Holomorphic spheres in loop groups, real holomorphic Bott periodicity, and stabilization of monopoles

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HOLOMORPHIC SPHERES IN LOOP GROUPS, REAL HOLOMORPHIC BOTT PERIODICITY, AND STABILIZATION OF MONOPOLES

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Abstract.

In this paper we study the topology of the spaces of base point preserving degree k holomorphic maps from the complex projective line $\mathbf{CP}^1$ into the infinite dimensional homogeneous manifold Sp/U and the loop groups $\Omega\mathrm{Sp}\,(n)$. Using the results and methods of [CLS] we prove a holomorphic analogue of one of the real Bott periodicity equivalences showing that $\mathrm{Hol}_k(\mathbf{CP}^1, \mathrm{Sp/U}) \simeq \mathrm{Hol}_k(\mathbf{CP}^1, \Omega\mathrm{Sp}) \simeq B\mathrm{O}(k)$. We apply these results to study the stabilized space of $\mathrm{Sp}\,(n)$-monopoles on S^4 proving an analogue of the Kirwan-Sander's theorem for instantons [K],[S].

Introduction.

One of the most fundamental theorems in Topology is the Bott Periodicity Theorem [B59]. This theorem identifies the topology of the loop spaces of the classical infinite rank Lie groups U, Sp and O. In the case of $\mathrm{U} = \lim_{\to n} \mathrm{U}\,(n)$ is states that there is a natural homotopy equivalence,

$$\beta\colon \mathbf{Z} \times B\mathrm{U} \xrightarrow{\simeq} \Omega\mathrm{U}$$

where $\Omega X = C^\infty_\bullet(S^1, X)$ denotes the space of smooth basepoint preserving maps from the unit circle S^1 to X.

In [A68] Atiyah uses elliptic operators to define a homotopy inverse

$$\bar\partial\colon \Omega\mathrm{U} \simeq C^\infty_\bullet(S^2, B\mathrm{U}) \to \mathbf{Z} \times B\mathrm{U}.$$

What he does is to use the $\bar\partial$ operator coupled to each element of $C^\infty_{\bullet,k}(S^2, B\mathrm{U})$ and takes the index. This implies that there exist a natural homotopy equivalences for the degree k components of the mapping spaces, and the classifying space $B\mathrm{U}$

$$B\mathrm{U} \xleftarrow{\bar\partial} C^\infty_{\bullet,k}(S^2, B\mathrm{U}) \xrightarrow{\beta} C^\infty_{\bullet,k}(S^2, \Omega\mathrm{U}). \tag{0.1}$$

In [CLS] R. Cohen, G. Segal and the author prove an analogue holomorphic version of this theorem,

Theorem A. *[CLS] There are natural homotopy equivalences*

$$B\mathrm{U}(k) \xleftarrow{\bar\partial} \mathrm{Hol}_k(\mathbf{CP}^1, B\mathrm{U}) \xrightarrow{\beta} \mathrm{Hol}_k(\mathbf{CP}^1, \Omega\mathrm{U}) \tag{0.2}$$

for every natural number k.

The maps $\bar\partial$ and β extend to the smooth mapping spaces to give (0.1). In fact, as we let $k \to \infty$ (0.2) becomes (0.1). In [CLS] a strongest result is proved determining the homotopy type of $\mathrm{Hol}_k(\mathbf{CP}^1, B\mathrm{U}(n))$ as that of the Mitchell-Segal filtration for the loop group $\Omega\mathrm{U}\,(n)$ [M86]. This theorem is proved there identifying the holomorphic mapping space as a moduli space of negative holomorphic bundles (as defined later) over $\mathbf{CP}^1$ with some additional data, then we identify a corresponding portion of the loop group with other moduli space of holomorphic bundles with additional data and compare both. We sketch the main argument for the proof of Theorem A in section 1, we refer the reader to [CLS] for the details.

In [C97] this theorem was used to understand periodicity properties of Cohen's holomorphic K-theory.

The periodicity theorem has a more complicated statement for the case of the orthogonal group O. The "Real" Bott periodicity Theorem establishes several homotopy equivalences and in particular that we have natural homotopy equivalences

$$B\mathrm{O} \xleftarrow{\bar\partial} C^\infty_{\bullet,k}(S^2, \mathrm{Sp/U}) \xrightarrow{\beta} C^\infty_{\bullet,k}(S^2, \Omega\mathrm{Sp}). \tag{0.3}$$

In this case the $\bar\partial$ operator has a real structure.

The main theorem of this paper is the following holomorphic version of (0.3)

Theorem B. *The spaces $BO(k)$,$\mathrm{Hol}_k(\mathbf{CP}^1,\mathrm{Sp/U})$ and $\mathrm{Hol}_k(\mathbf{CP}^1,\Omega\mathrm{Sp})$ are homotopy equivalent for all natural numbers k. Moreover there are natural homotopy equivalences*

$$BO(k)\xleftarrow{\delta}\mathrm{Hol}_k(\mathbf{CP}^1,\mathrm{Sp/U})\xrightarrow{\beta}\mathrm{Hol}_k(\mathbf{CP}^1,\Omega\mathrm{Sp}) \tag{0.4}$$

for every odd natural number k.

We prove this theorem in chapter 2 using the constructions and methods of [CLS] to reduce it to the complex case of Theorem A.

Observe that this theorem implies that the inclusion that forgets complex structures

$$\mathrm{Hol}_k(\mathbf{CP}^1,\mathrm{Sp/U})\hookrightarrow C^\infty_{\bullet,k}(S^2,\mathrm{Sp/U})$$

and

$$\mathrm{Hol}_k(\mathbf{CP}^1,\Omega\mathrm{Sp})\hookrightarrow C^\infty_{\bullet,k}(S^2,\Omega\mathrm{Sp})$$

are simply (up to homotopy) the inclusion

$$BO(k)\hookrightarrow BO.$$

Let $\mathcal{I}_{n,k}$ denote the moduli space of based $SU(n)$-instantons with charge k over S^4. There is a natural map

$$\delta\colon \mathcal{I}_{n,k}\to B\mathrm{U}(k) \tag{0.5}$$

given by the index bundle of the Dirac operator coupled to the instanton.

The following theorem due to F. Kirwan [K] and M. Sanders [S] answer the question of stabilization with respect to the rank n of the Lie group

Theorem C. *(Kirwan [K], Sanders [S]) There is a homotopy equivalence*

$$\delta\colon \mathcal{I}_{\infty,k}\xrightarrow{\simeq}B\mathrm{U}(k) \tag{0.6}$$

given by the Dirac operator. The composition

$$\mathcal{I}_{n,k}\to\mathcal{I}_{n+1,k}\to\cdots\to\mathcal{I}_{\infty,k}\simeq B\mathrm{U}(k) \tag{0.7}$$

is homotopy equivalent to (0.5) for every $k\in\mathbf{R}$.

Let $\mathcal{M}_{n,k}$ denote the moduli space of $\mathrm{Sp}(n)$-monopoles on $\mathbf{R}^3$. In this case the Dirac operator has a real structure as proved in [CJ93] giving a map

$$\delta\colon \mathcal{M}_{n,k}\to BO(k) \tag{0.8}$$

Using Theorem B together with results of Sanders we prove in chapter 3 the following analogue of the Kirwan-Sanders' result

Theorem D. *There is a homotopy equivalence*

$$\delta\colon \mathcal{M}_{\infty,k}\xrightarrow{\simeq}BO(k) \tag{0.9}$$

given by the Dirac operator. The composition

$$\mathcal{M}_{n,k}\to\mathcal{M}_{n+1,k}\to\cdots\to\mathcal{M}_{\infty,k}\simeq BO(k) \tag{0.10}$$

is homotopy equivalent to (0.8) for every odd natural number k.

In [CCMM91] it is proven that the space of $\mathrm{Sp}(1)$-monopoles of charge k, $\mathcal{M}_k$ is stably homotopy equivalent to the classifying space of the braid group

$$\mathcal{M}_k\simeq_s B\beta_{2k}.$$

In [CJ93] R. Cohen and J. Jones investigate further the relation between monopoles and braids. In [FC] Fred Cohen defines braid cobordism and computes the spectrum for it. He proves that stable braid cobordism has as associated spectrum the Eilenberg-McLane spectrum $K\mathbf{Z}_2$. In chapter 3 we define stable monopole cobordism and we use a theorem of Mahowald and elementary arguments to prove

Theorem E. *For any space X we have a natural isomorphism*

$$\Omega_*^{\mathcal{M}_\infty} X \cong H_*(X, \mathbf{Z}_2)$$

in particular any element of $H_*(X,\mathbf{Z}_2)$ is represented by a monopole oriented manifold uniquely up to monopole cobordism.

Finally we use Theorems of R. Cohen [C76] and the results of section 3 to prove the following unstable version of Theorem E.

Theorem F. *Consider the monopole filtration for $\Omega^2 S^2$ given by*

$$\mathcal{M}_1 \to \mathcal{M}_2 \to \cdots \to \mathcal{M}_k \to \cdots \to \mathcal{M}_\infty \simeq \Omega^2 S^2 \xrightarrow{\gamma} BO$$

where γ is the unique 2-fold that sends the generator $S^1 \subset \Omega^2 S^3$ of $\pi_1(\Omega^2 S^2)$ to the generator of $\pi_1 BO = \mathbf{Z}_2$. Let X_k be the 2-localization of the Thom spectrum of the stable vector bundle γ_k over $\mathcal{M}_k$ which is classified by the map

$$\gamma_k\colon \mathcal{M}_k \to \Omega^2 S^2 \xrightarrow{\gamma} BO.$$

Then each $Y_k \simeq_2 B_k$ where B_k is the 2-local Brown-Gitler spectrum [BG].

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§1. COMPLEX HOLOMORPHIC BOTT PERIODICITY.

§1.1. The Loop Group.

In this section we recall some basic results from the theory of loop groups. We follow the notation of our main reference for this, [PS86].

Let G be a classical finite dimensional Lie group, and let G^c be its complexification. The loop group $\mathcal{L}G$ is the group of smooth maps from the circle S^1 into G, with group law coming from that in G. The group $\mathcal{L}G$ has several important subgroups,

i) the group $\mathcal{L}^+G$ of maps $\gamma: S^1 \to G$ which extend to holomorphic maps of the closed unit disk to G,

ii) the group $\mathcal{L}_{\mathrm{pol}}G$ of loops whose matrix entries are finite Laurent polynomials in z and z^{-1} i.e. the loops of the form

$$\gamma(z) = \sum_{k=-N}^{N} A_k z^k$$

for some N, where the A_k are $n \times n$ matrices.

iii) the base group ΩG of based maps $\gamma: (S^1, \star) \to (G, \star)$; and its subgroup $\Omega_{\mathrm{pol}}G$.

We specialize now to the group $G = SU(n)$, even when some of the results can be stated in a more general manner. We have the following relation between some of those subgroups.

Theorem 1.1.1. *(cf. [A84 (2.10)], [PS86, 8.1.1]) Given a loop $\gamma \in \mathcal{L}\mathrm{Gl}_n(\mathbf{C})$ we can write it as*

$$\gamma = \gamma_n \cdot \gamma_+,$$

with $\gamma_n \in \Omega\mathrm{U}(n)$ and $\gamma_+ \in \mathcal{L}^+\mathrm{Gl}_n(\mathbf{C})$ in a unique way. Moreover the product map

$$\Omega\mathrm{U}(n) \times \mathcal{L}^+\mathrm{Gl}_n(\mathbf{C}) \to \mathcal{L}\mathrm{Gl}_n(\mathbf{C})$$

is a diffeomorphism, giving a natural identification

$$\Omega\mathrm{U}(n) = \mathcal{L}\mathrm{Gl}_n(\mathbf{C}) / \mathcal{L}^+\mathrm{Gl}_n(\mathbf{C}).$$

Actually this theorem can be further refined to get the Birkhoff Factorization Theorem, which in turn rapidly implies the following result, due to Grothendieck, that classifies holomorphic vector bundles over $\mathbf{CP}^1$, and that we will use later.

Theorem 1.1.2. *([G57], cf. [PS86, (8.2.4)]) Any holomorphic vector bundle over $\mathbf{CP}^1$ is isomorphic to a sum $H^{k_1} \oplus \cdots \oplus H^{k_m}$ where H^k is the k-th tensor power of the Hopf bundle. The set of integers $\{k_1, \ldots, k_n\}$ is a complete invariant for this classification.*

To understand better Theorem 1.1.1 we need to introduce the Grassmannian model for the loop group. Let $H = H^{(1)} = L^2(S^1; \mathbf{C})$ be the Hilbert space of square Lebesgue integrable complex functions on S^1, and let $H^{(n)} = L^2(S^1; \mathbf{C}^n)$ be the analogous space of vector valued functions. We will identify H with $H^{(n)}$ in the following way, given $f = (f_1, \ldots, f_n) \in H^{(n)}$ we associate to it the function

$$\tilde{f}(\zeta) = f_1(\zeta^n) + \zeta f_2(\zeta^n) + \cdots + \zeta^{n-1} f_n(\zeta^n). \tag{1.2.1}$$

Conversely, given the function $\tilde{f} \in H$ we recover $(f_i) \in H^{(n)}$ with the formula

$$f_{i+1}(z) = \frac{1}{n} \sum_{\zeta^n = z} \zeta^{-i} \tilde{f}(\zeta).$$

For more on this identification we refer the reader to [PS86 §6.5].

We will be using the standard polarization of $H^{(n)}$, i.e. we will decompose $H^{(n)}$ as $H_+^{(n)} \oplus H_-^{(n)}$ where $H_+^{(n)}$ and $H_-^{(n)}$ are the closed subspaces

$$H_+^{(n)} = \{\text{ functions whose negative Fourier coefficients vanish }\}$$

$$= \{ f \in H^{(n)} : f \text{ is the boundary value of a holomorphic function in the unit disk}\}$$

$$H_-^{(n)} = (H_+^{(n)})^\perp.$$

Taking the local coordinate θ on S^1 given by $z = e^{i\theta}$ for $z \in S^1$, let J be the infinitesimal rotation operator $-id/d\theta$. Then the polarization $H = H_+ \oplus H_-$ corresponds to the $+1$ and -1 eigenspaces of J respectively. Let's write $GL(H^{(n)})$ to denote the Banach Lie group of all invertible bounded operators of $H^{(n)}$. We are able to write now the following definition.

Definition. (cf. [PS86 §6.2]) The restricted general linear group of H, written $\mathbf{Gl}_{\text{res}}(H)$ is the subgroup of $GL(H)$ consisting of operators A such that $[J, A]$ is a Hilbert-Schmidt operator.

In other words, if we write $A \in GL(H)$ as

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with respect to the polarization $H = H_+ \oplus H_-$, then $A \in \mathbf{Gl}_{\text{res}}(H)$ if and only if b and c are Hilbert-Schmidt operators. In that case a and d are Fredholm operators. This fact is very related to a manifestation of the Bott Periodicity.

Given an element $\gamma \in \mathcal{L}\mathrm{Gl}_n(\mathbf{C})$ we form the multiplication operator M_γ on $H^{(n)}$ given by $f \mapsto \gamma f$. The operator M_γ turns out to be an element of $\mathbf{Gl}_{\text{res}}(H^{(n)})$.

Definition. The restricted Grassmannian $\mathrm{Gr}(H) = \mathrm{Gr}_\infty(H)$ is the set of all closed subspaces W of H such that

i) the orthogonal projection $\mathrm{pr}_+ : W \to H_+$ is a Fredholm operator and its image consists of smooth functions, and
ii) the orthogonal projection $\mathrm{pr}_- : W \to H_-$ is a Hilbert-Schmidt operator and its image consists of smooth functions.

We have as well the following submanifolds of $\mathrm{Gr} = \mathrm{Gr}(H)$,

a) $\mathrm{Gr}_0 = \{ W \in \mathrm{Gr} : \exists k \geq 0 \text{ such that } \zeta^k H_+ \subseteq W \subseteq \zeta^{-k} H_+ \}$,
b) $\mathrm{Gr}_0^{(n)} = \{ W \in \mathrm{Gr}_0 : \zeta^n W \subseteq W \}$,
c) $\mathrm{Gr}^{(n)} = \{ W \in \mathrm{Gr} : \zeta^n W \subseteq W \}$.

