log} (k\langle A\rangle, ⧢) \xrightarrow{T} (k\langle A\rangle, ⧢) \xrightarrow{\exp} (k\langle A\rangle, *)$$

of homomorphisms (that T is an endomorphism of $(k\langle A\rangle, ⧢)$ follows by taking $\diamond$ to be the zero product in Proposition 3.1). □

In fact, the following is a commutative diagram of algebra isomorphisms:



(14)

Corollary 3.4. *The involutions* $\Sigma T : (k\langle A\rangle, *) \to (k\langle A\rangle, *)$ *and* $T\Sigma : (k\langle A\rangle, \star) \to (k\langle A\rangle, \star)$ *are algebra automorphisms.*

Proof. Immediate from Proposition 3.1 and Corollary 3.3. □

3.2 A one-parameter family of automorphisms

Let $p \neq 0$ be an element of k, and set

$$H_p = \exp \Psi_{pt} \log$$

Evidently $H_1 = \text{id}$ and $H_p H_q = H_{pq}$, so this is a one-parameter family of isomorphisms of the vector space $k\langle A\rangle$. We can write $H_p = \Psi_{(1+t)^p - 1}$, where $(1+t)^p - 1$ is the power series

$$\sum_{n \geq 1} \binom{p}{n} t^n = pt + \frac{p(p-1)}{2!} t^2 + \frac{p(p-1)(p-2)}{3!} t^3 + \cdots .$$

From Corollary 3.2 $H_{-1} = \Sigma T$, so

$$H_{-1}(w * v) = \Sigma T(w * v) = \Sigma(T(w) \star T(v)) = H_{-1}(w) * H_{-1}(v)$$

for any words w, v. In fact, this property holds for all p.

Theorem 3.2. *For all* $p \neq 0$ *and words* w, v, $H_p(w * v) = H_p(w) * H_p(v)$.

Proof. Since

$$\Psi_{pt}(w) = p^{\ell(w)} w$$

for all words w, it follows that

$$\Psi_{pt}(w \sqcup\!\sqcup v) = \Psi_{pt}(w) \sqcup\!\sqcup \Psi_{pt}(v)$$

for all words w, v. Hence, since $\log(w * v) = \log w \sqcup\!\sqcup \log v$,

$$H_p(w * v) = \exp(\Psi_{pt}(\log w \sqcup\!\sqcup \log v)) = \exp(\Psi_{pt}(\log w) \sqcup\!\sqcup \Psi_{pt}(\log v))$$
$$= H_p(w) * H_P(v).$$

□

Thus, H_p is an automorphism of the algebra $(k\langle A\rangle, *)$. Note also that $TH_pT = \Psi_{1-(1-t)^p}$ is an automorphism of the algebra $(k\langle A\rangle, \star)$.

4 Hopf algebra structures

As in [5] we define a coproduct Δ on $k\langle A\rangle$ by

$$\Delta(w) = \sum_{uv=w} u \otimes v,$$

for words w, where the sum is over all pairs (u, v) of words with $uv = w$ including $(1, w)$ and $(w, 1)$, and a counit $\epsilon : k\langle A\rangle \to k$ by $\epsilon(1) = 1$ and $\epsilon(w) = 0$ for $\ell(w) > 0$. It will also be convenient to define the reduced coproduct $\tilde{\Delta}$ by $\tilde{\Delta}(1) = 0$ and $\tilde{\Delta}(w) = \Delta(w) - w \otimes 1 - 1 \otimes w$ for nonempty words w.

The coproduct can be used to define a convolution product on the set $\mathrm{Hom}_k(k\langle A\rangle, k\langle A\rangle)$ of k-linear maps from $k\langle A\rangle$ to itself, which we denote by $\odot$: for $L_1, L_2 \in \mathrm{Hom}_k(k\langle A\rangle, k\langle A\rangle)$ and words w of $k\langle A\rangle$,

$$L_1 \odot L_2(w) = \sum_{uv=w} L_1(u)L_2(v).$$

(The reader is warned that this is *not* the usual convolution for either of the Hopf algebras defined below.) The convolution product $\odot$ has unit element $\eta\epsilon$, where $\eta : k \to k\langle A\rangle$ is the unit map (i.e., it sends $1 \in k$ to $1 \in k\langle A\rangle$). It is easy to show that any $L \in \mathrm{Hom}_k(k\langle A\rangle, k\langle A\rangle)$ with $L(1) = 1$ has a convolutional inverse, which we denote by $L^{\odot(-1)}$.

We call $C \in \mathrm{Hom}_k(k\langle A\rangle, k\langle A\rangle)$ a contraction if $C(1) = 0$ and $C(w)$ is primitive for all words w, and $E \in \mathrm{Hom}_k(k\langle A\rangle, k\langle A\rangle)$ an expansion if $E(1) = 1$ and E is a coalgebra map. If C is a contraction and E is an expansion, we say (E, C) is an inverse pair if

$$E = (\eta\epsilon - C)^{\odot(-1)} = \eta\epsilon + C + C \odot C + C \odot C \odot C + \cdots \tag{15}$$

or equivalently

$$C = \eta\epsilon - E^{\odot(-1)} \tag{16}$$

Proposition 4.1. *Suppose* $C \in \mathrm{Hom}_k(k\langle A\rangle, k\langle A\rangle)$ *is a contraction and* E *is given by equation (15). Then* (E, C) *is an inverse pair. Conversely, if* $E \in \mathrm{Hom}_k(k\langle A\rangle, k\langle A\rangle)$ *is an expansion and* C *is given by equation (16), then* (E, C) *is an inverse pair.*

Proof. Suppose first that C is a contraction. Evidently $E(1) = 1$ from equation (15), so it suffices to show E a coalgebra map. Now equation (15) implies

$$E(w) = \sum_{u_1\cdots u_n = w} C(u_1)\cdots C(u_n)$$

for words $w \neq 1$, where the sum is over all decompositions $w = u_1\cdots u_n$ into subwords $u_i \neq 1$. Hence

$$\Delta E(w) = E(w)\otimes 1 + 1 \otimes E(w) + \\ \sum_{u_1\cdots u_n = w, n\geq 2}\sum_{i=1}^{n-1} C(u_1)\cdots C(u_i)\otimes C(u_{i+1})\cdots C(u_n),$$

which can be seen to agree with $(E\otimes E)\Delta(w)$.

