EINSTEIN-KÄHLER FORMS,
FUTAKI INVARIANTS AND CONVEX GEOMETRY
ON TORIC FANO VARIETIES

by

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0. INTRODUCTION.

Throughout this paper, we assume that X is a nonsingular n-dimensional toric Fano variety (defined over $\mathbb{C}$), i.e., X is an n-dimensional connected projective algebraic manifold satisfying the following conditions:

(a) X admits an effective almost homogeneous algebraic group action of $(\mathbb{G}_m)^n$ $(\cong(\mathbb{C}^*)^n$ as a complex Lie group).

(b) The set $\mathcal{K}$ of all Kähler forms on X in the De Rham cohomology class $2\pi\, c_1(X)_{\mathbb{R}}$ is non-empty.

For each $\omega\in\mathcal{K}$, by writing it as $\omega = \sqrt{-1}\sum g(\omega)_{\alpha\bar{\beta}}\, dz^{\alpha}\wedge dz^{\bar{\beta}}$ in terms of holomorphic local coordinates $(z^1, z^2, \ldots, z^n)$ of X, we have the corresponding Ricci form $\mathrm{Ric}(\omega)$ cohomologous to ω:

$$\mathrm{Ric}(\omega) := \sqrt{-1}\,\bar{\partial}\partial \log \det(g(\omega)_{\alpha\bar{\beta}}).$$

Then an element ω of $\mathcal{K}$ is called an Einstein-Kähler form if $\mathrm{Ric}(\omega) = \omega$. We now pose the following:

(0.1) PROBLEM*): Classify all X which admit, at least, one Einstein-Kähler form.

Obviously, the Fubini-Study form on $\mathbb{P}^n(\mathbb{C})$ is a typical Einstein-Kähler form. This settles Problem (0.1) for n = 1, because

*) This is also posed by T. Oda and Y. T. Siu.

the only possible X with $n = 1$ is $\mathbb{P}^1(\mathbb{C})$. However, the real difficulty comes up even at $n = 2$: Let S_i be the projective algebraic surface obtained from $\mathbb{P}^2(\mathbb{C})$ by blowing up i points in general position (where $1 \leqq i \leqq 3$). Then, in spite of lots of efforts of differential geometers, it is still unknown whether or not the nonsingular toric Fano variety S_3 admits an Einstein-Kähler form.

The purpose of this paper is to give a brief survey of recent progress on Problem (0.1) together with our related new results. Especially, in Sections 1$\sim$6 (though they are somewhat of expository nature), several key ideas are introduced often without proofs, while technical details are given in the subsequent four appendices. In particular, in Appendix C (see (9.2.3) for the most general statement), we shall show that the Futaki invariants of an anti-canonically (relatively) polarized toric bundle Y over W can be regarded as the barycentre of $\mathbf{m}(Y)$ in terms of "Duistermaat-Heckman's measure", where $\mathbf{m} : Y \to \mathbb{R}^n$ ($n = \dim_{\mathbb{C}} Y - \dim_{\mathbb{C}} W$) denotes the associated "relative" moment map defined, in Appendix B, without any ambiguity of translations (cf. (8.2)). Finally, in Appendix D, a very explicit description of Einstein-Kähler metrics for Sakane-Koiso's examples will be given (cf. (10.3.2), Step 4 of (10.3)).

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1. NOTATION, CONVENTIONS AND PRELIMINARIES.

Let $\mathbb{Z}_+$ (resp. $\mathbb{Z}_o$) be the set of positive (resp. non-negative) integers and $\mathbb{R}_+$ (resp. $\mathbb{R}_o$) be the set of positive (resp. non-negative) real numbers. We now put:

$$G := (\mathbb{G}_m)^n = \{(t_1, t_2, \ldots , t_n) \mid t_i \in \mathbb{C}^*\},$$
$$M := \{\mathbf{a} = (a_1, a_2, \ldots , a_n) \mid a_i \in \mathbb{Z}\} \ (\cong \mathbb{Z}^n),$$
$$N := \left\{ \mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \middle| \, b_j \in \mathbb{Z} \right\} \ (\cong \mathbb{Z}^n).$$

For $\mathbf{a} \in M$ and $\mathbf{b} \in N$ as above, we define $(\mathbf{a},\mathbf{b}) \in \mathbb{Z}$, $\chi^{\mathbf{a}} \in \mathrm{Hom}_{\mathrm{alg\ gp}}(G,\mathbb{G}_m)$ and $\lambda_{\mathbf{b}} \in \mathrm{Hom}_{\mathrm{alg\ gp}}(\mathbb{G}_m,G)$ by

$$(\mathbf{a},\mathbf{b}) := \textstyle\sum_{i=1}^n a_i b_i \ ,$$
$$\chi^{\mathbf{a}}((t_1, t_2, \ldots , t_n)) := t_1^{a_1} t_2^{a_2} \cdots t_n^{a_n} \ ,$$
$$\lambda_{\mathbf{b}}(t) := (t^{b_1}, t^{b_2}, \ldots , t^{b_n}),$$

where $t, t_1, \ldots , t_n \in \mathbb{G}_m$ $(= \mathbb{C}^*)$. Then the correspondence $\mathbf{a} \mapsto \chi^{\mathbf{a}}$ (resp. $\mathbf{b} \mapsto \lambda_{\mathbf{b}}$) canonically induces an isomorphism between the additive group M (resp. N) and the multiplicative group $\mathrm{Hom}_{\mathrm{alg\ gp}}(G,\mathbb{G}_m)$ (resp. $\mathrm{Hom}_{\mathrm{alg\ gp}}(\mathbb{G}_m,G)$). Note that

$$\chi^{\mathbf{a}}(\lambda_{\mathbf{b}}(t)) = t^{(\mathbf{a},\mathbf{b})} \quad \text{for all } t \in \mathbb{G}_m \ (= \mathbb{C}^*).$$

(1.1) DEFINITION: A non-empty subset σ of N is called a <u>cone</u> if the following conditions are satisfied:

(a) If $\mathbf{b} \in N$ satisfies $\beta\mathbf{b} \in \sigma$ for some $\beta \in \mathbb{Z}_+$, then $\mathbf{b} \in \sigma$.

(b) If $0 \neq \mathbf{b} \in \sigma$, then $-\mathbf{b} \notin \sigma$.

(c) $0 \in \sigma$.

(d) In terms of the natural additive structure of N, σ is a semigroup generated by its finite subset.

For a cone σ, there exists a unique irredundant finite subset $\{ \mathbb{b}^1, \mathbb{b}^2, \dots, \mathbb{b}^m \}$ of σ such that $\sigma = \sum_{k=1}^{m} \mathbb{Z}_0 \mathbb{b}^k$. These $\mathbb{b}^1$, $\mathbb{b}^2, \dots, \mathbb{b}^m$ are called the <u>fundamental generators</u> of the cone σ.

(1.2) DEFINITION: A non-empty subset τ of a cone σ is called a <u>face</u> of σ, denoted by $\tau \leqq \sigma$, if there exists an element $\mathbb{a}$ of M such that $(\mathbb{a},\mathbb{b}) \geqq 0$ for all $\mathbb{b}$ in σ and that $\tau = \{ \mathbb{b} \in \sigma \mid (\mathbb{a},\mathbb{b}) = 0 \}$. A <u>finite polyhedral decomposition</u> of N is a finite set Δ of cones in N such that

(a) if $\tau \leqq \sigma \in \Delta$, then $\tau \in \Delta$;

(b) if $\sigma, \tau \in \Delta$, then $\sigma \cap \tau \leqq \sigma$ and $\sigma \cap \tau \leqq \tau$;

(c) $N = \bigcup_{\sigma \in \Delta} \sigma$.

For every finite polyhedral decomposition Δ of N, we put

$$\Delta(i) := \{ \sigma \in \Delta \mid \dim \sigma = i \}, \qquad 0 \leqq i \leqq n,$$

where dim σ denotes the dimension of the real vector space spanned by σ in $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$.

(1.3) DEFINITION: A finite polyhedral decomposition Δ of N is said to be <u>nonsingular</u> if for each $\sigma \in \Delta(n)$, the set of fundamental generators of σ consists of n elements and forms a $\mathbb{Z}$-basis for N. For every nonsingular Δ, the set of fundamental generators of each element of $\Delta(i)$ consists of exactly i elements and is completed to a $\mathbb{Z}$-basis for N.

We shall now quote the following fundamental results due to Demazure [6], Miyake and Oda [18], and Mumford et al. [19] :

(1.4) THEOREM: <u>To every nonsingular finite polyhedral decomposition</u> Δ <u>of</u> N, <u>one can uniquely associate an</u> n-<u>dimensional irreducible nonsingular</u> G-<u>equivariant compactification</u> G_Δ <u>of</u> G <u>possessing the following two properties</u>:

(a) <u>To each</u> $\sigma \in \Delta(i)$, $0 \leqq i \leqq n$, <u>there corresponds a unique</u> (n-i)-<u>dimensional</u> G-<u>orbit</u>, <u>denoted by</u> $\mathbb{O}^\sigma$, <u>such that</u> G_Δ <u>is expressible as</u>

$$G_\Delta = \bigcup_{\sigma\in\Delta} \mathbb{O}^\sigma \qquad (\text{disjoint union}).$$

<u>Furthermore, the closure</u> $D(\sigma)$ <u>of</u> $\mathbb{O}^\sigma$ <u>in</u> G_Δ <u>is an irreducible nonsingular</u> (n-i)-<u>dimensional</u> G-<u>stable subvariety of</u> G_Δ <u>written in the form</u>

$$D(\sigma) = \bigcup_{\tau \geqq \sigma} \mathbb{O}^\tau \qquad (\text{disjoint union}).$$

(b) <u>For each</u> $\sigma \in \Delta(n)$, $U_\sigma := \bigcup_{\tau \leqq \sigma} \mathbb{O}^\tau$ <u>forms an affine open</u> G-<u>stable neighbourhood of</u> $\mathbb{O}^\sigma$ <u>in</u> G_Δ. <u>satisfying the conditions</u>

$$G \subsetneqq U_\sigma \cong \mathbb{A}^n(\mathbb{C})$$

<u>and</u>

$$G_\Delta = \bigcup_{\sigma\in\Delta(n)} U_\sigma .$$

<u>Let</u> $\{\mathbb{b}(\sigma)^1, \mathbb{b}(\sigma)^2, \ldots, \mathbb{b}(\sigma)^n\}$ <u>be the set of fundamental generators of</u> σ (<u>which forms a</u> $\mathbb{Z}$-<u>basis for</u> N), <u>and let</u> $\{\mathbb{a}(\sigma)^1, \mathbb{a}(\sigma)^2, \ldots, \mathbb{a}(\sigma)^n\}$ <u>be the dual basis for</u> M <u>defined by the relation</u> $(\mathbb{a}(\sigma)^i, \mathbb{b}(\sigma)^j) = \delta_{ij}$. <u>Then the corresponding characters</u>

$$\chi_{\sigma;i} := \chi^{\mathbb{a}(\sigma)^i} \in \mathrm{Hom}_{\mathrm{alg\ gp}}(G, \mathbb{G}_m), \quad 1 \leqq i \leqq n,$$

<u>extend to rational functions on</u> G_Δ , <u>which are all regular</u>

on U_σ, forming a system of coordinate functions on U_σ by the isomorphism

$$U_\sigma \cong \mathbb{A}^n(\mathbb{C})$$
$$u \longmapsto (\chi_{\sigma;1}(u), \chi_{\sigma;2}(u), \ldots, \chi_{\sigma;n}(u)).$$

In terms of these coordinates, the G-action on U_σ is described by

$$(\chi_{\sigma;1}(g\cdot u), \chi_{\sigma;2}(g\cdot u), \ldots, \chi_{\sigma;n}(g\cdot u))$$
$$= (\chi_{\sigma;1}(g)\cdot\chi_{\sigma;1}(u), \chi_{\sigma;2}(g)\cdot\chi_{\sigma;2}(u), \ldots, \chi_{\sigma;n}(g)\cdot\chi_{\sigma;n}(u)),$$

where both $g \in G$ and $u \in U_\sigma$ are arbitrary.

(1.5) THEOREM: Every n-dimensional irreducible nonsingular complete variety endowed with an effective regular G-action is G-equivariantly isomorphic to G_Δ for some nonsingular finite polyhedral decomposition Δ of N.

Finally, we remark the following:

(1.6) In terms of the holomorphic coordinates $(t_1, t_2, \ldots, t_n)$ for $G = \{(t_1, \ldots, t_n) \mid t_i \in \mathbb{C}^*\}$, the G-invariant vector fields

$$t_i\partial/\partial t_i, \quad i = 1,2,\ldots,n,$$

on G form a $\mathbb{C}$-basis for Lie(G). Furthermore, these naturally extend to holomorphic vector fields on G_Δ.

2. DEMAZURE'S RESULTS ON TORIC VARIETIES.

Throughout this section, we fix a nonsingular finite polyhedral decomposition Δ of N. Put $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$. Furthermore, for each $\rho \in \Delta(1)$, let b_ρ denote the unique fundamental generator of ρ. We now consider the divisor

$$K := -\sum_{\rho \in \Delta(1)} D(\rho)$$

on G_Δ. Recall the following fact due to Demazure [6]:

(2.1) THEOREM: K *is a canonical divisor of* G_Δ. *Moreover, the following are equivalent*:

(a) G_Δ *is a toric Fano variety*.

(b) $-K$ *is ample*.

(c) $-K$ *is very ample*.

(d) $\Sigma_{-K} := \{ a \in M_{\mathbb{R}} \mid (a, b_\rho) \leq 1$ *for all* $\rho \in \Delta(1)\}$ *is an n-dimensional compact convex polyhedron whose vertices are exactly* $\{a_\tau \mid \tau \in \Delta(n)\}$, *where each* a_τ *denotes the unique element of* M *such that* $(a_\tau, b) = 1$ *for all fundamental generators* b *of* τ.

(2.2) REMARK: It is easily seen that $\mathbb{P}^2(\mathbb{C})$, $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$, S_i $(1 \leq i \leq 3)$ are the only possible 2-dimensional nonsingular toric Fano varieties. Recently, for dimension three also, all nonsingular toric Fano varieties are completely classified (cf. Batyrev [4], K. Watanabe and M. Watanabe [23]).

(2.3) DEFINITION (Demazure [6; p.571]): An element a of M is called a *root* if there exists $\rho \in \Delta(1)$ such that $(a, b_\rho) = 1$ and that $(a, b_\sigma) \leq 0$ for all $\sigma \in \Delta(1)$ with $\sigma \neq \rho$. Let $R(\Delta)$ be the set of all roots in M.

Now, as an immediate consequence of a result of Demazure [6; p. 581], one obtains:

(2.4) THEOREM: <u>Let</u> $Aut(G_\Delta)$ <u>be the group of all holomorphic automorphisms of</u> G_Δ . <u>Then</u> $Aut(G_\Delta)$ <u>is a reductive algebraic group if and only if</u> $-R(\Delta) := \{-a \mid a \in R(\Delta)\}$ <u>coincides with</u> $R(\Delta)$.

