I. Generating Functions in Algebraic Geometry and Sums over Trees
II. Mordell-Weil Problem for Cubic Surfaces

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GENERATING FUNCTIONS IN ALGEBRAIC GEOMETRY AND SUMS OVER TREES

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§0. Summary of results

0.1. Introduction. In this paper we adress the following three problems.

A. Calculate the Betti numbers and Euler characteristics of moduli spaces $\overline{M}_{0,n}$ of stable n-pointed curves of genus zero (see e.g. [Ke]), or rather an appropriate generating function for these numbers.

B. The same for the space $X[n]$, a natural compactification of the space of n pairwise distinct labelled points on a non-singular compact algebraic variety X constructed for dim $X = 1$ in [BG] and in general in [FMPh]. (Beilinson and Ginzburg called this space "Resolution of Diagonals", Fulton and MacPherson use the term "Configuration Spaces").

C. Calculate the contribution of multiple coverings in the problem of counting rational curves on Calabi–Yau threefolds (see [AM], [Ko], and more detailed explanations below).

All these problems are united by the fact that available algebro-geometric information allows us to represent the corresponding numbers as a sum over trees with markings. M. Kontsevich in [Ko] invoked a general formula of perturbation theory in order to reduce the calculation of the relevant generating functions to the problem of finding the critical value of an appropriate formal potential. We solve problems A and B by applying this formalism in a simpler geometric context than that of [Ko]. Problem C is taken from [Ko]; we were able to directly complete Kontsevich's calculation in this case and obtain a simple closed answer.

We will now describe our results (0.3—0.5) and technique (0.6) in some detail.

0.2. General setup. Let Y be an algebraic variety over $\mathbf{C}$, possibly non-smooth and non-compact. Following [FMPh] we denote by $P_Y(q)$ the virtual Poincaré polynomial of Y which is uniquely defined by the following properties.

a). If Y is smooth and compact, then

$$P_Y(q) = \sum_j \dim H^j(Y) q^j. \tag{0.1}$$

In particular

$$\chi(Y) = P_Y(-1). \tag{0.2}$$

b). If $X = \coprod_i X_i$ is a finite union of pairwise disjoint locally closed strata, then

$$P_Y(q) = \sum_i P_{Y_i}(q). \tag{0.3}$$

c). $P_{Y\times Z}(q) = P_Y(q)P_Z(q)$. It follows that if Y is a fibration over base B with fiber F locally trivial in Zariski topology, then $P_Y(q) = P_B(q)P_F(q)$.

A definition of $P_Y(q)$ can be given using the weight filtration on the cohomology with compact support:

$$P_Y(q) = \sum_{i,j}(-1)^{i+j}\dim\,(\mathrm{gr}^j_W H^i_c(Y,\mathbf{Q}))q^j. \tag{0.4}$$

We apply the additivity formula (0.3) to the strata of the natural stratifications of $\overline{M}_{0,n}$ and $X[n]$ in Problems A, B. These strata can be indexed by marked trees describing various coalescing patterns of n-point configurations.

In [Ko], the role of Y is played by a compactification $M(W)$ of the space of parametrised rational curves in some manifold W. The relevant trees describe Gromov type degenerations of these curves. Kontsevich calculates certain Chern numbers of vector bundles over $M(W)$ and uses Bott's fixed point formula instead of (0.3) in order to represent them as a sum of local contributions. To make Bott's formula applicable, Kontsevich assumes that W is endowed with a torus action and lifts this action to $M(W)$. (Actually, his $M(W)$ is not a manifold but a smooth stack).

0.3. Moduli spaces. We put

$$\varphi(q,t) := t + \sum_{n=2}^{\infty} P_{\overline{M}_{0,n+1}}(q)\frac{t^n}{n!} \in \mathbf{Q}[q][[t]], \tag{0.5}$$

$$\chi(t) := \varphi(-1,t) = t + \sum_{n=2}^{\infty} \chi(\overline{M}_{0,n+1})\frac{t^n}{n!} \in \mathbf{Q}[[t]]. \tag{0.6}$$

0.3.1. Theorem. *a). $\varphi(q,t)$ is the unique root in $t + t^2\mathbf{Q}[q][[t]]$ of any one of the following functional/differential equations in t with parameter q :*

$$(1+\varphi)^{q^2} = q^4\varphi - q^2(q^2-1)t + 1, \tag{0.7}$$

$$(1+q^2t - q^2\varphi)\varphi_t = 1+\varphi. \tag{0.8}$$

b). χ is the unique root in $t + t^2\mathbf{Q}[[t]]$ of any one of the similar equations

$$(1+\chi)\,\log(1+\chi) = 2\chi - t, \tag{0.9}$$

$$(1+t-\chi)\chi_t = 1+\chi. \tag{0.10}$$

Equations (0.8) and (0.10) are equivalent to the following recursive formulas for the Poincaré polynomials. Put $p_n = p_n(q) = P_{\overline{M}_{0,n+1}}/n!$.

0.3.2. Corollary. *We have for $n \geq 1$:*

$$(n+1)p_{n+1} = p_n + q^2 \sum_{\substack{i+j=n+1\\ i\geq 2}} jp_ip_j, \tag{0.11}$$

$$P_{\overline{M}_{0,n+2}}(q) = P_{\overline{M}_{0,n+1}}(q) + q^2 \sum_{\substack{i+j=n+1 \\ i\geq 2}} \binom{n}{i} P_{\overline{M}_{0,i+1}}(q) P_{\overline{M}_{0,j+1}}(q). \qquad (0.12)$$

One can compare (0.11) with recursive formulas in [Ke], p. 550.

From (0.10) one sees that the function inverse to χ has a critical point at $t = e-2$. Don Zagier has shown me how to derive from this the following asymptotical formula:

$$\chi(\overline{M}_{0,n+1}) \cong \frac{1}{\sqrt{n}} \left(\frac{n}{e^2 - 2e} \right)^{n-\frac{1}{2}}.$$

We will prove Theorem 0.3.1 in §1. We will also discuss the ramification properties of φ as a function of t for $q^2 \neq 1$.