Now suppose E is an expansion. Equation (16) implies $C(1) = 0$, so it suffices to show $C(w)$ primitive for words w. We proceed by induction on $\ell(w)$. Suppose C primitive on all words of length $< n$, and let $\ell(w) = n$. Then equation (16) implies

$$C(w) = E(w) - \sum_{uv = w, v\neq 1} C(u)E(v),$$

and by the induction hypothesis it follows that $\Delta C(w)$ can be written

$$\Delta E(w) - \sum_{uv=w, v\neq 1} (C(u)\otimes 1)\Delta E(v) - \sum_{uv=w,v\neq 1} 1\otimes C(u)E(v) = \\ C(w)\otimes 1 + \sum_{\substack{uv=w\\ u\neq 1\neq v}} E(u)\otimes E(v) - \sum_{\substack{uv_1v_2 = w\\ v_2\neq 1}} C(u)E(v_1)\otimes E(v_2) + 1\otimes C(w).$$

Then

$$\tilde{\Delta}C(w) = \sum_{uv=w, u\neq 1\neq v}\left[E(u) - \sum_{u_1u_2=u} C(u_1)E(u_2)\right]\otimes E(v),$$

and the quantity in brackets is zero by equation (16). □

Now let

$$f = c_1 t + c_2 t^2 + \cdots, \quad c_1 \neq 0$$

be a formal power series, and let Ψ_f be the corresponding linear map of $k\langle A\rangle$ as defined in §3. Define the linear map $C_f : k\langle A\rangle \to kA$ by $C_f(1) = 0$ and $C_f(a_1 a_2 \cdots a_n) = c_n a_1 \diamond a_2 \diamond \cdots \diamond a_n$ for $a_1, a_2, \dots a_n \in A$. Then we have the following result.

Theorem 4.1. *For any $f \in \mathcal{P}$, (Ψ_f, C_f) is an inverse pair.*

Proof. It is evident from definitions that, for $w = a_1 \cdots a_n$, $a_i \in A$,

$$\Psi_f(w) = \sum_{k=1}^{n} C_f(a_1 \cdots a_k)\Psi_f(a_{k+1} \cdots a_n).$$

Stated in terms of the convolution product, this is

$$\Psi_f = C_f \odot \Psi_f + \eta\epsilon,$$

from which equation (15) (with $E = \Psi_f$, $C = C_f$) follows. Since evidently C_f is a contraction, the result follows. □

For a word w of $k\langle A\rangle$, say $w = a_1 \cdots a_n$ with the $a_i \in A$, the "reverse" of w is $R(w) = a_n a_{n-1} \cdots a_1$. If we set $R(1) = 1$, then R extends to a linear map from $k\langle A\rangle$ to itself, which is evidently an involution. While R is not a coalgebra map for Δ (despite the incorrect statment in [6]), we do have the following result.

Proposition 4.2. *R is an automorphism of both $(k\langle A\rangle, *)$ and $(k\langle A\rangle, \star)$.*

Proof. We prove the result for $*$, as the proof for $\star$ is almost identical. It suffices to show that $R(w_1 * w_2) = R(w_1) * R(w_2)$ for any words w_1, w_2. We proceed by induction on $\ell(w_1) + \ell(w_2)$. The result is trivial if w_1 or w_2 is 1, so we can write $w_1 = ua$ and $w_2 = vb$ for letters $a, b \in A$. From [11, Theorem 9] we have

$$ua * vb = (u * vb)a + (ua * v) + (u * v)(a \diamond b)$$

so

$$\begin{aligned} R(w_1 * w_2) &= R(ua * vb) \\ &= R((u * vb)a + (ua * v)b + (u * v)(a \diamond b)) \\ &= aR(u * vb) + bR(ua * v) + (a \diamond b)R(u * v) \end{aligned}$$

By the induction hypothesis, the latter sum is

$$a(R(u) * R(bv)) + b(R(au) * R(v)) + (a \diamond b)(R(u) * R(v)) = aR(u) * bR(v) \\ = R(w_1) * R(w_2).$$

□

Theorem 4.2. *$(k\langle A\rangle, *, \Delta)$ and $(k\langle A\rangle, \star, \Delta)$ are Hopf algebras, with respective antipodes $S_* = \Sigma TR$ and $S_\star = T\Sigma R$.*

Proof. The inductive argument in [5, Theorem 3.1] that Δ is a homomorphism for $*$ works equally well for $\star$, so $(k\langle A\rangle, *, \Delta)$ and $(k\langle A\rangle, \star, \Delta)$ are bialgebras. Although these bialgebras are not necessarily graded, they are filtered by word length: $k\langle A\rangle^n$ is the subalgebra generated by words of length at most n. Since $k\langle A\rangle^0 = k1$, these bialgebras are filtered connected, and thus automatically Hopf algebras (see, e.g., [10]). In fact, the proof of the explicit formula for S_* in [5, Theorem 3.2] (by induction on word length) carries over to this setting, giving

$$S_*(w) = (-1)^n \sum_{I \in \mathbf{C}(n)} I[a_n a_{n-1} \cdots a_1]$$

for a word $w = a_1 a_2 \cdots a_n$ in $k\langle A\rangle$, i.e., $S_*(w) = \Sigma TR(w)$. The antipode $S_\star$ of the Hopf algebra $(k\langle A\rangle, \star, \Delta)$ is uniquely determined by the conditions that $S_\star(1) = 1$ and

$$S_\star(1) = 1 \quad \text{and} \quad \sum_{uv=w} S_\star(u) \star v = 0 \quad \text{for words } w \neq 1. \tag{17}$$