(2.5) REMARK: In view of this theorem and (2.2), it is now possible to determine all 3-dimensional nonsingular toric Fano varieties G_Δ with reductive $Aut(G_\Delta)$. Such a G_Δ is, actually, isomorphic to one of the following (we owe the computation to Dr. T. Ashikaga):

$$\mathbb{P}^3(\mathbb{C}),\ \mathbb{P}^2(\mathbb{C})\times\mathbb{P}^1(\mathbb{C}),\ \mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C}),$$
$$\mathbb{P}^1(\mathbb{C})\times S_3,\ \mathbb{P}(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,-1)),\ F_1^5\ ,$$

where we used the notation of K. Watanabe and M. Watanabe [23]. Obviously, the first three varieties admit an Einstein-Kähler form. Note that, for the last three varieties, $Aut(G_\Delta)$ cannot act transitively on G_Δ . However, $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,-1))$ still admits an Einstein-Kähler form by virtue of a result of Sakane [22], partly because in this case, every maximal compact subgroup of $Aut(G_\Delta)$ acts on G_Δ with principal orbits of real codimension one (cf. Appendix D).

The importance of (2.4) comes from the following theorem in differential geometry due to Matsushima [17]:

(2.6) THEOREM: <u>Let</u> Y <u>be a compact complex connected manifold with</u> $\dim_{\mathbb{C}} Aut^0(Y) > 0$ (<u>where</u> $Aut^0(Y)$ <u>denotes the identity component of the group</u> $Aut(Y)$ <u>of holomorphic automorphisms of</u> Y). <u>If</u> Y <u>admits an Einstein-Kähler form, then</u> $Aut(Y)$ <u>is a reductive alge-</u>

<u>braic group and furthermore, the group of holomorphic isometries in</u> $\mathrm{Aut}^{o}(Y)$ <u>is a maximal compact subgroup of</u> $\mathrm{Aut}^{o}(Y)$.

3. EINSTEIN EQUATIONS.

For X as in Introduction, there exists a nonsingular finite polyhedral decomposition Δ of N such that $X = G_\Delta$ and that Δ satisfies the condition (d) of (2.1) (see (1.5) and (2.1)). In view of the inclusion

$$\{(t_1, \ldots, t_n) \mid t_i \in \mathbb{C}^*\} = G \subset G_\Delta ,$$

we may regard each t_i as a rational function on G_Δ. Consider the real-valued C^∞ functions $x_1, x_2, \ldots, x_n$ on G defined by

$$(*) \qquad t_i \bar{t}_i = |t_i|^2 = \exp(-x_i), \qquad 1 \leqq i \leqq n.$$

Since $\partial t_i = dt_i$, we have $\partial x_i = -dt_i/t_i$ and $\bar{\partial} x_i = -d\bar{t}_i/\bar{t}_i$. Therefore, for each C^∞ function $u = u(x_1, \ldots, x_n)$ defined on $\mathbb{R}^n = \{(x_1, \ldots, x_n) \mid x_i \in \mathbb{R}\}$, the following identity holds:

$$(3.1) \qquad \partial\bar{\partial} u = \sum_{i,j} (\partial^2 u/\partial x_i \partial x_j)(dt_i/t_i) \wedge (d\bar{t}_j/\bar{t}_j).$$

Let G_c be the maximal compact subgroup

$$\{(t_1, \ldots, t_n) \in (\mathbb{C}^*)^n \mid |t_i| = 1\} \ (\cong (S^1)^n)$$

of G. Since the anti-canonical bundle K_X^{-1} of X is ample, there exists a G_c-invariant fibre metric Ω for K_X^{-1} such that the corresponding first Chern form is a positive definite (1,1)-form. Namely, there exists a real-valued C^∞ function $u = u(x_1, \ldots, x_n)$ on $\mathbb{R}^n$ such that:

(3.2) $\exp(-u) \prod_{i=1}^n (\sqrt{-1}\, dt_i \wedge d\bar{t}_i / |t_i|^2)$ extends to a volume form on the whole $X = G_\Delta$;

(3.3) $\sqrt{-1}\, \partial\bar{\partial} u$ extends to a Kähler form on G_Δ.

Note that the volume form in (3.2) is naturally identified with Ω above (and is denoted by the same Ω). In view of (3.1), the statement (3.3) in particular implies:

(3.4) At each point of $\mathbb{R}^n$, the matrix $(\partial^2 u/\partial x_i \partial x_j)$ is positive definite.

Suppose now that X admits an Einstein-Kähler form $\omega \in \mathcal{K}$. Then by Theorem (2.6), we may assume that ω is G_c-invariant. Applying the above argument to $\Omega = \omega^n$, we obtain a real-valued C^∞ function $u = u(x_1, \ldots, x_n)$ on $\mathbb{R}^n$ satisfying the conditions (3.2), (3.4) and furthermore, by $\mathrm{Ric}(\omega) = \omega$,

(3.5) $\det(\partial^2 u/\partial x_i \partial x_j) = \exp(-u)$ on $\mathbb{R}^n$.

Conversely, suppose that a real-valued C^∞ function u on $\mathbb{R}^n$ satisfies (3.2), (3.4) and (3.5), where we return to our original situation that X $(= G_\Delta)$ is just a nonsingular n-dimensional toric Fano variety without any assumption of the existence of Einstein-Kähler forms. Then $\omega := \sqrt{-1}\,\partial\bar{\partial} u$ is still shown to be an Einstein-Kähler form on X. We now define:

(3.6) DEFINITION: The equation (3.5) above (together with the "boundary" condition (3.2) and the convexity (3.4) for u) is called the <u>Einstein equation</u> for the toric Fano variety $X = G_\Delta$.

4. MOMENT MAPS ON TORIC VARIETIES.

Fix a nonsingular finite polyhedral decomposition Δ of N. In this section, we study the moment map (cf. Atiyah [1], Guillemin and Steinberg [11]) of the toric variety G_Δ in terms of a suitable Kähler metric, if any, on G_Δ.

(4.1) We first assume that G_Δ is a (toric) Fano variety. Then in view of Section 3, there exists a real-valued C^∞ function u on $\mathbb{R}^n$ satisfying (3.2) and (3.3). Now, by the relation (*) of that section, we write each x_i as $x_i(\mathbf{t})$ with $\mathbf{t} = (t_1, \ldots, t_n) \in G$. Hence, every C^∞ function $f = f(x_1, \ldots, x_n)$ on $\mathbb{R}^n$ is regarded as a C^∞ function on G by setting $f(\mathbf{t}) := f(x_1(\mathbf{t}), \ldots, x_n(\mathbf{t}))$ for $\mathbf{t} \in G$. Recall that $M_\mathbb{R}$ is naturally identified with $\mathbb{R}^n$ (cf. Section 1). We now define the mapping $m_u : G \to M_\mathbb{R}(=\mathbb{R}^n)$ by

$$m_u(\mathbf{t}) := ((\partial u/\partial x_1)(\mathbf{t}), \ldots, (\partial u/\partial x_n)(\mathbf{t})), \quad \mathbf{t} \in G.$$

Then the work of Atiyah [1] is reformulated in the following slightly stronger form:

(4.2) THEOREM*): _Assume that G_Δ is a nonsingular toric Fano variety. Let Q be the closure of the image $m_u(G)$ in $M_\mathbb{R}$. Then $Q = \Sigma_{-K}$ (cf. (2.1)). Furthermore, $m_u : G \to M_\mathbb{R}$ continuously extends to a C^∞ map $\bar{m}_u : G_\Delta \to M_\mathbb{R}$. This $\bar{m}_u$ satisfies_

(a) _the inverse image $\bar{m}_u^{-1}(\sigma)$ of each open face σ of Σ_{-K} is a single G-orbit;_

(b) _$\bar{m}_u$ induces a diffeomorphism (including boundaries) between manifolds G_Δ/G_c and Σ_{-K} with corners._

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*) A more general statement will be proven in (8.2).

(4.3) REMARK: (i) It is easily checked that $\bar{m}_u$ above coincides with the moment map: $G_\Delta \to \mathrm{Lie}(G_c)^* \cong M_{\mathbb{R}}$ (cf. Atiyah [1], Guillemin and Steinberg [11]) associated with the Kähler form $\sqrt{-1}\,\partial\bar\partial u \in \mathcal{K}$. (See Appendix B for the proof.)

(ii) Consider the subgroup $G_{\mathbb{R}} := \{(t_1, \ldots, t_n) \in G \mid t_i \in \mathbb{R}_+\} (\cong (\mathbb{R}_+)^n)$ of G. Then by the natural inclusions $G_{\mathbb{R}} \subset G \subset G_\Delta$, we may regard $G_{\mathbb{R}}$ as a subset of G_Δ. Then the closure $\bar{G}_{\mathbb{R}}$ of $G_{\mathbb{R}}$ in G_Δ is a <u>manifold with corners</u> in the sense of Borel-Serre (cf. Oda [20]) and has a natural differentiable structure as described in Step 3 of (8.2). Note that G_Δ/G_c above is endowed with such a structure via the natural identification of G_Δ/G_c with $\bar{G}_{\mathbb{R}}$.

(iii) A difference of (4.2) from Atiyah's result [1; Theorem 2] is that the mapping between G_Δ/G_c and Q is, in our case, a diffeomorphism (instead of a homeomorphism) even along their boundaries. This diffeomorphism is essentially obtained from the ampleness of $K_{G_\Delta}^{-1}$ by the fact that a combination of (3.2) and (3.3) keeps the Jacobian of $\bar{m}_u|_{\bar{G}_{\mathbb{R}}} : \bar{G}_{\mathbb{R}} \to M_{\mathbb{R}}$ nonvanishing also along the boudary $\bar{G}_{\mathbb{R}} - G_{\mathbb{R}}$.

(4.4) We now assume that G_Δ is a projective variety (where G_Δ is not necessarily a Fano variety). Note that the corresponding hyperplane bundle $L := \mathcal{O}_{G_\Delta}(1)$ is written as $\mathcal{O}_{G_\Delta}(\sum_{\sigma \in \Delta(1)} \nu_\sigma D(\sigma))$ for some $\nu_\sigma \in \mathbb{Z}_0$. Then

$$\Sigma_L := \left\{ a \in M_{\mathbb{R}} \;\middle|\; (a, b_\sigma) \leq \nu_\sigma \text{ for all } \sigma \in \Delta(1) \right\}$$

is an n-dimensional compact convex polyhedron (cf. Oda [21]). Since L is ample, there exists a G_c-invariant fibre metric h for L such that the corresponding first Chern form is positive definite.

Therefore, we obtain a real valued C^∞ function u on $\mathbb{R}^n$ satisfying the condition (3.3) and also

$$h|_G = \exp(-u)\xi^*\otimes\bar{\xi}^* ,$$

where ξ denotes the unique holomorphic section to L over Y identified, over G, with the trivial section of constant value 1 in $\mathcal{O}_G$ via the natural isomorphism $\mathcal{O}_{G_\Delta}(\sum_{\sigma\in\Delta(1)}\nu_\sigma D(\sigma))|_G \cong \mathcal{O}_G$. Then by exactly the same formula as in (4.1), we have a mapping $m_{u,L} : G \to M_{\mathbb{R}}$ (we put L as a subscript to emphasize the line bundle L). Now, in Theorem (4.2), replace the assumption of ampleness of $K_{G_\Delta}^{-1}$ by that of L. Then (4.2) is still valid when we further replace m_u , $\bar{m}_u$, Σ_{-K} , respectively by $m_{u,L}$, $\bar{m}_{u,L}$, Σ_L (cf. (8.2)).

5. FUTAKI INVARIANTS FOR TORIC VARIETIES.

In [10], Futaki introduced an obstruction to the existence of Einstein-Kähler forms as follows: Let Y be a compact connected complex manifold and ω be a Kähler form on Y, if any, in the cohomology class $2\pi c_1(Y)_{\mathbb{R}}$. Note that the space $\mathfrak{X}(Y)$ of all holomorphic vector fields on Y forms a Lie algebra. Then a fundamental theorem of Futaki [10] states the following:

(5.1) THEOREM: Let f_ω be the real-valued C^∞ function on Y defined uniquely, up to constant, by $\mathrm{Ric}(\omega) - \omega = \sqrt{-1}\,\bar\partial\partial f_\omega$. Put $c := ((2\pi c_1(Y))^n[Y])^{-1}$, where $n = \dim_{\mathbb{C}} Y$. We further define a linear map $F = F_Y : \mathfrak{X}(Y) \to \mathbb{R}$ by

$$F(V) := c\int_Y (Vf_\omega)\omega^n, \qquad V \in \mathfrak{X}(Y).$$

Then this map F does not depend on the choice of ω. Moreover,

(a) F is trivial on $[\mathfrak{X}(Y), \mathfrak{X}(Y)]$.

(b) If Y admits an Einstein-Kähler form, then F is trivial.

In order to compute this F for toric varieties, we introduce the following quantities:

(5.2) DEFINITION: Let Δ be a nonsingular finite polyhedral decomposition of N. If G_Δ is a Fano variety (resp. a projective variety with its hyperplane bundle L), then we define an element a_Δ (resp. $a_{\Delta,L}$) of $M_{\mathbb{R}}$ as the barycentre of the polyhedron Σ_{-K} (resp. Σ_L). Namely, the i-th component of the vector a_Δ (resp. $a_{\Delta,L}$) in the vector space $M_{\mathbb{R}}$ $(= \mathbb{R}^n)$ is

$$\int_{\Sigma_{-K}} x_i\, dx_1\wedge dx_2\wedge \dots \wedge dx_n \Big/ \int_{\Sigma_{-K}} dx_1\wedge dx_2\wedge \dots \wedge dx_n\,,$$

$$(\text{resp. } \int_{\Sigma_L} x_i\, dx_1\wedge dx_2\wedge \dots \wedge dx_n \Big/ \int_{\Sigma_L} dx_1\wedge dx_2\wedge \dots \wedge dx_n),$$

where $(x_1, x_2, \dots, x_n)$ is the system of standard coordinates of $M_{\mathbb{R}}$ $(= \mathbb{R}^n)$. Obviously, a_Δ (resp. $a_{\Delta,L}$) is in $M_{\mathbb{Q}} := M \otimes_{\mathbb{Z}} \mathbb{Q}$.

For toric Fano varieties, we can deduce from (4.2) the following simple formula:

(5.3) THEOREM: Let G_Δ be a nonsingular toric Fano variety. In terms of the notation of (1.6) and (5.1), we put $\tilde{a}_i := F(t_i \partial/\partial t_i)$ for each $i = 1, 2, \dots, n$. Then

$$a_\Delta = (\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n).$$

(5.4) REMARK: (i) In Appendix C, we shall prove a more general version of (5.3) above (cf. (9.2.3)).

(ii) We identify each element $a = (a_1, a_2, \dots, a_n)$ of $M_{\mathbb{R}}$ with $\sum_{i=1}^n a_i dt_i/t_i \in \mathrm{Lie}(G)^*$. Then Theorem (5.3) shows that, for any nonsingular toric Fano variety G_Δ, the restriction $F|_{\mathrm{Lie}(G)}$ of $F : \mathfrak{X}(G_\Delta) \to \mathbb{R}$ to $\mathrm{Lie}(G)$ coincides with a_Δ .

In view of (5.3) and (5.4), we call the element a_Δ of $M_{\mathbb{R}}$ the Futaki invariant of the toric Fano variety G_Δ . Now, (a) of (5.1) together with (5.3) implies

(5.5) COROLLARY: Let G be a nonsingular toric Fano variety such that $\mathrm{Aut}(G_\Delta)$ is reductive. Then $F : \mathfrak{X}(G_\Delta) \to \mathbb{R}$ is trivial if and only if $a_\Delta = 0$.