0.4. Configuration spaces. For a compact smooth algebraic manifold X of dimension m, set

$$\psi_X(q,t) = 1 + \sum_{n\geq 1} P_{X[n]}(q) \frac{t^n}{n!} \in \mathbf{Q}[q][[t]], \qquad (0.13)$$

$$\chi_X(t) = \psi_X(-1,t) = 1 + \sum_{n\geq 1} \chi(X[n]) \frac{t^n}{n!} \in \mathbf{Q}[[t]]. \qquad (0.14)$$

Put also

$$\kappa_m = \frac{q^{2m}-1}{q^2-1} = P_{\mathbf{P}^{m-1}}(q).$$

0.4.1. Theorem. *Denote by* $y^0 = y^0(q,t)$ *the unique root in* $t + t^2\mathbf{Q}[q^2][[t]]$ *of any one of the following equations:*

$$\kappa_m(1+y^0)^{q^{2m}} = q^{2m}(q^{2m} + \kappa_m - 1)y^0 - q^{2m}(q^{2m}-1)t + \kappa_m, \qquad (0.15)$$

$$\left[q^{2m}t + 1 - (q^{2m} - 1 + \kappa_m)y^0\right] y^0_t = 1 + y^0. \qquad (0.16)$$

Then we have in $\mathbf{Q}[q][[t]]$*:*

$$\psi_X(q,t) = (1+y^0)^{P_X(q)}. \qquad (0.17)$$

0.4.2. Theorem. *Denote by* $\eta = \eta(t)$ *the unique root in* $t + t^2\mathbf{Q}[[t]]$ *of any one of the following equations:*

$$m(1+\eta)\log(1+\eta) = (m+1)\eta - t, \qquad (0.18)$$

$$(t + 1 - m\eta)\eta_t = 1 + \eta. \qquad (0.19)$$

Then we have in $\mathbf{Q}[[t]]$*:*

$$\chi_X(t) = (1+\eta)^{\chi(X)}. \qquad (0.20)$$

Theorems 0.4.1 and 0.4.2 are proved in §2.

I am grateful to C. Soulé who remarked that (0.17) follows from a less neat identity which I deduced initially. He has also informed me that he and H. Gillet constructed a map $X \mapsto [h^*(X)]$ from varieties to the K_0–ring of Grothendieck's motives having all the formal properties of the virtual Poincaré polynomial. We can more or less mechanically use it in all our constructions; in particular, q^2 will be replaced by Tate's motive $[h^2(\mathbf{P}^1)]$.

For the reader's convenience, we list the first terms of the generating series we have considered:

$$\varphi(q,t) = t + \frac{t^2}{2!} + \frac{t^3}{3!}(q^2+1) + \frac{t^4}{4!}(q^4+5q^2+1) + \frac{t^5}{5!}(q^6+16q^4+16q^2+1) + \frac{t^6}{6!}(q^8+42q^6+127q^4+42q^2+1) + \dots,$$

$$P^{-1}(\psi_X(q,t)-1) = t + \frac{t^2}{2!}(\kappa_m + P - 1) + \frac{t^3}{3!}\left[(P-1)(P-2) + \kappa_m(q^{2m}-2) + 3(P-1)\kappa_m + 3\kappa_m^2\right] + \frac{t^4}{4!}[P^3 - 6P^2 + 11P - 6 + \kappa_m(6P^2 - 26P + 26 + 4Pq^{2m} - 9q^{2m} + q^{4m}) + \kappa_m^2(15P + 10q^{2m} - 35) + 15\kappa_m^3] + \dots$$

where we put $P = P_X(q)$.

0.5. Multiple coverings. Consider the following general problem of enumerative geometry.

Problem $P_{g,k}(X,\beta,\mathcal{I})$. *Given a projective algebraic manifold X, find the number of parametrised algebraic curves of genus g in X, in the homology class β, with k marked points, satisfying some incidence conditions $\mathcal{I}$.*

Notice that in this vaguely stated problem we implicitly assume that the number of solutions is only "virtually" finite, and look for the number of virtual solutions.

In [Ko], Maxim Kontsevich suggested a general scheme allowing him to simultaneously define this number for a wide class of problems and to calculate it in many cases using Bott's residue formula. In the three examples he considered in full detail we have $X = \mathbf{P}^n$ for some n, $g = 0$, and β is $d[\mathbf{P}^1]$ for some $d \geq 1$. The remaining data is as follows.

(i) $n = 2:\ X = \mathbf{P}^2,\ k = 3d-1$. The problem is to find the number of rational curves of degree d in $\mathbf{P}^2$ passing through $3d-1$ points in general positions.

(ii) $n = 4:\ X = \mathbf{P}^4,\ k = 0$. The problem is to find the number of rational curves of degree d lying in a quintic hypersurface V.

(iii) $n = 1:\ X = \mathbf{P}^1,\ k = 0$. Here we additionally assume that X is a rational curve embedded in the quintic threefold (or a more general Calabi–Yau threefold) with normal sheaf $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$, and the problem is to calculate the contribution of maps of degree d, $\mathbf{P}^1 \to X$, to the number of solutions of problem (ii).

Using a different definition of the last contribution which we denote m_d P. Aspinwall and D. Morrison [AM] calculated it and confirmed an earlier prediction by P. Candelas et al.

In this note we show that Kontsevich's formula gives the same answer:

0.5.1. Theorem. $m_d = d^{-3}$.

0.6. Summation over trees. A *tree* τ here is a finite connected simply connected CW–complex. We denote by V_τ the set of its vertices, E_τ the set of its edges. Valency $|v|$ of a vertex $v \in V_\tau$ is the number of edges adjoining v. A *flag* of τ is a pair (v, e) where v is a vertex, and e is an adjoining edge.

A *marking* of a tree τ is a vaguely defined notion. It may consist of a family of marks of given type(s) put onto vertices, edges, flags, and satisfying certain restrictions. Below we will describe pecisely a family of markings which we will call *standard* ones.

The generating functions φ studied above and in [Ko] are calculated in three steps.

STEP 1. Represent φ as an (infinite) sum of certain weights $w_\varphi(\tau, \mu)$ taken over isomorphism classes of marked trees (τ, μ):

$$\varphi = \sum_{(\tau,\mu)/(iso)} w_\varphi(\tau, \mu). \tag{0.21}$$

This stage involves a combinatorial encoding of the raw algebro–geometric data, determining type of marking and weights.

STEP 2. Try to rewrite (0.21) in a standard form of the following type. Choose a set A (finite or countable) and a family of *symmetric* tensors indexed by A: g^{ab}, $a, b \in A$; $C_{a_1,\dots,a_k}$, $a_i \in A$, $k \geq 1$. The coordinates g^{ab}, $C_{a_1,\dots,a_k}$ must be elements of a topological commutative ring.

The *standard marking* corresponding to this data is a map $f: F_\tau \to A$.

The *standard weight* of a marked tree (τ, f) corresponding to this data is

$$w(\tau, f) := \frac{1}{|\mathrm{Aut}\ \tau|} \prod_{\alpha \in E_\tau} f^{(\partial\alpha)} \prod_{v \in V_\tau} C_{f(\sigma v)}. \tag{0.22}$$

Here we use the following notation. For an edge α, $\partial\alpha$ denotes the set of two flags of this edge, and $f(\partial\alpha)$ is the set of two marks (a, b) put on these flags by f. Similarly, for a vertex v, σv denotes the set of all flags containing v, and $f(\sigma v)$ is the respective family of marks.