This definition permits us to direct our attention to the natural action of $\mathcal{L}\mathrm{Gl}_n(\mathbf{C})$ (and of $\mathcal{L}\mathrm{U}(n)$) on $\mathrm{Gr}^{(n)}$ via the multiplication operator. This action is transitive and its isotropy group is $\mathcal{L}^+\mathrm{Gl}_n(\mathbf{C})$ (the constant loops in $\mathrm{U}(n)$ respectively.) This immediately implies the identification

$$\Omega\mathrm{U}(n) = \mathcal{L}\mathrm{U}(n)/\mathrm{U}(n) = \mathcal{L}\mathrm{Gl}_n(\mathbf{C})/\mathcal{L}^+\mathrm{Gl}_n(\mathbf{C}) = \mathrm{Gr}^{(n)}.$$

This identification is what we call the Grassmannian Model for $\Omega\mathrm{U}(n)$.

Theorem 1.1.3. *(cf. [PS86, (8.3.3)]) Under the identification $\mathrm{Gr}^{(n)} \leftrightarrow \Omega\mathrm{U}(n)$ the submanifold $\mathrm{Gr}_0^{(n)}$ corresponds to $\Omega_{\text{pol}}\mathrm{U}(n)$.*

The Bott periodicity theorem can be stated (and proved) very naturally in this context [G75],[PS86 §8.8]. We have the following results.

Proposition 1.1.4. *(cf. [PS86, (8.6.6)]) The inclusion map $\mathrm{Gr}_0^{(n)} \hookrightarrow \mathrm{Gr}^{(n)}$, or equivalently $\Omega_{\text{pol}}\mathrm{U}(n) \hookrightarrow \Omega\mathrm{U}(n)$, is a homotopy equivalence.*

Theorem 1.1.5. *(Bott Periodicity Theorem) The inclusion*

$$\Omega_{\text{pol}}\mathrm{U}(n) = \mathrm{Gr}_0^{(n)} \hookrightarrow \mathrm{Gr}_0$$

induces an isomorphism of homotopy groups up to dimension $2n-2$.

Definition. The **virtual dimension** of an element $W \in \mathrm{Gr}$ is the Fredholm index of the orthogonal projection $\mathrm{pr}_+ : W \to H_+$, in particular if $W \in \mathrm{Gr}_0$ we have

$$\text{virt.dim } W = \dim(W/W \cap H_+) - \dim(H_+/W \cap H_+).$$

Observe that we can write

$$\mathrm{Gr}_0 = \bigcup_{p \leq q} \mathrm{Gr}(\zeta^p H_+/\zeta^q H_+)$$

and therefore we can identify Gr_0 with $\mathbf{Z} \times BU$ where the $\mathbf{Z}$ factor indexes the virtual dimension, corresponding with the degree of the corresponding element in $\Omega_{\mathrm{pol}}\mathrm{U}(n)$.

§1.2. The Mitchell Filtration.

In this section we will describe a filtration of $\Omega_{\mathrm{pol}}S\mathrm{U}(n)$ (that is homotopy equivalent to $\Omega S\mathrm{U}(n)$,) due to Mitchell [M86], we follow Segal for this [S89].

First we need to define a very useful map $\lambda: \mathrm{Gr}_m(\mathbf{C}^n) \to \Omega_{\mathrm{pol}}\mathrm{U}(n)$. For a given m-dimensional subspace V of $\mathbf{C}^n$ let pr_V and $\mathrm{pr}_{V^\perp}$ be the orthogonal projections of $\mathbf{C}^n$ onto V and $V^\perp$ respectively, then we define $\lambda_V: S^1 \to \mathrm{U}(n)$ by

$$\lambda_V(z) = z\mathrm{pr}_V + \mathrm{pr}_{V^\perp}. \tag{2.1}$$

This map has the property that

$$\lambda_{V\oplus W} = \lambda_V \lambda_W \tag{2.2}$$

when V and W are orthogonal.

We have in particular a map $\lambda: \mathbf{CP}^n = \mathrm{Gr}_1(\mathbf{C}^n) \to \Omega_{\mathrm{pol}}S\mathrm{U}(n)$, and it is well known that λ exhibits $H_*(\Omega S\mathrm{U}(n); \mathbf{Z})$ as the symmetric algebra on the reduced homology of $\mathbf{CP}^{n-1}$, i.e. a polynomial algebra on generators $b_1, \ldots, b_{n-1}$ with $\dim b_i = 2i$. Let $R_{n,k}$ be the abelian group of polynomials in $b_1, \ldots, b_{n-1}$ of degree at most k. The main result of [M86] is then

Theorem 1.2.1. *The loop space $\Omega_{\mathrm{pol}}S\mathrm{U}(n)$ has a filtration $F_{n,k}$ that realizes the filtration $R_{n,k}$ in homology. Furthermore, $F_{n,k}$ fits in the commutative diagram*

$$\begin{array}{ccc} \Omega_{\mathrm{pol}}S\mathrm{U}(n) & \xrightarrow{j} & B\mathrm{U} \\ \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle h} \\ F_{n,k} & \xrightarrow{\gamma_{n,k}} & B\mathrm{U}(k) \end{array}$$

here j and h are the standard maps, and the quotient $F_{n,k}/F_{n,k-1}$ is homotopic to the Thom complex of $\gamma_{n-1,k}$ over $F_{n-1,k}$.

In the case $n = \infty$, j becomes the Bott Periodicity map and $\gamma_{\infty,k}$ is a homotopy equivalence, and then we recover the stable splitting of Snaith [Sn79].

The following loop group description of this filtration will prove useful later. Denote by $\breve{\Omega}_k$ the loops of winding number k in $\Omega_{\mathrm{pol}}\mathrm{U}(n)$. Let V_0 be the subspace of $\mathbf{C}^n$ consisting of vectors of the form $(a, 0, \ldots, 0)$, and let $\tilde{\lambda}$ denote the loop λ_{V_0} defined by (2.1). Then for all $k \geq 1$ we can write

$$F_{n,k} = \tilde{\lambda}^{-k}\breve{\Omega}_k. \tag{2.4}$$

Observe that via the identification (1.3) we have that $\breve{\Omega}_k$ is a subspace of $\mathrm{Gr}_k(H_+) \simeq B\mathrm{U}(k)$. Identifying $F_{n.k}$ with $\breve{\Omega}_k$ the map $\gamma_{n,k}$ is the natural inclusion into $\mathrm{Gr}_k(H_+)$ and the bundle that it classifies has fibre H_+/W at W.

Remark 1.2.2. There is a interesting grassmannian model for the space $\Omega\mathrm{U}(n) / \vee n$, and a corresponding filtration as was pointed out by Crabb and Mitchell [CM87]. It is obtained as follows. For a given involution τ in $\mathrm{U}(n)$ one can associate a corresponding involution in the loop group $\Omega\mathrm{U}(n)$ by

$$(\sigma\gamma)(z) = \tau(\gamma(\bar{z}))$$

and we will have

$$(\Omega\mathrm{U}(n))^\sigma \simeq \Omega(\mathrm{U}(n) / \mathrm{U}(n)^\tau).$$

In particular if we take τ to be complex conjugation we see that we can think of $\Omega\mathrm{U}(n) / \vee n$ as a subspace of the loop group $\Omega\mathrm{U}(n)$, that is, the fixed point set under σ,

$$(\sigma\gamma)(z) = \overline{\gamma(\bar{z})}.$$

This in turn produces the desired grassmannian model and filtration. Only restrict yourself to those $\gamma(z)$ with real fourier coefficients, and to those subspaces on the grassmannian model that are real. The notions of polynomial loops and polynomial grassmannian are similarly defined in this case. The corresponding filtration is written $F^{\mathbf{R}}_{n,k}$. Observe that the inclusion

$$F^{\mathbf{R}}_{n,k} \subset F_{n,k}$$

becomes in the limit over n the classical inclusion

$$BO(k) \subset BU(k).$$

We refer the reader to [CM87] for details.

§1.3. Construction of the Map β.

We will proceed now to construct the map $\beta\colon \mathbf{Z} \times B\mathrm{U} \to \Omega\mathrm{U}$ with the promised properties. The map will be very related to the loop-group version of Bott periodicity (theorem 1.1.5 above,) but we have to be careful. We state the result that we want to prove.

Proposition 1.3.1. *There exists a realization of the Bott periodicity map*

$$\beta\colon \mathbf{Z} \times B\mathrm{U} \to \Omega\mathrm{U}$$

that restricts to a map

$$\beta\colon B\mathrm{U} \to \Omega S\mathrm{U},$$

with the following "holomorphic" property. If X is a compact complex manifold and we have a holomorphic map $X \to \mathrm{Gr}_m(\mathbf{C}^n)$, then the composition

$$X \to \mathrm{Gr}_m(\mathbf{C}^n) \hookrightarrow B\mathrm{U}(n) \to B\mathrm{U} \xrightarrow{\beta} \Omega S\mathrm{U}$$

is also holomorphic. Moreover we have the following formula for β. If $(k, V) \in k \times \mathrm{Gr}_m(\mathbf{C}^n) \subset k \times B\mathrm{U}$ then

$$\beta(k, V) = \tilde{\lambda}^{m-n+k}\lambda_{V^\perp}.$$

Remark 1.3.2. The map $\Omega S\mathrm{U}(n) \to \Omega S\mathrm{U}(n+1)$ is holomorphic. If X is a complex compact manifold, the space of holomorphic maps $\mathrm{Hol}(X, \Omega S\mathrm{U})$ is the limit of the spaces $\mathrm{Hol}(X, \Omega S\mathrm{U}(n))$. Is in this sense that we refer to the map $X \to \Omega S\mathrm{U}$ being holomorphic in the previous proposition.

We prove this theorem in several steps.

Lemma 1.3.3. *For a given loop $\xi \in \Omega_{\mathrm{pol}}\mathrm{U}(n)$ the map*

$$\beta_{\xi,k,n}\colon \mathrm{Gr}_k(\mathbf{C}^n) \to \Omega_{\mathrm{pol}}\mathrm{U}(n)$$

given by

$$V \mapsto \xi \cdot \lambda_{V^\perp}$$

is holomorphic.

Proof: We use the twisted Cauchy-Riemann equations that appear in the proof of proposition 1.3.1 of [V91] (in our case $\alpha = 0$.) Let $\bar{\partial}$ be the $(0,1)$ component of the d operator in $\mathrm{Gr}_k(\mathbf{C}^n)$. Let $f\colon \mathrm{Gr}_k(\mathbf{C}^n) \to \Omega_{\mathrm{pol}}\mathrm{U}(n)$ be any holomorphic function. Let $A(z)$ the holomorphic extension of $f^{-1}\bar{\partial}f$ to the unit disk in $\mathbf{C}$. Then Valli proves that the function $V \to f(V)\lambda_{V^\perp}$ is holomorphic if and only if

$$\mathrm{pr}_{V^\perp}\bar{\partial}\mathrm{pr}_V + \mathrm{pr}_{V^\perp}A(0)\mathrm{pr}_V = 0.$$

If f is constant $\bar{\partial}f = 0$, and this condition is equivalent to

$$\mathrm{pr}_{V^\perp}\bar{\partial}\mathrm{pr}_V = 0$$

that in turn is equivalent to $V \to \lambda_{V^\perp}$ being holomorphic and this is well known (see [V91] Lemma 1.1.)∎

Corollary 1.3.4. *The map*

$$\beta_{k,n}\colon \mathrm{Gr}_k(\mathbf{C}^n) \to \Omega_{\mathrm{pol}}\mathrm{U}(n)$$

given by

$$V \mapsto \tilde{\lambda}^{m-n+k} \cdot \lambda_{V^\perp}$$

is holomorphic.

Now we define a natural inclusion $i\colon \mathrm{Gr}^{(n)} \hookrightarrow \mathrm{Gr}^{(n+1)}$. First we see $H^{(n)}$ as $H \otimes \mathbf{C}^n$, and $H_+^{(n)}$ as $H_+ \otimes \mathbf{C}^n$. Then For $W \in \mathrm{Gr}^{(n)}$ we define

$$i(W) = W \oplus H_+.$$

We have

Lemma 1.3.5. *The following diagram commutes*

$$\begin{array}{ccc} \Omega_{\mathrm{pol}}\mathrm{U}(n) & \xrightarrow{\psi} & \mathrm{Gr}_0^{(n)} \\ \downarrow j & & \downarrow i \\ \Omega_{\mathrm{pol}}\mathrm{U}(n+1) & \xrightarrow{\psi} & \mathrm{Gr}_0^{(n+1)} \end{array}$$

Here j is the canonical inclusion, and ψ is the identification (1.3).

And also.

Lemma 1.3.6. *The following diagram commutes*

$$\begin{array}{ccc} m \times \mathrm{Gr}_k(\mathbf{C}^n) & \xrightarrow{\beta} & \Omega_{\mathrm{pol}_m}\mathrm{U}(n) \\ \downarrow & & \downarrow i \\ m \times \mathrm{Gr}_{k+1}(\mathbf{C}^{n+1}) & \xrightarrow{\beta} & \Omega_{\mathrm{pol}_m}\mathrm{U}(n+1) \end{array}$$

The proofs are straightforward. We need more lemmas.

Lemma 1.3.7. *The composition*

$$\mathrm{Gr}_k(\mathbf{C}^n)\xrightarrow{\beta}\Omega_{\mathrm{pol}}\mathrm{U}\,(n)\xrightarrow{\psi}\mathrm{Gr}_0^{(n)}\hookrightarrow \mathrm{Gr}_0=\mathbf{Z}\times B\mathrm{U}$$

classifies the tautological stable bundle over the Grassmannian.

Proof. It is easy to check that the composition

$$\mathrm{Gr}_k(\mathbf{C}^n)\xrightarrow{\lambda_{V^\perp}}\Omega_{\mathrm{pol}}\mathrm{U}\,(n)\xrightarrow{\psi}\mathrm{Gr}_0^{(n)}\hookrightarrow \mathrm{Gr}_0=\mathbf{Z}\times B\mathrm{U}$$

classifies the tautological bundle (here $\lambda_{V^\perp}$ represents $V\mapsto\lambda_{V^\perp}$.)

Claim. Multiplication by $\tilde{\lambda}^t$ seen as a map

$$\mathbf{Z}\times B\mathrm{U}=\mathrm{Gr}_0\xrightarrow{\tilde{\lambda}^t}\mathrm{Gr}_0=\mathbf{Z}\times B\mathrm{U}$$

classifies the canonical stable bundle over $\mathbf{Z}\times B\mathrm{U}$.

This claim is valid since multiplication by $\tilde{\lambda}^t$ seen as an operator in H is just a reordering of the canonical orthonormal basis.

Taking $t=m-n+k$ and composing finishes the proof.∎

Lemma 1.3.8. *The following diagram is homotopy-commutative*

$$\begin{array}{ccc}\mathrm{Gr}_0^{(n)} & \xrightarrow{i} & \mathrm{Gr}_0^{(n+1)}\\ & \searrow & \downarrow\\ & & \mathrm{Gr}_0\end{array}$$

Note: *This diagram is not commutative.*

Proof. The maps

$$\mathrm{Gr}_0^{(n)}\xrightarrow{i}\mathrm{Gr}_0^{(n+1)}\hookrightarrow\mathrm{Gr}_0=\mathbf{Z}\times B\mathrm{U}$$

and

$$\mathrm{Gr}_0^{(n)}\hookrightarrow\mathrm{Gr}_0=\mathbf{Z}\times B\mathrm{U}$$

represent the same element in $K(Gr_0^{(n)})$ since

$$\frac{W}{W\cap H_+^{(n)}}-\frac{H_+^{(n)}}{W\cap H_+^{(n)}}=\frac{W\oplus H_+}{(W\oplus H_+)\cap H_+^{(n+1)}}-\frac{H_+^{(n+1)}}{(W\oplus H_+)\cap H_+^{(n+1)}}.$$

They are therefore homotopic.∎

The next lemma will be very useful.