Finally, note the following:

(5.6) REMARK: Suppose that G_Δ is a nonsingular projective variety with the corresponding very ample line bundle L (where G_Δ is not necessarily a Fano variety). Even in this case, we have a theorem similar to (5.3). Actually, $a_{\Delta,L}$ coincides with

$$((2\pi c_1(L))^n[G_\Delta])^{-1}(r_L)_*\big|_{\mathrm{Lie}(G)}$$

in terms of the notation in Appendix A (see also (9.2.4)).

6. CONCLUDING REMARKS.

A finite polyhedral decomposition Δ of N is called <u>canonically symmetric</u> if the following conditions are satisfied:

(i) Δ is nonsingular;

(ii) Δ has the property (d) of (2.1);

(iii) $-R(\Delta) = R(\Delta)$;

(iv) $a_\Delta = 0$.

Now, combining (1.5), (2.1), (2.4), (2.6), (b) of (5.1), (5.5), we obtain:

(6.1) THEOREM: *Let X be as in Introduction. If X admits an Einstein-Kähler form, then there exists a canonically symmetric finite polyhedral decomposition Δ of N such that X is G-equivariantly isomorphic to G_Δ.*

In view of this theorem, (0.1) in Introduction is divided into the following two problems:

(6.2) PROBLEM: *Classify all canonically symmetric finite polyhedral decompositions of N (up to isomorphism).*

(6.3) PROBLEM: *Let Δ be a canonically symmetric finite polyhedral decomposition of N. Then does G_Δ admit an Einstein-Kähler metric?*

For (6.2), if $n \geqq 4$, no definitive results are known so far. In the case $n \leqq 3$, we can classify all canonically symmetric finite polyhedral decompositions Δ of N. Namely, the corresponding G_Δ is one of the following:

(a) For n = 1: $\mathbb{P}^1(\mathbb{C})$.

(b) For n = 2: $\mathbb{P}^2(\mathbb{C})$, $\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})$, S_3 .

(c) For n = 3: $\mathbb{P}^3(\mathbb{C})$, $\mathbb{P}^2(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})$, $\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})$, $\mathbb{P}^1(\mathbb{C})\times S_3$,
$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,-1))$.

If n = 3, for instance, this classification easily follows from (2.5), since we can eliminate the possibility of F_1^5 as follows: Let $\mathbb{b}'$, $\mathbb{b}''$, $\mathbb{b}^{(k)}$ $(0 \leqq k \leqq 6)$ be vectors in $N(= \mathbb{R}^3)$ defined as

$$\mathbb{b}' = \begin{pmatrix}1\\0\\0\end{pmatrix},\ \mathbb{b}'' = \begin{pmatrix}-1\\0\\-1\end{pmatrix},\ \mathbb{b}^{(0)} = \mathbb{b}^{(6)} = \begin{pmatrix}0\\1\\0\end{pmatrix},\ \mathbb{b}^{(1)} = \begin{pmatrix}0\\0\\1\end{pmatrix},$$

$$\mathbb{b}^{(2)} = \begin{pmatrix}0\\-1\\1\end{pmatrix},\ \mathbb{b}^{(3)} = \begin{pmatrix}0\\-1\\0\end{pmatrix},\ \mathbb{b}^{(4)} = \begin{pmatrix}0\\0\\-1\end{pmatrix},\ \mathbb{b}^{(5)} = \begin{pmatrix}0\\1\\-1\end{pmatrix}.$$

In terms of these vectors, Δ for F_1^5 is characterized by

$$\Delta(3) = \left\{ \mathbb{Z}_0\mathbb{b}'+\mathbb{Z}_0\mathbb{b}^{(k-1)}+\mathbb{Z}_0\mathbb{b}^{(k)},\ \mathbb{Z}_0\mathbb{b}''+\mathbb{Z}_0\mathbb{b}^{(k-1)}+\mathbb{Z}_0\mathbb{b}^{(k)} \,\middle|\, 1 \leqq k \leqq 6 \right\},$$

and hence the associated compact convex polyhedron Σ_{-K} has exactly 12 vertices:

(1,1,1), (1,0,1), (1,-1,0), (1,-1,-1), (1,0,-1), (1,1,0),
(-2,1,1), (-2,0,1), (-1,-1,0), (0,-1,-1), (0,0,-1), (-1,1,0).

It then follows that $a_\Delta \neq 0$.

For (6.3), we have some results on S_3 and $\mathbb{P}^1(\mathbb{C})\times S_3$ (cf. [7]) by the method of Section 3, though we do not go into details.

7. APPENDIX A.

We here fix, once for all, a holomorphic line bundle L over a d-dimensional compact complex connected manifold Y. Assume that a complex Lie subgroup S of Aut(Y) acts holomorphically on L as bundle isomorphisms covering the S-action on Y. (If $L = K_Y^{-1}$, then our S-action on L is always assumed to be the standard one on K_Y^{-1}.) Let H be the set of all C^∞ Hermitian fibre metrics of the line bundle L over Y. For each $h \in H$, we denote by $c_1(L;h)$ the first Chern form $(\sqrt{-1}/2\pi)\,\bar{\partial}\partial \log(h)$ of the metric h. Furthermore, note that S acts on H (from the right) by

$$H \times S \ni (h,\ s) \longmapsto s^*h \in H,$$

where s^*h is defined by $(s^*h)(\ell_1,\ell_2) := h(s(\ell_1),s(\ell_2))$ for all $\ell_1,\ \ell_2 \in L$ in the same fibres of L over Y. Now, to each pair $(h',h'') \in H\times H$, we associate the real number $R_L(h',h'') \in \mathbb{R}$ by

$$R_L(h',h'') := \int_a^b \left(\frac{1}{2}\int_Y h_t^{-1}\frac{\partial h_t}{\partial t}(2\pi c_1(L;h_t))^d\right)dt,$$

$\{h_t \mid a \leqq t \leqq b\}$ being an arbitrary piecewise smooth path in H such that $h_a = h'$ and $h_b = h''$. Then by a result of Donaldson*) applied to the line bundle L, the number $R_L(h',h'')$ above is independent of the choice of the path $\{h_t \mid a \leqq t \leqq b\}$ and therefore well-defined. Moreover, R_L is S-invariant, i.e.,

$$R_L(s^*h',s^*h'') = R_L(h',h'') \quad \text{for all } s \in S \text{ and all } h',\ h'' \in H,$$

and satisfies the 1-cocycle condition, i.e.,

(i) $R_L(h',h'') + R_L(h'',h') = 0$ and

(ii) $R_L(h,h') + R_L(h',h'') + R_L(h'',h) = 0$,

for all $h,\ h',\ h'' \in H$. In particular, the number $R_L(h,s^*h)$

*) See Proposition 6 of S. K. Donaldson's paper "Anti-self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles", Proc. London Math. Soc. 50 (1985), 1-26.

depends only on s and is independent of the choice of $h \in H$.

Now, by setting

$$r_L(s) := \exp(R_L(h, s^*h)), \quad s \in S,$$

one easily obtains (see, for instance, [14 ; §5]):

(7.1) PROPOSITION: $r_L : S \to \mathbb{R}_+$ _is a Lie group homomorphism from_ S _to the multiplicative group_ $\mathbb{R}_+$ _of positive real numbers_.

Let $(r_L)_* : \mathrm{Lie}(S) \to \mathbb{R}$ be the Lie algebra homomorphism associated with r_L , where we always regard Lie(S) as a Lie subalgebra of $\mathfrak{X}(Y)$ (cf. §5). For each holomorphic vector field $V \in \mathfrak{X}(Y)$, we denote by $V_{\mathbb{R}}$ the corresponding real vector field $V + \bar{V}$ on Y. Then,

(7.2) PROPOSITION: (i) _Let_ D ($\subsetneq$ Y) _be an_ S-_stable closed analytic subset of_ Y. _Suppose there exists an_ S-_invariant holomorphic section_ b _over_ Y - D _to the dual bundle_ L^* _of_ L. _For each_ $h \in H$, _let_ u_h _be the real-valued_ C^∞ _function on_ Y - D _such that_ $h = \exp(-u_h)\, b \otimes \bar{b}$ _on_ Y - D. _Then_

$$(7.2.1) \qquad (r_L)_*(V) = -\frac{1}{2}\int_{Y-D} V_{\mathbb{R}}(u_h)(\sqrt{-1}\,\partial\bar{\partial}u_h)^d$$

for all $h \in H$ _and all_ $V \in \mathrm{Lie}(S)$.

(ii) _Under the same assumption as in_ (i) _above, we consider the case where_ $L = K_Y^{-1}$. _Suppose further that_ L _is ample. Then the restriction_ $F_Y|_{\mathrm{Lie}(S)}$ _of_ F_Y (cf. (5.1)) _to_ Lie(S) _satisfies_

$$(7.2.2) \qquad F_Y|_{\mathrm{Lie}(S)} = ((2\pi c_1(L))^d[Y]\ )^{-1}(r_L)_* .$$

PROOF: Since (7.2.1) is straightforward from the definition of R_L , it suffices to show (7.2.2). From the assumption of ampleness of L, there exists a metric $h \in H$ for $L = K_Y^{-1}$ such that $\omega := \sqrt{-1}\,\partial\bar{\partial}u_h$ extends to a Kähler form on Y in the cohomology class $2\pi c_1(Y)_{\mathbb{R}}$. Put $\Omega := (\sqrt{-1})^d(-1)^{d(d-1)/2}\exp(-u_h)\, b \wedge \bar{b}$.

Then Ω is a volume form on Y satisfying

$$\mathrm{Ric}(\omega) - \omega = \sqrt{-1}\,\partial\bar{\partial} f,$$

where $f := \log(\Omega/\omega^d)$. In view of $\omega^d = \exp(-f)\Omega$, we obtain

$$\begin{aligned} 0 &= -\int_Y (\text{Lie deriv. of } \exp(-f)\Omega \text{ w.r.t. } V_{\mathbb{R}}) \\ &= \int_Y V_{\mathbb{R}}(f)\,\omega^d - \int_Y \exp(-f)(\text{Lie deriv. of } \Omega \text{ w.r.t. } V_{\mathbb{R}}) \\ &= \int_Y V_{\mathbb{R}}(f)\,\omega^d + \int_Y V_{\mathbb{R}}(u_h)\,\omega^d = 2\int_Y V(f)\,\omega^d + \int_Y V_{\mathbb{R}}(u_h)\,\omega^d . \end{aligned}$$

This together with (7.2.1) implies (7.2.2).

(7.3) REMARK: In a forthcoming paper (cf. Bando and Mabuchi [3]), we shall give a little more systematic treatment of (7.2) above.

(7.4) REMARK: In view of the definition of R_L , it is easy to extend the formula (7.2.1) to the following slightly general case:

FACT: Let D, b, h, u_h be the same as in (i) of (7.2). We further assume that there exists an S-invariant morphism $\xi : Y \to W$ of Y into a complex manifold W. Fix an arbitrary line bundle L' on W and let h' be a C^∞ Hermitian metric for L'. Put $L'' := \xi^* L' \otimes L$. Then for all $h \in H$ and all $V \in \mathrm{Lie}(S)$, we have:

(7.4.1) $$(r_{L''})_*(V) = -\frac{1}{2}\int_{Y-D} V_{\mathbb{R}}(u_h)(\sqrt{-1}\,\partial\bar{\partial} u_h + 2\pi \xi^* c_1(L',h'))^d .$$

(7.5) REMARK: We here denote $(r_L)_*$ by $(r_{L,Y})_*$ to emphasize the base space Y. Furthermore, assume that there exists a surjective S-equivariant morphism $\lambda : \tilde{Y} \to Y$ from a compact complex connected manifold $\tilde{Y}$ endowed with a holomorphic S-action. Put $\tilde{L} := \lambda^* L$. Note that the S-action on L naturally induces the one on $\tilde{L}$. Then obviously,

(7.5.1) $$(r_{\tilde{L},\tilde{Y}})_* = (\deg \lambda)(r_{L,Y})_* .$$

8. APPENDIX B.

The purpose of this appendix is to prove a relative version of (4.2) and (4.4). Let G (resp. G_c) be as in Section 1 (resp. 3), and P be a holomorphic principal bundle over a complex connected manifold W with structure group G. (Recall that, by standard definition, G acts on P from the right.) In our case, however, G acts on P from the left by

$$G \times P \ni (g, p) \longmapsto g \cdot p := p \cdot g \in P.$$

(Since G is abelian, there is no essential difference between left and right G-actions.) Note that P is locally trivial, i.e., W is written as a union of its open neighbourhoods W_α, $\alpha \in A$, such that for each α, we have a G-equivariant isomorphism

$$\tau_\alpha : P|_{W_\alpha} \cong W_\alpha \times G.$$

Let $pr_2 : W_\alpha \times G \to G$ be the natural projection to the second factor and write G as $\{(t_1, \ldots, t_n) \mid t_i \in \mathbb{C}^*\}$ (cf. Section 1).

(8.1) Let Y be a complex manifold with an effective holomorphic G-action containing P as a G-stable Zariski-open dense subset. We further assume that there exists a G-invariant morphism $\zeta : Y \to W$ satisfying the following conditions:

(8.1.1) The restriction $\zeta|_P : P \to W$ coincides with the original principal bundle P over W;

(8.1.2) $P_w := (\zeta|_P)^{-1}(w)$ is Zariski-open and dense in $Y_w := \zeta^{-1}(w)$ for each $w \in W$;

(8.1.3) f is a projective morphism with the corresponding f-very ample line bundle $L := \mathcal{O}_Y(1) \in \mathrm{Pic}(Y)$;

(8.1.4) L is expressible as $\mathcal{O}_Y(D)$ for some effective divisor D on Y with $\mathrm{Supp}(D) \subset Y - P$.

We first observe that the G-action on Y naturally lifts to a linear G-action on the line bundle L such that the following holds:

(8.1.5) Let ξ be the holomorphic section*) to L over Y which is identified, over P, with the trivial section of constant value 1 in $\mathcal{O}_P$ via the natural isomorphism $\mathcal{O}_Y(D)|_P \cong \mathcal{O}_P$. Then G acts identically on ξ .

Note also that the cohomology class $2\pi\, c_1(L)_{\mathbb{R}}$ is represented by a G_c-invariant C^∞ (1,1)-form ω on Y such that the pullback of ω to Y_w , denoted by ω_w , is a Kähler form on Y_w for each $w \in W$. Then there exists a G_c-invariant Hermitian C^∞ metric h for L satisfying

(8.1.6) $h|_P = \exp(-u)\, \xi^* \otimes \bar{\xi}^*$, and

(8.1.7) $\omega|_P = \sqrt{-1}\, \partial\bar{\partial} u$

for some G_c-invariant C^∞ function u on P. We shall now define $\mathbf{m}: P \to M_{\mathbb{R}}$, $\Delta = \Delta_w$, $\Sigma = \Sigma_w$ $(w \in W)$ as follows: For each $\alpha \in A$, put

$$t_i^{(\alpha)} := (\mathrm{pr}_2 \circ \iota_\alpha)^* (t_i), \qquad 1 \leqq i \leqq n,$$

and consider the real-valued C^∞ functions $x_1^{(\alpha)}, x_2^{(\alpha)}, \ldots , x_n^{(\alpha)}$ on $P|_{W_\alpha}$ defined by

$$t_i^{(\alpha)} \bar{t}_i^{(\alpha)} = |t_i^{(\alpha)}|^2 = \exp(-x_i^{(\alpha)}), \qquad 1 \leqq i \leqq n.$$

---

*) This section ξ vanishes along Supp(D) so that zero(ξ) = D.