Now S_* satisfies

$$\sum_{uv=w} S_*(Tu) * Tv = 0$$

for $w \neq 1$; apply T both sides to get

$$\sum_{uv=w} TS_*T(u) \star v = 0,$$

from which we see that $S_\star = TS_*T$ satisfies (17). Since T commutes with R, this means that $S_\star = T\Sigma R$. □

Since S_* and $S_\star$ are antipodes of commutative Hopf algebras, they are involutions and algebra automorphisms of $(k\langle A\rangle, *)$ and $(k\langle A\rangle, \star)$ respectively. Since R commutes with Σ and T, this gives another proof of Corollary 3.4. Note also that $S_* S_\star = \Sigma^2$ and $S_\star S_* = \Sigma^{-2}$.

For any $f \in \mathcal{P}$, Ψ_f is a coalgebra map by Theorem 4.1. In particular, the maps H_p of the last section are automorphisms of the Hopf algebra $(k\langle A\rangle, *, \Delta)$, and (14) is a commutative diagram of Hopf algebra isomorphisms.

Recall that $(k\langle A\rangle, \diamond)$ is a noncommutative algebra. We will now show that $(k\langle A\rangle, \diamond, \tilde{\Delta})$ is an infinitesimal Hopf algebra (see [1] for definitions).

Theorem 4.3. $(k\langle A\rangle, \diamond, \tilde{\Delta})$ *is an infinitesimal Hopf algebra, with antipode* $S_\diamond = -\Sigma^{-1}$.

Proof. First we show that $(k\langle A\rangle, \diamond, \tilde{\Delta})$ is an infinitesimal bialgebra, i.e., that

$$\tilde{\Delta}(w \diamond v) = \sum_v (w \diamond v_{(1)}) \otimes v_{(2)} + \sum_w w_{(1)} \otimes (w_{(2)} \diamond v), \tag{18}$$

for words w, v, where

$$\tilde{\Delta}(w) = \sum_w w_{(1)} \otimes w_{(2)} \quad \text{and} \quad \tilde{\Delta}(v) = \sum_v v_{(1)} \otimes v_{(2)}.$$

Equation (18) is immediate if w or v is 1, so we can assume both are nonempty. Write $w = a_1 \cdots a_n$ and $v = b_1 \cdots b_m$, where the a_i and b_i are letters. If $n = 1$ we have $\tilde{\Delta}(w) = 0$, so equation (18) becomes

$$\tilde{\Delta}(a_1 \diamond v) = \sum_v (a_1 \diamond v_{(1)}) \otimes v_{(2)},$$

which is evidently true. The case $m = 1$ is similar, so we can assume that $n, m \geq 2$. Then

$$\tilde{\Delta}(w \diamond v) = \sum_{i=1}^{m-2} a_1 \cdots a_{n-1}(a_n \diamond b_1) b_2 \cdots b_{m-i} \otimes b_{m-i+1} \cdots b_m + \sum_{j=1}^{n-2} a_1 \cdots a_j \otimes a_{j+1} \cdots a_{n-1}(a_n \diamond b_1) b_2 \cdots b_m, \tag{19}$$

and the right-hand side evidently agrees with that of equation (18).

To show $(k\langle A\rangle, \diamond, \tilde{\Delta})$ an infinitesimal Hopf algebra we now need to show that it has an antipode, i.e., a function $S_\diamond \in \mathrm{Hom}_k(k\langle A\rangle, k\langle A\rangle)$ with

$$\sum_w S_\diamond(w_{(1)}) \diamond w_{(2)} + S_\diamond(w) + w = 0 = \sum_w w_{(1)} \diamond S_\diamond(w_{(2)}) + w + S_\diamond(w)$$

for any $w \in k\langle A\rangle$, where $\tilde{\Delta}(w) = \sum_w w_{(1)} \otimes w_{(2)}$. This follows from [1, Prop. 4.5], but we shall prove that $S_\diamond = -\Sigma^{-1}$ by showing that $-\Sigma^{-1}$ satisfies the defining property. We prove that the equation

$$\sum_w \Sigma^{-1}(w_{(1)}) \diamond w_{(2)} = -\Sigma^{-1}(w) + w \tag{20}$$

holds for all words w by induction on the length of w. Evidently equation (20) is true if $\ell(w) \le 1$. Now suppose equation (20) holds for $w \ne 1$: we prove it for aw, $a \in A$. Since $\tilde{\Delta}(aw) = a\tilde{\Delta}(w) + a \otimes w$, we must show that

$$\sum_w \Sigma^{-1}(aw_{(1)}) \diamond w_{(2)} + \Sigma^{-1}(a) \diamond w = -\Sigma^{-1}(aw) + aw$$

Using equation (13), this is

$$a\sum_w \Sigma^{-1}(w_{(1)}) \diamond w_{(2)} - a \diamond \sum_w \Sigma^{-1}(w_{(1)}) \diamond w_{(2)} + a \diamond w$$
$$= -a\Sigma^{-1}(w) + a \diamond \Sigma^{-1}(w) + aw.$$