Lemma 1.3.9. *Let $p(z)=z^m q(z)$ where $q(z)$ is a polynomial of one complex variable and m is any integer. Suppose that $p(z)$ has the following properties,*

i) $p(1)=1$,

ii) $|p(z)|=1$ *for all z such that $|z|=1$. If k is the degree of the polynomial $q(z)$ then we can write*

$$p(z)=z^{k+m}.$$

Proof. First we consider the case in which $m=0$, that is the case in which $p(z)$ is a polynomial. We only need to prove that all the roots of the polynomial are equal to 0. Let $\alpha_1,\ldots,\alpha_k$ be the roots of the polynomial. Let $r(z)$ be the following Blaschke product,

$$r(z)=\prod_i\frac{z-\alpha_i}{1-\bar{\alpha}_i z},$$

and let $u = 1/r(1)$. We now consider the following function

$$f(z) = \frac{p(z)}{ur(z)}.$$

It is a holomorphic function from the Riemann sphere to itself. Now by the maximum modulus principle $p(z)$ has also the following property,
iii) $|p(z)| \leq 1$ for all $|z| \leq 1$.

It is clear that f has also properties i), ii) and iii). But f has no zero in the closed unit disk. Considering the function $1/f$ and using again the maximum modulus principle we conclude easily that f must be a constant function, but by property ii), it must be $f(z) = 1$. Then

$$p(z) = ur(z).$$

But the only way in which $r(z)$ can be a polynomial is if all the α_i's are 0.

Now for the general case $m \in \mathbf{Z}$, we apply the case $m = 0$ to $q(z)$, and then we multiply by z^m.∎

Corollary 1.3.10. *Let $\Omega_{\mathrm{pol}_0}\mathrm{U}(n)$ be the set of loops in $\Omega_{\mathrm{pol}}\mathrm{U}(n)$ of degree 0. Then*

$$\Omega_{\mathrm{pol}}S\mathrm{U}(n) = \Omega_{\mathrm{pol}_0}\mathrm{U}(n).$$

Proof. Let $\gamma \in \Omega_{\mathrm{pol}_0}\mathrm{U}(n)$. Apply the previous lemma to the following polynomial $f(z) = \det\gamma(z)$.∎

We are now able to prove the main result of this section.

Proof of Proposition 1.3.1. We obtain maps

$$\mathbf{Z} \times B\mathrm{U} \to \Omega_{\mathrm{pol}}\mathrm{U} \to \mathrm{Gr}_0^{(\infty)} \to \mathrm{Gr}_0 = \mathbf{Z} \times B\mathrm{U}$$

The first arrow is the limit of lemma 1.3.6, the second arrow is the limit of lemma 1.3.5, and the third is the limit of lemma 1.3.8.

That the composition

$$\mathbf{Z} \times B\mathrm{U} \to \mathbf{Z} \times B\mathrm{U}$$

is homotopic to the identity is a consequence of lemma 1.3.7.

The composition

$$\Omega_{\mathrm{pol}}\mathrm{U} \to \mathrm{Gr}_0^{(\infty)} \to \mathrm{Gr}_0 = \mathbf{Z} \times B\mathrm{U}$$

is the Bott periodicity map due to the theorem 1.5.

¿From this we can see that the composition

$$\mathbf{Z} \times B\mathrm{U} \to \Omega_{\mathrm{pol}}\mathrm{U} \to \Omega\mathrm{U}$$

is the Bott periodicity equivalence.

That this map restricts to

$$B\mathrm{U} \to \Omega S\mathrm{U}$$

follows from corollary 1.3.10.

The holomorphicity is the content of the corollary 1.3.4.

Finally the explicit formula follows from the construction.∎

Corollary 1.3.11. *The map β induces a map*

$$\mathrm{Hol}_k(\mathbf{CP}^1, B\mathrm{U}) \to \mathrm{Hol}_k(\mathbf{CP}^1, \Omega S\mathrm{U})$$

that we call again β, that fits in the following commutative diagram

$$\begin{array}{ccc} \mathrm{Hol}_k(\mathbf{CP}^1, B\mathrm{U}) & \hookrightarrow & \Omega^2_k B\mathrm{U} \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\simeq} \\ \mathrm{Hol}_k(\mathbf{CP}^1, \Omega S\mathrm{U}) & \hookrightarrow & \Omega^3_k S\mathrm{U} \end{array}$$

§1.4. The Spaces $\mathrm{Hol}_k(\mathbf{CP}^1, \mathrm{Gr}_n(\mathbf{C}^{n+m}))$.

We denote by $\mathcal{M}^m_{n,k}$ the space $\mathrm{Hol}_k(\mathbf{CP}^1, \mathrm{Gr}_n(\mathbf{C}^{n+m}))$ of degree k base point preserving holomorphic maps from the Riemann sphere $\mathbf{CP}^1$ to the complex grassmannian manifold $\mathrm{Gr}_n(\mathbf{C}^{n+m})$. (cf. Definitions 1.5.1 next section for notion of degree.) In this section we describe some results of B. Mann and J. Milgram [MM91]. We will use later the description of $\mathcal{M}^m_{n,k}$ explained there.

The spaces $\mathcal{M}^m_{n,k}$ have been extensively studied, for example due to its natural appearance in control theory. Some of the fundamental papers describing the geometry and topology of these spaces are [C76], [S79], [CCMM] and [MM91].

In [C76] it is proved that $\mathcal{M}^m_{n,k}$ is a smooth complex manifold of complex dimension $k(n+m)$. Clark gives there explicit charts for this manifold. However for us it will be more convenient to use the chart structure described in [MM91].

Remember that the grassmannian $\mathrm{Gr}_n(\mathbf{C}^{n+m})$ can be seen as equivalence classes matrices of rank n of the form $[A : B]$ where $A \in \mathrm{Mat}_{n,m}(\mathbf{C})$ and $B \in \mathrm{Mat}_{n,m}(\mathbf{C})$. Here $[A : B] \sim [UA, UB]$ for every $U \in \mathrm{Gl}_n(\mathbf{C})$. The equivalence class $[A : B]$ is identified with subspace of $\mathbf{C}^{n+m}$ that is the image of the linear map $\mathbf{C}^n \to \mathbf{C}^{n+m}$ given by the matrix $[A : B]$ written on the canonical basis.

A holomorphic map $f : \mathbf{CP}^1 \to \mathrm{Gr}_n(\mathbf{C}^{n+m})$ of degree k can always be written in the form

$$f(z) = [D(z) : N(z)]$$

where

a) $N(z) \in \mathrm{Mat}_{n,m}(\mathbf{C}[z])$ and $D(z) \in \mathrm{Mat}_{n,n}(\mathbf{C}[z])$ are both polynomial matrices.
b) The matrices $N(z)$ and $D(z)$ are *coprime*, that is, there are polynomial matrices $A(z) \in \mathrm{Mat}_{n,n}(\mathbf{C}[z])$ and $B(z) \in \mathrm{Mat}_{m,n}(\mathbf{C}[z])$ such that

$$N(z)B(z) + D(z)A(z) = I$$

In consequence the matrix $[D(u) : N(u)]$ has full rank n for all $u \in \mathbf{C}$.
c) $\deg(\det(D(z))) = k$
d) We have the base-point condition

$$\lim_{z\to\infty} D^{-1}(z)N(z) = 0$$

coming from the condition $f(\infty) = [I : 0]$.

The following is proved in [MM91].

Proposition 1.4.1. *After left multiplication by a unitary unimodular polynomial matrix $U \in \mathrm{Mat}_{n,n}(\mathbf{C}[z])$ one can always bring $f \in \mathcal{M}^m_{n,k}$ to the unique canonical form*

$$f(z) = [P(z) : Q(z)] =$$

$$\begin{pmatrix} p_{11}(z) & p_{12}(z) & \dots & p_{1n}(z) & q_{11}(z) & \dots & q_{1m}(z) \\ 0 & p_{22}(z) & \dots & p_{2n}(z) & q_{21}(z) & \dots & q_{2m}(z) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & p_{nn}(z) & q_{n1}(z) & \dots & q_{nm}(z) \end{pmatrix}$$

where in addition to conditions a),b)c) and d) above, one also has

i) *P is upper triangular and each $p_{ii}(z)$ is monic.*
ii) $\deg(p_{ji}(z)) < \deg(p_{ii}(z)) = k_i$
iii) $\sum_{i=1}^n k_i = k$
iv) $\deg(q_{nj}(z)) < \deg(p_{nn}(z))$
v) *In general the entries of Q are well defined modulo polynomials of degrees that are determined inductively row by row by the degrees of the polynomials $p_{i,j}(z)$ and the earlier determined rows of Q.*

A sequence $K = (k_1, \dots, k_n)$ such that $\sum_{k+1}^n k_i = k$, $k_i \geq 0$ is called a partition of k. We order such partitions lexicographically as follows, if $K' = (k'_1, \dots, k'_n)$ is another such partition we say that $K' \prec K$ if there is a j such that $k'_j < k_j$ and $k'_i = k_i$ for $i > j$. We denote the set of all such partitions by $\mathcal{K}$

For every partition $K \in \mathcal{K}$ of k we let $X_K = X(k_1, \dots, k_n)$ be the subspace of $\mathcal{M}^m_{n,k}$ of all elements such that the integers appearing as $\deg(p_{ii})$ in the canonical form are precisely $k_1, \dots, k_n$ in that order. Mann and Milgram define the following stratification of $\mathcal{M}^m_{n,k}$ by open manifolds,

$$\mathcal{F}_r(\mathcal{M}^m_{n,k}) = \bigcup_{k_n \geq r} X(k_1, \dots, k_n)$$

then they prove the following

Proposition 1.4.2. *$X(k_1, \dots, k_n)$ is a complex submanifold of $\mathcal{M}^m_{n,k}$ of complex dimension $(m+1)k + \sum_{i=2}^{n}(i-1)k_i$, moreover the normal bundle ν_K of $X(k_1, \dots, k_n)$ inside $\mathcal{M}^m_{n,k}$ is trivial.*

The family $\{\nu_K\}_{K \in \mathcal{K}}$ is a set of open charts for $\mathcal{M}^m_{n,k}$. In fact ν_K is obtained by carefully adding a lower triangular matrix to P which degrees hold similar conditions to b) above. The "change of chart" gluing maps are given then by the algorithm that takes the resulting matrices to the canonical form.

Example 1.4.3.

The structure of $\mathcal{M}^m_{k,1}$ for $n = 1$ is given by the following proposition due to Mann and Milgram.

Proposition 1.4.4. *Let $m\xi_n^{-1}$ be the complex vector bundle over $\mathbf{CP}^n$ given by*

$$(a_o, \dots, a_n; w_1, \dots, w_m) \sim (za_0, \dots, za_n, z^{-1}w_1, \dots, z^{-1}w_n)$$

$$z \in \mathbf{C}^*,\ [a_0, \dots, a_n] \in \mathbf{CP}^{n-1},\ \sum_{i=0}^{n} |a_i|^2 \neq 0$$

then

$$\mathcal{M}^m_{n,1} \approx \mathbf{C} \times (m\xi_{n-1}^{-1})^*$$

is a homeomorphism where $(m\xi_{n-1}^{-1})^*$ is the complement of the zero section of $m\xi_{n-1}^{-1}$.

Proof. We will give the map

$$T : \mathbf{C} \times (m\xi_{n-1}^{-1})^* \to \mathcal{M}^m_{1,n}$$

realizing this homeomorphism. First observe that $\mathbf{C} \times (m\xi_{n-1}^{-1})^*$ can also be described as $\mathbf{C} \times (\mathbf{C}^n)^* \times (\mathbf{C}^m)^* / \sim$ where

$$(A; a_1, \dots, a_n; w_1, \dots, w_m) \sim (A; za_0, \dots, za_{n-1}; z^{-1}w_1, \dots, z^{-1}w_m)$$

$$z \in \mathbf{C}^*,\ \sum |a_i|^2 \neq 0,\ \sum |w_j|^2 \neq 0$$

We also know that

$$\mathcal{M}^m_{n,1} = (\mathrm{Mat}_{n,n}(\mathbf{C}[z]) \times \mathrm{Mat}_{n,m}(\mathbf{C}[z]))^\bullet / \sim'$$

where the bullet indicates conditions a)-d) above.

We give a map $\mathbf{C} \times (\mathbf{C}^n)^* \times (\mathbf{C}^m)^* \to (\mathrm{Mat}_{n,n}(\mathbf{C}[z]) \times \mathrm{Mat}_{n,m}(\mathbf{C}[z]))^\bullet$ that descends to $T : \mathbf{C} \times (m\xi_{n-1}^{-1})^* \to \mathcal{M}^m_{1,n}$ and represents the desired homeomorphism.

If $a_1 \neq 0$ then

$T(A; a_1, \dots, a_n; w_1, \dots, w_m) =$

$$\begin{pmatrix} z - A & 0 & \dots & 0 & 0 & a_1 w_1 & \dots & a_1 w_m \\ a_2/a_1 & -1 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1}/a_1 & 0 & \dots & -1 & 0 & 0 & \dots & 0 \\ a_n/a_1 & 0 & \dots & 0 & -1 & 0 & \dots & 0 \end{pmatrix}$$

and in general if $a_i \neq 0$ then
$T(A; a_1, \ldots, a_n; w_1, \ldots, w_m) =$

$$\begin{pmatrix} -1 & 0 & \ldots & a_1/a_i & \ldots & 0 & 0 & \ldots & 0 \\ 0 & -1 & \ldots & a_2/a_i & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & z-A & \ldots & 0 & a_i w_1 & \ldots & a_i w_m \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_n/a_i & \ldots & -1 & 0 & \ldots & o \end{pmatrix}$$

∎

Here the chart structure of $\mathcal{M}^m_{1,n}$ is clear from the previous proposition.
Observe that $\mathcal{M}^m_{1,n} \simeq S(m\xi^{-1}_{n-1})$ the associated sphere bundle. Therefore we have

$$S^{2m-1} \to \mathcal{M}^m_{1,n} \to \mathbf{C}\mathrm{P}^{n-1}$$

and as we let $m \to \infty$ we get

$$\mathcal{M}_{1,n} \simeq \mathbf{C}\mathrm{P}^{n-1}$$

To conclude this section we state the main result of [MM91].

Theorem 1.4.5. *Let $t(K) = (n-1)k - \sum_{k=2}^{n}(j-1)k_j$ for all $K \in \mathcal{K}$. Then for every coefficient ring A we have*

$$H_*(\mathcal{M}^m_{k,n}; A) \cong \bigoplus_{K \in \mathcal{K}} H_*(\Sigma^{2t(K)}(\mathcal{M}^m_{k_1,1} \times \cdots \times \mathcal{M}^m_{k_n,1})_+; A),$$

moreover, the map induced by the inclusion

$$\iota^{k,n,m}: \mathcal{M}^m_{k,n} = \mathrm{Hol}_k(S^2; \mathrm{Gr}_n(\mathbf{C}^{n+m})) \to \Omega^2_k(\mathrm{Gr}_n(\mathbf{C}^{n+m}))$$

namely

$$\iota^{k,n,m}_*: H_*(\mathcal{M}^m_{k,n}; A) \to H_*(\Omega^2_k(\mathrm{Gr}_n(\mathbf{C}^{n+m})); A)$$

is an injection in homology for all coefficients.

The case $n = 1$ is taken care of in [CCMM]. Let $F_k(\mathbf{C})$ be the configuration space of k distinct points in $\mathbf{C}$ and

$$D_j = F_k(\mathbf{C})_+ \wedge_{\Sigma_k} S^k$$

then F. Cohen, R. Cohen, B. Mann and J. Milgram prove

Theorem 1.4.6. *The following spaces are stably homotopy equivalent*

$$\mathcal{M}^m_{1,k} \simeq_s \Sigma^{(2n-2)j} D_j$$

We refer the reader to the original paper and also [CCMM2] for details.

§1.5. Stabilization for High Rank Instantons.

The space of G-instantons over the S^4 is important for us due to the following theorem of Atiyah and Donaldson [A84a], [D84a],

Theorem 1.5.1. *The Space of charge k G-instantons over the four-sphere is homeomorphic to*

$$\mathrm{Hol}_k(\mathbf{CP}^1; \Omega G).$$

The problem of studying the topology of moduli spaces of instantons over 4-manifolds have been studied in various cases by Cohen [CR94], Kirwan [K], Sanders [S], Bryan-Sanders [BS97], and Norbury-Sanders [NS]. In particular in [S] the following is proved.