Finally, the standard sum over trees, or in physics speak, a *partition function* is

$$Z := \sum_{\tau/(iso)} \sum_{f: F_\tau \to A} w(\tau, f). \tag{0.23}$$

The passage from (0.21) to (0.23) is not completely automatic and indeed not always possible. Luckily, in can be made for all the problems discussed in [Ko] and here. I cannot explain conceptually why this is so. In particular, the factor $1/|\mathrm{Aut}\ \tau|$ in the Problems A, B, resp. C, occurs for different geometric reasons.

If we managed to represent (0.21) in the form (0.23), then we can try to complete the calculation of $\varphi = Z$ with the help of the following identity.

Assume that the matrix (g^{ab}) has an inverse matrix (g_{ab}).

STEP 3. Consider an auxiliary family of independent variables (fields) $\varphi = \{\varphi_a \mid a \in A\}$. Construct the formal function (potential)

$$S(\varphi) = -\sum_{a,b\in A} g_{ab}\frac{\varphi_a\varphi_b}{2} + \sum_{k\geq 1, a_i\in A} \frac{1}{k!} C_{a_1,\dots a_k}\varphi_{a_1}\dots\varphi_{a_k}. \qquad (0.24)$$

Denote by $\varphi^0 = \{\varphi^0_a \mid a \in A\}$ an appropriate critical point of $S(\varphi)$ that is, a solution of equations $\frac{\partial S}{\partial \varphi_a}|_{\varphi=\varphi^0} = 0,\ a \in A$.

0.6.1. Claim.

$$Z = S^{crit} = S(\varphi^0). \qquad (0.25)$$

This remains a "physical" statement until we specify the relevant topological ring containing g and C, prove the existence and uniqueness of φ^0, and the convergence of $S(\varphi^0)$. (See [Ko] for the standard physical argument "proving" 0.6.1). For example, considering $(g^{ab},\ C_{a_1,\dots,a_k})$ as independent formal variables, one can treat (0.22) as a formal series in these variables, and prove (0.6.1) as an identity in a localization of this ring.

Anyway, STEP 3 involves three calculational difficulties.

a). We must be able to sum $S(\varphi)$. In our problems A,B this is easy. In [Ko], a partial success is achieved, reducing $S(\varphi)$ to a new potential which is quadratic in φ_a but highly non–linear in a finite set of new auxiliary variables.

b). We must be able to solve $dS = 0$ and to find φ^0.

c). We must be able to calculate $S(\varphi^0)$.

The following trick, also well known to physicists, will allow us in certain cases to avoid the last unpleasant calculation.

We will deform the data $(g^{ab},\ C_{a_1,\dots,a_k})$ by introducing independent parameters $t = \{t_a \mid a \in A\}$ and replacing C_a by $t_a C_a$. The rest of the data $A,\ g^{ab},\ C_{a_1,\dots,a_k}$ for $k \geq 2$ remains unchanged. Let Z^t, S^t, φ^{0t} be respectively the deformed partition function, potential, and the critical point. Then we have

0.6.2. Claim. *For all $a \in A$, we have*

$$\frac{\partial Z^t}{\partial t_a} = C_a\varphi_a^{0t}. \qquad (0.26)$$

From the view point of generating functions, we lose no information replacing (0.25) by (0.26).

To deduce (0.26) from (0.25), one applies Claim 0.6.1 to Z^t and differentiates in t:

$$\frac{\partial Z^t}{\partial t_a} = \frac{\partial}{\partial t_a}\left(S^t(\varphi)|_{\varphi=\varphi^{0t}}\right) =$$

$$\sum_b \frac{\partial S^t(\varphi)}{\partial \varphi_b}|_{\varphi=\varphi^{0t}}\frac{\partial \varphi_b^{0t}}{\partial t_a} + \frac{\partial S^t}{\partial t_a}|_{\varphi=\varphi_a^{0t}} = C_a\varphi_a^{0t}$$

because S^t depends on t only via linear terms $\sum t_a C_a \varphi_a$.

On the other hand, to prove (0.26) in a formal context, one can totally bypass Claim 0.6.1 and simply apply a universal inversion formula to the formal map $(\varphi_a) \mapsto (\partial S^t/\partial\varphi_a)$ giving simultaneously existence, uniqueness, and expression for φ^{0t} as a sum over trees. Such inversion formulas are classical. The version closest to our needs is given in [GK]; the only difference is that $\partial S^t/\partial\varphi_a$ at 0 does not vanish. We leave details to the reader.

Functional equations (0.7), (0.9), (0.15), (0.18) are essentially relations for coordinates of the critical point. Differential equations are obtained from them by differentiating in t.

Acknowledgements. I am grateful to M. Kontsevich for many enlightening explanations, and to Don Zagier for teaching me PARI. After this work was written, I learned that E. Getzler proved (0.7) and (0.9) by essentially the same method.

§1. Moduli spaces

In this section, we prove the Theorem 0.3.1 following the three step procedure described in 0.6.

1.1. Marked trees and strata. A tree is called *stable* if $|v| \neq 2$ for all vertices v. If $|v| = 1$ we call v end vertex. Let V_τ^1 be the set of end vertices. An n–marking of τ is a bijection $\mu : V_\tau^1 \to \{1, \dots, n\}$. We also put $V_\tau^0 = V \setminus V_\tau^1$ and refer to it as the set of interior vertices.

Let now $(C; x_1, \dots, x_n)$ be a compact connected curve of arithmetical genus zero with $n \geq 3$ labelled non–singular points. The combinatorial structure of this curve is described by the following stable tree with n–marking (τ, μ): $V_\tau^0 =$ {irreducible components of C}, $V_\tau^1 = \{x_1, \dots x_n\}$; $\mu : x_i \mapsto i$; an edge connects two interior vertices if the respective components of C have non–empty intersection; an edge connects an interior vertex to an end vertex if the respective point belongs to the respective component.

Denote now by $M(\tau, \mu) \subset \overline{M}_{0,n}$ the set of points parametrising stable curves of the type (τ, μ). If τ has only one interior vertex, $M(\tau, \mu) := M_{0,n}$ is the big cell. The following statement summarises the main properties of these sets; for a proof, see [Ke].

1.1.1. Proposition. *a). $M(\tau, \mu)$ is a locally closed subset of $\overline{M}_{0,n}$ depending only on (the isomorphism class of) (τ, μ).*

b). $\overline{M}_{0,n}$ is the union of pairwise disjoint strata $M(\tau, \mu)$ for all marked stable n–trees (τ, μ).

c). For any (τ, μ),

$$M(\tau, \mu) \cong \prod_{v \in V_\tau^0} M_{0,|v|}.$$

Notice that there exists exactly one stable tree $\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet$ which does not correspond to any stable curve.