The conclusion then follows by use of the induction hypothesis (20). The proof that

$$\sum_w w_{(1)} \diamond \Sigma^{-1}(w_{(2)}) = w - \Sigma^{-1}(w)$$

is similar, except that in place of equation (13) one needs the identity

$$\Sigma^{-1}(wa) = \Sigma^{-1}(w)a - \Sigma^{-1}(w) \diamond a$$

for words w and letters a. □

The algebra $(k\langle A\rangle, \diamond)$ has the canonical derivation $D = \diamond\tilde{\Delta}$, i.e. $D(w) = 0$ for words w with $\ell(w) \le 1$ and

$$D(a_1 a_2 \cdots a_n) = \sum_{i=1}^{n-1} a_1 \cdots a_i \diamond a_{i+1} \cdots a_n$$

for letters $a_1, \dots, a_n$, $n \geq 2$. We note that $D^n(w) = 0$ whenever $n \geq \ell(w)$, so that

$$e^D = \sum_{n=0}^{\infty} \frac{D^n}{n!} = \mathrm{id} + D + \frac{D^2}{2!} + \cdots$$

makes sense as an element of $\mathrm{Hom}_k(k\langle A\rangle, k\langle A\rangle)$, and similarly for e^{-D}. By [1, Prop. 4.5], $\Sigma^{-1} = -S_\diamond = e^{-D}$. In fact, this can be sharpened as follows.

Corollary 4.1. *For any $r \in k$, $\Sigma^r = e^{rD}$.*

Proof. By definition

$$\Sigma^r(w) = \Psi_{\frac{t}{1-rt}}(w) = \sum_{|I|=\ell(w)} r^{|I|-\ell(I)} I[w] \tag{21}$$

for any word w of $k\langle A\rangle$. On the other hand, by [1, Prop. 4.4]

$$\frac{D^k}{k!} = \diamond^{(k)} \tilde{\Delta}^{(k)},$$

where $\diamond^{(k)} : k\langle A\rangle^{\otimes(k+1)} \to k\langle A\rangle$ and $\tilde{\Delta}^{(k)} : k\langle A\rangle \to k\langle A\rangle^{\otimes(k+1)}$ are respectively the iterated $\diamond$-product and coproduct maps. Now for a word w of $k\langle A\rangle$,

$$\diamond^{(k)} \tilde{\Delta}^{(k)}(w) = \sum_{\ell(I)=k+1, |I|=\ell(w)} I\langle w\rangle = \sum_{\ell(I)=k+1, |I|=\ell(w)} I^*[w].$$

and so

$$\begin{aligned} e^{rD}(w) &= \sum_{k\geq 0} r^k \diamond^k \tilde{\Delta}^{(k)}(w) = \sum_{k\geq 0} r^k \sum_{\ell(I)=k+1, |I|=\ell(w)} I^*[w] \\ &= \sum_{|I|=\ell(w)} r^{\ell(I)-1} I^*[w] = \sum_{|I|=\ell(w)} r^{\ell(I^*)-1} I[w] = \sum_{|I|=\ell(w)} r^{|I|-\ell(I)} I[w], \end{aligned}$$

which agrees with the right-hand side of equation (21). □

Corollary 4.2. *For any $r \in k$, Σ^r is an automorphism of $(k\langle A\rangle, \diamond)$.*

Proof. The exponential of a derivation is an automorphism [9, sect. I.2], so this follows from the preceding result. □

5 Exponentials and logarithms

Let

$$f = c_1 t + c_2 t^2 + \dots$$

be a formal power series in $\mathcal{P}$. Let λ be a formal parameter, and $\bullet$ any of the symbols $*$, $\star$, $\sqcup\!\sqcup$, or $\diamond$. We define

$$f_\bullet(\lambda w) = \sum_{i \geq 1} \lambda^i c_i w^{\bullet i} \in k\langle A\rangle[[\lambda]],$$

where $w \in k\langle A\rangle$. We write $\exp_\bullet(\lambda w)$ for $1 + g_\bullet(\lambda w)$ and $\log_\bullet(1 + \lambda w)$ for $f_\bullet(\lambda w)$, where

$$g = t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots = e^t - 1,$$
$$f = t - \frac{t}{2} + \frac{t}{3} - \cdots = \log(1 + t).$$

Then for any $w \in k\langle A\rangle$,

$$\log_\bullet(\exp_\bullet(\lambda w)) = \lambda w \quad \text{and} \quad \exp_\bullet(\log_\bullet(1 + \lambda w)) = 1 + \lambda w;$$

and for $w, v \in k\langle A\rangle$ for $\bullet = *$ or $\bullet = \star$, and $w, v \in kA$ for $\bullet = \diamond$,

$$\exp_\bullet(\lambda(w + v)) = \exp_\bullet(\lambda w) \bullet \exp_\bullet(\lambda v). \tag{22}$$

We extend the automorphisms Ψ_f of $k\langle A\rangle$ to $k\langle A\rangle[[\lambda]]$ by setting $\Psi_f(\lambda) = \lambda$.

The following result generalizes Lemma 3 of [7] .