Let $\mathcal{I}_k^{SU(n)}$ denote the moduli space of based $SU(n)$-instantons with charge k over S^4. There is a natural map

$$\delta: \mathcal{I}_k^{\mathrm{SU}(n)} \to B\mathrm{U}(k)$$

given by the index bundle of the Dirac operator coupled to the instanton. For $\mathcal{I}_k^{Spn}$ there is correspondingly a map

$$\delta: \mathcal{I}_k^{\mathrm{Sp}(n)} \to BO(k)$$

Then the following is true.

Theorem 1.5.2. *There are homotopy equivalences*

$$\delta: \mathcal{I}_k^{SU(n)} \xrightarrow{\simeq} B\mathrm{U}(k)$$

$$\delta: \mathcal{I}_k^{\mathrm{Sp}(n)} \xrightarrow{\simeq} BO(k)$$

given by the Dirac operators. The compositions

$$\mathcal{I}_k^{\mathrm{SU}(n)} \to \mathcal{I}_k^{\mathrm{SU}(n+1)} \to \cdots \to \mathcal{I}_k^{\mathrm{SU}} \simeq \mathrm{BU}(k)$$

$$\mathcal{I}_k^{\mathrm{Sp}(n)} \to \mathcal{I}_k^{\mathrm{Sp}(n+1)} \to \cdots \to \mathcal{I}_k^{\mathrm{Sp}} \simeq BO(k)$$

are homotopy equivalent to the dirac maps for every $k \in \mathbf{N}$.

§1.6. Holomorphic Bott Periodicity: The complex case.

In this paragraph we sketch the argument used in [CLS] to prove Theorem A in the introduction. In fact in [CLS] a more refined result is proved involving the identification of the topology of $\mathrm{Hol}_k(\mathbf{CP}^1, B\mathrm{U}(n))$ as that of the Mitchell-Segal filtration $F_{n,k}$. We only state this result at the end. In fact we only use Theorem A in what follows and we don't need the full strength of the method. We originally wanted to relate the Mitchell filtration of $S\mathrm{U}/\mathrm{O}$ with some spaces of holomorphic maps, but it seems that in the real case the relation is not as direct.

The general strategy of the proof is as follows. We define a subspace $\mathcal{C}^-_{n,k}$ of the loop group $\Omega\mathrm{U}(n)$ that is the analogue to $F_{n,k}$ inside $\Omega_{\mathrm{pol}}\mathrm{U}(n)$. This space $\mathcal{C}^-_{n,k}$ has the property that in the limit over n we have

$$\lim_{n\to\infty} \mathcal{C}^-_{n,k} \simeq B\mathrm{U}(k).$$

Then we define an itermediate space $\chi_{n,k}$ with maps

$$\mathcal{M}_{n,k} \leftarrow \chi_{n,k} \rightarrow \mathcal{C}^-_{n,k}$$

and prove that both arrows are homotopy equivalences.

Proposition 1.6.1. *There is a homotopy equivalence*

$$\mathcal{M}_{\infty,k} \simeq B\mathrm{U}(k)$$

Once we have that using corollary 1.3.11, the usual Bott periodicity theorem and the fact that the map $B\mathrm{U}(k) \to B\mathrm{U}$ is injective in homology Theorem A follows directly from this proposition.

Now we sketch the proof of proposition 1.6.1. For this we set some notation and definitions.

Definitions 1.6.2. The space of all basepoint preserving holomorphic maps from $\mathbf{CP}^1$ to $B\mathrm{U}$, $\mathrm{Hol}(\mathbf{CP}^1, B\mathrm{U})$ is the direct limit $\lim_{\to} \mathrm{Hol}(\mathbf{CP}^1, B\mathrm{U}(n))$ of holomorphic maps from $\mathbf{CP}^1$ to the grassmannian of all n-dimensional complex vector subspaces of $\mathbf{C}^\infty$, $B\mathrm{U}(n) = \mathrm{Gr}_n(\mathbf{C}^\infty)$. In this section we write $V = \mathbf{C}^\infty$ a fix infinite dimensional complex vector space. We write Γ_n to denote the universal canonical n-dimensional holomorphic vector bundle over $\mathrm{Gr}_n(V)$.

The space $\mathrm{Hol}(\mathbf{CP}^1, \mathrm{Gr}_n(\mathbf{C}^{n+m}))$ has countably many components indexed by the *degree* $k \in \mathbf{Z}$ where k is the value of the first Chern class $c_1(f^*\Gamma^m_n)$ of the pull-back under f of the canonical bundle Γ^m_n over $\mathrm{Gr}_n(\mathbf{C}^{n+m})$ evaluated on the basic generator $[\mathbf{CP}^1] \in H_2(\mathbf{CP}^1)$,

$$k = -c_1(f^*\Gamma^m_n)[\mathbf{CP}^1].$$

We denote the k-th component by

$$\mathcal{M}^m_{n,k} = \mathrm{Hol}_k(\mathbf{CP}^1, \mathrm{Gr}_n(\mathbf{C}^{n+m})).$$

If we omit one of the indices on the notation $\mathcal{M}^m_{n,k}$ we mean that we are stabilizing that index to infinity. For example

$$\mathcal{M}_{n,k} = \lim_{m\to\infty} \mathrm{Hol}_k(\mathbf{CP}^1, \mathrm{Gr}_n(\mathbf{C}^{n+m})) = \mathrm{Hol}_k(\mathbf{CP}^1, B\mathrm{U}(n))$$

and

$$\mathcal{M}_k = \mathrm{Hol}_k(\mathbf{CP}^1, B\mathrm{U}).$$

Our first remark is that $\mathcal{M}_{n,k}$ can also be interpreted as a space of holomorphic bundles over $\mathbf{CP}^1$ with additional structure. Let

$$\Upsilon^c_{n,k} = \left\{ \begin{array}{ccc} E & \xrightarrow{\iota} & \epsilon^{n+m}_{\mathbf{C}} \\ \downarrow & & \downarrow \\ \mathbf{CP}^1 & = & \mathbf{CP}^1 \end{array} : \iota \text{ is holom.}, c_1(E) = -k, \iota(E_\infty) = (\mathbf{C}^n \oplus 0)_\infty \right\}$$

consisting of isomorphism classes of pairs (E, ι) where E is an n-dimensional complex holomorphic bundle over $\mathbf{CP}^1$ and i is a holomorphic inclusion into the trivial complex holomorphic vector bundle $\epsilon_{\mathbf{C}}^{n+m}$ of dimension $n+m$ over $\mathbf{CP}^1$ such that $\iota(E_\infty) = \mathbf{C}^n \oplus 0^m \subseteq \mathbf{C}^{n+m} = (\epsilon_{\mathbf{C}}^{n+m})_\infty$.

Proposition 1.6.3. *The spaces $\mathcal{M}_{n,k}$ and $\Upsilon_{n,k}^c$ are homeomorphic.*

Proof. Given an element $f\colon \mathbf{CP}^1 \to \mathrm{Gr}_n(\mathbf{C}^{n+m})$ in $\mathcal{M}_{n,k}$ we define the pair (E, ι) as follows,

$$E = f^*\Gamma_n^m$$

and ι is defined by the commutative diagram

$$\begin{array}{ccccc} f^*\Gamma_n^m & \xrightarrow{\iota} & f^*\epsilon_{\mathbf{C}}^{n+m} & = & \epsilon_{\mathbf{C}}^{n+m} \\ \big\downarrow f^* & & \big\downarrow & & \\ \Gamma_n^m & \longrightarrow & \bar{\epsilon}_{\mathbf{C}}^{n+m} & & \end{array}$$

where $\bar{\epsilon}_{\mathbf{C}}^{n+m}$ is the trivial bundle over $\mathrm{Gr}_n(\mathbf{C}^{n+m})$.

Conversely given the pair $[E, \iota]$ we define f by the rule

$$f(z) = \iota(E_z) \subseteq \mathbf{C}^{n+m}$$

where E_z is the fiber of E at $z \in \mathbf{CP}^1$. ∎

Definition 1.6.4. We define the spaces $\mathcal{C}_{n,k}$ and $\mathcal{C}_{n,k}^-$ as follows.

a) The space $\mathcal{C}_n$ is defined as the moduli space of isomorphism classes of pairs (E, θ) where E is a holomorphic bundle of rank n over $\mathbf{CP}^1$ with, and θ is a holomorphic trivialization of $E|_{D_\infty}$.
b) The space $\mathcal{C}_{n,k}$ will be the moduli space of isomorphism classes of pairs (E, θ) where E is a holomorphic bundle of rank n over $\mathbf{CP}^1$ with $c_1(E) = -k$, and θ is a holomorphic trivialization of $E|_{D_\infty}$.
c) The space $\mathcal{C}_{n,k}^- \subset \mathcal{C}_{n,k}$ is defined as the space of pairs (E, θ) such that E is negative (i.e. $E \cong \mathcal{O}(k_1) \oplus \cdots \oplus \mathcal{O}(k_n)$ with all $k_j \leq 0$.)

Both spaces $\mathcal{C}_{n,k}$ and $\mathcal{C}_{n,k}^-$ can be interpreted as subspaces of the loop group due to the following proposition taken from [PS86].

Proposition 1.6.5. *([PS86 (8.10.1)]) The loop group $\Omega\mathrm{U}(n)$ is homeomorphic to $\mathcal{C}_n$.*

Proof. To define the homeomorphism $\mathcal{F} : \mathcal{C}_n \to \Omega\mathrm{U}(n)$, given any element $(E, \theta) \in \mathcal{C}_n$ choose any trivialization τ of $E|_{D_0}$, the corresponding attaching map $\gamma(z) \in \mathcal{L}\mathrm{Gl}_n(\mathbf{C})$ is well defined up to an element $\tau \in \mathcal{L}^+\mathrm{Gl}_n(\mathbf{C})$ therefore $\gamma(z) \in \mathcal{L}\mathrm{Gl}_n(\mathbf{C})/\mathcal{L}^+\mathrm{Gl}_n(\mathbf{C}) = \Omega\mathrm{U}(n)$ defines $\mathcal{F}(E, \theta) \in \Omega\mathrm{U}(n)$. The inverse $\mathcal{F}^{-1}(\gamma)$ is given by the isomorphism class of the pair (E, θ) where the bundle $E = (D_0 \times \mathbf{C}^n) \cup_\gamma (D_\infty \times \mathbf{C}^n)$ and θ is the obvious trivialization. ∎

Observe that the set of isomorphism classes of bundles E is a the discrete set $\mathcal{K}$ defined in section 1.4, and that the map $\mathcal{C}_n \to \mathcal{K}$ is discontinuous.

Corollary 1.6.6. *The space $\mathcal{C}_n$ can be identified with the grassmannian* $\mathrm{Gr}^{(n)}$.

Now observe that for elements in $\mathcal{C}_{n,k}^-$ $H^0(\mathbf{CP}^1; E^* \otimes \mathcal{O}(-1))$ is a k-dimensional complex space (by Kodaira's vanishing theorem and the Riemman-Roch theorem for example) that can be interpreted as pairs of polynomials on z and z^{-1} (that is on D_0 and D_∞,) related by $\gamma(z)$. Let $W = \gamma(z)H_+^{(n)}$ be the corresponding element on the grassmannian model. With this interpretation and the Birkhoff Factorization theorem one sees that we have the following isomorphism

$$H^0(\mathbf{CP}^1; E^* \otimes \mathcal{O}(-1)) \cong W \cap H_-$$

. Therefore the corresponding interpretation on the grassmannian model for $\mathcal{C}_{n,k}^-$ is

Corollary 1.6.7. *The space $\mathcal{C}^-_{n,k}$ is homeomorphic to the subspace of* $\mathrm{Gr}^{(n)}$ *consisting of those W such that*

$$\dim(W \cap H_-) = k$$

In particular as we let $n \to \infty$ we have

$$\mathcal{C}^-_{\infty,k} \approx BU(k)$$

Now we make a very important definition.

Definition 1.6.8. The space X is defined as the space of isomorphism pairs of triples (E, ι, θ) where E is a holomorphic bundle of rank n over $\mathbf{CP}^1$ with, θ is a holomorphic trivialization of $E|_{D_\infty}$ and $\iota: E \to \mathbf{CP}^1 \times V$ is a holomorphic bundle immersion of E into the trivial infinite dimensional holomorphic bundle over $\mathbf{CP}^1$.

We have obvious maps

$$\chi_{n,k} \xrightarrow{\pi_{n,k}} \Upsilon^c_{n,k} \approx \mathcal{M}_{n,k} = \mathrm{Hol}_k(\mathbf{CP}^1, \mathrm{Gr}_n(V))$$
$$(E, \iota, \theta) \mapsto (E, \iota)$$

and

$$\chi_{n,k} \xrightarrow{p_{n,k}} \mathcal{C}^-_{n,k} \subset \mathcal{L}\mathrm{Gl}_n(\mathbf{C})/\mathcal{L}^+\mathrm{Gl}_n(\mathbf{C})$$
$$(E, \iota, \theta) \mapsto (E, \theta).$$

Proposition 1.6.9. *The maps*

$$\mathcal{X}_{n,k} \xrightarrow{\pi_{n,k}} \mathcal{M}_{n,k} = \mathrm{Hol}_k(\mathbf{CP}^1, \mathrm{Gr}_n(V))$$

and

$$\mathcal{X}_{n,k} \xrightarrow{p_{n,k}} \mathcal{C}^-_{n,k} \subset \mathcal{L}\mathrm{Gl}_n(\mathbf{C})/\mathcal{L}^+\mathrm{Gl}_n(\mathbf{C})$$

are quasifibrations.

Proof.
We can identify $\mathcal{X}_{n,k}$ as a set with the family of commutative diagrams,

$$\mathcal{X}_{n,k} = \left\{ \begin{array}{ccc} D_\infty & \xrightarrow{\theta} & \mathrm{Fr}_n(V) \\ \downarrow & & \downarrow \\ \mathbf{CP}^1 & \xrightarrow{f} & \mathrm{Gr}_n(V) \end{array} : \theta, f \text{ are holomorphic} \right\}.$$

Observe that $\mathcal{M}_{n,k}$ is the limit of the connected complex manifolds

$$\mathcal{M}^m_{n,k} = \mathrm{Hol}_k(\mathbf{CP}^1, \mathrm{Gr}_n(\mathbf{C}^{n+m})).$$

We define

$$\mathcal{X}^m_{n,k} = \left\{ \begin{array}{ccc} D_\infty & \xrightarrow{\theta} & \mathrm{Fr}_n(\mathbf{C}^{n+m}) \\ \downarrow & & \downarrow \\ \mathbf{CP}^1 & \xrightarrow{f} & \mathrm{Gr}_n(\mathbf{C}^{n+m}) \end{array} : \theta, f \text{ are holomorphic} \right\}.$$

Now we prove that

$$\mathcal{X}^m_{n,k} \xrightarrow{\pi_m} \mathcal{M}^m_{n,k}$$

is a local fibration. That is enough to conclude that π is a local fibration.

In [MM91] Mann and Milgram prove that $\mathcal{M}^m_{n,k}$ is a connected complex manifold of dimension $k(n+m)$ giving explicit charts [cf. §1.4]. To prove that π_m is a local fibration we take a chart $\mathcal{U}$ and construct a bijection

$$\Psi : \mathcal{U} \times \mathrm{Hol}(D_\infty, \mathrm{Gl}_n(\mathbf{C})) \to \pi_m^{-1}(\mathcal{U})$$

and define the topology in $\mathcal{X}^m_{n,k}$ by asking those maps to be homeomorphisms. Then π_m is a fibration along $\mathcal{U}$ for all $\mathcal{U}$. We proceed now to define Ψ.