We can now calculate Poincaré polynomials.

1.1.2. Proposition. *We have*

$$P_{M(\tau,\mu)}(q) = \prod_{v\in V_\tau^0} P_{M_{0,|v|}}(q), \tag{1.1}$$

$$P_{M_{0,k}}(q) = \binom{q^2-2}{k-3}(k-3)!. \tag{1.2}$$

Proof. (1.1) follows from the Proposition 1.1.1 and the multiplicativity of Poincaré polynomials.

To prove (1.2), one can use the following geometric facts. First, the morphism $\pi:\ \overline{M}_{0,n+1}\to\overline{M}_{0,n}$ forgetting the last marked point is (canonically isomorphic to) the universal curve. Second, the infinity of the source consists of structure sections and fibers at infinity of the target. Therefore, over the big cell $M_{0,n}$ this morphism is a Zariski locally trivial fibration with fiber $\mathbf{P}^1$, and $M_{0,n+1} = \pi^{-1}(M_{0,n})\setminus\{$*union of structure sections*$\}$.

From the addivity of Poincaré polynomials it follows that

$$P_{M_{0,n+1}}(q) = P_{M_{0,n}}(q)P_{\mathbf{P}^1}(q) - nP_{M_{0,n}}(q) = (q^2+1-n)P_{M_{0,n}}(q).$$

Since $P_{M_{0,3}}(q)=1$, we get (1.2).

Summarizing, we have for $n\geq 3$:

$$P_{\overline{M}_{0,n}}(q)t^n = \sum_{\substack{(\tau,\mu)/(iso)\\ |V_\tau^1|=n}} \prod_{v\in V_\tau^0}\binom{q^2-2}{|v|-3}(|v|-3)!\prod_{v\in V_\tau^1} t, \tag{1.3}$$

where t is a new formal variable, and the sum is taken over n-marked stable trees.

1.2. Passage to the standard marking. Comparing (1.3) to (0.22) and (0.23) we are more or less compelled to choose $A=\{*\}$ (one element set), $g^{**}=1$, $C_*=t$, $C_{**}=0$ (this gives weight zero to non–stable trees), and finally, denoting by C_k the component with $k\geq 3$ indices,

$$C_k = \binom{q^2-2}{k-3}(k-3)!. \tag{1.4}$$

In particular, we can forget about $f:\ F_\tau\to\{*\}$.

This makes the weight of (τ,μ) depend only on $\tau/(iso)$, but not μ. Now, if $|V_\tau^1|=n$, the set of all n–markings of τ consists of $n!$ elements and is effectively acted upon by the group Aut τ. Therefore,

$$\text{card }\{(\tau,\mu)\}/(iso) = \frac{n!}{|\text{Aut }\tau|}\text{card }\{\tau\}/(iso).$$

Putting together (1.3), (0.22), and (0.23), we see finally that $\Phi(q,t)=Z^t$ where

$$\Phi(q,t) := \frac{t^2}{2!} + \sum_{n\geq 3}\frac{t^n}{n!}P_{\overline{M}_{0,n}}(q), \tag{1.5}$$

$$Z^t := \sum_{\tau/(iso)} \frac{1}{|\mathrm{Aut}\ \tau|} \prod_{v\in V_\tau} C_{|v|}. \tag{1.6}$$

The summation in (1.6) is now taken over all trees, the term $t^2/2$ in (1.5) comes from the two–vertex tree, and the generating function argument t in (1.5) corresponds precisely to the deformation parameter t introduced at the end of the subsection 0.6.

We will now use (0.26) in order to calculate

$$\frac{\partial Z^t}{\partial t} = \frac{\partial \Phi(q,t)}{\partial t} := \varphi(q,t).$$

1.3. Potential. From (0.24) and (1.4) one sees that

$$S^t(\varphi) = -\frac{\varphi^2}{2} + t\varphi + \sum_{k\geq 3} C_k\varphi^k =$$

$$-\frac{\varphi^2}{2} + t\varphi + \sum_{k\geq 3} \binom{q^2-3}{k-3} \frac{\varphi^k}{k(k-1)(k-2)}.$$

This can easily be summed. We need only the derivative.

1.3.1. Proposition. *For generic q we have*

$$\frac{\partial}{\partial\varphi} S^t(\varphi) = \frac{(1+\varphi)^{q^2} - 1 - q^4\varphi}{q^2(q^2-1)} + t, \tag{1.7}$$

and for $q=-1$,

$$\frac{\partial}{\partial\varphi} S^t(\varphi) = (1+\varphi)\,\log(1+\varphi) - 2\varphi + t. \tag{1.8}$$

1.4. End of the proof. We see now that (0.7), resp. (0.9), are equations for the critical point $d_\varphi S^t = 0$. Differentiating them in t and eliminating $(1+\varphi)^{q^2}$, resp log $(1+\varphi)$, we get (0.8), resp. (0.10).

1.5. Ramification of $\varphi(q,t)$ as a function of t. If q^2 is rational but $\neq 1$, we see from (0.7) that φ is an algebraic function of t of genus 0. Otherwise it is transcendental and infinitely valued. In order to understand its topology, we can use the following classical trick.

Consider the differential equation for a function $y = y(x)$:

$$yy_x = ax + by;\ a,b \in \mathbf{C}. \tag{1.9}$$

Let $w_{1,2}$ be roots of its *characteristic equation*

$$w^2 - bw - a = 0. \tag{1.10}$$

Assume that $w_1 \neq w_2$ and put

$$A_1 = \frac{w_1}{w_2 - w_1},\ A_2 = \frac{w_2}{w_1 - w_2} \tag{1.11}$$

so that $A_1 + A_2 = -1$. A direct calculation shows:

Proposition 1.5.1.. *Put $w(x) = y(x)/x$. Then the general solution of (1.9) is given by the implicit equation*

$$Cx = (w - w_1)^{A_1}(w - w_2)^{A_2}, \tag{1.12}$$

where C is an arbitrary constant.

We can apply this to (0.8) putting

$$y = 1 + q^2 t - q^2\varphi,\ x = q^2 t + q^2 + 1.$$

Then we find

$$w_1 = 1,\ w_2 = q^{-2}, A_1 = \frac{q^2}{1-q^2},\ A_2 = \frac{1}{q^2-1}.$$

One can calculate C evaluating (1.12) at the point $t = 0$ where we have $x = q^2 + 1,\ y = 1,\ w = (q^2+1)^{-1}$.

§2. Configuration spaces

In this section, we prove Theorems 0.4.1 and 0.4.2.