Now, on $P|_{W_\alpha}$, u above is regarded as a function $u(w, x_1^{(\alpha)}, \ldots, x_n^{(\alpha)})$ in $w, x_1^{(\alpha)}, \ldots, x_n^{(\alpha)}$. By the same argument as in Section 3,

(8.1.8) $\quad \partial\bar{\partial} u_w = \sum_{i,j} (\partial^2 u/\partial x_i^{(\alpha)} \partial x_j^{(\alpha)}) (dt_i^{(\alpha)}/t_i^{(\alpha)}) \wedge (d\bar{t}_j^{(\alpha)}/\bar{t}_j^{(\alpha)})$ on P_w $(w \in W)$,

where $u_w := u|_{P_w}$. Let $\mathbf{m}^{(\alpha)}: P|_{W_\alpha} \to M_{\mathbb{R}} (= \mathbb{R}^n)$ be the mapping defined by

$$\mathbf{m}^{(\alpha)}(p) := ((\partial u/\partial x_1^{(\alpha)})(p), \ldots, (\partial u/\partial x_n^{(\alpha)})(p)), \qquad p \in P.$$

Then it is easily seen that $\mathbf{m}^{(\alpha)}$, $\alpha \in A$, are glued together defining a global mapping $\mathbf{m}: P \to M_{\mathbb{R}} (= \mathbb{R}^n)$ such that the restriction of $\mathbf{m}$ to each $P|_{W_\alpha}$ coincides with $\mathbf{m}^{(\alpha)}$. Now, let w be an arbitrary point of W and choose an $\alpha \in A$ such that $w \in W_\alpha$. We can then regard Y_w as a nonsingular toric variety by

$$G \ni (t_1^{(\alpha)}(p), \ldots, t_n^{(\alpha)}(p)) \overset{\cong}{\longleftrightarrow} p \in P_w \subset Y_w .$$

Hence, there exists a unique nonsingular finite polyhedral decomposition $\Delta = \Delta_w$ of N such that

(1) Δ can depend only on w and is independent of the choice of α.

(2) $Y_w \cong G_\Delta$ as a toric variety.

Furthermore, $L_w := L|_{Y_w}$ is written in the form

$$L_w = \mathcal{O}_{G_\Delta}\left(\sum_{\rho \in \Delta(1)} \nu_\rho D(\rho)\right) \qquad \text{for some } \nu_\rho\text{'s in } \mathbb{Z}_0 ,$$

via the identification of Y_w with G_Δ. Letting $\mathbf{b}_\rho$ be as in Section 2, we now define an n-dimensional compact convex polyhedron $\Sigma = \Sigma_w$ in $M_{\mathbb{R}}$ by

(8.1.9) $\quad \Sigma := \{\mathbf{a} \in M_{\mathbb{R}} \mid (\mathbf{a}, \mathbf{b}_\rho) \leq \nu_\rho \text{ for all } \rho \in \Delta(1)\}$.

Since L_w is ample, the vertices of Σ are exactly $\{\mathbf{a}_\sigma \mid \sigma \in \Delta(n)\}$, where each $\mathbf{a}_\sigma$ denotes the unique element of M such that $(\mathbf{a}_\sigma, \mathbf{b}_\rho) = \nu_\rho$ for all $\rho \in \Delta(1)$ with $\rho \leq \sigma$ (cf. Oda [21]). Then we have:

(8.2) THEOREM: Let Q be the closure of the image $m(P)$ in $M_{\mathbb{R}}$. Then $Q = \sum_w$ for all $w \in W$. (In particular, $\Sigma = \Sigma_w$ and $\Delta = \Delta_w$ are both independent of w.) Furthermore, $m: P \to M_{\mathbb{R}}$ naturally extends to a C^∞ map $\bar{m} : Y \longrightarrow M_{\mathbb{R}}$. Let w be an arbitrary point of W. Then $\bar{m}$ satisfies

(a) $\bar{m}^{-1}(\sigma) \cap Y_w$ is a single G-orbit for each open face σ of Σ;

(b) $\bar{m}$ induces a diffeomorphism (including boundaries) between manifolds Y_w/G_c and Σ $(= \Sigma_w)$ with corners;

(c) $\bar{m}|_{Y_w} : Y_w \to M_{\mathbb{R}}$ coincides with the mapping $\bar{m}_{u_w, L_w}$ in (4.4) via the identification of Y_w with G_Δ and is just the moment map: $Y_w \to \mathrm{Lie}(G_c)^* (\cong M_{\mathbb{R}})$ associated with the Kähler form ω_w $(= \sqrt{-1}\,\partial\bar{\partial} u_w)$ on Y_w.

(8.2.1) REMARK: Consider the case where W consists of a single point. Then (8.2) above implies (4.4). If we further assume $L = K_Y^{-1}$, then (8.2) shows nothing but (4.2) and (4.3).

PROOF OF (8.2): Step 1. Fix an $\alpha \in A$ such that $w \in W_\alpha$. For simplicity, put $z_i := t_i^{(\alpha)}$ and $x_i := x_i^{(\alpha)}$, $i=1,2,\ldots,n$. Let $0 \leqq \theta_i < 2\pi$ be such that $z_i = \exp((-x_i/2)+\sqrt{-1}\,\theta_i)$. Then $(z_1, \ldots, z_n)$ (resp. $(x_1, \ldots, x_n, \theta_1, \ldots, \theta_n)$) forms a system of holomorphic local coordinates (resp. real local coordinates) of Y_w. Note that

$$(8.2.2) \qquad z_i\,\partial/\partial z_i + \bar{z}_i\,\partial/\partial \bar{z}_i = -\,2\,\partial/\partial x_i\,, \qquad 1 \leqq i \leqq n.$$

We now write the Kähler form ω_w as $\sqrt{-1}\sum_{i,j} u_{i\bar{j}}\, dz_i \wedge d\bar{z}_j$ on P_w, where $u_{i\bar{j}} := \partial_i \partial_{\bar{j}}(u_w)$. Put

$$V_i := t_i\,\partial/\partial t_i \in \mathrm{Lie}(G) \subsetneq \mathfrak{X}(Y), \qquad 1 \leqq i \leqq n,$$

in terms of the coordinates $t_1, \ldots, t_n$ for $G = \{(t_1, \ldots, t_n) \mid t_i \in \mathbb{C}^*\}$. Then there exist real-valued C^∞ functions $\varphi_{w,i}$, $i=1,2, \ldots,n$, on Y_w such that

(8.2.3) $V_i|_{Y_w} = \sum_{j,k} u^{\bar{j}k}(\partial_{\bar{j}} \varphi_{w,i}) \partial/\partial z_k$, $1 \leqq i \leqq n$,

$(u^{\bar{j}k})$ being the inverse matrix of $(u_{i\bar{j}})$ (see, for instance, Kobayashi [12 ; p.94]). On the other hand, by (8.2.2), the real vector field $(V_i)_{\mathbb{R}}$ (cf. Appendix A) is written as

(8.2.4) $(V_i)_{\mathbb{R}} = - 2\, \partial/\partial x_i$, $1 \leqq i \leqq n$,

on Y_w . Now, on P_w , (8.2.3) above implies

(Lie deriv. of ω_w w.r.t. $(V_i)_{\mathbb{R}}$) $= 2\sqrt{-1}\, \partial\bar{\partial}\varphi_{w,i}$.

Moreover, by (8.2.4),

(Lie deriv. of ω_w w.r.t. $(V_i)_{\mathbb{R}}$) $= -2\sqrt{-1}\, \partial\bar{\partial}(\partial u_w/\partial x_i)$.

Therefore, $\partial u_w/\partial x_i = -\varphi_{w,i} + C_{w,i}$ on P_w for some real constant $C_{w,i} \in \mathbb{R}$. Hence $m|_{P_w}$ and $-(\varphi_{w,1}, \ldots, \varphi_{w,n})$ coincide up to translation, which implies the latter half of (c). Since the former half of (c) is obvious, this proves (c).

Step 2. Put $\tilde{\varphi}_{w,i} := -\varphi_{w,i} + C_{w,i}$. Note that, for each i, $\tilde{\varphi}_{w,i}$ depends smoothly on w, because both $\partial\bar{\partial}\tilde{\varphi}_{w,i}$ (= Lie deriv. of $-2^{-1}\omega_w$ w.r.t. $(V_i)_{\mathbb{R}}$) and $\tilde{\varphi}_{w,i}|_{P_w}$ $(= \partial u_w/\partial x_i)$ depend smoothly on w. We then have a natural extension of m to a C^∞ mapping $\bar{m} : Y \to M_{\mathbb{R}}$ by setting, for each fibre Y_w $(w \in W)$, as follows:

$$\bar{m}(y) := (\tilde{\varphi}_{w,1}(y), \ldots, \tilde{\varphi}_{w,n}(y)), \quad y \in Y_w .$$

Let Q_w be the image $\bar{m}(Y_w)$ of Y_w under this mapping $\bar{m}$. Then by a result of Atiyah [1 ; Theorem 2] applied to the compact Kähler manifold (Y_w, ω_w), our Q_w forms a compact convex polyhedron in $M_{\mathbb{R}}$ such that

(a)' $\bar{m}^{-1}(\sigma) \cap Y_w$ is a single G-orbit for each open face σ of Q_w;

(b)' $\bar{m}$ induces a homeomorphism of Y_w/G_c onto Q_w .

(Without using Atiyah's result, we can prove this by modifying the arguments in Steps 3 and 4.) We now observe that $\sum_w$ is an n-dimensional compact convex polyhedron in $M_{\mathbb{R}}$ only with integral vertices $\in M$. Therefore, if $Q_w = \sum_w$ $(w \in W)$, then the C^∞ dependence of $m|_{Y_w}$ on w implies that $\sum_w$ does not depend on w at all. Thus, the proof of (8.2) is reduced to showing the following:

(a)" $Q_w = \sum_w$;

(b)" $\bar{m}$ induces a diffeomorphism (including boundaries) between manifolds Y_w/G_c and Q_w with corners.

<u>Step</u> 3. We may now assume without loss of generality that W consists of a single point. Therefore, we may further assume $P = G$ and $Y = G_\Delta$. Let $G_{\mathbb{R}}$ and $\bar{G}_{\mathbb{R}}$ be the same as in (ii) of (4.3). Then $\bar{G}_{\mathbb{R}}$ is naturally identified with Y/G_c. Note that

$$\bar{G}_{\mathbb{R}} = \bigcup_{\sigma \in \Delta(n)} U_\sigma^{\mathbb{R}}$$

in terms of the notation in (1.4), where $U_\sigma^{\mathbb{R}} := U_\sigma \cap \bar{G}_{\mathbb{R}}$ is a coordinate open subset of $\bar{G}_{\mathbb{R}}$ (diffeomorphically) identified with the product $(\mathbb{R}_0)^n$ of n-copies of $\mathbb{R}_0$ by

$$U_\sigma^{\mathbb{R}} \cong (\mathbb{R}_0)^n , \quad y \mapsto (|\chi_{\sigma;1}(y)|^2, |\chi_{\sigma;2}(y)|^2, \dots , |\chi_{\sigma;n}(y)|^2).$$

Now, fix an arbitrary element σ of $\Delta(n)$. Recall that the real-valued C^∞ functions $x_i = x_i(\mathbf{t})$, $i=1,2,\dots,n$, on G are defined by $|t_i|^2 = \exp(-x_i)$ for $\mathbf{t} = (t_1, \dots , t_n) \in G$. Similarly, to the function $\chi_{\sigma;i} = \chi_{\sigma;i}(\mathbf{t})$, we associate a new function $\tilde{x}_i = \tilde{x}_i(\mathbf{t})$ on G by

$$|\chi_{\sigma;i}(\mathbf{t})|^2 = \exp(-\tilde{x}_i), \qquad \mathbf{t} \in G.$$

Then, in terms of the notation in (1.4), we have

(8.2.5) $\tilde{x}_i = (a(\sigma)^i, x)$, $1 \leqq i \leqq n$,

where

$$x := \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Furthermore, put

$$\rho^i := \mathbb{Z}_0 b(\sigma)^i \in \Delta(1), \quad 1 \leqq i \leqq n.$$

Since $\exp(-u)\xi^* \otimes \bar{\xi}^*$ (cf. (8.1.6)) extends to a C^∞ Hermitian metric for $L = \mathcal{O}_{G_\Delta}(\sum_{\rho \in \Delta(1)} \nu_\rho D(\rho))$, there exists a real-valued C^∞ function $H : (\mathbb{R}_0)^n \to \mathbb{R}$ such that

$$u = \sum_{i=1}^n \nu_i \tilde{x}_i + H(r_1, \ldots, r_n) \quad \text{on } U_\sigma^{\mathbb{R}},$$

where $r_i := |\chi_{\sigma;i}|^2$ $(= \exp(-\tilde{x}_i))$ and $\nu_i := \nu_{\rho_i}$. We can now give a closer study of the function $u = u(x_1, \ldots, x_n) = u(\tilde{x}_1, \ldots, \tilde{x}_n)$. For example, their first and second derivatives with respect to $\tilde{x}_1, \ldots, \tilde{x}_n$ are computed immediately:

(i) $\sum_{i=1}^n (\partial u/\partial x_i)(\partial x_i/\partial \tilde{x}_j) = \partial u/\partial \tilde{x}_j = \nu_j - (\partial H/\partial r_j) r_j$,

(ii) $\partial^2 u/\partial \tilde{x}_i \partial \tilde{x}_j = (\partial^2 H/\partial r_i \partial r_j) r_i r_j + \delta_{ij} (\partial H/\partial r_j) r_j$.

Recall that $(a(\sigma)^i, b(\sigma)^j) = \delta_{ij}$. Hence, combining (i) with (8.2.5), we obtain

(i)' $(\bar{m}, b(\sigma)^j) = \nu_j - (\partial H/\partial r_j) r_j$, $1 \leqq j \leqq n$.

Let p_σ be the point $\in U_\sigma^{\mathbb{R}}$ corresponding to the origin of $(\mathbb{R}_0)^n$ (i.e., $r_1(p_\sigma) = r_2(p_\sigma) = \ldots = r_n(p_\sigma) = 0$). Then by (i)', $(\bar{m}(p_\sigma), b(\sigma)^j) = \nu_j$ for all j. Thus,

(8.2.6) $\bar{m}(p_\sigma) = a_\sigma$.

Now, fix an arbitrary point y of $U_\sigma^{\mathbb{R}}$ and put $I := \{ i \in \{1,2,\ldots,n\} \mid r_i(y) = 0 \}$. Then we may assume without loss of generality that $I = \{1,2,\ldots,q\}$ for some q with $0 \leqq q \leqq n$ (where if $q = 0$, we always assume $I = \phi$). In view of (8.1.7) and (8.1.8),

$$\omega = \sqrt{-1}\sum_{i,j=1}^{n} (\partial^2 u/\partial\tilde{x}_i\partial\tilde{x}_j)(d\chi_{\sigma;i}/\chi_{\sigma;i})\wedge(d\bar{\chi}_{\sigma;j}/\bar{\chi}_{\sigma;j})$$

on U_σ in terms of holomorphic local coordinates $(\chi_{\sigma;1}, \ldots, \chi_{\sigma;n})$. Rewrite this identity, using (ii) above. Then, when evaluated at y,

$$\omega(y) = \sqrt{-1}\sum_{i\in I}(\partial H/\partial r_i)(y)\,d\chi_{\sigma;i}\wedge d\bar{\chi}_{\sigma;i} + \sqrt{-1}\sum_{i,j>q}(\partial^2 u/\partial\tilde{x}_i\partial\tilde{x}_j)(y)(d\chi_{\sigma;i}/\chi_{\sigma;i})\wedge(d\bar{\chi}_{\sigma;j}/\bar{\chi}_{\sigma;j}),$$

where the last summation is taken over all i, $j \in \{1,2,\ldots,n\}$ such that $i > q$ and $j > q$. Since ω is a Kähler form, it follows that:

(8.2.7) $\quad (\partial H/\partial r_i)(y) > 0 \quad$ for all $i \in I$, and

(8.2.8) $\quad ((\partial^2 u/\partial\tilde{x}_i\partial\tilde{x}_j)(y))_{q<i,j\leqq n}$ is a positive definite matrix.