Theorem 5.1. *For any $f = c_1 t + c_2 t^2 + \cdots \in \mathcal{P}$ and $z \in kA[[\lambda]]$,*

$$\Psi_f \left(\frac{1}{1 - \lambda z} \right) = \frac{1}{1 - f_\diamond(\lambda z)}.$$

Proof. In fact, we shall show that

$$E \left(\frac{1}{1 - \lambda z} \right) = \frac{1}{1 - C(\lambda z + \lambda^2 z^2 + \cdots)} \tag{23}$$

for any inverse pair (E, C): the conclusion then follows by Theorem 4.1, noting that $f_\diamond(\lambda z) = C_f(\lambda z + \lambda^2 z^2 + \cdots)$. We can write the left-hand side of equation (23) as

$$(\eta\epsilon + C + C \odot C + \cdots)\,(1 + \lambda z + \lambda^2 z^2 + \cdots) = 1 + \sum_{n \geq 1} \sum_{k \leq n} C^{\odot k}(\lambda^n z^n),$$

which we will denote by $\square$. Evidently each term except 1 in $\square$ has an initial factor of form $C(\lambda^k z^k)$, so

$$\square - 1 = C(\lambda z)\square + C(\lambda^2 z^2)\square + \cdots = C(\lambda z + \lambda z^2 + \cdots)\square,$$

and equation (23) follows. $\square$

Since $\exp : (k\langle A\rangle, ⧢) \to (k\langle A\rangle, *)$ is an algebra isomorphism, we have $\exp f_{⧢} = f_* \exp$ for any $f \in \mathcal{P}$. For such f we also have $\Sigma f_\star = f_* \Sigma$ and $T f_* = f_\star T$. In particular, for $z \in kA[[\lambda]]$, $\Sigma f_\star(\lambda z) = f_*(\lambda z)$ and $T f_*(\lambda z) = f_\star(-\lambda z)$. For $z \in kA[[\lambda]]$ we also have

$$\exp_*(\lambda z) = \exp(\exp_{⧢}(\lambda z)) = \exp\left(\frac{1}{1-\lambda z}\right), \tag{24}$$

where we have used the identity

$$\exp_{⧢}(\lambda z) = 1 + \lambda z + \lambda^2 z^2 + \lambda^3 z^3 + \cdots = \frac{1}{1-\lambda z},$$

which in turn follows from $z^{⧢ n} = n! z^n$ for $z \in kA[[\lambda]]$. We can now give a quick proof of the following result (cf. [8, Prop. 4] and [7, Prop. 3]).

Theorem 5.2. *For* $z \in kA[[\lambda]]$,

$$\exp_*(\log_\diamond(1+\lambda z)) = \frac{1}{1-\lambda z} \quad \textit{and} \quad \exp_\star(-\log_\diamond(1+\lambda z)) = \frac{1}{1+\lambda z}.$$

Proof. In view of equation (24), the first identity is equivalent to

$$\frac{1}{1-\log_\diamond(1+\lambda z)} = \log\left(\frac{1}{1-\lambda z}\right), \tag{25}$$

which is just Theorem 5.1 applied to the formal power series $f = \log(1+t)$. To get the second identity, apply T to both sides of the first. $\square$

Remark. By applying Theorem 5.1 to the formal power series $e^t - 1$, we get

$$\exp\left(\frac{1}{1-\lambda z}\right) = \frac{1}{1-(\exp_\diamond(\lambda z) - 1)},$$

or $\exp_*(\lambda z) = (2 - \exp_\diamond(\lambda z))^{-1}$, as in [8, Prop. 4].

Here are some corollaries of Theorem 5.2.

Corollary 5.1. *For $p \in k$ and $z \in kA[[\lambda]]$,*

$$H_p\left(\frac{1}{1-\lambda z}\right) = \left(\frac{1}{1-\lambda z}\right)^{*p}.$$

Proof. Using Theorems 5.2 and 3.2, the left-hand side of the identity can be written

$$H_p(\exp_*(\log_\diamond(1+\lambda z))) = \exp_*(H_p(\log_\diamond(1+\lambda z))).$$

Now $\log_\diamond(1+\lambda z) \in kA[[\lambda]]$, so the latter quantity is

$$\exp_*(p\log_\diamond(1+\lambda z)) = (\exp_*(\log_\diamond(1+\lambda z)))^{*p} = \left(\frac{1}{1-\lambda z}\right)^{*p}.$$

□

Corollary 5.2. *For any $f = c_1t + c_2t^2 + \cdots \in \mathcal{P}$ and $z \in kA[[\lambda]]$,*

$$\Psi_f\left(\frac{1}{1-\lambda z}\right) * \Psi_g\left(\frac{1}{1+\lambda z}\right) = 1,$$

where $g(t) = (1+f(-t))^{-1} - 1$, i.e., g is the composition $\frac{t}{1-t} \circ (-t) \circ f \circ (-t)$.

Proof. By Theorem 3.1 $\Psi_g = \Sigma T \Psi_f T = H_{-1}\Psi_f T$, so the conclusion can be written as $\xi * H_{-1}(\xi) = 1$ for

$$\xi = \Psi_f\left(\frac{1}{1-\lambda z}\right).$$

Now Theorem 5.1 says that $\xi = \frac{1}{1-\lambda u}$ for

$$u = c_1z + \lambda c_2 z^{\diamond 2} + \lambda^2 c_3 z^{\diamond 3} + \cdots \in kA[[\lambda]],$$

so we can apply Corollary 5.1 with $p = -1$ to obtain the conclusion. □

Remark. In particular, taking $f = \frac{t}{1-pt}$ in the preceding result gives

$$\Sigma^p\left(\frac{1}{1-\lambda z}\right) * \Sigma^{1-p}\left(\frac{1}{1+\lambda z}\right) = 1 \tag{26}$$

for any $p \in k$, generalizing Corollary 1 of [7].