2.1. Nests and strata. Let X be a smooth compact algebraic variety. The configuration space $X[n]$, $n \geq 2$, is defined in [FMPh] as the closure of its big cell $X^n \setminus (\cup_{i<j}\Delta_{ij})$ (Δ_{ij} is the diagonal $x_i = x_j$) in $X^n \times \prod_S \widetilde{X}^S$, where S runs over subsets $S \subset \{1, \dots, n\}$, $|S| \geq 2$; X^S denotes the respective partial product of X's, and $\widetilde{X}^S$ is the blow up of the small diagonal Δ_S in X^S.

Every S determines a divisor at infinity $D(S) \subset X[n]$. Namely, let $\pi_S : X[n] \to X^S$ be the canonical projection. Then $\pi_S^{-1}(\Delta_S) = \cup_{T \supset S} D(T)$.

The natural stratification of $X[n]$ described in [FMPh] consists of (open subsets of) intersections $\overline{X(\mathcal{S})} = \cap_{i=1}^r D(S_i)$ corresponding to sets $\mathcal{S} = \{S_1, \dots, S_r\}$ of subsets in $\{1, \dots, n\}$ called *nests.*

2.1.1. Definition. *a). $\mathcal{S} = \{S_1, \dots, S_r\}$ is a nest (or n-nest) if $|S_i| \geq 2$ for all i, and either $S_i \subset S_j$ or $S_j \subset S_i$ for all i, j such that $S_i \cap S_j \neq \emptyset$.*

In particular, $\mathcal{S} = \emptyset$ is a nest, and $\mathcal{S} = \{S\}$ is a nest, if $|S| \geq 2$.

b). A nest $\mathcal{S}$ is called whole (resp. broken) if $\{1, \dots, n\} \in \mathcal{S}$ (resp. $\{1, \dots, n\} \notin \mathcal{S}$).

Denote by $X(\mathcal{S}) \subset \overline{X(\mathcal{S})} = \cap_{S \in \mathcal{S}} D(S)$ the subset of points not belonging to smaller closed strata. The following facts are proved in [FMPh].

2.1.2. Proposition. *a). For any $n \geq 2$ and n-nest $\mathcal{S}$, $X(\mathcal{S})$ is a locally closed subset of $X[n]$.*

b). $X[n]$ is the union of pairwise disjoint strata $X(\mathcal{S})$ for all n-nests $\mathcal{S}$.

2.2. From nests to marked trees. As in 1.1 we consider a bijection $\mu : V_\tau^1 \to \{1, \dots, n\}$ as a part of the appropriate marking for our problem. The remaining data is supplied by choosing *orientation of all edges.*

2.2.1. Definition. *A tree τ marked in this way is called admissible iff:*

a). Every vertex of τ except of one has exactly one incoming edge.

b). The exceptional vertex has only outgoing edges, and their number is ≥ 2. This vertex is called source.

c). All interior vertices with possible exception of source have valency ≥ 3.

2.2.2. Proposition. *The following maps are (1,1):*

$$\{broken\ n-nests\} \to \{whole\ n-nests\} \to \{admissible\ marked\ n-trees\}/(iso),$$

$$\mathcal{S} \mapsto \mathcal{S} \cup \{\{1,\dots,n\}\} \mapsto \tau(\mathcal{S}) = \tau(\mathcal{S} \cup \{\{1,\dots,n\}\}).$$

Here τ is defined by its sets of vertices and edges: if $\mathcal{S} = \{S_1,\dots,S_r\}$, then

$$V_\tau = \{\widetilde{S}_1,\dots,\widetilde{S}_{n+r}\} := \{S_1,\dots,S_r,\{1\},\dots,\{n\}\},$$

and an edge oriented from $\widetilde{S}_i$ to $\widetilde{S}_j$ connects these two vertices iff $\widetilde{S}_j \subset \widetilde{S}_i$ and no $\widetilde{S}_k$ lies strictly in between these two subsets.

This is proved by direct observation. The following facts are worth mentioning.

a). $\{1,\dots,n\}$ is the source of $\tau(\mathcal{S})$ for any $\mathcal{S}$.

b). $\{1\},\dots,\{n\}$ are all end vertices.

c). $i \in S_j$ iff one can pass from $S_j \in V_\tau$ to $\{i\} \in V_\tau$ in τ by going always in positive direction.

A reader is advised to convince him– or herself that the source has valency ≥ 2 and all other interior vertices have valency ≥ 3.

Denote the source by s and the set of the remaining interior vertices V_τ^0.

2.2.3. Proposition ([FMPh]). *The virtual Poincaré polynomials of strata $X(\mathcal{S})$ are given by the following formulas (we add a formal variable t).*

If $\mathcal{S}$ is a broken n-nest, $s \in V_{\tau(\mathcal{S})}$:

$$t^n P_{X(\mathcal{S})}(q) = \binom{P_X(q)}{|s|}|s|! \times \prod_{v\in V^0_{\tau(\mathcal{S})}} \kappa_m \binom{q^{2m}-2}{|v|-3}(|v|-3)! \times \prod_{v\in V^1_{\tau(\mathcal{S})}} t. \qquad (2.1)$$

If $\mathcal{S}$ is a whole n-nest:

$$t^n P_{X(\mathcal{S})}(q) = P_X(q)\kappa_m \binom{q^{2m}-2}{|s|-2}(|s|-2)! \times$$

$$\prod_{v\in V^0_{\tau(\mathcal{S})}} \kappa_m \binom{q^{2m}-2}{|v|-3}(|v|-3)! \times \prod_{v\in V^1_{\tau(\mathcal{S})}} t. \qquad (2.2)$$

Comparing (2.1) and (2.2) one sees that one can express the joint contribution of two nests corresponding to an admissible marked tree τ as a product of local weights corresponding to all vertices of τ. The local weight of the source will be

$$\binom{P_X(q)}{|s|}|s|!+P_X(q)\kappa_m\binom{q^{2m}-2}{|s|-2}(|s|-2)!$$

and the remaining local weights in (2.1) and (2.2) coincide and depend only on the valency.

2.3. Passage to the standard marking. We make the following choices.

Put $A=\{+,-\}$. Interpret a mark $+$ (resp. $-$) on a flag as incoming (resp. outgoing) orientation of this flag. Thus, $f:\ F_\tau\to A$ is a choice of orientation of all flags.

Put $g^{+-}=g^{-+}=1,\ g^{++}=g^{--}=0$. This makes the standard weight of (τ,f) vanish unless all *edges* are unambiguously oriented by f.

Put $C_+=t$ (see (2.1) and (2.2)) and $C_-=0$. The last choice makes the standard weight vanish unless all end edges are oriented outwards.

Put $C_{+-}=C{-+}=0$. This excludes vertices of the type $\to\bullet\to$.

Put also $C_{a_1,\dots,a_k}=0$ if $\{+,+\}\subset\{a_1,\dots,a_k\}$. This eliminates vertices with ≥ 2 incoming edges.