On the other hand, the Jacobian $J(\bar{m})_y$ of the mapping $\bar{m} : U_\sigma^{\mathbb{R}} \to M_{\mathbb{R}}$ at the point y in terms of the coordinates $(r_1, \ldots, r_n)$ for $U_\sigma^{\mathbb{R}}$ is computed as follows:

$$J(\bar{m})_y = \det\left(\frac{\partial(\partial u/\partial x_i)}{\partial r_j}(y)\right)_{1\leqq i,j\leqq n} = \pm\det\left(\frac{\partial(\partial u/\partial\tilde{x}_i)}{\partial r_j}(y)\right)_{1\leqq i,j\leqq n}$$

$$= \pm\det\left(\begin{array}{c|c} \begin{matrix} -(\partial H/\partial r_1)(y) & & 0 \\ & -(\partial H/\partial r_2)(y) & \\ 0 & \ddots & -(\partial H/\partial r_q)(y) \end{matrix} & * \\ \hline 0 & \left(\dfrac{-1}{r_j}\dfrac{\partial^2 u}{\partial\tilde{x}_i\partial\tilde{x}_j}(y)\right)_{q<i,j\leqq n} \end{array}\right),$$

where the last identity follows from

$$\frac{\partial(\partial u/\partial x_i)}{\partial r_j}(y) = -(\partial^2 H/\partial r_i \partial r_j) r_i - \delta_{ij}(\partial H/\partial r_j), \quad (\text{cf. (ii)}).$$

Now, in view of (8.2.7) and (8.2.8), we obtain $J(\bar{m})_y \neq 0$. This together with (b)' (cf. Step 2) yields (b)". Hence, it suffices to show (a)", i.e., $Q = \Sigma$. For each j, let y_j be the point in $U_\sigma^{\mathbb{R}}$ such that $r_i(y_j) = (1 - \delta_{ij}) r_i(y)$, $1 \leqq i \leqq n$. Then by (i)', $(\bar{m}(y_j), b(\sigma)^j) = \nu_j$. On the other hand, by (i), (i)' and (8.2.8),

$$- r_j \frac{\partial(\bar{m}, b(\sigma)^j)}{\partial r_j} \left(= \frac{\partial(\bar{m}, b(\sigma)^j)}{\partial \tilde{x}_j} = \partial^2 u/\partial \tilde{x}_j^{\,2}\right) \geqq 0 \quad \text{on } U_\sigma^{\mathbb{R}}.$$

Therefore, we have

$$(8.2.9) \qquad (\bar{m}(y), b(\sigma)^j) \leqq (\bar{m}(y_j), b(\sigma)^j) = \nu_j, \qquad 1 \leqq j \leqq n.$$

<u>Step</u> 4. In this final step, we complete the proof of $Q = \Sigma$, assuming that W is a single point. Let y be an arbitrary point of $G_{\mathbb{R}}$. Then $y \in U_\sigma^{\mathbb{R}}$ for all $\sigma \in \Delta(n)$. Hence, by (8.2.9), $(\bar{m}(y), b(\sigma)^j) \leqq \nu_j$ for all σ and j, i.e., $\bar{m}(y) \in \Sigma$. Since Q is the closure of $\bar{m}(G_{\mathbb{R}}) (= m(G))$ in $M_{\mathbb{R}}$, we now obtain $Q \subseteq \Sigma$. Recall that Q is a compact convex polyhedron in $M_{\mathbb{R}}$ (cf. Step 2). Therefore, (8.2.6) immediately implies $Q = \Sigma$.

9. APPENDIX C.

In this appendix, by using a measure $d\mu$ of Duistermaat-Heckman's type (cf. [7]), we shall generalize the integral formula of Koiso and Sakane [13] on Futaki invariants. Our present result includes, at the same time, (5.3) and (5.6) in the earlier section as special cases.

(9.1.1) DEFINITION: Let Y be a complex connected manifold endowed with an effective holomorphic G-action, and Δ be a nonsingular finite polyhedral decomposition of N. Furthermore, let $\xi: Y \to W$ be a proper G-invariant morphism of Y onto a connected complex manifold W. Then a pair $(\xi: Y \to W,\ G_\Delta)$ is called a <u>toric bundle</u> if the following conditions are satisfied:

(a) ρ is locally trivial, i.e., W is a union $\bigcup_{\alpha \in A} W_\alpha$ of its open subsets W_α, $\alpha \in A$, such that for each α, there exists a G-equivariant isomorphism $\iota_\alpha : \xi^{-1}(W_\alpha) \cong W_\alpha \times G_\Delta$.

(b) If $\alpha, \beta \in A$ are such that $W_\alpha \cap W_\beta \neq \phi$, then there exists a holomorphic G-valued function $t_{\alpha\beta} = t_{\alpha\beta}(w)$ on $W_\alpha \cap W_\beta$ such that
$$\iota_\alpha \circ \iota_\beta^{-1}(w,\ x) = (w,\ t_{\alpha\beta}(w) \cdot x)$$
for all $w \in W_\alpha \cap W_\beta$ and all $x \in G_\Delta$.

(9.1.2) REMARK: In the above, let $pr_{1,\alpha}: W_\alpha \times G_\Delta \to G_\Delta$ be the natural projection to the second factor. Put $P := \bigcup_{\alpha \in A} (pr_{1,\alpha} \circ \iota_\alpha)^{-1}(G)$. Then $\xi|_P : P \to W$ is naturally regarded as a principal bundle with structure group G.

(9.1.3) DEFINITION: Let $(\xi: Y \to W,\ G_\Delta)$ be a toric bundle and L a line bundle over Y. Then a triple $(\xi: Y \to W,\ G_\Delta,\ L)$ is called a <u>polarized toric bundle</u> if there exists an effective divisor D on Y such that

(a) $L = \mathcal{O}_Y(D)$;

(b) $\mathrm{Supp}(D) \subset Y - P$, where P is as in (9.1.2);

(c) $D|_{Y_w}$ is an ample (or equivalently, very ample) divisor on Y_w for each $w \in W$.

(9.1.4) REMARK: For a polarized toric bundle $(\xi: Y \to W, G_\Delta, L)$, one can easily check that Y, W, P, L, D above always satisfy the conditions (8.1.1)$\sim$(8.1.4) in Appendix B. Conversely, let Y, W, P, L, D be as in Appendix B (satisfying the conditions (8.1.1)$\sim$(8.1.4)). Then by Theorem (8.2), the corresponding $\Delta = \Delta_w$ is independent of w, and it easily follows that the associated triple $(\xi: Y \to W, G_\Delta, L)$ forms a polarized toric bundle.

(9.2) We now fix a polarized toric bundle $(\xi: Y \to W, G_\Delta, L)$. Then for each $\rho \in \Delta(1)$, the subsets $(pr_{1,\alpha} \circ \iota_\alpha)^{-1}(D(\rho))$, $\alpha \in A$, of Y are glued together defining a global prime divisor, denoted by $\tilde{D}(\rho)$, on Y. Hence, the divisor D (cf. (a) of (9.1.3)) is written as $\sum_{\rho \in \Delta(1)} \nu_\rho \tilde{D}(\rho)$ for some ν_ρ's in $\mathbb{Z}_0$. We thus have the corresponding n-dimensional compact convex polyhedron Σ in $M_{\mathbb{R}}$ defined by (8.1.9).

(9.2.1) REMARK: Let a_k, $k=0,1,\ldots,s$, be the integral points in Σ, i.e., $\Sigma \cap M = \{a_k \mid 0 \leqq k \leqq s\}$. Furthermore, put

$$\chi_k := \chi^{-a_k}, \qquad 0 \leqq k \leqq s,$$

where on the right-hand side, we used the notation in Section 1. Then the mapping

$$G \ni t \longmapsto (\chi_0(t) : \chi_1(t) : \ldots : \chi_s(t)) \in \mathbb{P}^s(\mathbb{C})$$

extends to an embedding: $G_\Delta \hookrightarrow \mathbb{P}^s(\mathbb{C})$ such that the corresponding hyperplane bundle on G_Δ is $\mathcal{O}_{G_\Delta}(\sum_{\rho \in \Delta(1)} \nu_\rho D(\rho))$ (cf. Oda [21]).

In particular, the pullback $(= \sqrt{-1}\,\partial\bar\partial \log(\sum_{k=0}^{s} |\chi_k|^2))$ of the Fubini-Study form on $\mathbb{P}^s(\mathbb{C})$ to G_Δ is positive definite everywhere on G_Δ.

(9.2.2) DEFINITION: Since $G = (\mathbb{C}^*)^n$, we can componentwise express $t_{\alpha\beta} = t_{\alpha\beta}(w)$ in (b) of (9.1.1) in the form

$$t_{\alpha\beta}(w) = (\, t^{(1)}_{\alpha\beta}(w),\ t^{(2)}_{\alpha\beta}(w),\ \dots,\ t^{(n)}_{\alpha\beta}(w)\,), \quad w \in W_\alpha \cap W_\beta .$$

Hence for each i, the system of transition functions $\{t^{(i)}_{\alpha\beta}\}_{\alpha,\beta\in A}$ defines a holomorphic line bundle $L^{(i)}$ over W. Let $P^{(i)}$ (:= $L^{(i)}$ - (zero section)) be the $\mathbb{C}^*$-bundle over W corresponding to $L^{(i)}$. Then, in terms of the natural identification

$$P = P^{(1)} \times_W P^{(2)} \times_W \dots \times_W P^{(n)} ,$$

we can write each point p of P as

$$p = (p^{(1)}, p^{(2)}, \dots, p^{(n)})$$

with $p^{(i)} \in P^{(i)}$, $i=1,2,\dots,n$. For each i, fix an arbitrary C^∞ Hermitian metric h_i on $L^{(i)}$ and define a C^∞ function $\tilde{x}_i = \tilde{x}_i(p)$ on P by

$$\exp(-\tilde{x}_i(p)) = h_i(p^{(i)}, p^{(i)}), \qquad p \in P.$$

We shall now show the following formula:

(9.2.3) THEOREM: <u>Put</u> $e := \dim_{\mathbb{C}} W$ <u>and</u> $\gamma_{n,e} := (n+e)!/e!$. <u>Let</u> L' <u>be an arbitrary line bundle over</u> W <u>and put</u> $L'' := \zeta^* L' \otimes L$. <u>We now assume that</u> W <u>is compact</u>. <u>Furthermore, let</u> $x = (x_1, x_2, \dots, x_n)$ <u>be the system of standard coordinates on</u> $M_{\mathbb{R}} (= \mathbb{R}^n)$, <u>and</u> $T = T(x)$ <u>be the polynomial in</u> $x_1, \dots, x_n$ <u>defined by</u> $T(x) := \gamma_{n,e}(c_1(L') + \sum_{j=1}^{n} x_j c_1(L^{(j)}))^e[W]$. <u>Then in terms of the notation in</u> (1.6) <u>and Appendix</u> A, <u>we have</u>:

(a) $(r_{L''})_*(t_i\partial/\partial t_i) = (2\pi)^{n+e}\int_{\Sigma} x_i\, d\mu, \quad 1 \leqq i \leqq n,$

(b) $c_1(L'')^{n+e}[Y] = \int_{\Sigma} d\mu,$

<u>where</u> $d\mu := T(x)dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n$.

(9.2.4) REMARK: In (9.2.3) above, assume that W is a single point. Then by $e = 0$, $T(x)$ is nothing but the constant function 1 on M_R . Hence, (5.6) is straightforward from (9.2.3) above. We further obtain (5.3) by setting $L = K_Y^{-1}$ (see also (7.2.2)).

(9.2.5) REMARK: Note that $d\mu$ is a polynomial measure on M_R . If L is ample on the whole space Y, then this fact is already observed by Duistermaat and Heckman [7] (see especially their formula (1.11)).

PROOF OF (9.2.3): <u>Step</u> 1. Let $u = u(\tilde{x}_1(p), \ldots, \tilde{x}_n(p))$ be the C^∞ function in $\tilde{x}_1 = \tilde{x}_1(p), \ldots, \tilde{x}_n = \tilde{x}_n(p)$ defined by

$$u := \log\left(\sum_{k=0}^{s} \exp(a_k, \tilde{\mathbf{x}}(p))\right),$$

where

$$\tilde{\mathbf{x}}(p) := \begin{pmatrix} \tilde{x}_1(p) \\ \tilde{x}_2(p) \\ \vdots \\ \tilde{x}_n(p) \end{pmatrix} \qquad (p \in P).$$

Let ξ be the holomorphic section to L over Y as in (8.1.5). Then, in view of (9.2.1), the metric $\exp(-u)\xi^* \otimes \bar{\xi}^*$ for $L|_P$ extends to a G_c-invariant C^∞ Hermitian metric, denoted by h, for the whole line bundle L such that the pullback of $c_1(L,h)$ to each fibre Y_w is positive definite. We now have the corresponding $m : P \to M_R$ as in (8.1). Note that, for each $w \in W$, the image $m(P_w)$ is just the interior of Σ. Furthermore, one can easily check that the mapping m is given by

$$m(p) = ((\partial u/\partial \tilde{x}_1)(p), \ldots, (\partial u/\partial \tilde{x}_n)(p)), \qquad p \in P.$$

Step 2. Fix an arbitrary point w' of W, and let U be its sufficiently small neighbourhood in W. Over this U, choose a holomorphic local base s_i for each line bundle $L^{(i)}$ and write $h^{(i)}$ as $f_i(w)\, s_i^* \otimes \bar{s}_i^*$ for some positive C^∞ function $f_i = f_i(w)$ on U. Note that, by a suitable choice of s_i's, we may assume

$$f_i(w') = 1 \quad \text{and} \quad (df_i)(w') = 0 \quad \text{for all } i.$$

We now choose a system $(w_1, \dots, w_e)$ of holomorphic local coordinates on U and write each point w of U as $w = (w_1, \dots, w_e)$ in terms of these coordinates. Then by the isomorphism

$$P|_U (= P^{(1)} \times_W \cdots \times_W P^{(n)}|_U) \cong U \times G$$
$$(t_1 s_1(w), \dots, t_n s_n(w)) \longleftrightarrow (w,\ t=(t_1, \dots, t_n)),$$

we may regard $(w_1, \dots, w_e, t_1, \dots, t_n)$ as a system of holomorphic local coordinates on $P|_U$. Since

$$\partial \tilde{x}_j = -(dt_j/t_j) - \xi^*(\partial f_j/f_j) \quad \text{and} \quad \bar{\partial} \tilde{x}_j = -(d\bar{t}_j/\bar{t}_j) - \xi^*(\bar{\partial} f_j/f_j),$$

the following holds at each point of the fibre $P_{w'}$:

$$\partial\bar{\partial} u = \partial\Big\{\sum_{j=1}^n (\partial u/\partial \tilde{x}_j)(-(d\bar{t}_j/\bar{t}_j) - \xi^*(\bar{\partial} f_j/f_j))\Big\}$$
$$= \sum_{i,j} (\partial^2 u/\partial\tilde{x}_i \partial\tilde{x}_j)(dt_i/t_i)\wedge(d\bar{t}_j/\bar{t}_j) + \sum_{j=1}^n (\partial u/\partial\tilde{x}_j)\xi^* \bar{\partial}\partial \log(f_j).$$