Given $f \in \mathcal{M}^m_{n,k}$ Mann and Milgram use results from control theory to show that f has a unique canonical representation as a polynomial matrix of the form

$$f(z) = \begin{pmatrix} p_{11}(z) & p_{12}(z) & \dots & p_{1n}(z) & q_{11}(z) & \dots & q_{1m}(z) \\ 0 & p_{22}(z) & \dots & p_{2n}(z) & q_{21}(z) & \dots & q_{2m}(z) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & p_{nn}(z) & q_{n1}(z) & \dots & q_{nm}(z) \end{pmatrix}$$

with certain additional conditions. When we vary the coefficients of the polynomials within these conditions we get a chart $\mathcal{U}$ containing f with complex dimension $k(n+m)$ as explained in section 1.4. This, of course, endows $\mathcal{U}$ with a local section $\psi : \mathcal{U} \to \pi^{-1}(\mathcal{U})$ simply by taking

$$\theta(z) = \begin{pmatrix} p_{11}(z) & p_{12}(z) & \dots & p_{1n}(z) & q_{11}(z) & \dots & q_{1m}(z) \\ 0 & p_{22}(z) & \dots & p_{2n}(z) & q_{21}(z) & \dots & q_{2m}(z) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & p_{nn}(z) & q_{n1}(z) & \dots & q_{nm}(z) \end{pmatrix}$$

and

$$\psi(f) = (f, \theta).$$

To define Ψ we write

$$\Psi(f, \tau) = \tau \cdot \psi(f)$$

in other words, if $f(z) = [P(z), Q(z)]$, then $\Psi(f, \tau)$ will be the pair

$$(f(z), \theta(z)) = ([P(z), Q(z)], \tau(z) \cdot (P(z), Q(z))).$$

Now we prove that p is a continuous map, we keep the notations. Given a point $(f, \theta) \in \mathcal{X}_{n,k}$ then there exists a neighborhood $\tilde{\mathcal{U}}$ around it in $\mathcal{X}_{n,k}$ so that $p|_{\tilde{\mathcal{U}}}$ can be factored as

$$\tilde{\mathcal{U}} \xrightarrow{\Theta} \mathcal{L}\mathrm{Gl}_n(\mathbf{C}) \to \mathcal{L}\mathrm{Gl}_n(\mathbf{C}) / \mathcal{L}^+\mathrm{Gl}_n(\mathbf{C})$$

and hence we only need to prove that the map Θ is continuous.

We choose $\tilde{\mathcal{U}} = \pi^{-1}(\mathcal{U})$ as above and then we write

$$\Theta : \mathcal{U} \times \mathrm{Hol}(D_\infty, \mathrm{Gl}_n(\mathbf{C})) \to \mathcal{L}\mathrm{Gl}_n(\mathbf{C})$$

$$\Theta(f(z), \tau(z)) = \tau(z) \cdot P(z)$$

this is just the attaching map induced by the "canonical" trivialization given by the local section ψ. This implies that Θ is continuous.

Finally we need to show that p is a quasifibration. This is a consequence of the proof of proposition (8.10.1) in [PS86] (or of the factorization theorem). For given $\gamma(z) \in \mathcal{C}^k_{m,n}$ this proof gives a canonical way of choosing an isomorphism $E \cong \mathcal{O}(k_1) \oplus \cdots \oplus \mathcal{O}(k_n)$ in a neighborhood of γ fixing the Grothendieck type where E is the holomorphic bundle associated to the principal bundle P of that proposition. Then the fiber is of course canonically identified with the linear embeddings of $\mathcal{O}_{k_1,\dots,k_n}$ into the infinite dimensional trivial bundle in that neighborhood of γ proving the proposition. That can be said more explicitly, γ gives a bundle with a trivialization at infinity, with this trivialization we can identify $\Gamma(E^*)^* \cong \mathbf{C}^{n+k}$ with the classic spaces of polynomials in z^{-1} in a coherent way close to $\gamma(z)$. This proves that we have a fibration at every strata and this implies that we have a quasifibration. We refer the reader to [CLS] for details. ∎

Proposition 1.6.10. *The maps $p_{n,k}$ and $\pi_{n,k}$ are homotopy equivalences.*

Proof. The homotopy fiber for $\pi_{n,k}$ is the space $\mathrm{Hol}(D_\infty, \mathrm{Gl}_n(\mathbf{C}))$ which is contractible. The homotopy fiber for the map $p_{n,k}$ is the space of bundle injections $\iota: E \to \epsilon^\infty_{\mathbf{C}}$. Let's denote this space by $\mathcal{I}_\infty = \lim_\to \mathcal{I}_n$. But given a point space $\mathcal{I}_n$ contracts to a point inside $\mathcal{I}_{2n}$. This is true for if $i_1: \epsilon^n_{\mathbf{C}} \to \epsilon^n_{\mathbf{C}} \times \epsilon^n_{\mathbf{C}} \to \epsilon^{2n}_{\mathbf{C}}$ is the inclusion into the first n components and i_2 is similarly the inclusion into the second n components, then for any two elements $\iota_1, \iota_2 \in \mathcal{I}_n$ the line

$$t i_1 \iota_1 + (1-t) i_2 \iota_2$$

is completely contained in $\mathcal{I}_{2n}$. Thus $\mathcal{I}_\infty$ is contractible.∎

Proof of Proposition 1.6.1. We have

$$\mathcal{M}_{n,k} \leftarrow \chi_{n,k} \to \mathcal{C}^-_{n,k}$$

and both arrows are homotopy equivalences. Moreover

$$\mathcal{C}_{\infty,k} \simeq B\mathrm{U}(k)$$

∎

Remark 1.6.11. In [A68] Atiyah defines a map

$$\bar\partial: C^\infty_\bullet(S^2, B\mathrm{U}) \to \mathbf{Z} \times B\mathrm{U}.$$

twisting the $\bar\partial$ elliptic operator with each element of $C^\infty_{\bullet,k}(S^2, B\mathrm{U})$ tensored with $\mathcal{O}(-1)$ and taking the index. He also points out that

$$\mathrm{Ker}(\bar\partial_E) \cong H^0(\mathbf{CP}^1; E^* \otimes \mathcal{O}(-1))$$

and

$$\mathrm{Coker}(\bar\partial_E) \cong H^1(\mathbf{CP}^1; E^* \otimes \mathcal{O}(-1))$$

He then proves that this map is a homotopy equivalence. By Kodaira's vanishing theorem and an elementary Riemann-Roch computation this map restricts to the holomorphic mapping space as follows

$$\bar\partial: \mathrm{Hol}_k(\mathbf{CP}^1, B\mathrm{U}) \to B\mathrm{U}(k)$$

and from the arguments above we know that this map is a homotopy equivalence.

Observe that $F_{n,k}$ the Mitchell filtration is a polynomial version of what we have called $\mathcal{C}^-_{n,k}$. In [CLS] we use a morse theoretic argument to prove the analogue of Proposition 1.1.4.

Proposition 1.6.12. *$F_{n,k}$ and $\mathcal{C}^-_{n,k}$ are homotopy equivalent.*

This proves the following conjecture of Mann and Milgram

$$F_{n,k} \simeq \mathrm{Hol}_k(\mathbf{CP}^1, B\mathrm{U}(n)).$$

§2. REAL HOLOMORPHIC BOTT PERIODICITY.

In this section we prove Theorem B using the results and methods of the previous section.

§2.1 Quaternionic Holomorphic Bundles.

In this section we define and investigate several interpretations of the space of holomorphic maps $\mathrm{Hol}_k(\mathbf{CP}^1, \mathrm{Sp}(n)/\mathrm{U}(n))$, and some spaces related to it. We begin with the definitions.

Definitions 2.1.1. As before, E always represents a negative holomorphic bundle over $\mathbf{CP}^1$.

A quaternionic linear trivialization Ψ of $E \otimes \mathbf{H}$

$$\Psi : E \otimes \mathbf{H} \to \epsilon^n_{\mathbf{H}}$$

is said to be $\mathbf{H}$-holomorphic if the map

$$E \to E \oplus \bar{E} \cong E \otimes \mathbf{H} \xrightarrow{\Psi} \epsilon^n_{\mathbf{H}} = \mathbf{CP}^1 \times \mathbf{H}^n$$

is holomorphic. We define $\mathcal{H}_{n,k}$ as the space of base point preserving degree k holomorphic maps from $\mathbf{CP}^1$ to $\mathrm{Sp}(n)/\mathrm{U}(n)$. For this we write

$$\mathcal{H}_{n,k} = \mathrm{Hol}_k(\mathbf{CP}^1, \mathrm{Sp}(n)/\mathrm{U}(n)).$$

We define $\mathcal{N}_{n,k}$ is the space of isomorphism classes of pairs (E, Ψ) where E is a negative holomorphic bundle of rank n over $\mathbf{CP}^1$ with first chern class $c_1(E) = -k$ and Ψ is a quaternionic linear trivialization that is $\mathbf{H}$–holomorphic.

We say that the pair (E_1, Ψ_1) is isomorphic to the pair (E_2, Ψ_2) if there exits an isomorphism $T: E_1 \to E_2$ such that the following diagram is commutative

$$\begin{array}{ccc} E_1 \otimes \mathbf{H} & \xrightarrow{\Psi_1} & \epsilon^n_{\mathbf{H}} \\ \downarrow{\scriptstyle T\otimes 1} & & \| \\ E_2 \otimes \mathbf{H} & \xrightarrow{\Psi_2} & \epsilon^n_{\mathbf{H}} \end{array}$$

We also define as an auxiliary device the following space,

$\Upsilon_{n,k} =$

$$\left\{ \begin{array}{ccc} E & \xrightarrow{i} & \epsilon^n_{\mathbf{H}} \\ \downarrow & & \downarrow \\ \mathbf{CP}^1 & = & \mathbf{CP}^1 \end{array} : i \text{ is holom.}, c_1(E) = -k, jE = E^{\perp}, i(E_\infty) = (\mathbf{C}^n \oplus 0)_\infty \right\}$$

in other words, this space consists of isomorphism classes of pairs (E, i) where E is as before, and i is a holomorphic inclusion into the trivial bundle $\epsilon^n_{\mathbf{H}}$ (of complex dimension $2n$) such that once we see E as a sub-bundle $i(E)$ of $\epsilon^n_{\mathbf{H}}$ it has the property that the fiber at $z \in \mathbf{CP}^1$ $i(E_z) \subset \mathbf{H}^n$ holds

$$j \cdot i(E_z) = i(E_z)^{\perp}.$$

We prove now that these three spaces are actually the same.

Proposition 2.1.2. *The spaces $\mathcal{H}_{n,k}$, $\mathcal{N}_{n,k}$ and $\Upsilon_{n,k}$ are homeomorphic.*

Proof. The proof consist of two parts.

Part 1. The spaces $\mathcal{H}_{n,k}$ and $\Upsilon_{n,k}$ are homeomorphic.

That this is the case is given by the fact that we can use the following model for the homogeneous space $\mathrm{Sp}(n)/\mathrm{U}(n)$,

$$\mathrm{Sp}(n)/\mathrm{U}(n) = \{\text{Complex subspaces } V \subset \mathbf{H}^n = \mathbf{C}^{2n} : jV = V^\perp\} \subset \mathrm{Gr}_n(\mathbf{C}^{2n}).$$

where if $A \in \mathrm{Sp}(n)$ then the class $[A] \in \mathrm{Sp}(n)/\mathrm{U}(n)$ is represented by the subspace of $\mathbf{H}^n$ given by $V = A(\mathbf{C}^n \oplus 0)$. Here of course j is the quaternionic imaginary unit, and $\mathbf{H} = \mathbf{C} \oplus j\mathbf{C}$.

When we are given an element $f\colon \mathbf{CP}^1 \to \mathrm{Sp}(n)/\mathrm{U}(n)$ in $\mathcal{H}_{n,k}$ we define E by

$$E = f^*\iota^*\Gamma_n$$

where ι is the canonical inclusion $\mathrm{Sp}(n)/\mathrm{U}(n) \xrightarrow{\iota} \mathrm{Gr}_n(\mathbf{C}^{2n})$ given above and Γ_n is the tautological n-dimensional complex holomorphic vector bundle over $\mathrm{Gr}_n(\mathbf{C}^{2n})$. With this definition E has an associated inclusion i into the trivial bundle.

Conversely given a pair $[E, i]$, we define f by

$$f(z) = i(E_z) \subset \mathbf{C}^{2n}$$

where E_z denotes the fiber of E at $z \in \mathbf{CP}^1$.

Part 2. The spaces $\mathcal{N}_{n,k}$ and $\Upsilon_{n,k}$ are homeomorphic.

We construct a bijection $\mathcal{N}_{n,k} \leftrightarrow \Upsilon_{n,k}$. First, given $[E, \Psi] \in \mathcal{N}_{n,k}$ we define i by

$$i\colon E \to E \oplus \bar{E} \cong E \otimes \mathbf{H} \xrightarrow{\psi} \epsilon^n_{\mathbf{H}}.$$

Conversely given $[E, i] \in \Upsilon_{n,k}$ we define Ψ by

$$\Psi = i \otimes 1\colon E \otimes \mathbf{H} \to \epsilon^n_{\mathbf{H}} \otimes \mathbf{H} = \epsilon^n_{\mathbf{H}}.$$

This finishes the proof of the proposition.∎

Remark 2.1.3. The $\mathbf{H}$–holomorphicity of Ψ is a somewhat strong condition and has to be taken with care. Given two of those trivializations $\Psi_1\colon E \otimes \mathbf{H} \to \epsilon^n_{\mathbf{H}}$ and $\Psi_2\colon E \otimes \mathbf{H} \to \epsilon^n_{\mathbf{H}}$ then the composition $\Psi_2 \circ \Psi_1^{-1}\colon \epsilon^n_{\mathbf{H}} \cong \epsilon^n_{\mathbf{H}}$ is an isomorphism but in general it is not holomorphic – so it is not necessarily given by a constant matrix in $GlCn$.

Example 2.1.4. Here we take a closer look at the case $n = 1$, $k = 1$. In this case $\mathrm{Gl}_1(\mathbf{H}) = \mathbf{H}^*$, $\mathrm{Gl}_1(\mathbf{C}) = \mathbf{C}^*$, $\mathrm{Sp}(1)$ is the group of unit quaternions and $\mathrm{U}(1)$ is the unit circle in $\mathbf{C}$. The map

$$\mathrm{U}(1) \to \mathrm{Sp}(1) \to \mathrm{Sp}(1)/\mathrm{U}(1)$$

is the Hopf fibration and $\mathrm{Sp}(1)/\mathrm{U}(1) \approx S^2$, in fact we have that it is a complex manifold, and therefore $\mathbf{CP}^1 = \mathrm{Sp}(1)/\mathrm{U}(1)$. The inclusion $\mathrm{Sp}(1)/\mathrm{U}(1) \subseteq \mathrm{Gr}_1(\mathbf{C}^2) = \mathbf{CP}^1$ is actually an equality. We can therefore write

$$\mathcal{H}_{1,1} = \mathrm{Hol}_1(\mathbf{CP}^1, \mathbf{CP}^1).$$

If we choose the base point condition $f(\infty) = \infty$, we know that all such maps are given by

$$f(z) = \alpha z + \beta$$

where α is a nonzero complex number and β is any complex number. That fact implies that we have the following homeomorphism

$$\mathcal{H}_{1,1} \approx \mathbf{C}^* \times \mathbf{C} \simeq S^1.$$

This space Now we write an explicit formula for Ψ in terms of α and β. Observe that in this example there is only one possible bundle E for all the elements in $\mathcal{H}_{1,1}$, namely the canonical Hopf bundle $\mathcal{O}(-1)$ over $\mathbf{CP}^1$.

We start with an element $f \in \mathcal{H}_{1,1}$

$$f(z) = \alpha z + \beta$$

We can realize E as follows

$$E = \{(z; \lambda(z+j)) \in \mathbf{CP}^1 \times \mathbf{H} : \lambda \in \mathbf{C}\}$$

Then Ψ will be given by left multiplication by $(z+j)^{-2}(f(z)+j)$, and we have that

$$f(z) = \Psi_z(E_z)$$

In this case we can encode Ψ as a function

$$\Psi: \mathbf{CP}^1 \to \mathrm{Gl}_1(\mathbf{H}) = \mathbf{H}^*$$

$$\Psi(z) = (z+j)^{-1}(f(z)+j)$$

We define $\Psi(\infty) = \alpha$. More explicitly we can write

$$\Psi(z) = \frac{\alpha|z|^2 + \beta\bar{z} + 1}{|z|^2+1} + j\frac{-\alpha z + z - \beta}{|z|^2+1}$$

this is neither holomorphic nor constant.

Even with those consideration Ψ keeps some holomorphicity properties. We have, for example, the following identity property.