For tensors with $k\geq 2$ minuses among the indices we put

$$C_{-\dots -}=\binom{P_X(q)}{k}k!+\kappa_mP_X(q)\binom{q^{2m}-2}{k-2}(k-2)! \tag{2.3}$$

(because only the source has all outgoing edges), and

$$C_{+-\dots -}=\kappa_m\binom{q^{2m}-2}{k-2}(k-2)! \tag{2.4}$$

(cf. (2.1) and (2.2)).

The standard weight of a marked tree defined by this data again is independent on the part $\mu:\ V_\tau^1\to\{1,\dots,n\}$ of the initial marking which accounts for the factor $\dfrac{n!}{|\mathrm{Aut}\ \tau|}$ below.

Summarizing, we put

$$\Phi_X(q,t):=\sum_{n\geq 2}\frac{t^n}{n!}P_{X[n]}(q), \tag{2.5}$$

$$Z^t:=\sum_{\tau/(iso)}\frac{1}{|\mathrm{Aut}\ \tau|}\sum_{f:F_\tau\to\{+,-\}}\prod_{\alpha\in E_\tau}g^{f(\partial\alpha)}\prod_{v\in V_\tau}C_{f(\sigma v)}, \tag{2.6}$$

and get from the previous discussion

$$Z^t=\Phi_X(q,t),\qquad \frac{\partial}{\partial t}Z^t:=\phi_X(q,t). \tag{2.7}$$

2.4. Potential. We change notation: $\varphi_+ = x$,$\varphi_- = y$. From 2.3 we see that (already t-deformed) potential is

$$S^t(x,y) = -xy + tx + \kappa_m \sum_{k=2}^{\infty} \binom{q^{2m}-2}{k-2} \frac{xy^k}{k(k-1)} +$$

$$\sum_{k=2}^{\infty} \binom{P_X(q)}{k} y^k + \kappa_m P_X(q) \sum_{k=2}^{\infty} \binom{q^{2m}-2}{k-2} \frac{y^k}{k(k-1)} \tag{2.8}$$

(we have two arguments x, y but only one $t = t_+$ because $C_- = 0$).

We must solve the system

$$\frac{\partial S^t}{\partial x}|_{x^0,y^0} = \frac{\partial S^t}{\partial y}|_{x^0,y^0} = 0, \tag{2.9}$$

and (0.26) then tells us that

$$\frac{\partial}{\partial t} Z^t = \varphi_X(q,t) = x^0. \tag{2.10}$$

Again, $S^t(x,y)$ can be easily summed. To write down the functional equation, we need only x–derivative which for general q is

$$\frac{\partial S^t}{\partial x} = -y + t + \kappa_m \frac{(1+y)^{q^{2m}} - 1 - q^{2m}y}{q^{2m}(q^{2m}-1)}. \tag{2.11}$$

For $q = -1$:

$$\frac{\partial S^t}{\partial x} = -y + t + m[(1+y)\log(1+y) - y]. \tag{2.12}$$

2.5. End of the proof. We now see that (0.15), resp (0.18), are the equations defining y^0. Differentiating in t we get (0.16) and (0.19). And since $S^t(x,y)$ is linear in x, the vanishing of the y-derivative gives an explicit expression of x^0 via y^0:

$$\varphi_X(q,t) = P_X(q) \frac{(1+y^0)^{P_X(q)} + (q^{2m} + \kappa_m - 1)y^0 - q^{2m}t - 1}{1 + (1 - q^{2m} - \kappa_m)y^0 + q^{2m}t}.$$

To see that this is equivalent to (0.17) one can differentiate (0.17) in t and use (0.16).

2.6. Ramification of y^0. Replaying the game of 1.5, we put (changing the meaning of x, y in favor of those in 1.5):

$$y = y(q,t) := q^{2m}t + 1 - (q^{2m} + \kappa_m - 1)y^0(q,t),$$

$$x := t + \frac{q^{2m} + \kappa_m}{q^{2m}}, \qquad w(q,t) = y/x.$$

Then (0.16) becomes

$$yy_x = -q^{2m}x + (q^{2m}+1)y$$

so that in the notation of 1.5

$$w_1 = 1,\ w_2 = q^{2m},\ A_1 = \frac{1}{q^{2m}-1},\ A_2 = \frac{q^{2m}}{1-q^{2m}},$$

and finally

$$Cx = (w-w_1)^{A_1}(w-w_2)^{A_2}$$

for some C.

§3. Multiple coverings

3.1. Kontsevich's formula for Problem C. Kontsevich represents m_d as a rational function of two variables λ_1, λ_2 which is formally homogeneous of degree zero and actually is expected to be a constant.

Geometrically, this statement must be a corollary of Bott's fixed point formula for *smooth stacks*. The λ-variables in this context are coordinates of a toric vector field on the target $\mathbf{P}^1$. Until this has been worked out, we simply go ahead with Kontsevich and take this independence for granted.

The function in question is a sum of contributions indexed by isomorphism classes of connected trees τ endowed with markings: each vertex v is marked by $f_v = 1$ or 2 so that no neighbors have the same mark; each edge α is marked by a positive integer d_α. Only those marked trees contribute to m_d for which $\deg\tau := \sum_\alpha d_\alpha = d$.

We introduce the following notation for a marked tree τ: $F=$ the number of vertices marked by 2; $\sigma_v = \sum_{\alpha: v\in\alpha} d_\alpha$; $w_i = \sum_{v: f_v = i}(|v|-1),\ i=1,2$.

Then we have

$$m_d = (\lambda_1-\lambda_2)^{2-2d}\sum_{\tau:\deg\tau=d}\frac{1}{|\mathrm{Aut}\,\tau|}(-1)^{d+F}\lambda_1^{2w_1}\lambda_2^{2w_2}V(\tau)E(\tau),$$

$$V(\tau) = \prod_v \sigma_v^{|v|-3},\ E(\tau) = \prod_\alpha \frac{d_\alpha^3}{d_\alpha!^2}\prod_{a+b=d_\alpha;a,b\geq 1}(a\lambda_1+b\lambda_2)^2.$$

3.2. Theorem. $m_d = d^{-3}$.

Proof. We will calculate the value of m_d at $\lambda_1 = 1,\ \lambda_2 = 0$. The drastic simplification results from the fact that the factor $\lambda_2^{2w_2}$ vanishes unless $w_2 = 0$. Now, $w_2 = 0$ implies that τ has no vertices of multiplicity ≥ 2 marked by 2. Hence τ either has only one edge, or is a star with central vertex marked 1, and end vertices marked by 2. We will consider the first case as one ray star as well.