Now, define real-valued functions $0 \leqq \theta_j < 2\pi$ on $P_{w'}$ by

$$t_j = \exp((-\tilde{x}_j/2) + \sqrt{-1}\,\theta_j), \quad j=1,2,\dots,n,$$

and set $V^i := t_i \partial/\partial t_i$. Furthermore, let h' be a C^∞ Hermitian metric for L' and put:

$$\tau' := \gamma_{n,e}\Big\{c_1(L',h') + \sum_{j=1}^n (\partial u/\partial\tilde{x}_j) c_1(L^{(j)}, h^{(j)})\Big\}^e ,$$
$$\tau'' := \gamma_{n,e}\Big\{c_1(L',h') + \sum_{j=1}^n x_j\, c_1(L^{(j)}, h^{(j)})\Big\}^e .$$

Then in view of (cf. (8.2.2))

$$dt_j \wedge d\bar{t}_j / |t_j|^2 = \sqrt{-1}\, d\tilde{x}_j \wedge d\theta_j \quad \text{and} \quad (V^i)_{\mathbb{R}}(u) = -2\, \partial u/\partial\tilde{x}_i ,$$

we have:

(c) $$(-1/2)\int_{P_{w'}} (V^i)_{\mathbb{R}}(u)(\sqrt{-1}\,\partial\bar\partial u + 2\pi\xi^*c_1(L',h'))^{n+e}$$
$$= (2\pi)^e\int_{P_{w'}} (\partial u/\partial\tilde{x}_i)\det(\partial^2 u/\partial\tilde{x}_k\partial\tilde{x}_\ell)(\textstyle\prod_{j=1}^n(\sqrt{-1}dt_j\wedge d\bar{t}_j/|t_j|^2))\wedge\xi^*(\tau')$$
$$= (2\pi)^{n+e}\int_{\tilde{x}\in R^n}\{(\partial u/\partial\tilde{x}_i)\det(\partial^2 u/\partial\tilde{x}_k\partial\tilde{x}_\ell)\,\tau'(w')\}\,d\tilde{x}_1\wedge d\tilde{x}_2\wedge\cdots\wedge d\tilde{x}_n$$
$$= (2\pi)^{n+e}\int_{\Sigma}\{x_i\tau''(w')\}\,dx_1\wedge dx_2\wedge\cdots\wedge dx_n\ ,$$

where the last identity is obtained by setting $x_j = \partial u/\partial\tilde{x}_j$, $j=1,2,..,n$. Similar computations also show that:

(d) $$\int_{P_{w'}}((\sqrt{-1}/2\pi)\,\partial\bar\partial u + \xi^* c_1(L',h'))^{n+e} = \int_{\Sigma}\tau''(w')dx_1\wedge dx_2\wedge\cdots\wedge dx_n\ .$$

<u>Step</u> 3. In view of (7.4.1), an integration of (c) over W yields (a). Since $(\sqrt{-1}/2\pi)\partial\bar\partial u + c_1(L',h')$ represents $c_1(L'')$, we obtain (b) by integrating (d) over W.

(9.3) We here assume that n = 1, i.e., $G = \mathbb{C}^*$. Fix a holomorphic line bundle L_1 over a compact complex connected manifold W and consider the vector bundle $E := \mathcal{O}_W \oplus L_1$ of rank 2 over W (where vector bundles and locally free sheaves are used interchangeably if there is no fear of confusion). We now put $Y := \mathbb{P}(E^*)$ and let $\xi: Y \to W$ be the natural projection. Then Y = (E - (zero section))/$\mathbb{C}^*$ and L_1 is regarded as a Zariski-open subset of Y by

$$L_1 \hookrightarrow \mathbb{P}(E^*)(=Y),\quad \ell \longmapsto (1\oplus\ell) \text{ modulo } \mathbb{C}^*.$$

Via this inclusion, the zero section of L_1 defines an effective prime divisor, denoted by D_0 , on Y. Note that we have another divisor $D_\infty := Y - L_1 \in \mathrm{Div}(Y)$ on Y. Put $P := L_1 - D_0$. Then the natural $\mathbb{C}^*$-action on the line bundle L_1 extends to a holomorphic action of $G = \mathbb{C}^*$ on Y with the fixed point set $D_0 \cup D_\infty$.

Furthermore, P is regarded as a principal bundle over W with structure group G. Let (n', n'') $(\neq (0,0))$ be a pair of non-negative integers which will be specified later. Put $D := n'D_0 + n''D_\infty \in \mathrm{Div}(Y)$. Then $L := \mathcal{O}_Y(D)$ is a ξ-very ample line bundle on Y. We thus have a polarized toric bundle $(\xi : Y \to W, \mathbb{P}^1(\mathbb{C}), L)$.

(9.3.1) REMARK: Fix an arbitrary C^∞ Hermitian metric h_1 for the line bundle L_1. Now, recall the arguments in Step 1 of the proof of (9.2.3). Then, in view of (9.2.2), we can define real-valued C^∞ functions $\tilde{x} = \tilde{x}(p)$ and $u = u(p)$ on P by

$$\exp(-\tilde{x}(p)) := h_1(p, p) \qquad (p \in P),$$

$$u(p) := \log\Big(\sum_{k=-n''}^{n'} \exp(k\,\tilde{x}(p))\Big) \qquad (p \in P).$$

We also have the corresponding mapping $m : P \to M_{\mathbb{R}} (= \mathbb{R})$ as in (8.1) and moreover, it is given by

$$m(p) = (\partial u / \partial \tilde{x})(p), \qquad p \in P.$$

Note that, for each $w \in W$, the image $m(P_w)$ is the interior of the closed interval $\Sigma = [-n'', n']$.

(9.3.2) DEFINITION: Let $Y^{(1)}$ (resp. $Y^{(2)}$) be a compact complex connected manifold on which G acts holomorphically and effectively with the corresponding fixed point set $D^{(1)}$ (resp. $D^{(2)}$). Furthermore, let $\{D_i^{(2)} \mid i \in I\}$ be the set of all connected components of $D^{(2)}$. Then a surjective G-equivariant morphism $\lambda : Y^{(1)} \to Y^{(2)}$ is called a G-collapsing if the following conditions are satisfied:

(1) λ maps $Y^{(1)} - D^{(1)}$ isomorphically onto $Y^{(2)} - D^{(2)}$.

(2) There exists a (possibly empty) subset J of I such that $\lambda : Y^{(1)} \to Y^{(2)}$ is the monoidal transformation of $Y^{(2)}$ with centre $\bigcup_{j\in J} D_j^{(2)}$. (If J is empty, then λ is nothing but an isomorphism of $Y^{(1)}$ onto $Y^{(2)}$.)

We now fix an arbitrary G-collapsing $\lambda: Y \to \tilde{Y}$ for Y above, and let n', n'' be respectively the (complex) codimension of $\lambda(D_0)$, $\lambda(D_\infty)$ in Y. Write G as $\{t \mid t \in \mathbb{C}^*\}$. Then, Theorem (9.2.3) allows us to obtain the following refinement of the integral formula of Koiso and Sakane [13] on Futaki invariants:

(9.3.3) THEOREM: <u>Put</u> $e := \dim_{\mathbb{C}} W$. <u>Writing for brevity</u> $K_{\tilde{Y}}^{-1}$ <u>as</u> $\tilde{L}$, <u>we have</u>:

(a) $(r_{\tilde{L},\tilde{Y}})_*(t\partial/\partial t) = (2\pi)^{e+1}(e+1)\int_{-n''}^{n'} x(c_1(W) + x\, c_1(L_1))^e[W]dx.$

<u>Suppose now that</u> $\tilde{Y}$ <u>is a Fano manifold</u>, <u>i.e.</u>, $\tilde{L}$ <u>is ample</u>. <u>Let</u> $F_{\tilde{Y}}|_{Lie(G)}$ <u>be the restriction of</u> $F_{\tilde{Y}}$: $\mathfrak{X}(\tilde{Y}) \to \mathbb{R}$ <u>to</u> Lie(G) (cf. (5.1)). <u>Then</u>

(b) $F_{\tilde{Y}}|_{Lie(G)} = 0$ <u>if and only if</u> $\int_{-n''}^{n'} x(c_1(W) + x c_1(L_1))^e[W]dx = 0.$

PROOF: Note that $\mathcal{O}_Y(\lambda^*\tilde{L}) = \mathcal{O}_Y(K_Y^{-1}) \otimes_{\mathcal{O}} \mathcal{O}_Y((n'-1)D_0 + (n''-1)D_\infty)$ $= \mathcal{O}_Y((\xi^* K_W^{-1}) \otimes L)$. Hence by (9.2.3) applied to $L' = K_W^{-1}$, the right-hand side of (a) is $(r_{\lambda^*\tilde{L},Y})_*(t\partial/\partial t)$. This together with (7.5.1) yields (a). Now, (b) is strightforward from (a) in view of (7.2.2) applied to S = G.

(9.4) Now, let Y be a q-dimensional compact complex connected manifold endowed with a holomorphic effective action of $G = (\mathbb{C}^*)^n$. Assume that there exists an ample line bundle L on Y endowed with a linear holomorphic G-action which covers the action on Y. Then we have a Kähler form ω on Y representing $2\pi c_1(L)_{\mathbb{R}}$. Express ω as $\sqrt{-1}\sum g_{\alpha\bar{\beta}} dz^{\alpha}\wedge dz^{\bar{\beta}}$ in terms of holomorphic local coordinates $(z^1, z^2, \ldots, z^q)$ on Y. Let $V_i \in \mathfrak{X}(Y)$ be the image of $t_i\partial/\partial t_i \in \mathrm{Lie}(G)$ under the natural inclusion $\mathrm{Lie}(G) \hookrightarrow \mathfrak{X}(Y)$. Now, for each i, there exists a real-valued C^∞ function φ_i (which is unique up to an additive constant) such that

$$V_i = \sum_{\alpha,\beta} g^{\bar{\beta}\alpha}\partial_{\bar{\beta}}\varphi_i \,\partial/\partial z^{\alpha} \quad \text{(cf. Step 1 of the proof of (8.2)).}$$

For each $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n (= M_{\mathbb{R}})$, we define a mapping $m^{\mathbf{a}} : Y \to M_{\mathbb{R}}$ by

$$m^{\mathbf{a}}(y) = (-\varphi_1(y)+a_1, -\varphi_2(y)+a_2, \ldots, -\varphi_n(y)+a_n), \qquad y \in Y.$$

Then the image $\Sigma^{\mathbf{a}} := m^{\mathbf{a}}(Y)$ is an n-dimensional compact convex polyhedron in $M_{\mathbb{R}}$ (cf. Atiyah [1]). Recall that the push-forward by m^{a} of the symplectic measure $(\omega/2\pi)^q$ is a piecewise polynomial measure, denoted by $d\mu$, on $M_{\mathbb{R}}$ of finite total volume $c_1(L)^q[Y]$ (cf. Duistermaat and Heckman [7], Atiyah and Bott [2]).

(9.4.1) DEFINITION: Let $\mathbf{a}$ be the unique element of $M_{\mathbb{R}}$ such that

$$(2\pi)^q \int_{\Sigma^{\mathbf{a}}} x_i \, d\mu = (r_L)_*(t_i\partial/\partial t_i), \quad 1 \leqq i \leqq n,$$

where $(x_1, \ldots, x_n)$ are the standard coordinates on $M_{\mathbb{R}} (= \mathbb{R}^n)$. We then denote $m^{\mathbf{a}}$ by m. Now, the mapping $m : Y \to M_{\mathbb{R}}$ is called the <u>strict moment map</u> associated with the Hodge metric ω on Y. Note that, in view of Theorem (9.2.3), this m is compatible with the one defined in Appendix B.

(9.4.2) REMARK: Suppose that the Kähler form ω represents $2\pi c_1(Y)_{\mathbb{R}}$. In this special case, one has the following fact (which is essentially pointed out to us by A. Futaki): Let $\widetilde{\omega}$ be the Kähler form on Y such that $\mathrm{Ric}(\widetilde{\omega}) = \omega$ and that $\widetilde{\omega}$ is cohomologous to ω. Then the strict moment map $\mathbf{m} : Y \to M_{\mathbb{R}} (= \mathbb{R}^n)$ associated with ω is characterized by

$$\mathbf{m}(y) = (-\widetilde{\varphi}_1(y), -\widetilde{\varphi}_2(y), \dots, -\widetilde{\varphi}_n(y)), \qquad y \in Y,$$

where each $\widetilde{\varphi}_i$ is a real-valued C^∞ function on Y such that the following conditions are satisfied:

(a) $\widetilde{\varphi}_i$ coincides with φ_i up to an additive constant;

(b) $\int_Y \widetilde{\varphi}_i \, \widetilde{\omega}^n = 0$.

10. APPENDIX D.

In [22], Sakane constructed examples of Einstein-Kähler metrics on nonhomogeneous Fano manifolds. Afterwards, these were reformulated and generalized by Koiso and Sakane [13; Theorem 4.2], where almost at the same time, the author found a very simple proof for their results. (A little later, Bando also obtained a similar proof independently.) Since this new proof has the advantage of describing Einstein-Kähler metrics very explicitly, we here explain the detail.

Assume now that $n = 1$, i.e., $G = \mathbb{C}^*$. Let $\tilde{Y}$ be a compact complex connected manifold endowed with a holomorphic effective G-action such that the corresponding fixed point set consists of just two connected components $\tilde{D}_0$ and $\tilde{D}_\infty$. Furthermore, assume that Y is of class $\mathcal{C}$, i.e., Y is bimeromorphic to a compact Kähler manifold. Note that, via isotropy representation, our G-action on $\tilde{Y}$ naturally induces a G-action on the normal bundle $N(\tilde{D}_0:\tilde{Y})$ (resp. $N(\tilde{D}_\infty:\tilde{Y})$) of $\tilde{D}_0$ (resp. $\tilde{D}_\infty$) in $\tilde{Y}$. We finally assume that each element of G acts on both $N(\tilde{D}_0:\tilde{Y})$ and $N(\tilde{D}_\infty:\tilde{Y})$ as a scalar multiplication of the vector bundles.