Proposition 2.1.5. *Given $[E, \Psi_1], [E, \Psi_2] \in \mathcal{N}_{n,k}, y$ if Ψ_1 and Ψ_2 coincide over an open set U in $\mathbf{CP}^1$ then they are identical.*

Proof. Observe that

$$\Psi_z: E_z \otimes \mathbf{H} \to \mathbf{H}^n$$

determines completely the corresponding $f(z)$, for writing $E_z \subset E_z \otimes \mathbf{H}$ we have

$$f(z) = \Psi_z(E_z).$$

Therefore $\Psi|_U$ defines $f|_U$ and vice versa. ∎

We consider some additional definitions.

Definition 2.1.6. We define the space $\mathcal{Y}_{n,k}$ as the space of isomorphism classes $[E, \Psi, \theta, g]$ where

E is a negative holomorphic bundle of rank n over $\mathbf{CP}^1$ with first chern class $-k$.
Ψ is an $\mathbf{H}$-holomorphic trivialization $\Psi: E \otimes \mathbf{H} \cong \epsilon^n_{\mathbf{H}}$.
θ is a holomorphic trivialization of $E|_{D_\infty}$, $\theta: E|_{D_\infty} \cong D_\infty \times \mathbf{C}^n$.
g is an isomorphism $g: D_\infty \times \mathbf{H}^n \to D_\infty \times \mathbf{H}^n$ such that

$$\Psi|_{D_\infty} = g \circ (\theta \otimes 1). \tag{2.1}$$

Remark 2.1.7. Observe that g is not necessarily holomorphic.

We have a very natural map

$$\mathcal{X}_{n,k} \to \mathcal{N}_{n,k}$$

$$[E, \Psi, \theta, g] \mapsto [E, \Psi]$$

we would like to understand this map.

Example 2.1.8. The case $n = 1$, $k = 1$. In this case we know that Ψ can be identified with a map of the form

$$\Psi : \mathbf{CP}^1 \to \mathbf{H}^*$$
$$\Psi(z) = (z + j)^{-1}(\alpha z + \beta + j)$$

where $\alpha \in \mathbf{C}^*$, $\beta \in \mathbf{C}$. Similarly θ can be identified with a holomorphic map $\tilde{\theta}: D_\infty \to \mathbf{C}^*$ as follows,

$$\theta(z; \lambda(z + j)) = \left(z; \tilde{\theta}(z)\frac{\lambda}{z}\right) \in D_\infty \times \mathbf{C}$$

or with the change of variable $y = 1/z$, $\mu = \lambda z$

$$\theta\left(\frac{1}{y}; \mu(1 + j\bar{y})\right) = \left(\frac{1}{y}; \tilde{\theta}\left(\frac{1}{y}\right)\mu\right)$$

Then $\theta \otimes 1$ can be encoded as a map

$$\theta \otimes 1 : D_\infty \to \mathbf{H}^*$$
$$(\theta \otimes 1)(z) = \tilde{\theta}(z)$$

we have that g in turn can be thought of as a map

$$g : \mathbf{CP}^1 \to \mathbf{H}^*$$

and in fact the condition (2.1) can be written in this case as

$$\Psi(z) = \theta(z)g(z)$$

therefore for any pair $[E, \Psi]$ and any θ there is a unique g with this property. As we can see any two of Ψ, θ, g determine the remaining one completely in view of the last proposition.

From this we see that the map $\mathcal{X}_{1,1} \to \mathcal{N}_{1,1}$ is a fibration with fiber homeomorphic to the space $\mathrm{Hol}(D_\infty, \mathrm{Gl}_1(\mathbf{C}))$. Hence we have a homotopy equivalence

$$\mathcal{X}_{1,1} \simeq \mathcal{N}_{1,1}.$$

This holds in general.

Proposition 2.1.9. *The maps*

$$\mathcal{X}_{n,k} \to \mathcal{N}_{n,k}$$

are homotopy equivalences.

Proof. The argument is essentially the same as above, g can be identified naturally with a map

$$g : D_\infty \to \mathrm{Gl}_n(\mathbf{H})$$

and then we have

$$\Psi_z = g(z) \cdot (\theta_z \otimes 1)$$

therefore by the previous proposition any two of Ψ, θ and g determine the third. Hence the map $\mathcal{X}_{n,k} \to \mathcal{N}_{n,k}$ is a fibration with fiber $\mathrm{Hol}(D_\infty, \mathrm{Gl}_n(\mathbf{C}))$ therefore is a homotopy equivalence.∎

We make one more definition.

Definition 2.1.10. We define $\mathcal{K}_{n,k}$ as the space of equivalence classes $[E, \theta, g]$ where

E is a negative holomorphic bundle of rank n over $\mathbf{CP}^1$ with first chern class $-k$.

θ is a holomorphic trivialization of $E|_{D_\infty}$, $\theta: E|_{D_\infty} \cong D_\infty \times \mathbf{C}^n$.

g is an isomorphism $g: D_\infty \times \mathbf{H}^n \to D_\infty \times \mathbf{H}^n$ such that

$$\Psi|_{D_\infty} = g \circ (\theta \otimes 1).$$

We have the following proposition concerning the map

$$\mathcal{X}_{n,k} \to \mathcal{K}_{n,k}$$

$$[E, \Psi, \theta, g] \mapsto [E, \theta, g]$$

Proposition 2.1.11. *The map*

$$\mathcal{X}_{n,k} \to \mathcal{K}_{n,k}$$

is a homeomorphism.

Proof. The formula

$$\Psi_z = g(z) \cdot (\theta_z \otimes 1)$$

prove that θ and g completely determines Ψ.∎

§2.2. Real Holomorphic Bott Periodicity.

In this section we prove the following theorem

Theorem 2.2.1. *There are homotopy equivalences*

$$BO(k) \longleftarrow \mathrm{Hol}_k(\mathbf{CP}^1, \mathrm{Sp/U}) \longrightarrow \mathrm{Hol}_k(\mathbf{CP}^1, \Omega\mathrm{Sp})$$

for every natural number k.

The first step in doing so is to prove the following

Theorem 2.2.2. *There is a holomorphic map* $\mathrm{Sp/U} \to \Omega\mathrm{Sp}$ *that realizes the Bott homotopy equivalence* $\mathrm{Sp/U} \simeq \Omega\mathrm{Sp}$.

Proof. We set some conventions first. Here we interpret $\mathrm{Sp}(n)/\mathrm{U}(n)$ as the fixed set $\mathrm{Gr}_n(\mathbf{C}^{2n})$ under the involution $V \mapsto jV^{\perp}$ where j is the quaternionic unit when we identify $\mathbf{C}^{2n}$ with $\mathbf{H}^n$ in the usual way $\mathbf{H}^n = \mathbf{C}^n \oplus j\mathbf{C}^n$. We can do that because the spaces belonging to that fixed set constitute the orbit of $\mathbf{C}^n \oplus 0 \subset \mathbf{H}^n$ under the standard action of $\mathrm{Sp}(n)$ on $\mathbf{H}^n$. By Sp/U we mean the direct limit $\lim_{\to} \mathrm{Sp}(n)/\mathrm{U}(n)$, when we take the stabilization maps $\mathrm{Sp}(n)/\mathrm{U}(n) \to \mathrm{Sp}(n+1)/\mathrm{U}(n+1)$ given by $V \mapsto V \oplus (\mathbf{C} \oplus j0)$. Now we use the Grassmannian model of section 8.5 in [PS86] for the algebraic loops $\Omega_{\mathrm{pol}}\mathrm{Sp}(m)$. Again we define $\Omega_{\mathrm{pol}}\mathrm{Sp}$ as the direct limit $\lim_{\to} \Omega_{\mathrm{pol}}\mathrm{Sp}(m)$ under the obvious stabilization maps. We claim that both spaces $\Omega_{\mathrm{pol}}\mathrm{Sp}$ and Sp/U are homeomorphic. To see this what we do is to define maps

$$\mathcal{I}_n \colon \mathrm{Sp}(n(2n-1))/\mathrm{U}(n(2n-1)) \to \Omega_{\mathrm{pol}}\mathrm{Sp}(n)$$

that defines the homeomorphism in the limit. For that we write down maps

$$\hat{\mathcal{I}}_n \colon \mathbf{C}^{n(2n-1)} = \mathbf{H}^{n(2n-1)} \to L^2(S^1; \mathbf{H}^n) = L^2(S^1; \mathbf{C}^{2n})$$

inducing diagrams in which $\mathcal{I}_n$ fits

$$\begin{array}{ccccc} \mathrm{Gr}_{2n-1}(\mathbf{C}^{2n(2n-1)}) & \longrightarrow & \mathrm{Gr}^{(2n)} & & \\ \uparrow\cup & & \uparrow\cup & & \\ \mathrm{Sp}(n(2n-1))/\mathrm{U}(n(2n-1)) & \xrightarrow{\mathcal{I}_n} & (\mathrm{Gr}^{(2n)}_{\mathbf{H}})_0 & = & \Omega_{\mathrm{pol}}\mathrm{Sp}(n) \end{array}$$

We choose the following basis over $\mathbf{H}$ for the spaces $\mathbf{H}^{n(2n-1)}$ and $L^2(S^1; \mathbf{H}^n)$ respectively

$$\mathcal{F}_n = \{\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_{n(2n-1)}\}$$

and

$$\mathcal{Z}_n = \left\{ \begin{matrix} \cdots & z^{-2}\vec{e}_1 & z^{-1}\vec{e}_1 & z^0\vec{e}_1 & z^1\vec{e}_1 & z^2\vec{e}_1 & \cdots \\ \cdots & z^{-2}\vec{e}_2 & z^{-1}\vec{e}_2 & z^0\vec{e}_2 & z^1\vec{e}_2 & z^2\vec{e}_2 & \cdots \\ \cdots & z^{-2}\vec{e}_3 & z^{-1}\vec{e}_3 & z^0\vec{e}_3 & z^1\vec{e}_3 & z^2\vec{e}_3 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & z^{-2}\vec{e}_n & z^{-1}\vec{e}_n & z^0\vec{e}_n & z^1\vec{e}_n & z^2\vec{e}_n & \cdots \end{matrix} \right\}.$$

Here the quaternionic structure on $\mathbf{H}^{n(2n-1)}$ is the usual one, but on the space $L^2(S^1;\mathbf{H}^n)$ we define it as $J = z^{-1}j$. The following formulas describe the maps $\mathcal{I}_n$. Here we indicate where every element of the basis $\mathcal{F}_n$ is sent to in $\mathcal{Z}_n$ – that is enough to completely specify $\mathcal{I}_n$.

$$\hat{\mathcal{I}}_1\colon [\vec{f}_1] \mapsto [z^0\vec{e}_1]$$

$$\hat{\mathcal{I}}_2\colon \begin{bmatrix} \vec{f}_6 & \vec{f}_1 & \vec{f}_2 \\ \vec{f}_5 & \vec{f}_4 & \vec{f}_3 \end{bmatrix} \mapsto \begin{bmatrix} z^{-1}\vec{e}_1 & z^0\vec{e}_1 & z^1\vec{e}_1 \\ z^{-1}\vec{e}_2 & z^0\vec{e}_2 & z^1\vec{e}_2 \end{bmatrix}.$$

$$\hat{\mathcal{I}}_3\colon \begin{bmatrix} \vec{f}_7 & \vec{f}_6 & \vec{f}_1 & \vec{f}_2 & \vec{f}_{15} \\ \vec{f}_8 & \vec{f}_5 & \vec{f}_4 & \vec{f}_3 & \vec{f}_{14} \\ \vec{f}_9 & \vec{f}_{10} & \vec{f}_{11} & \vec{f}_{12} & \vec{f}_{13} \end{bmatrix} \mapsto \begin{bmatrix} z^{-2}\vec{e}_1 & z^{-1}\vec{e}_1 & z^0\vec{e}_1 & z^1\vec{e}_1 & z^2\vec{e}_1 \\ z^{-2}\vec{e}_2 & z^{-1}\vec{e}_2 & z^0\vec{e}_2 & z^1\vec{e}_2 & z^2\vec{e}_2 \\ z^{-2}\vec{e}_3 & z^{-1}\vec{e}_3 & z^0\vec{e}_3 & z^1\vec{e}_3 & z^2\vec{e}_3 \end{bmatrix}.$$

Following that pattern we define $\hat{\mathcal{I}}_n$ inductively. Then we write

$$\mathcal{I}_n(V) = \hat{\mathcal{I}}_n(V) \oplus z^n H_+^{(2n)}$$

With that definition we have that for every $V \in \mathrm{Sp}\,(n(2n-1))\,/\mathrm{U}\,(n(2n-1))$ it is verified that

$$z^{-1}\mathcal{I}_n(V)^\perp = \mathcal{I}_n(V)$$

achieving the desired result.∎

Corollary 2.2.3. *There is a map*

$$\mathrm{Sp/U} \to \Omega\mathrm{Sp}$$

that is holomorphic, realizes the Bott homotopy equivalence and induces a commutative diagram

$$\begin{array}{ccc} \mathrm{Hol}_k(\mathbf{CP}^1, \mathrm{Sp/U}) & \longrightarrow & \Omega^2\mathrm{Sp/U} \\ \downarrow & & \downarrow \scriptstyle\mathrm{Bott.} \\ \mathrm{Hol}_k(\mathbf{CP}^1, \Omega\mathrm{Sp}) & \longrightarrow & \Omega^3\mathrm{Sp} \end{array}$$

The map $\mathrm{Sp/U} \to \Omega\mathrm{Sp}$ that we are referring to is the inclusion of $\Omega_{\mathrm{pol}}\mathrm{Sp}$ into $\Omega\mathrm{Sp}$.

Theorem 2.2.4. *The spaces $BO(k)$, $\mathrm{Hol}_k(\mathbf{CP}^1,\mathrm{Sp/U})$ and $\mathrm{Hol}_k(\mathbf{CP}^1,\Omega\mathrm{Sp})$ are homotopy equivalent. Moreover there are natural homotopy equivalences*

$$BO(k) \xleftarrow{\delta} \mathrm{Hol}_k(\mathbf{CP}^1, \mathrm{Sp/U}) \xrightarrow{\beta} \mathrm{Hol}_k(\mathbf{CP}^1, \Omega\mathrm{Sp})$$

for every odd natural number k.

Proof. With all the preliminaries ready we proceed to prove the theorem. The idea is to look at diagrams we use in section 1.6,

$$\mathrm{Hol}_k(\mathbf{CP}^1, B\mathrm{U}) \xleftarrow{\simeq} \chi_{\infty,k} \xrightarrow{\simeq} \mathcal{C}_{\infty,k}$$

and

$$\mathcal{C}_{\infty,k} \simeq B\mathrm{U}(k)$$

and find $\mathbf{Z}_2$-involutions that make both diagrams commutative. Then we observe that all the equivalences above are $\mathbf{Z}_2$-homotopy equivalences, and finally taking fixed points to the diagrams above we get the desired result.