Corollary 5.3. *For $y, z \in kA[[\lambda]]$,*

$$\frac{1}{1-\lambda y} * \frac{1}{1-\lambda z} = \frac{1}{1-\lambda y-\lambda z-\lambda^2 y \diamond z}$$

and

$$\frac{1}{1+\lambda y} \star \frac{1}{1+\lambda z} = \frac{1}{(1+\lambda y)\diamond(1+\lambda z)}.$$

Proof. Using Theorem 5.2, the left-hand side of the first identity is

$$\begin{aligned} &\exp_*(\log_\diamond(1+\lambda y)) * \exp_*(\log_\diamond(1+\lambda z)) = \exp_*(\log_\diamond(1+\lambda y)+\log_\diamond(1+\lambda z)) \\ &= \exp_*(\log_\diamond((1+\lambda y)\diamond(1+\lambda z))) = \exp_*(\log_\diamond(1+\lambda y+\lambda z+\lambda^2 y\diamond z)) \\ &= \frac{1}{1-\lambda y-\lambda z-\lambda^2 y\diamond z}, \end{aligned}$$

so the identity follows. To get the second identity, apply T to both sides of the first. □

We also have the following result, which is proved in [7, Prop. 4] by another method.

Theorem 5.3. *For $a, b \in A$,*

$$\Sigma\left(\frac{1}{1-\lambda ab}\right) = \left(\frac{1}{1-\lambda ab}\right) * \Sigma\left(\frac{1}{1-\lambda a\diamond b}\right).$$

Proof. Using equation (26) with $p = 1$, the conclusion can be written as

$$\Sigma\left(\frac{1}{1-\lambda ab}\right) * \left(\frac{1}{1+\lambda a\diamond b}\right) = \left(\frac{1}{1-\lambda ab}\right).$$

Now use Corollary 3.2 and apply log to both sides to make this

$$T\log\left(\frac{1}{1-\lambda ab}\right) \sqcup\log\left(\frac{1}{1+\lambda a\diamond b}\right) = \log\left(\frac{1}{1-\lambda ab}\right). \tag{27}$$

Now

$$\log\left(\frac{1}{1-\lambda ab}\right) = 1 + \sum_{i\ge 1}\lambda^i \sum_{\substack{I=(i_1,\dots,i_n)\\ |I|=2i}} \frac{(-1)^n}{i_1 i_2\cdots i_n} I[(ab)^i],$$

and applying T simply eliminates the signs. Further,

$$\log\left(\frac{1}{1+\lambda a\diamond b}\right) = 1 + \sum_{i\geq 1}\lambda^i \sum_{\substack{J=(j_1,\dots,j_k)\\ |J|=i}} \frac{(-1)^k}{j_1 j_2\cdots j_k} J[(a\diamond b)^i],$$

so to prove (27) and hence the conclusion it suffices to show

$$\sum_{i=0}^{m}\left(\sum_{\substack{I=(i_1,\dots,i_n)\\ |I|=2i}} \frac{1}{i_1\cdots i_n} I[(ab)^i]\right) \sqcup \left(\sum_{\substack{J=(j_1,\dots,j_k)\\ |J|=m-i}} \frac{(-1)^k}{j_1\cdots j_k} J[(a\diamond b)^{m-i}]\right)$$

$$= \sum_{\substack{I=(i_1,\dots,i_n)\\ |I|=2m}} \frac{(-1)^n}{i_1\cdots i_n} I[(ab)^m]. \quad (28)$$

To prove the latter equation, we consider an arbitrary term of the form

$$(i_1, i_2, \dots, i_n)[(ab)^m], \quad i_1+\cdots+i_n = 2m, \quad (29)$$

and note that every even $i_h = 2j$ produces a factor $(a\diamond b)^{\diamond j}$. Write $(i_1,\dots,i_n)$ as $(t_1^{p_1},\dots,t_s^{p_s})$, where the exponents mean repetition, and let $(t_{u_1},\dots,t_{u_f}) = (2j_{u_1},\dots,2j_{u_f})$ be the subsequence of even t_i's. Then (29) appears on the right-hand side of equation (28) with coefficient

$$\frac{(-1)^{p_1+\cdots+p_s}}{t_1^{p_1}\cdots t_s^{p_s}} = \frac{(-1)^{p_{u_1}+\cdots+p_{u_f}}}{t_1^{p_1}\cdots t_s^{p_s}}, \quad (30)$$

and on the left-hand side of equation (28) with coefficient

$$\sum_{0\leq q_{u_h}\leq p_{u_h}} \frac{1}{t_1^{p_1'}\cdots t_s^{p_s'}} \frac{(-1)^{q_{u_1}+\cdots+q_{u_f}}}{j_{u_1}^{q_{u_1}}\cdots j_{u_f}^{q_{u_f}}} \binom{p_{u_1}}{q_{u_1}}\cdots\binom{p_{u_f}}{q_{u_f}},$$

where

$$p_i' = \begin{cases} p_i, & \text{if } t_i \text{ is odd;}\\ p_i - q_i, & \text{if } t_i \text{ is even.}\end{cases}$$

Since $j_{u_h}^{q_{u_h}} = 2^{-q_{u_h}} t_{u_h}^{q_{u_h}}$ for $h = 1,2,\dots,f$, we can write the latter coefficient as

$$\frac{1}{t_1^{p_1}\cdots t_s^{p_s}} \sum_{0\leq q_{u_h}\leq p_{u_h}} (-2)^{q_{u_1}+\cdots+q_{u_f}} \binom{p_{u_1}}{q_{u_1}}\cdots\binom{p_{u_f}}{q_{u_f}},$$

which by the binomial theorem agrees with (30). □

6 Applications

To demonstrate the scope of applications of quasi-shuffle products, in this section we will outline four types of objects that are homomorphic images of quasi-shuffle algebras: multiple zeta values, (finite) multiple harmonic sums, multiple q-zeta values, and values of multiple polylogarithms at roots of unity. In each case we show how Theorem 5.2 can be applied.