(10.1) REMARK: Blow up $\tilde{Y}$ along $\tilde{D}_0$ and $\tilde{D}_\infty$. We then have a G-collapsing $\lambda : Y \to \tilde{Y}$ (cf. (9.3.2)) such that $D_0 := \lambda^{-1}(\tilde{D}_0)$ and $D_\infty := \lambda^{-1}(\tilde{D}_\infty)$ are nonsingular irreducible divisors on Y fixed by the G-action. Put $P := Y - (D_0 \cup D_\infty)$. Then by the generalized Bialynicki-Birula's decomposition of Fujiki [8] (see also Fujiki [9; (6.10)], Carrell and Sommese [5]), we have a natural G-equivariant identification of $P \cup D_0$ (resp. $P \cup D_\infty$)

with $N(D_0:Y)$ (resp. $N(D_\infty:Y)$) (cf. [15]). Hence, by reversing the G-action, one can view $N(D_0:Y)$ - (zero section) as the same $\mathbb{C}^*$-bundle as $N(D_\infty:Y)$ - (zero section) over $W := P/\mathbb{C}^* \cong D_0 \cong D_\infty$. There now exists a line bundle L_1 over W such that $L_1 = N(D_0:Y)$ and that $L_1^{-1} = N(D_\infty:Y)$. Put $E := \mathcal{O}_W \oplus L_1$. We can thus regard Y as $\mathbb{P}(E^*)$ and furthermore, exactly the same situation as in (9.3) happens. (Therefore, until the end of this appendix, we freely use the notation of (9.3).) Let $e := \dim_{\mathbb{C}} Y - 1$. Then by (b) of (9.3.3),

(10.1.1) $F_{\tilde{Y}}\big|_{Lie(G)} = 0$ <u>if and only if</u> $\int_{-n''}^{n'} x(c_1(W) + x c_1(L_1))^e [W]\,dx = 0$,

where n' and n" are respectively the (complex) codimension of $\tilde{D}_0$ and $\tilde{D}_\infty$ in $\tilde{Y}$.

(10.2) DEFINITION: For simplicity, put $\hat{P} := \lambda(P)$. Recall that every element of G acts on both $N(\tilde{D}_0:\tilde{Y})$ and $N(\tilde{D}_\infty:\tilde{Y})$ as a scalar multiplication. Hence, applying again the generalized Bialynicki-Birula's decomposition of Fujiki [8] (see also Fujiki [9; (6.10)]), we have a natural G-equivariant identification of $\tilde{P} \cup \tilde{D}_0$ (resp. $\tilde{P} \cup \tilde{D}_\infty$) with $N(\tilde{D}_0:\tilde{Y})$ (resp. $N(\tilde{D}_\infty:\tilde{Y})$). Now, let h be an arbitrary C^∞ Hermitian metric on L_1. Note that this h naturally induces a Hermitian metric, denoted by h^{-1}, on the dual bundle L_1^{-1} of L_1. In view of the identifications

$$(L_1 - (\text{zero section})) = P \cong \tilde{P} = (N(\tilde{D}_0:\tilde{Y}) - (\text{zero section}))$$

and

$$(L_1^{-1} - (\text{zero section})) = P \cong \tilde{P} = (N(\tilde{D}_\infty:\tilde{Y}) - (\text{zero section})),$$

the Hermitian norm $\| \ \|_h$ (resp. $\| \ \|_{h^{-1}}$) on L_1 (resp. L_1^{-1}) induces a norm on $N(\tilde{D}_0:\tilde{Y})$ (resp. $N(\tilde{D}_\infty:\tilde{Y})$). Then for a Kähler form ω on W, (h, ω) is said to be a <u>tight pair</u> if the following conditions are satisfied:

(1) The norms on $N(\widetilde{D}_0:\widetilde{Y})$ and $N(\widetilde{D}_\infty:\widetilde{Y})$ induced from h are C^∞ Hermitian norms of respective vector bundles.

(2) ω is an Einstein-Kähler form satisfying $\mathrm{Ric}(\omega) = \omega$.

(3) The eigenvalues of $c_1(L_1;h)$ with respect to ω are constant on W.

(4) $\lambda^{-1*}\{\rho^{2(n'-1)}(\xi^*\omega)^e\wedge\partial\rho\wedge\bar\partial\rho\}$ (resp. $\lambda^{-1*}\{\tau^{2(n''-1)}(\xi^*\omega)^e\wedge\partial\tau\wedge\bar\partial\tau\}$) on $\widetilde{P}$ extends to a C^∞ (nonvanishing) $(e+1,e+1)$-form on $N(\widetilde{D}_0:\widetilde{Y})(=\widetilde{P}\cup\widetilde{D}_0)$ (resp. $N(\widetilde{D}_\infty:\widetilde{Y})(=\widetilde{P}\cup\widetilde{D}_\infty)$),

where $\xi : Y(=\mathbb{P}(E^*)) \to W$ is the natural projection and $\rho : L_1 \to \mathbb{R}$ (resp. $\tau : L_1^{-1} \to \mathbb{R}$) denotes the norm function defined by $\rho(x) := \|x\|_h$ (resp. $\tau(x) := |x|_{h^{-1}}$) for x in L_1 (resp. L_1^{-1}). In particular, if $n' = n'' = 1$, then (h,ω) is a tight pair if and only if (2) and (3) are satisfied.

We shall now give a slight modification of the result of Koiso and Sakane [13; Theorem 4.2]:

(10.3) THEOREM: <u>Assume that $\widetilde{Y}$ is a Fano manifold, i.e., $K_{\widetilde{Y}}^{-1}$ is ample. If there exists a tight pair (h,ω), then the following are equivalent</u>:

(a) $F_{\widetilde{Y}}|_{\mathrm{Lie}(G)} = 0$;

(b) <u>$\widetilde{Y}$ admits an Einstein-Kähler form</u>.

PROOF: In view of (5.1), it suffices to show that (a) implies (b) under the assumption that (h,ω) as above exists. The proof consists of four steps.

<u>Step</u> 1. Let $\mu_1 \leqq \mu_2 \leqq \dots \leqq \mu_e$ be the constant eigenvalues of $2\pi c_1(L_1;h)$ with respect to ω. Put $D := n'D_0 + n''D_\infty$ and $L := \mathcal{O}_Y(D)$.

Then $\lambda^* K_{\tilde{Y}}^{-1} = L \otimes \zeta^* K_W^{-1}$ (see the proof of (9.3.3)). Hence, via the identification of D_0 (resp. D_∞) with W, we have:

$$\lambda^* K_{\tilde{Y}}^{-1}\Big|_{D_0} = L \otimes \zeta^* K_W^{-1}\Big|_{D_0} = L_1^{\otimes n'} \otimes K_W^{-1},$$

$$(\text{resp. } \lambda^* K_{\tilde{Y}}^{-1}\Big|_{D_\infty} = (L_1^{-1})^{\otimes n''} \otimes K_W^{-1}).$$

Therefore, via the identification of W with D_0 (resp. D_∞), the cohomology class $n'c_1(L_1)_{\mathbb{R}} + c_1(W)_{\mathbb{R}}$ (resp. $-n''c_1(L_1)_{\mathbb{R}} + c_1(W)_{\mathbb{R}}$) in $H^2(D_0:\mathbb{R})$ (resp. $H^2(D_\infty:\mathbb{R})$) is represented by $\lambda^*\theta_0$ (resp. $\lambda^*\theta_\infty$) for some positive definite (1,1)-form θ_0 (resp. θ_∞) on $\tilde{D}_0$ (resp. $\tilde{D}_\infty$). On the other hand, $2\pi c_1(W)_{\mathbb{R}}$ is represented by the Kähler form ω. We now have the following:

1) If $-n'' < x < n'$, then $(\omega^e[W])\prod_{k=1}^{e}(1+\mu_k x) = \{(2\pi)(c_1(W)+xc_1(L_1))\}^e[W] > 0$ and in particular $1+\mu_k x > 0$ for all k.

2) The smallest nonnegative integer m such that $(c_1(W)+n'c_1(L_1))^{m+1}$ (resp. $(c_1(W)-n''c_1(L_1))^{m+1}$) is numerically trivial is $\dim_{\mathbb{C}} \tilde{D}_0$ (resp. $\dim_{\mathbb{C}} \tilde{D}_\infty$). Hence the order of zeroes of $\prod_{k=1}^{e}(1+\mu_k x)$ at $x = n'$ (resp. $x = -n''$) is $n'-1$ (resp. $n''-1$).

<u>Step</u> 2. Define a polynomial $A = A(x)$ in x by

$$A(x) := -\int_{-n''}^{x} s\prod_{k=1}^{e}(1+\mu_k s)\, ds.$$

Note that, by our condition (a), we have $A(n') = A(-n'') = 0$ (cf. (10.1.1)). In view of 2) of Step 1, the order of zeroes of $A(x)$ at $x = n'$ (resp. $x = -n''$) is n' (resp. n''). Furthermore, by 1) of Step 1, both $0 < A(x) \leqq A(0)$ and $A'(x)/x < 0$ hold for all nonzero x with $-n'' < x < n'$. In particular, the rational function $A'(x)/(xA(x))$ is free from poles and zeroes over the open interval $(-n'', n')$, and has a pole of order 1 at both $x = n'$ and $x = -n''$. Now,

$$B(x) := -\int_{0}^{x} A'(s)/(sA(s))\, ds$$

is monotone increasing over the interval $(-n'', n')$ and moreover, B maps $(-n'', n')$ diffeomorphically onto $\mathbb{R}$, because in a neighbourhood of $x = n'$ (resp. $x = -n''$), $B(x)$ is written as $-\log(n'-x)$ + a real analytic function (resp. $\log(x+n'')$ + a real analytic function). Let $B^{-1}: \mathbb{R} \to (-n'', n')$ be the inverse function of $B : (-n'', n') \to \mathbb{R}$, and define a real-valued C^∞ function $r = r(\tilde{p})$ on $\tilde{P}$ by

$$\exp(-r(\tilde{p})) = \{(\lambda^{-1*}\rho)(\tilde{p})\}^2 \ (= \{(\lambda^{-1*}\tau)(\tilde{p})\}^{-2}), \quad \tilde{p} \in \tilde{P}.$$

Note here that, since (h, ω) is a tight pair, (1) of (10.2) shows that $(\lambda^{-1*}\rho)^2$ (resp. $(\lambda^{-1*}\tau)^2$) extends to a C^∞ function on $\tilde{P} \cup \tilde{D}_0$ (resp. $\tilde{P} \cup \tilde{D}_\infty$). We now define a C^∞ function $x = x(r)$ in r by

$$x(r) := B^{-1}(r) \qquad (\text{i.e., } r = B(x(r))).$$

Then $u(r) := -\log(A(x(r)))$ satisfies (cf. (10.3.1))

$$(*) \quad u''(r)\prod_{k=1}^{e}(1+\mu_k u'(r)) = \exp(-u(r)),$$

since we have the identities $x'(r) = -x(r)A(x(r))/A'(x(r))$, $u'(r) = x(r)$ and $A'(x(r)) = -x(r)\prod_{k=1}^{e}(1+\mu_k x(r))$.

<u>Step</u> 3. Now, let η be the $(e+1,e+1)$-form on $\tilde{P}$ defined by

$$\eta := \sqrt{-1}\ 4(e+1)\exp(-u(r))\ \lambda^{-1*}((\xi^*\omega)^e \wedge \partial\rho \wedge \bar{\partial}\rho/\rho^2)$$
$$(= \sqrt{-1}\ 4(e+1)\exp(-u(r))\ \lambda^{-1*}(\overline{(\xi^*\omega)^e} \wedge \partial\tau \wedge \bar{\partial}\tau/\tau^2)\ ).$$

In this step, we shall show that η extends to a volume form on $\tilde{Y}$. First, in view of Step 2,

$$r = -\log(n'-x(r)) + \text{a real analytic function in } x(r),$$
$$(\text{resp. } r = \log(n''+x(r)) + \text{a real analytic function in } x(r)).$$

Hence, $(\lambda^{-1*}\rho)^2$ (resp. $(\lambda^{-1*}\tau)^2$) is written as a real analytic function in $x(r)$ with a simple zero at $x(r) = n'$ (resp. $-n''$).

On the other hand, Step 2 shows also that $\exp(-u(r))$ is a real analytic function in $x(r)$ with zeroes of order exactly n' (resp. n'') at $x(r) = n'$ (resp. $-n''$). Thus, in a neighbourhood of D_0 (resp. D_∞), $(\lambda^{-1*}\rho)^{-2n'}\exp(-u(r))$ (resp. $(\lambda^{-1*}\tau)^{-2n''}\exp(-u(r))$) is written as a nonvanishing real analytic function in $(\lambda^{-1*}\rho)^2$ (resp. $(\lambda^{-1*}\tau)^2$). Since (h, ω) is a tight pair, (4) of (10.2) now implies that η extends to a volume form on $\widetilde{Y}$.

<u>Step</u> 4. Regarding η as a volume form on $\widetilde{Y}$ (cf. Step 3), we shall finally show that $\widetilde{\omega} := \sqrt{-1}\,\bar{\partial}\partial \log \eta$ is an Einstein-Kähler form on $\widetilde{Y}$. Fix an arbitrary point w_0 of W. Then over a sufficiently small open neighbourhood U of w_0 in W, there exist a holomorphic local base σ for L_1 and a system $(z_1, z_2, \ldots, z_e)$ of holomorphic local coordinates on U such that

1) $h_{|U} = H(w)\sigma^*\otimes\bar{\sigma}^*$ for some positive real-valued C^∞ function $H = H(w)$ on U satisfying both $H(w_0) = 1$ and $(dH)(w_0) = 0$;

2) $\omega(w_0) = \sqrt{-1}\sum_{k=1}^{e} dz_k \wedge d\bar{z}_k$;

3) $(\bar{\partial}\partial H)(w_0) = \sqrt{-1}\sum_{k=1}^{e} \mu_k dz_k\wedge d\bar{z}_k$.

Via the identification

$$\begin{aligned} U\times\mathbb{C}^* &\cong P_{|U} \\ (w,t) &\leftrightarrow t\cdot\sigma(w), \end{aligned}$$

we regard $(z_1, z_2, \ldots, z_e, t)$ as a system of holomorphic local coordinates on the open subset $P_{|U}$ of Y. Then over $P_{|U}$,

$$\lambda^*\eta = \sqrt{-1}(e+1)\lambda^*(\exp(-u(r)))(\xi^*\omega)^e\wedge dt\wedge d\bar{t}/|t|^2 .$$

Note that $\mathrm{Ric}(\omega) = \sqrt{-1}\,\bar{\partial}\partial\log\omega^e = \omega$. Hence along the fibre P_{w_0} ,

$$\lambda^*\tilde{\omega} = \sqrt{-1}\,\partial\bar{\partial}\,\lambda^*(u(r)) + \xi^*\omega$$
$$= \sqrt{-1}\,\lambda^*(u''(r))dt\wedge d\bar{t}/|t|^2 + \sqrt{-1}\,\lambda^*(u'(r))\,\bar{\partial}\partial\log H + \xi^*\omega,$$

(see, for similar computations, Step 2 of the proof of (9.2.3)). Therefore, when restricted to $\lambda(P_{w_0})$, the (1,1)-form $\tilde{\omega}$ is written in the form

$$\sqrt{-1}\,u''(r)\,\lambda^{-1*}(dt\wedge d\bar{t}/|t|^2) + \sqrt{-1}\,\sum_{k=1}^{e}(1+\mu_k u'(r))\,\lambda^{-1*}(dz_k\wedge d\bar{z}_k),$$

which is positive definite in view of (*) of Step 2. Consequently, along $\lambda(P_{w_0})$, we can express $\tilde{\omega}^{e+1}$ as

$$\sqrt{-1}(e+1)u''(r)\left(\prod_{k=1}^{e}(1+\mu_k u'(r))\right)\lambda^{-1*}\left\{\left(\sum_{k=1}^{e}\sqrt{-1}dz_k\wedge d\bar{z}_k\right)^e\wedge dt\wedge d\bar{t}/|t|^2\right\},$$

and hence $\tilde{\omega}^{e+1} = \eta$ (cf. (*) of Step 2). Since w_0 is an arbitrary point of W, we now have $\mathrm{Ric}(\tilde{\omega}) = \tilde{\omega}$ everywhere on $\tilde{Y}$. Thus, $\tilde{\omega}$ is an Einstein-Kähler form on $\tilde{Y}$.