More explicitly, observe that we can define an involution ρ on the space $\mathrm{Hol}_k(\mathbf{CP}^1, \mathrm{Gr}_n(\mathbf{C}^{2n}))$ (if we fix an identification $\mathbf{C}^{2n} \cong \mathbf{H}^n$) by

$$\rho(f)(z) = j \cdot f(z)^{\perp}$$

The map

$$\rho\colon \mathrm{Hol}_k(\mathbf{CP}^1, \mathrm{Gr}_n(\mathbf{C}^{2n}) \to \mathrm{Hol}_k(\mathbf{CP}^1, \mathrm{Gr}_n(\mathbf{C}^{2n})$$

is well defined because the following maps are anti-holomorphic.

$$\mathrm{Gr}_n(\mathbf{C}^{2n}) \xrightarrow{j\cdot} \mathrm{Gr}_n(\mathbf{C}^{2n}), \mathrm{Gr}_n(\mathbf{C}^{2n}) \xrightarrow{\perp} \mathrm{Gr}_n(\mathbf{C}^{2n})$$

From the model used in the previous proposition for Sp/U we can see that the fixed point set for this involution ρ is

$$\mathrm{Hol}_k(\mathbf{CP}^1, \mathrm{Gr}_n(\mathbf{C}^{2n}))^{\rho} = \mathrm{Hol}_k(\mathbf{CP}^1, \mathrm{Sp}\,(n)\,/\mathrm{U}\,(n))$$

The involution σ that we will use in $\mathcal{C}_{\infty,k}$ is given by

$$\sigma(\gamma)(z) = \overline{\gamma(\bar{z})}$$

Now from the Crabb-Mitchell model (cf. remark 1.2.2) for the space $\Omega(\mathrm{O/U})$ we have that

$$\mathcal{C}^{\sigma}_{\infty,k} = B\mathrm{O}(k)$$

because as explained before we end with a real grassmannian model of k-planes in $\mathbf{R}^{\infty}$. The two involutions match nicely in $\chi^{2n}_{n,k}$ (here we use the notation of proposition 1.6.9). By this we mean that we can define an involution τ in $\chi^{2n}_{n,k}$ so that for a given element $(E, \iota, \theta(z))$ it is sent to $(E, j \cdot \iota^{\perp}, \overline{\theta(\bar{z})})$. The reason why this involution exists is that the function $j\overline{f(\bar{z})}^{\perp}$ is holomorphic, and the canonical forms from control theory (the same we used to prove the quasifibration properties in last chapter) are compatible with this action. In any case we define the involution τ as above and we obtain the following diagram of $\mathbf{Z}_2$-maps

$$\mathrm{Hol}_k(\mathbf{CP}^1, \mathrm{Gr}_n(\mathbf{C}^{2n})) \xleftarrow{\pi^n_{n,k}} \chi^n_{n,k} \xrightarrow{p^n_{n,k}} \mathcal{C}_{n,k}$$

of course only one of the arrows above is a homotopy equivalence. Restricting to the fixed points we get

$$\mathrm{Hol}_k(\mathbf{CP}^1, \mathrm{Gr}_n(\mathbf{C}^{2n}))^{\rho} \xleftarrow{\pi^n_{n,k}} (\chi^n_{n,k})^{\tau} \xrightarrow{p^n_{n,k}} \mathcal{C}^{\sigma}_{n,k}$$

We only need to prove that both arrows become homotopy equivalences in the limit over n. From the proof of proposition 1.6.9 and the fact that in the factorization theorem the factors of $\overline{\gamma(\bar{z})}$ are related to $\gamma(z)$ in a compatible way we conclude that in the previous diagram both arrows are quasifibrations. Now we observe that the proof of proposition 1.6.10 remains valid in this case. The homotopy fibers are now the space $\mathrm{Hol}(D_{\infty}, \mathrm{Gl}_n\,(\mathbf{R}))$ of real analytic maps, and the space of holomorphic bundle immersions with the additional property given by $j \cdot \iota^{\perp} = \iota$, this condition is convex in the space of bundle immersions. Therefore we can take fixed points and get the homotopy equivalence

$$\mathrm{Hol}_k(\mathbf{CP}^1, \mathrm{Sp/U}) \simeq B\mathrm{O}(k)$$

We also know from the previous argument that the map

$$\mathrm{Hol}_k(\mathbf{CP}^1, \mathrm{Sp/U}) \to \mathrm{Hol}_k(\mathbf{CP}^1, B\mathrm{U})$$

is homotopic to the map of Mitchell filtrations

$$F^{\mathbf{R}}_{\infty,k} \to F_{\infty,k}$$

that is in turn

$$B\mathrm{O}(k) \to B\mathrm{U}(k).$$

From what we pointed out in the remark 1.6.11, this implies that the $\bar{\partial}$ map

$$\mathrm{Hol}_k(\mathbf{CP}^1, \mathrm{Sp/U}) \xrightarrow{\bar{\partial}} B\mathrm{U}(k)$$

factors as

$$\mathrm{Hol}_k(\mathbf{CP}^1, \mathrm{Sp/U}) \to B\mathrm{O}(k) \to B\mathrm{U}(k).$$

This can be proved more directly, of course. In any case we write from now on

$$\mathrm{Hol}_k(\mathbf{CP}^1, \mathrm{Sp/U}) \xrightarrow{\bar{\partial}} B\mathrm{O}(k).$$

From the corollary before this theorem, Sander's theorem 1.5.2 we obtain the following commutative diagram

$$\begin{array}{ccccc} \mathrm{Hol}_k(\mathbf{CP}^1, \mathrm{Sp/U}) & \xrightarrow{\simeq} & B\mathrm{O}(k) & \to & B\mathrm{O} \\ \downarrow & & & & \downarrow = \\ \mathrm{Hol}_k(\mathbf{CP}^1, \Omega\mathrm{Sp}) & \xrightarrow{\simeq} & B\mathrm{O}(k) & \to & B\mathrm{O} \end{array} \qquad (\star)$$

It is known that for every homotopy commutative diagram

$$\begin{array}{ccc} B\mathrm{O}(k) & \hookrightarrow & B\mathrm{O} \\ \downarrow h & & \downarrow = \\ B\mathrm{O}(k) & \hookrightarrow & B\mathrm{O} \end{array}$$

the map h is homotopic to the identity whenever k is odd.∎

Combining this theorem with the results from section 2.1 we obtain,

Corollary 2.2.5. *The spaces $\mathcal{X}_{n,k}$, $\mathcal{N}_{n,k}$ and $\mathcal{K}_{n,k}$ in the limit as $n \to \infty$ have the homotopy type of the classifying space $B\mathrm{O}(k)$.*

3. THE SPACE OF MONOPOLES AND THE DIRAC OPERATOR.

In this chapter we study some properties of the dirac operator defined over the spaces of monopoles.

§3.1. Stabilization of Sp (n)-monopoles

Consider the space of configurations (A, ϕ) where

i) The gauge field A is a smooth connection on the trivial Sp (1) bundle over $\mathbf{R}^3$ denoted by P.

ii) The Higgs field ϕ is a smooth section of the vector bundle associated to P via the adjoint representation.

$$\phi: \mathbf{R}^3 \to \mathbf{sp}(n).$$

iii) Define $d_\phi: \mathbf{R}^3 \to \mathbf{R}$ by taking $d_\phi(x)$ to be the distance inside $\mathbf{sp}(n)$ from $\phi(x)$ to the adjoint orbit of iI_n in $\mathbf{sp}(n)$ (this orbit is homeomorphic to $\mathrm{Sp}(n)/\mathrm{U}(n)$). Then we require $d_\phi \in L^6(\mathbf{R}^3)$.

iv) The Yang-Mills-Higgs energy of the pair (A, ϕ) is finite

$$\mathcal{U}(A,\phi) = \frac{1}{2}\int_{\mathbf{R}^3}(|F_A|^2 + |D_A\phi|^2)dvol < \infty.$$

Here F_A is the curvature of A and D_A is the covariant derivative associated to A.

v) The pair (A, ϕ) satisfies the Bogomolnyi equation

$$*F_A = D_A\phi,$$

where $*$ is the Hodge star operator on $\mathbf{R}^3$.

vi) We define the charge $k \in \mathbf{Z}$ associated to every such pair and every number $R \in \mathbf{R}$. This number k will not depend on R if R is sufficiently large. Since the orbit $\mathrm{Sp}(n)/\mathrm{U}(n) \subset \mathbf{sp}(n)$ is compact for every point in the sphere centered at the origin of radius R in $\mathbf{R}^3$, $x \in S^2(0;R)$ we can choose the closest point $\psi(x) \in \mathrm{Sp}(n)/\mathrm{U}(n)$ to $\phi(x) \in \mathbf{sp}(n)$. Then

$$\psi: S^2 \to \mathrm{Sp}(n)/\mathrm{U}(n).$$

Let k be the class of ψ in $\pi_2(\mathrm{Sp}(n)/\mathrm{U}(n)) = \mathbf{Z}$.

Definition 3.1.1. The space of Sp (n)-monopoles of charge k denoted by $\mathcal{M}_k^n$ is defined as the space of configurations (A, ϕ) satisfying the conditions i) to vi), modulo the action of the gauge group of automorphisms of P whose restriction to the fiber over the origin is the identity.

The topology of these spaces has been extensively studied. Some of the fundamental results can be found in [A84b], [AH88], [BM88a], [D84b], [T84], [S79], [CCMM] [H89], [HM89] and [G]. It is known from that work that there is a homeomorphism between the space of monopoles $\mathcal{M}_k^n$ and the space of holomorphic maps $\mathrm{Hol}_k(\mathbf{CP}^1; \mathrm{Sp}(n)/\mathrm{U}(n))$. In [CJ90] it is proved that the Dirac operator has a real structure and defines a real dirac bundle over the space $\mathcal{M}_k^n$. In fact the results therein relating this bundle with the real Bott periodicity theorem imply that this bundle corresponds exactly with the k-dimensional real bundle over $\mathrm{Hol}_k(\mathbf{CP}^1; \mathrm{Sp}(n)/\mathrm{U}(n))$ defined by the $\bar{\partial}$ operator. Combining these results with theorem 2.2.4 we obtain the following theorem.

Theorem D. *The space $\mathcal{M}_k^\infty$ has the homotopy type of the classifying space $BO(k)$ Moreover, for every k odd there is a homotopy equivalence*

$$\delta: \mathcal{M}_{\infty,k} \xrightarrow{\simeq} BO(k) \qquad (*)$$

given by the Dirac operator. The composition

$$\mathcal{M}_{n,k} \to \mathcal{M}_{n+1,k} \to \cdots \to \mathcal{M}_{\infty,k} \simeq BO(k)$$

is homotopy equivalent to (*) for every odd natural number k.

§3.2. COMPUTING $\Omega_*^{\mathcal{M}_\infty}X$, THE MONOPOLE COBORDISM.

In this section we define and compute the monopole cobordism. In [D84b] Donaldson proved that

$$\mathcal{M}_k \approx \mathrm{Rat}_k$$

where $\mathrm{Rat}_k = \mathrm{Hol}_k(\mathbf{CP}^1; \mathbf{CP}^1)$. We will need to define a version of Segal's stabilization map

$$\mathcal{M}_k \approx \mathrm{Rat}_k \hookrightarrow \mathrm{Rat}_{k+1} \approx \mathcal{M}_{k+1}$$

as follows, by [Lemma 11.7, C91] we have the gluing map μ making the following diagram commutative,

$$\begin{array}{ccc} \mathrm{Rat}_k \times \mathrm{Rat}_r & \xrightarrow{\mu} & \mathrm{Rat}_{k+r} \\ \downarrow & & \downarrow \\ \Omega_k^2 S^2 \times \Omega_r^2 S^2 & \xrightarrow{\sigma} & \Omega_{k+r}^2 S^2 \end{array}$$

Here σ is the usual loop-sum operation in $\Omega^2 S^2$, and we are considering a rational function as a configuration of roots and poles. We glue them taking a fixed diffeomorphism between the complex plane, and the upper and lower open half planes in $\mathbf{C}$. Then we define

$$\begin{aligned} \mathrm{Rat}_k &\hookrightarrow \mathrm{Rat}_{k+1}, \\ \alpha &\mapsto \mu(\alpha, \beta_0). \end{aligned}$$

Here β_0 is the rational function that as a configuration of poles and roots in the lower half plane $\mathbf{C}_- = \{z : \Im(z) < 0\}$, is the pole $-i$ and the root $-(i+1)$. In other words, given $r(z) = p(z)/q(z) \in \mathrm{Rat}_k$, using the fixed diffeomorphism of $\mathbf{C}$ with $\mathbf{C}_+$ we think of it as having its original poles and roots in $\mathbf{C}_+$, and then we identify it with $p(z)(z+i+1)/q(z)(z+i) \in \mathrm{Rat}_k$. This gives us the desired definition for

$$\mathcal{M}_k \overset{j_k}{\hookrightarrow} \mathcal{M}_{k+1}$$

Lemma 3.2.1. *The Dirac bundle is compatible with this inclusion, i.e.* $j_k^* \delta_{k+1} = \delta_k \oplus \epsilon^1$, *as stable bundles.*

Remark. We will prove later that this lemma remains valid unstably, this is, the unstable bundles are compatible with the inclusion.

First Proof. What we need to show is that we have the homotopy commutative diagram

$$\begin{array}{ccc} \mathcal{M}_k & \xrightarrow{\delta_k} & BO \\ \downarrow & & \downarrow \\ \mathcal{M}_{k+1} & \xrightarrow{\delta_{k+1}} & BO \end{array}$$

Now, from [CJ90] we have the diagram

$$\begin{array}{ccccccccc} \mathcal{M}_k & = & \mathrm{Rat}_k & \xleftarrow{\psi} & \mathrm{Rat}_k^0 & = & B\beta_{2,k} & \simeq & \tilde{C}_k \times_{\Sigma_k} (\mathbf{C}^*)^k \\ & & \downarrow j_k & & \downarrow j_k & & & & \\ \mathcal{M}_{k+1} & = & \mathrm{Rat}_{k+1} & \xleftarrow{\psi} & \mathrm{Rat}_{k+1}^0 & & & & \end{array}$$

So that

$$\begin{array}{ccc} KO(\mathcal{M}_k) & \xrightarrow{\psi^*} & KO\left(\tilde{C}_k \times_{\Sigma_k} (\mathbf{C}^*)^k\right) \\ \uparrow j_k^* & & \uparrow j_k^* \\ KO(\mathcal{M}_{k+1}) & \xrightarrow{\psi^*} & KO\left(\tilde{C}_{k+1} \times_{\Sigma_{k+1}} (\mathbf{C}^*)^{k+1}\right) \end{array}$$

The horizontal arrows being injective.

In [CJ92] $\psi^* \delta_k$ is described as follows. Let $\xi_1 \to S^1$ be the real Hopf bundle over S^1, and $\xi \to \mathbf{C}^*$ its pull back to $\mathbf{C}^* \xrightarrow{\simeq} S^1$. Then,

$$\begin{array}{rcl}\psi^*\delta_k & = & \tilde{C}_k \times_{\Sigma_k} (\xi)^k \\ & & \downarrow \\ & & \tilde{C}_k \times_{\Sigma_k} (\mathbf{C}^*)^k\end{array}$$

We need to show then that

$$j_k^* \left(\tilde{C}_{k+1} \times_{\Sigma_{k+1}} (\xi)^{k+1}\right) = \tilde{C}_k \times_{\Sigma_k} (\xi)^k \oplus \epsilon^1 \tag{3.2.1}$$

Now, we can write explicitly the formula for j_k in the following terms. Consider a generic element in $\tilde{C}_k \times_{\Sigma_k} (\mathbf{C}^*)^k$,

$$r(z) = \sum_{i=1}^{k} \frac{a_i}{z - b_i} = [(a_1, b_1), \ldots, (a_k, b_k)] \in \tilde{C}_k \times_{\Sigma_k} (\mathbf{C}^*)^k$$

where $a_i \in \mathbf{C}^*$ and $b_i \in \mathbf{C}_+$ are k distinct points. Then, computing the appropriate residues we get, $j_k [(a_1, b_1), \ldots, (a_k, b_k)] =$

$$\left[\left(a_1 \left(\frac{b_1 + i + 1}{b_1 + i}\right), b_1\right), \ldots, \left(a_k \left(\frac{b_k + i + 1}{b_k + i}\right), b_k\right), (r(-i), -i)\right]$$

Considering the homotopy $J_k : \tilde{C}_k \times_{\Sigma_k} (\mathbf{C}^*)^k \times [0,1] \to \tilde{C}_{k+1} \times_{\Sigma_{k+1}} (\mathbf{C}^*)^{k+1}$, given by

$$J_k([(a_1, b_1), \ldots, (a_k, b_k)]; t) =$$

$$\left[\left(a_1 \left(\left(\frac{b_1 + i + 1}{b_1 - 1}\right) t + (1-t)\right), b_1\right), \ldots, \right.$$
$$\left.\left(a_k \left(\left(\frac{b_k + i + 1}{b_k - 1}\right) t + (1-t)\right), b_k\right), (r_t(-i), -i)\right]$$

where we define $r_t(-i)$ as follows. If $r(-i) = \rho e^{i\theta}$, then

$$r_t(-i) = \frac{\rho}{(1-t)\rho + t} e^{i\theta}.$$

First, we have to verify that the image of the homotopy J_k is contained, as we already claimed, in $\tilde{C}_{k+1} \times_{\Sigma_{k+1}} (\mathbf{C}^*)^{k+1}$. For this we only have to verify that for all $0 \leq t \leq 1$ the following quantities don't vanish

$$\left(\frac{b_m + i + 1}{b_m + i}\right) t + (1-t) \neq 0 \tag{3.2.2}$$

and

$$(1-t)\rho + t \neq 0 \tag{3.2.3}$$

Now, the quantity in (3) can't be zero since $\rho \geq 0$. On the other hand, the quantity in (2) can only be zero if the triangle with vertices $(b_m, -i-1, -1)$ is degenerate. But recall that b_i is in the upper half plane, and that can't happen. What all this shows is that up to homotopy we can write

$$j_k[(a_1, b_1), \ldots, (a_k, b_k)] = [(a_1, b_1), \ldots, (a_k, b_k), (e^{i\theta}, -i)]$$

and this immediately implies (1). This means that $\psi^*(j_k^*\delta_{k+1}) = j_k^*\psi^*\delta_{k+1} = (\psi^*\delta_k) \oplus \epsilon^1 = \psi^*(\delta_k \oplus \epsilon^1)$, but since ψ^* is injective in KO-theory, we have the equivalence of the stable bundles.∎

Second Proof of Lemma 3.2.1. This is an easy consequence of the last diagram in [T84] section A2.∎

In view of this lemma, we can consider

$$\mathcal{M}_\infty \xrightarrow{\delta_\infty} BO \tag{3.2.4}$$

Definition 3.2.2 We say that an unoriented smooth compact manifold M has a **monopole orientation** if there is a homotopy commutative diagram

$$\begin{array}{ccc} \mathcal{M}_\infty & \xrightarrow{\delta_\infty} & BO \\ \uparrow{\scriptstyle \bar\nu} & \nearrow{\scriptstyle \nu} & \\ M & & \end{array}$$

where ν classifies the stable normal bundle to M.