Now, let τ be such a star of degree d. The set $\{d_\alpha\}$ forms a partition of d into positive summands which uniquely defines the isomorphism class of τ. It is convenient to write this partition as the set of multiplicities $R = \{r_1, r_2, \dots\}$, where $r_i=$ the number of edges marked by i so that $\sum_i ir_i = d$. Obviously, $|\mathrm{Aut}\,\tau| = \prod_i r_i!$.

After some reshuffling, our assertion thus reduces to the following identity:

$$(?)\qquad \sum_R \frac{1}{\prod_i r_i!}\prod_i \left(-\frac{d}{i}\right)^{r_i} = (-1)^d.$$

Now, the left hand side of (?) can be obtained in the following way. Consider the formal series $e^{\sum_{i\geq 1} y_i t^i}$, take its terms of degree d in t and put in them $y_i = -d/i$. But we can clearly proceed in reverse order first making the substitution $y_i = -d/i$. Then the series in the exponent becomes $\sum_i (-d/i)t^i = d\log(1-t)$, so that finally we get the coefficient of t^d in $(1-t)^d$. QED

Remark. One can observe that $(-1)^d$ coincides with the contribution of just one trivial partition: $r_d = 1$. The remaining terms cancel. Geometrically, this means that degenerating configurations do not contribute with this choice of vector field. Algebraically, this can be rewritten as an equality of two sums, one over proper partitions with odd, another with even number of summands.

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MORDELL–WEIL PROBLEM FOR CUBIC SURFACES

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§0. Introduction

Let V be a plane non-singular geometrically irreducible cubic curve over a finitely generated field k. The Mordell–Weil theorem for V can be restated in the following geometric form: there is a finite subset $B \subset V(k)$ such that the whole $V(k)$ can be obtained from B by drawing secants (and tangents) through pairs of previously constructed points and consecutively adding their new intersection points with V.

In this note I address the question of validity of this statement for cubic surfaces. After reminding some constructions from the book [Ma], I analyze a numerical example, and then prove a different version of the Mordell–Weil statement for split cubic surfaces. A shameless change of the composition law allows me to reduce this problem to the classical theorem on the structure of abstract projective planes. Unfortunately, the initial question, which is more natural to ask for minimal surfaces, remains unanswered. I would like to call attention to this problem and its calculational aspects.

I am grateful to Don Zagier whose tables are quoted in §2, and to M. Rovinsky and A. Skorobogatov, discussions with whom helped me to state and prove the main theorem.

§1. A summary of known results

1.1. Notation. Let V be a cubic hypersurface without multiple components over a field k in $\mathbf{P}^d$, $d \geq 2$. If $x, y, z \in V(k)$ are three points (with multiplicities) lying on a line in $\mathbf{P}^d$ not belonging to V, we write $x = y \circ z$. Thus $\circ$ is a (partial and multivalued) composition law on $V(k)$. We will also consider its restriction on subsets of $V(k)$, e.g. that of smooth points.

If $x \in V(k)$ is smooth, and does not lie on a hyperplane component of V, the birational map $t_x : V \to V$, $y \mapsto x \circ y$, is well defined. Denote by Bir V the full group of birational automorphisms ov V.

The following two results summarize the properties of $\{t_x\}$ for curves and surfaces respectively. The first one is classical, and the second one is proved in [M].

1.2. Theorem. *Let V be a smooth cubic curve. Then:*

a). Bir V is a semidirect product of the group of projective automorphisms and the subgroup generated by $\{t_x \mid x \in V(k)\}$.

b). We have identically

$$t_x^2 = (t_x t_y t_z)^2 = 1 \tag{1.1}$$

for all $x, y, z \in V(k)$.

If in addition k is finitely generated over a prime field, then:

c). Bir V is finitely generated.

d). All points of $V(k)$ can be obtained from a finite subset of them by drawing secants and tangents and adding the intersection points.

1.3. Theorem. *Let V be a minimal smooth cubic surface over a perfect non-closed field k. Then:*

a). Bir V *is a semi-direct product of the group of projective automorphisms and the subgroup generated by*

$$\{t_x \mid x \in V(k)\} \text{ and } \{s_{u,v} \mid u, v \in V(K);\ [K:k] = 2;\ u, v \text{ are conjugate over } k\}$$

where

$$s_{u,v} := t_u t_{u \circ v} t_v.$$

b). We have identically

$$t_x^2 = (t_x t_{x \circ y} t_y)^2 = (s_{u,v})^2 = 1,\ \ s t_x s^{-1} = t_{s(x)}, \tag{1.2}$$

for all pairs u, v not lying on lines in V, and projective automorphisms s.

c). The relations (1.2) form a presentation of Bir V.

We remind that V is called *minimal* if one cannot blow down some lines of V by a birational morphism defined over k. The opposite class consists of *split* surfaces upon which all lines are k–rational.

1.4. Discussion. Although the two theorems are strikingly parallel, there is an important difference between finiteness properties in one– and two–dimensional cases.

Basically, (1.1) means only that $x + y := e \circ (x \circ y)$ is an abelian group law with identity e, whereas the statements c) and d) of the Theorem 1.2 additionally assert that this group is finitely generated. Therefore, (1.1) generally is not a complete system of relations between $\{t_x\}$.

Contrariwise, since (1.2) is complete, Bir$V$ in the twodimensional case cannot be finitely generated if $V(k)$ is infinite. This can be proved by a direct group theoretic argument establishing a canonical form of a word in $\{t_x, s_{u,v}\}$ (cf. [Ma], sections 39.8.1 and 39.8.2).

Therefore, if something like the statement d) of Theorem 1.2 is expected to be true for cubic surfaces, this must reflect a deep difference between relations among $\{t_x, s_{u,v}\}$ in Bir V and relations among $\{x\}$ in $(V(k), \circ)$. The latter are much less understood than the former. One reason is that exceptional subvarieties of birational automorphisms are rationally parametrized curves in V which presumably should be treated as a whole in a reasonable finiteness statement. In fact, a typical example of such subset is a cubic curve $C(x)$ with double point $x \in V(k)$ obtained as intersection of V with tangent plane at x. Now, the set $(C(x)(k) \setminus \{x\}, \circ)$ with a composition law $x + y = e \circ (x \circ y)$ is isomorphic to the group of k–points of a form of the multiplicative group. Such a group is not finitely generated even for $k = \mathbf{Q}$. On the other hand, in $(V(k), \circ)$ this whole set must be considered as the domain of multivalued expression $x \circ x$, because geometrically all its points can be obtained by drawing tangents with k–rational direction to x. Therefore finite generation is still conceivable.

This comment must also help the reader to accept the definition of a generalized operation $\circ_{(C,p)}$ in §3, which is another way to deal with the same difficulty.