6.1 Multiple zeta values

Suppose $A = \{z_1, z_2, \dots\}$ with the product $z_i \diamond z_j = z_{i+j}$. Then $(\mathbf{Q}\langle A\rangle, *)$ is the "harmonic algebra" of [4], and is in fact isomorphic to the algebra of quasi-symmetric functions. If we let $\mathfrak{H}^1 = k\langle A\rangle$ and $\mathfrak{H}^0 \subset \mathfrak{H}^1$ the subspace generated by monomials that don't start in z_1, then there is a homomorphism $\zeta : (\mathfrak{H}^0, *) \to \mathbf{R}$ given as in the introduction:

$$\zeta(z_{k_1} z_{k_2} \cdots z_{k_l}) = \sum_{m_1 > m_2 > \cdots > m_l \geq 1} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_l^{k_l}}.$$

(The restriction that $k_1 \neq 1$ is necessary for convergence of the series.) One also has the multiple star-zeta values (MSZVs)

$$\sum_{m_1 \geq m_2 \geq \cdots \geq m_l \geq 1} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_l^{k_l}}, \tag{31}$$

and if we define $\zeta^\star(z_{k_1} z_{k_2} \cdots z_{k_l})$ to be (31) then $\zeta^\star : (\mathfrak{H}^0, \star) \to \mathbf{R}$ is a homomorphism.

From Theorem 5.2 we have

$$\exp_* \left(\sum_{i \geq 1} \frac{(-1)^{i-1}}{i} \lambda^i z_k^{\diamond i} \right) = \sum_{i \geq 0} z_k^i \lambda^i \tag{32}$$

and applying ζ to both sides gives

$$\exp \left(\sum_{i \geq 1} \frac{(-1)^{i-1}}{i} \lambda^i \zeta(ik) \right) = 1 + \sum_{i \geq 1} \zeta(z_k^i) \lambda^i.$$

That is, the MZV $\zeta(k, \dots, k)$ (with r repetitions of $k \geq 2$) is the coefficient of λ^r in

$$\exp \left(\sum_{i \geq 1} \frac{(-1)^{i-1}}{i} \lambda^i \zeta(ik) \right).$$

This is a well-known result: it goes back at least to [2] (see equation (11)). To obtain the counterpart for zeta-star values, replace λ with $-\lambda$ in the second part of Theorem 5.2 and set $z = z_k$ to get

$$\exp_\star \left(\sum_{i\geq 1} \frac{\lambda^i z_k^{\diamond i}}{i} \right) = \sum_{i\geq 0} z_k^i \lambda^i. \tag{33}$$

Now apply $\zeta^\star$ to both sides:

$$\exp \left(\sum_{i\geq 1} \frac{\lambda^i \zeta(ik)}{i} \right) = 1 + \sum_{i\geq 1} \zeta^\star(z_k^i)\lambda^i,$$

so that

$$\zeta^\star(\underbrace{k, \dots, k}_{r}) = \text{coefficient of } \lambda^r \text{ in } \exp \left(\sum_{i\geq 1} \frac{\lambda^i \zeta(ik)}{i} \right).$$

Cf. [7, p. 203].

6.2 Multiple harmonic sums

If one defines, for fixed n, the finite sums

$$A_{(k_1,\dots,k_l)}(n) = \sum_{n\geq m_1 > m_2 > \dots > m_l \geq 1} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_l^{k_l}}$$

and

$$S_{(k_1,\dots,k_l)}(n) = \sum_{n\geq m_1 \geq m_2 \geq \dots \geq m_l \geq 1} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_l^{k_l}},$$

then there are homomorphisms $\zeta_{\leq n} : (\mathfrak{H}^1, *) \to \mathbf{R}$ and $\zeta^\star_{\leq n} : (\mathfrak{H}^1, \star) \to \mathbf{R}$ given by

$$\zeta_{\leq n}(z_{k_1} \cdots z_{k_l}) = A_{(k_1,\dots,k_l)}(n)$$

and

$$\zeta^\star_{\leq n}(z_{k_1} \cdots z_{k_l}) = S_{(k_1,\dots,k_l)}(n).$$

If we apply these homomorphisms to the equations (32) and (33) above, we obtain

$$\zeta_{\leq n}(\underbrace{k, \dots, k}_{r}) = \text{coefficient of } \lambda^r \text{ in } \exp \left(\sum_{i\geq 1} \frac{(-1)^{i-1}}{i} \lambda^i A_{ik}(n) \right).$$

and

$$\zeta^\star_{\le n}(\underbrace{k,\dots,k}_{r}) = \text{coefficient of } \lambda^r \text{ in } \exp\left(\sum_{i\ge 1}\frac{\lambda^i A_{ik}(n)}{i}\right). \tag{34}$$

Note that k can be 1 in these formulas since the sums involved are finite. Equation (34) can be compared to the explicit formula given by equation (21) of [6].

6.3 Multiple q-zeta values

As in the preceding examples let $A = \{z_1, z_2, \dots\}$, but now define the product $\diamond$ by

$$z_i \diamond z_j = z_{i+j} + (1-q)z_{i+j-1}. \tag{35}$$

Here we take as our ground field $k = \mathbf{Q}[1-q]$. Then we have homomorphisms $\zeta_q : (\mathfrak{H}^0, *) \to \mathbf{Q}[[q]]$ and $\zeta^\star_q : (\mathfrak{H}^0, \star) \to \mathbf{Q}[[q]]$ given by

$$\zeta_q(z_{k_1}z_{k_2}\cdots z_{k_l}) = \sum_{m_1>m_2>\cdots>m_l\ge 1}\frac{q^{m_1(k_1-1)+m_2(k_2-1)+\cdots+m_l(k_l-1)}}{[m_1]^{k_1}[m_2]^{k_2}\cdots[m_l]^{k_l}},$$

and

$$\zeta^\star_q(z_{k_1}z_{k_2}\cdots z_{k_l}) = \sum_{m_1\ge m_2\ge\cdots>m_l\ge 1}\frac{q^{m_1(k_1-1)+m_2(k_2-1)+\cdots+m_l(k_l-1)}}{[m_1]^{k_1}[m_2]^{k_2}\cdots[m_l]^{k_l}},$$

where $[m] = (1-q^m)/(1-q)$.