This defines $\Omega_*^{\mathcal{M}_\infty} X$ the bordism classes of maps of monopole oriented manifolds into X. Now we want to prove the following.

Theorem 3.2.3. *For any manifold X, we have a natural isomorphism*

$$\Omega_*^{\mathcal{M}_\infty} X \cong H_*(X, \mathbf{Z}_2).$$

Corollary 3.2.4. *Any element of $H_*(X, \mathbf{Z}_2)$ is represented by a monopole oriented manifold uniquely up to monopole cobordism.*

This will be easy once we have proved:

Proposition 3.2.5. *The Thom spectrum $\mathbf{M}(\delta)$ for the monopole Dirac bundle* (4) *is the Eilenberg-McLane spectrum $\mathbf{K}(\mathbf{Z}_2)$.*

For that we will use a beautiful result of Mahowald [M77], [MM74], [P78]. We know that $\pi_1 BO = \mathbf{Z}_2$, choose a generator $S^1 \xrightarrow{\eta} BO$, and using the fact that BO is a double loop space we get

$$\Omega^2 S^3 = \Omega^2\Sigma^2 S^1 \xrightarrow{\Omega^2\Sigma^2\eta} \Omega^2\Sigma^2 BO \to BO$$

$$\Omega^2 S^3 \xrightarrow{\bar\eta} BO$$

Theorem 3.2.6. *(Mahowald) The Thom spectrum obtained from $\bar\eta$ is $\mathbf{K}(\mathbf{Z}_2)$ localized at the prime 2.*

For a very short proof see [P78]. We will prove later the following key lemma,

Key Lemma 3.2.7. *There is a homotopy commutative diagram*

$$\begin{array}{ccc} \mathcal{M}_\infty & \xrightarrow{\delta_\infty} & BO \\ \downarrow{\scriptstyle \theta} & \nearrow{\scriptstyle \bar\eta} & \\ \Omega^2 S^3 & & \end{array}$$

Such that θ is a homotopy equivalence.

Proof of Proposition 3.2.5, Theorem 3.2.3 and Corollary 3.2.4.
We follow an argument of [CF78]. The diagram of Lemma 3.2.7, induces at the level of Thom spectra

$$\mathbf{M}(\theta) \to \mathbf{M}(\bar\eta)$$

that is an isomorphism on homology groups, hence, at the prime 2 we have

$$\mathbf{M}(\delta) =_{(2)} \mathbf{K}(\mathbf{Z}_2, 0).$$

We observe now that twice a monopole oriented manifold bounds. Therefore $\Omega_*^{\mathcal{M}_\infty} X$ is all 2-torsion, proving proposition 3.2.5. Theorem 3.2.3 and corollary 3.2.4 follow from standard results on cobordism theory.∎

Corollary 3.2.8. *If M^n is a closed monopole oriented manifold with $n \geq 1$, then M bounds a monopole oriented manifold.*

Proof. We have $\Omega_*^{\mathcal{M}_\infty}(\star) = \pi_n^s \mathbf{K}(\mathbf{Z}_2, 0) = 0.$ ∎

Let $h: S^3 \to S^3$ be the Hopf fibration. We need the following lemma, in the proof of the Key Lemma 3.2.7.

Lemma 3.2.9. *Consider the composition $\zeta = \zeta_k: \Omega^2 S^3 \to BO$ given by:*

$$\Omega^2 S^3 \xrightarrow{\Omega^2 h} \Omega_0^2 S^2 \xrightarrow{\sim} \Omega_k^2 S^2 \to BO$$

where the last arrow is the composition

$$\Omega_k^2 S^2 \hookrightarrow \Omega^2 S^2 \to \Omega^2\left(\mathrm{Sp}(1)/\mathrm{U}(1)\right) \to \Omega^2\left(\mathrm{Sp}/\mathrm{U}\right) \simeq \mathbf{Z} \times BO$$

and recall the Mahowald map,

$$\Omega^2 S^3 = \Omega^2 \Sigma^2 S^1 \xrightarrow{\Omega^2\Sigma^2 \eta} \Omega^2 \Sigma^2 BO \to BO$$

$$\Omega^2 S^3 \xrightarrow{\bar{\eta}} BO$$

then ζ and $\bar{\eta}$ are homotopic.

Remark: Here we are using the Bott periodicity equivalence $\Omega^2(\mathrm{Sp}/\mathrm{U}) \simeq \mathbf{Z} \times BO$, and the fact that $\mathrm{Sp}(1)/\mathrm{U}(1) = S^2$.

Proof of Lemma 9. First we maker an observation (cf. [M72] page 43). If $f: X \to \Omega^{n+1} Z$ is any continuous map, then there exist a unique continuous map of $(n+1)$-fold loop spaces $g: \Omega^{n+1}\Sigma^{n+1} X \to \Omega^{n+1} Z$ such that the following diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow{\text{std.inc.}} & \Omega^{n+1}\Sigma^{n+1} X \\ & \searrow f & \downarrow g \\ & & \Omega^{n+1} Z \end{array}$$

In our case $X = S^1$, $n = 2$ and $\Omega^2 Z = BO$. From this we see that it is enough to show that the composites

$$\tilde{\zeta}: S^1 \to \Omega^2 S^3 \xrightarrow{\zeta} BO$$

$$\tilde{\eta}: S^1 \to \Omega^2 S^3 \xrightarrow{\eta} BO$$

are homotopic. Observe that $\pi_1 BO = \pi_0 \mathbf{O} = \mathbf{Z}_2$ and by definition $\tilde{\eta}$ is a generator of $\pi_1 BO$. This reduces this lemma to prove that $\tilde{\zeta}$ is not null-homotopic (i.e. it is a generator for the fundamental group $\pi_1 BO$.) Let's study the composition $\tilde{\zeta}$ step by step:

$$S^1 \hookrightarrow \Omega^2 \Sigma^2 S^1 \xrightarrow{\Omega^2 h, \simeq} \Omega_0^2 S^2 \xrightarrow{\psi, \simeq} \Omega_k^2 S^2 \xrightarrow{\phi} BO = \Omega^2 Z.$$

Claim 1. The canonical inclusion $\gamma: S^1 \to \Omega^2\Sigma^2 S^1$ is a generator for the fundamental group $\pi_1 \Omega^2 \Sigma^2 S^1 = \pi_1 S^3 = \mathbf{Z}$.

To see this we let ξ: $S^1 \to \Omega^2\Sigma^2 S^1$ be a generator for the above mentioned fundamental group of $\Omega^2 \Sigma^2 S^1$, getting the diagram

$$\begin{array}{ccc} S^1 & \xrightarrow{\gamma} & \Omega^2\Sigma^2 S^1 \\ & \searrow \xi & \downarrow f, \text{ unique} \\ & & \Omega^2\Sigma^2 S^1 \end{array}$$

Writing the induced map at the level of the fundamental group for f, we obtain

$$f_*: \pi_1 \Omega^2 \Sigma^2 S^1 \to \pi_1 \Omega^2 \Sigma^2 S^1 \xrightarrow{\cong} \mathbf{Z},$$

and then we easily compute,

$$f_*[\gamma] = [f \circ \gamma] = [\xi] = 1 \in \mathbf{Z}.$$

This is telling us that $f\colon \mathbf{Z} \to \mathbf{Z}$ is a surjective group-homomorphism, hence an isomorphism (because $\mathbf{Z}/\ker f \cong \mathbf{Z}$,) this in turn implies $[\gamma] = \pm 1 \in \mathbf{Z}$, proving Claim 1.

Claim 2. The composition $\psi \circ \Omega^2 h \circ \gamma\colon S^1 \to \Omega^2_k S^2$ is a generator for $\pi_1 \Omega^2_k S^2 \cong \mathbf{Z}$.

This is because both $\Omega^2 h$ and ψ are homotopy equivalences (the first map $\Omega^2 h$ is so because we have $\star \simeq \Omega^2 S^1 \to \Omega^2 S^3 \xrightarrow{\Omega^2 h} \Omega^2_0 S^2$.)

Claim 3. This Lemma is valid for the case $k = 0$.

To prove the validity of this claim we need to understand the action on the fundamental group of the map

$$\Omega^2_0 S^2 \hookrightarrow \Omega^2 S^2 = \Omega^2 (\mathrm{Sp}(1)/\mathrm{U}(1)) \to \Omega^2 (\mathrm{Sp}/\mathrm{U}) \xrightarrow{\mathrm{Bott}, \simeq} \mathbf{Z} \times BO$$

Taking into consideration the previous claims, to prove that $\bar{\eta}$ is a generator for $\pi_1 BO$ we only need to verify that $\mathrm{Sp}(1)/\mathrm{U}(1) \to \mathrm{Sp}/\mathrm{U}$ sends a generator of $\pi_3(\mathrm{Sp}(1)/\mathrm{U}(1)) = \pi_1 \Omega^2 (\mathrm{Sp}(1)/\mathrm{U}(1))$ to a generator of $\pi_3(\mathrm{Sp}/\mathrm{U}) = \pi_1 \Omega^2 (\mathrm{Sp}/\mathrm{U})$.

For that, consider the homotopy sequence for the fibration

$$\mathrm{U}(1) \to \mathrm{Sp}(1) \to \mathrm{Sp}(1)/\mathrm{U}(1)$$

and remember that $\pi_3 \mathrm{U}(1) = 0$, $\pi_2 \mathrm{U}(1) = 0$, $\pi_3 \mathrm{Sp}(1) = \mathbf{Z}$, $\pi_3(\mathrm{Sp}(1)/\mathrm{U}(1)) = \mathbf{Z}$. From that we deduce that the map

$$\mathrm{Sp}(1) \to \mathrm{Sp}(1)/\mathrm{U}(1)$$

induces an isomorphism at the level of π_3.

We also have that the map $\mathrm{Sp}(n) \to \mathrm{Sp}$ induces the Bott isomorphisms

$$\pi_r(\mathrm{Sp}) \cong \pi_r(\mathrm{Sp}(n)), \qquad n \geq \left\lfloor \frac{r+2}{4} \right\rfloor.$$

Then the map

$$\mathrm{Sp}(1) \to \mathrm{Sp}$$

induces a π_3-isomorphism.

To see that $\mathrm{Sp} \to \mathrm{Sp}/\mathrm{U}$ sends a generator of $\pi_3 \mathrm{Sp}$ to a generator of $\pi_3(\mathrm{Sp}/\mathrm{U})$ look at the following sequence:

$$\begin{array}{ccccccc} \pi_3 \mathrm{U} & \longrightarrow & \pi_3 \mathrm{Sp} & \longrightarrow & \pi_3(\mathrm{Sp}/\mathrm{U}) & \longrightarrow & \pi_2 \mathrm{U} \\ \| & & \| & & \| & & \| \\ \mathbf{Z} & \longrightarrow & \mathbf{Z} & \longrightarrow & \mathbf{Z}_2 & \longrightarrow & 0 \end{array}$$

Combining that we conclude that the composition $\mathrm{Sp}(1) \to \mathrm{Sp} \to \mathrm{Sp}/\mathrm{U}$ sends a generator to a generator, but this last composition is the same as $\mathrm{Sp}(1) \to \mathrm{Sp}(1)/\mathrm{U}(1) \to \mathrm{Sp}/\mathrm{U}$. This completes the prove of the Claim 3.

This lemma thus reduces to the commutativity (up to homotopy) of the following diagram:

$$\begin{array}{ccccc} \Omega^2_0 S^2 & \hookrightarrow & \Omega^2 S^2 & \longrightarrow & \mathbf{Z} \times BO \\ \downarrow \{+k\} & & \downarrow \{+k\} & & \downarrow [+k] \\ \Omega^2_k S^2 & \hookrightarrow & \Omega^2 S^2 & \longrightarrow & \mathbf{Z} \times BO \end{array}$$

That is a known fact.∎

Consider the monopole filtration for $\Omega^2 S^2$ given by

$$\mathcal{M}_1 \to \mathcal{M}_2 \to \cdots \to \mathcal{M}_k \to \cdots \to \mathcal{M}_\infty \xrightarrow{\theta, \simeq} \Omega^2 S^3 \xrightarrow{\bar{\eta}} BO$$

Let X_k be the 2-localization of the Thom spectrum of the stable vector bundle γ_k over $\mathcal{M}_k$ which is classified by the map

$$\gamma_k\colon \mathcal{M}_k \to \Omega^2 S^3 \xrightarrow{\bar{\eta}} BO.$$

We state now the following unstable version of the previous results.

Theorem F. *Consider the monopole filtration for $\Omega^2 S^2$ given by*

$$\mathcal{M}_1 \to \mathcal{M}_2 \to \cdots \to \mathcal{M}_k \to \cdots \to \mathcal{M}_\infty \xrightarrow{\theta,\simeq} \Omega^2 S^3 \xrightarrow{\bar{\eta}} BO$$

Let X_k be the 2-localization of the Thom spectrum of the stable vector bundle γ_k over $\mathcal{M}_k$ which is classified by the map

$$\gamma_k \colon \mathcal{M}_k \to \Omega^2 S^3 \xrightarrow{\bar{\eta}} BO.$$

Then each $X_k \simeq_2 B_k$ where B_k is the 2-local Brown-Gitler spectrum [BG].

The strategy to prove this is to apply the following theorem of R. Cohen [C79]

Theorem. *(R. Cohen) Suppose $\{X_k; k \geq 0\}$ is a family of 2-local spectra. Then each $X_k \simeq_2 B_k$ if and only if the family satisfies the following properties:*

(1) $H^*(X_k; \mathbf{Z}_2) = M_k$ generated by a class $u_k \in H^0(X_k; \mathbf{Z}_2)$.

(2) For every pair of integers $r, s \geq 0$ there exists a pairing

$$\mu_{r,s} \colon X_r \wedge X_s \to X_{r+s}$$

such that

$$\mu^*_{r,s}(u_{r+s}) = u_r \otimes u_s \in H^0(X_r \wedge X_s; \mathbf{Z}_2).$$

(3) For every $i \geq 0$ there exists a "cup-1 product":

$$\zeta_i \colon S^1 \times_{\mathbf{Z}_2} X^{(2)}_{2^i} \to X_{2^{i+1}}$$

such that $\zeta_i^*(u_{2^{i+1}}) = e_0 \otimes_{\mathbf{Z}_2} u_{2^i} \otimes u_{2^i} \in H^0(S^1 \times_{\mathbf{Z}_2} X^{(2)}_{2^i}; \mathbf{Z}_2)$.

Details of this proof will appear elsewhere.

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