§2. Minimal cubic surfaces: some numerical data

2.1. The structure of data. Let V be a smooth cubic surface over a field k such that $V(k)$ is infinite. Let $h: V(k) \to \mathbf{R}_+$ be a counting function (i.e. for all $H > 0$, the set $V_H := \{x \in V(k) \mid h(x) < H\}$ is finite). In order to find a generating subset in $(V(k), \circ)$, one can proceed as follows.

A. Choose a large H and compile the list of all elements of V_H. Let points x in it be ordered by increasing $h(x)$. We will write $x < y$ if x precedes y, and use the number of a point in this list as its name.

B. For every x and every $y < x$, calculate points $x \circ y$ and choose among them those $z = x \circ y$ for which $z < x$. Rewrite every such relation as $x = y \circ z,\ y, z < x$, and register it at the same line as (coordinates and number of) x. Notice that if by chance $y = z$, the last relation means exactly that x lies in the tangent section of V with double point x.

If such a relation exists for x, we will call x *strongly decomposable.*

If all points x with sufficiently large $h(x)$ were strongly decomposable, then the ones which are not would form a finite generating set. This is the case for cubic curves with height as counting function. For cubic surfaces the tables strongly indicate that it is not the case.

Therefore we have to consider decompositions of length ≥ 3, $x = M(x_1, \ldots, x_n)$, $x_i < x$, where M is a non–associative monomial w.r.t. $\circ$. We will call *weakly decomposable* points admitting such a decomposition.

A direct search of such decompositions is very time–consuming (as well as a direct search of points). One problem is that intermediate results can have height much larger than H; another is that we have no a priori bound for the length of decomposition.

In the example discussed below we used simple search algorithms allowing to list those monomials $M(x_1, \ldots, x_n) < H$ for which there is a computation scheme representing it as an iteration of double compositions with all intermediate results registered in V_H. For example, if we have two strong decompositions $x = y \circ z = u \circ v$ with, say, $y > z, u, v$, then we get a weak decomposition $y = z \circ (u \circ v)$.

2.2. An example. D. Zagier produced a table of all primitive solutions of $\sum_{i=1}^4 i x_i^3 = 0$ with $h(x) := \sum_{i=1}^4 |x_i| \leq 1100$. He found 379 such points and strong decompositions of 339 among them.

By the search described above we found weak decompositions of further 24 points. This left us with 16 generators for 379 points, probably too many to state a finiteness conjecture. However, there remains a possibility that this number will diminish if decompositions with larger intermediate results are taken into account.

Here are some numerical illustrations. The first three points $1 = (1, 0, 1, -1)$, $2 = (1, 1, -1, 0)$, $3 = (1, -1, -1, 1)$ are indecomposable. The next 26 points are strongly decomposable, e.g.

$$24 = (1, 28, -19, -18) = 2 \circ 2 = 13 \circ 13 = 14 \circ 21 = 5 \circ 23.$$

Points 27, 28, and 29 are only weakly decomposable, and $30 = (15, -37, 5, 29)$ stubbornly resisted decomposition.

One of the longest decompositions found is

$$77 = 5 \circ (1 \circ (35 \circ (2 \circ (33 \circ ((2 \circ 11) \circ (12 \circ (21 \circ 70))))))).$$

§3. Birationally trivial cubic surfaces: a finiteness theorem

3.1. Modified composition. Let V be a smooth cubic surface, and $x, y \in V(k)$. Let $C \subset V$ be a curve on V passing through x, y, and $p: C \to \mathbf{P}^2$ an embedding of C into a projective plane such that $p(C)$ is cubic, and $p(x) \circ p(y)$ is defined in $p(C)$. We assume that C and p are defined over k.

In this situation we will put

$$x \circ_{(C,p)} y := p^{-1}(p(x) \circ p(y)).$$

Example 1. Choose $C =$ a plane section of V containing x, y. If p is the embedding of C into the secant plane, then $x \circ_{(C,p)} y = x \circ y$ in the standard notation. Notice that the result does not depend on C if $x \neq y$. If $x = y$, then the choice of C is equivalent to the choice of a tangent line to V at x so that the multivaluedness of $\circ$ is taken care of by the introduction of this new parameter.

Example 2. Assume now that V admits a birational morphism $p: V \to \mathbf{P}^2$ defined over k (e.g., V is split). We will choose and fix p once for all. Then any plane section C of V not containing one of the blown down lines as a component is embedded by p into $\mathbf{P}^2$ as a cubic curve. Therefore we can apply to (C, p) the previous construction. Notice that this time $x \circ_{(C,p)} y$ depends on C even if $x \neq y$.

The following Theorem is the main result of this note:

3.2. Theorem. *Assume that k is a finitely generated field. In the situation of Example 2, the complement $U(k)$ to the blown down lines in $V(k)$ is finitely generated with respect to operations $\circ_{(C,p)}$.*

Proof. Let us start with the following auxiliary construction. Choose a k-rational line $l \subset \mathbf{P}^2$. Then $\Gamma := p^{-1}(l)$ is a twisted rational cubic in V. The family of all such cubics reflects properties of that of lines: a) any two different points a, b of $U(k)$ belong to a unique $\Gamma(a, b)$; b) any two different Γ's either have one common k-point, or intersect a common blown down line.

Define now a (partial) quaternary operation on $U(k)$:

$$*(a, b; c, d) := \Gamma(a, b) \cap \Gamma(c, d).$$

It is defined for a Zariski dense open subset in $U(k)^4$.

Claim 1. If $x = *(a, b; c, d)$ is well-defined, then there exists a plane section C of V such that

$$*(a, b; c, d) = a \circ_{(C,p)} b.$$

In fact, choose C containing a, b, and x. Then p maps $\Gamma(a, b)$ to a line intersecting $p(C)$ at a, b, x.

It suffices now to establish the following fact:

Claim 2. $U(k)$ is finitely generated with respect to $*$.

To prove this, it suffices to demonstrate that $\mathbf{P}^2(k)$ is finitely generated with respect to the similar quaternary operation

$$*(a,b;c,d) := l(a,b) \cap l(c,d)$$

where $l(a,b)$ is the line containing a, b.

In fact, start with four points in general position in $\mathbf{P}^2(k)$. Introduce projective coordinates using these four poits as basic. Generate all points starting with these four and adding intersections of lines passing through pairs of constructed points. Obviously, the resulting set will be an abstract projective plane satisfying the Desargues axiom. Hence it will coincide with $\mathbf{P}^2(k_0)$ where k_0 is the prime subfield. Represent k as $k_0(t_1, \dots, t_n)$. Add to the initial four points the ones with coordinates $(1 : t_i : 0)$ and generate a new abstract projective plane as earlier. It will contain $\mathbf{P}^1(k)$ and hence coincide with $\mathbf{P}^2$, by a classical reasoning: cf. [H].

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