Formulas like those obtained in the last two examples are complicated by presence of the extra term in equation (35). Iteration of (35) gives

$$z_k^{\diamond i} = \sum_{j=0}^{i-1}\binom{i-1}{j}(1-q)^j z_{ik-j}.$$

Then, as in [7, Ex. 4], we can apply ζ_q and $\zeta^\star_q$ to equations (32) and (33) to get, for $k \ge 2$,

$$\zeta_q(\underbrace{k,\dots,k}_{r}) =$$

$$\text{coefficient of } \lambda^r \text{ in } \exp\left[\sum_{i\ge 1}\frac{(-1)^{i-1}\lambda^i}{i}\left(\sum_{j=0}^{i-1}\binom{i-1}{j}(1-q)^j\zeta_q(ik-j)\right)\right]$$

and

$$\zeta_q^\star(\underbrace{k,\ldots,k}_{r}) =$$

$$\text{coefficient of } \lambda^r \text{ in } \exp\left[\sum_{i\geq 1}\frac{\lambda^i}{i}\left(\sum_{j=0}^{i-1}\binom{i-1}{j}(1-q)^j\zeta_q(ik-j)\right)\right].$$

6.4 Multiple polylogarithms at roots of unity

Fix $r \geq 2$, and let $\omega = e^{\frac{2\pi i}{r}}$. Then for an integer composition $I = (i_1, \ldots, i_k)$, the values of the multiple polylogarithm Li_I at rth roots of unity are given by

$$\mathrm{Li}_I(\omega^{j_1}, \ldots, \omega^{j_k}) = \sum_{n_1 > \cdots > n_k \geq 1} \frac{\omega^{n_1 j_1} \cdots \omega^{n_k j_k}}{n_1^{i_1} \cdots n_k^{i_k}},$$

and the series converges provided $\omega^{j_1} i_1 \neq 1$. We can define the "star-multiple" polylogarithms by

$$\mathrm{Li}_I^\star(\omega^{j_1}, \ldots, \omega^{j_k}) = \sum_{n_1 \geq \cdots \geq n_k \geq 1} \frac{\omega^{n_1 j_1} \cdots \omega^{n_k j_k}}{n_1^{i_1} \cdots n_k^{i_k}}.$$

Here we let $A = \{z_{i,j} : i \geq 1, 0 \leq j \leq r-1\}$ and $z_{i,j} \diamond z_{p,q} = z_{i+p,j+q}$, where the second subscript is understood mod r. The algebra $(k\langle A\rangle, *)$ is called the Euler algebra in [5] (see Example 2). Let $\mathfrak{E}_r = k\langle A\rangle$, $\mathfrak{E}_r^0$ the subalgebra of $k\langle A\rangle$ generated by words not starting in $z_{1,0}$. Then there is a homomorphism Z from $(\mathfrak{E}_r^0, *)$ to $\mathbf{C}$ given by

$$Z(z_{i_1,j_1} \cdots z_{i_k,j_k}) = \mathrm{Li}_{(i_1,\ldots,i_k)}(\omega^{j_1}, \ldots, \omega^{j_k}),$$

and a homomorphism $Z^\star : (\mathfrak{E}_r^0, \star) \to \mathbf{C}$ given by

$$Z^\star(z_{i_1,j_1} \cdots z_{i_k,j_k}) = \mathrm{Li}^\star_{(i_1,\ldots,i_k)}(\omega^{j_1}, \ldots, \omega^{j_k}).$$

From Theorem 5.2 we have

$$\exp_*\left(\sum_{i\geq 1}\frac{(-1)^{i-1}\lambda^i}{i} z_{s,t}^{\diamond i}\right) = \sum_{i\geq 0} z_{s,t}^i \lambda^i.$$

Now $z_{s,t}^{\diamond i} = z_{si,ti}$, and if t is relatively prime to r the preceding equation is

$$\exp_* \left(\sum_{j=0}^{r-1} \sum_{\substack{i \geq 1 \\ ti \equiv j \mod r}} \frac{(-1)^{i-1}\lambda^i}{i} z_{is,j} \right) = \sum_{i \geq 0} z_{s,t}^i \lambda^i,$$

with fewer terms on the left-hand side if t has factors in common with r. Applying Z to both sides, we have

$$\mathrm{Li}_{\underbrace{(s,\ldots,s)}_{k}}(\underbrace{\omega^t,\ldots,\omega^t}_{k}) = \text{coefficient of } \lambda^k \text{ in } \exp \left(\sum_{j=0}^{r-1} \sum_{\substack{i \geq 1 \\ ti \equiv j \mod r}} \frac{(-1)^{i-1}\lambda^i}{i} \mathrm{Li}_{is}(\omega^j) \right).$$

In the case $r = 2$, $t = 1$, and $s > 1$, this simplifies to equation (12) from [2]. The counterpart for star-multiple polylogarithms is

$$\mathrm{Li}^{\star}_{\underbrace{(s,\ldots,s)}_{k}}(\underbrace{\omega^t,\ldots,\omega^t}_{k}) = \text{coefficient of } \lambda^k \text{ in } \exp \left(\sum_{j=0}^{r-1} \sum_{\substack{i \geq 1 \\ ti \equiv j \mod r}} \frac{\lambda^i}{i} \mathrm{Li}_{is}(\omega^j) \right).$$

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