(10.3.1) REMARK: Let $K \in \mathbb{R}_+$ and $\mu_k \in \mathbb{R}$ $(k=1,2,\ldots,e)$. Furthermore, let a, b, c be real numbers such that $1+\mu_k c \neq 0$ for all k. Then, for a sufficiently small $\varepsilon > 0$, we can here give a complete solution of the ordinary differential equation

(1) $$y''(x)\prod_{k=1}^{e}(1+\mu_k y'(x)) = K\exp(-y(x)), \qquad a-\varepsilon < x < a+\varepsilon,$$

with the initial conditions

$$y(a) = b \quad\text{and}\quad y'(a) = c.$$

In order to solve this, we put $s := y'(x)$ and $A := \exp(-y(x))$. Since $y''(x)$ does not change its sign over the interval $(a-\varepsilon, a+\varepsilon)$, the inverse function theorem allows us to regard x as a C^∞ function x(s) in s. Consequently, A is also regarded as a C^∞ function A(s) in s. Then

$$A'(s)y''(x) = (dA/ds)(ds/dx) = dA/dx = -s\,A(s).$$

In particular, multiplying both sides of (1) by $A'(s)/A(s)$, we have

$$-s\prod_{k=1}^{e}(1+\mu_k s) = K\cdot A'(s).$$

Thus, x and y(x) are written in terms of the parameter s as follows:

(2) $$y(x) = -\log A(s),$$

where $A(s)$ is the polynomial $\exp(-b)-K^{-1}\int_c^s t\prod_{k=1}^{e}(1+\mu_k t)dt$ in s.
As for x, we have

$$ds/dx = y''(x) = \left(\prod_{k=1}^{e}(1+\mu_k s)\right)^{-1}K\cdot A(s), \quad (\text{cf. }(1)),$$

and therefore,

(3) $$x = a + \int_c^s \left(\prod_{k=1}^{e}(1+\mu_k t)\right)K^{-1}A(t)^{-1}dt.$$

Now, $(x, y(x))$ moves along the curve parametrized by (2) and (3) above.

(10.3.2) REMARK: We apply the above construction of Einstein-Kähler metrics to the case where $Y = \widetilde{Y} = \mathbb{P}(E^*)$ with $E := \mathcal{O}_W \oplus \mathcal{O}_W(k,-k)$ and $W := \mathbb{P}^m(\mathbb{C}) \times \mathbb{P}^m(\mathbb{C})$ ($m \in \mathbb{Z}_+$, $1 \leqq k \leqq m$). Note that $L_1 := \mathcal{O}_W(k,-k)$ denotes the line bundle $pr_1{}^*\mathcal{O}_{\mathbb{P}^m}(k) \oplus pr_2{}^*\mathcal{O}_{\mathbb{P}^m}(-k)$ over W, where pr_i : $\mathbb{P}^m(\mathbb{C}) \times \mathbb{P}^m(\mathbb{C}) \to \mathbb{P}^m(\mathbb{C})$ is the natural projection to the i-th factor (i=1,2). Now, let $\sigma : Q_0(\mathbb{C}^{m+1}) \to \mathbb{C}^{m+1}$ be the blowing-up of $\mathbb{C}^{m+1}$ at the origin $0 = (0, \ldots, 0)$ of $\mathbb{C}^{m+1}$, and let

$$p : \mathbb{C}^{m+1} - \{0\} \longrightarrow \mathbb{P}^m(\mathbb{C})$$
$$(z_0, z_1, \ldots, z_m) \mapsto (z_0 : z_1 : \ldots : z_m)$$

be the natural projection. Then the rational map $p\circ\sigma : Q_0(\mathbb{C}^{m+1}) \to \mathbb{P}^m(\mathbb{C})$ easily turns out to be a morphism, and via this morphism, we can regard $Q_0(\mathbb{C}^{m+1})$ as the line bundle $F := \mathcal{O}_{\mathbb{P}^m}(-1)$ over $\mathbb{P}^m(\mathbb{C})$. Hence, via the identification of $\mathbb{C}^{m+1} - \{0\}$ with F - (zero section), the function

$$\mathbb{C}^{m+1} - \{0\} \ni (z_0, z_1, \ldots, z_m) \longmapsto \sqrt{|z_0|^2 + |z_1|^2 + \ldots + |z_m|^2} \in \mathbb{R}$$

is viewed as a Hermitian norm of the line bundle F. Since $L_1 = pr_1{}^*(F^{\otimes -k}) \otimes pr_2{}^*(F^{\otimes k})$, this Hermitian norm on F induces a natural norm $\| \ \|_h$ on L_1 associated with a Hermitian metric h for L_1. We can now define $\rho : L_1 \to \mathbb{R}$ by $\rho(\ell) := \|\ell\|_h$ ($\ell \in L_1$). Note moreover that the Fubini-Study form ω_0 on $\mathbb{P}^m(\mathbb{C})$ is defined by

$$p^*\omega_0 = \sqrt{-1}\,\partial\bar{\partial} \log(\textstyle\sum_{i=0}^m |z_i|^2).$$

Then, $\omega := (m+1)(pr_1{}^*\omega_0 + pr_2{}^*\omega_0)$ is an Einstein-Kähler form on W such that (h, ω) is a tight pair (cf. (10.2)), because the eigenvalues $\mu_1 \leqq \mu_2 \leqq \ldots \leqq \mu_{2m}$ of $2\pi c_1(L_1;h)$ with respect to ω are all constant. In fact, we have

$$-\mu_1 = -\mu_2 = \ldots = -\mu_m = \mu_{m+1} = \mu_{m+2} = \ldots = \mu_{2m} = k/(m+1).$$

Recall that $G(:= \mathbb{C}^*)$ acts on the line bundle L_1 by scalar multiplication and that $Y(= \tilde{Y})$ is naturally a G-equivariant compactification of L_1 (cf. (9.3)). Now by

$$\int_{-1}^{1} v(c_1(W) + v\, c_1(L_1))^{2m}[W]\,dv = (c_1(W))^{2m}[W] \int_{-1}^{1} v(1-k^2v^2/(m+1)^2)^m dv = 0,$$

we have $F_Y|_{\mathrm{Lie}(G)} = 0$. Hence we can find an Einstein-Kähler metric on Y as constructed in the proof of (10.3) (see also Sakane [22]). Let A(s) be the polynomial in s defined by

$$A(s) := -\int_{-1}^{s} v(1-k^2v^2/(m+1)^2)^m dv .$$

Furthermore, define a C^∞ function $x = x(\rho)$ in ρ by

$$\rho^2 = \exp\left\{-\int_0^x (1-k^2s^2/(m+1)^2)^m/A(s)\,ds\right\}.$$

Then $\eta := \sqrt{-1}(8m+4)A(x(\rho))(\xi^*\omega)^{2m}\wedge\partial\rho\wedge\bar{\partial}\rho/\rho^2$ extends to a volume form on Y, where $\xi : L_1 \rightarrow W$ denotes the natural projection (cf. Step 3 of the proof of (10.3)). Then in view of Step 4 of the proof of (10.3), we can now conclude that $\tilde{\omega} := \sqrt{-1}\,\bar{\partial}\partial\log\eta$ is an Einstein-Kähler form on Y such that $\tilde{\omega}^{2m+1} = \eta$.

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(4.3) REMARK: (i) It is easily checked that $\bar{m}_u$ above coincides with the moment map: $G_\Delta \to \mathrm{Lie}(G_c)^* \cong M_{\mathbb{R}}$ (cf. Atiyah [1], Guillemin and Steinberg [11]) associated with the Kähler form $\sqrt{-1}\,\partial\bar\partial u \in \mathcal{K}$. (See Appendix B for the proof.)

(ii) Consider the subgroup $G_{\mathbb{R}} := \{(t_1, \ldots, t_n) \in G \mid t_i \in \mathbb{R}_+\} (\cong (\mathbb{R}_+)^n)$ of G. Then by the natural inclusions $G_{\mathbb{R}} \subset G \subset G_\Delta$, we may regard $G_{\mathbb{R}}$ as a subset of G_Δ. Then the closure $\bar{G}_{\mathbb{R}}$ of $G_{\mathbb{R}}$ in G_Δ is a <u>manifold with corners</u> in the sense of Borel-Serre (cf. Oda [20]) and has a natural differentiable structure as described in Step 3 of (8.2). Note that G_Δ/G_c above is endowed with such a structure via the natural identification of G_Δ/G_c with $\bar{G}_{\mathbb{R}}$.

(iii) A difference of (4.2) from Atiyah's result [1; Theorem 2] is that the mapping between G_Δ/G_c and Q is, in our case, a diffeomorphism (instead of a homeomorphism) even along their boundaries. This diffeomorphism is essentially obtained from the ampleness of $K_{G_\Delta}^{-1}$ by the fact that a combination of (3.2) and (3.3) keeps the Jacobian of $\bar{m}_u|_{\bar{G}_{\mathbb{R}}} : \bar{G}_{\mathbb{R}} \to M_{\mathbb{R}}$ nonvanishing also along the boudary $\bar{G}_{\mathbb{R}} - G_{\mathbb{R}}$.

(4.4) We now assume that G_Δ is a projective variety (where G_Δ is not necessarily a Fano variety). Note that the corresponding hyperplane bundle $L := \mathcal{O}_{G_\Delta}(1)$ is written as $\mathcal{O}_{G_\Delta}(\sum_{\sigma \in \Delta(1)} \nu_\sigma D(\sigma))$ for some $\nu_\sigma \in \mathbb{Z}_0$. Then

$$\Sigma_L := \{a \in M_{\mathbb{R}} \mid (a, b_\sigma) \leq \nu_\sigma \text{ for all } \sigma \in \Delta(1)\}$$

is an n-dimensional compact convex polyhedron (cf. Oda [21]). Since L is ample, there exists a G_c-invariant fibre metric h for L such that the corresponding first Chern form is positive definite.

EINSTEIN-KÄHLER FORMS, FUTAKI INVARIANTS AND CONVEX GEOMETRY ON TORIC FANO VARIETIES

by

Toshiki Mabuchi

0. INTRODUCTION.

Throughout this paper, we assume that X is a nonsingular n-dimensional toric Fano variety (defined over $\mathbb{C}$), i.e., X is an n-dimensional connected projective algebraic manifold satisfying the following conditions:

(a) X admits an effective almost homogeneous algebraic group action of $(\mathbb{G}_m)^n$ ($\cong(\mathbb{C}^*)^n$ as a complex Lie group).

(b) The set $\mathcal{K}$ of all Kähler forms on X in the De Rham cohomology class $2\pi\, c_1(X)_{\mathbb{R}}$ is non-empty.

For each $\omega \in \mathcal{K}$, by writing it as $\omega = \sqrt{-1} \sum g(\omega)_{\alpha\bar{\beta}}\, dz^{\alpha} \wedge dz^{\bar{\beta}}$ in terms of holomorphic local coordinates $(z^1, z^2, \dots, z^n)$ of X, we have the corresponding Ricci form $\mathrm{Ric}(\omega)$ cohomologous to ω:

$$\mathrm{Ric}(\omega) := \sqrt{-1}\, \bar{\partial}\partial \log \det(g(\omega)_{\alpha\bar{\beta}}).$$

Then an element ω of $\mathcal{K}$ is called an Einstein-Kähler form if $\mathrm{Ric}(\omega) = \omega$. We now pose the following:

(0.1) PROBLEM*): <u>Classify all</u> X <u>which admit, at least, one Einstein-Kähler form</u>.

Obviously, the Fubini-Study form on $\mathbb{P}^n(\mathbb{C})$ is a typical Einstein-Kähler form. This settles Problem (0.1) for n = 1, because

*) This is also posed by T. Oda and Y. T. Siu.

the only possible X with n = 1 is $\mathbb{P}^1(\mathbb{C})$. However, the real difficulty comes up even at n = 2: Let S_i be the projective algebraic surface obtained from $\mathbb{P}^2(\mathbb{C})$ by blowing up i points in general position (where $1 \leqq i \leqq 3$). Then, in spite of lots of efforts of differential geometers, it is still unknown whether or not the nonsingular toric Fano variety S_3 admits an Einstein-Kähler form.

The purpose of this paper is to give a brief survey of recent progress on Problem (0.1) together with our related new results. Especially, in Sections 1~6 (though they are somewhat of expository nature), several key ideas are introduced often without proofs, while technical details are given in the subsequent four appendices. In particular, in Appendix C (see (9.2.3) for the most general statement), we shall show that the Futaki invariants of an anti-canonically (relatively) polarized toric bundle Y over W can be regarded as the barycentre of $\mathbf{m}(Y)$ in terms of "Duistermaat-Heckman's measure", where $\mathbf{m} : Y \to \mathbb{R}^n$ ($n = \dim_{\mathbb{C}} Y - \dim_{\mathbb{C}} W$) denotes the associated "relative" moment map defined, in Appendix B, without any ambiguity of translations (cf. (8.2)). Finally, in Appendix D, a very explicit description of Einstein-Kähler metrics for Sakane-Koiso's examples will be given (cf. (10.3.2), Step 4 of (10.3)).

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1. NOTATION, CONVENTIONS AND PRELIMINARIES.

Let $\mathbb{Z}_+$ (resp. $\mathbb{Z}_0$) be the set of positive (resp. non-negative) integers and $\mathbb{R}_+$ (resp. $\mathbb{R}_0$) be the set of positive (resp. non-negative) real numbers. We now put:

$$G := (\mathbb{G}_m)^n = \{(t_1, t_2, \dots, t_n) \mid t_i \in \mathbb{C}^*\},$$
$$M := \{\mathbb{a} = (a_1, a_2, \dots, a_n) \mid a_i \in \mathbb{Z}\} \ (\cong \mathbb{Z}^n),$$
$$N := \left\{ \mathbb{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \middle| \ b_j \in \mathbb{Z} \right\} \ (\cong \mathbb{Z}^n).$$

For $\mathbb{a} \in M$ and $\mathbb{b} \in N$ as above, we define $(\mathbb{a},\mathbb{b}) \in \mathbb{Z}$, $\chi^{\mathbb{a}} \in \mathrm{Hom}_{\mathrm{alg\ gp}}(G,\mathbb{G}_m)$ and $\lambda_{\mathbb{b}} \in \mathrm{Hom}_{\mathrm{alg\ gp}}(\mathbb{G}_m,G)$ by

$$(\mathbb{a},\mathbb{b}) := \textstyle\sum_{i=1}^{n} a_i b_i ,$$
$$\chi^{\mathbb{a}}((t_1, t_2, \dots, t_n)) := t_1^{a_1} t_2^{a_2} \cdots t_n^{a_n} ,$$
$$\lambda_{\mathbb{b}}(t) := (t^{b_1}, t^{b_2}, \dots, t^{b_n}),$$

where $t, t_1, \dots, t_n \in \mathbb{G}_m$ $(= \mathbb{C}^*)$. Then the correspondence $\mathbb{a} \mapsto \chi^{\mathbb{a}}$ (resp. $\mathbb{b} \mapsto \lambda_{\mathbb{b}}$) canonically induces an isomorphism between the additive group M (resp. N) and the multiplicative group $\mathrm{Hom}_{\mathrm{alg\ gp}}(G,\mathbb{G}_m)$ (resp. $\mathrm{Hom}_{\mathrm{alg\ gp}}(\mathbb{G}_m,G)$). Note that

$$\chi^{\mathbb{a}}(\lambda_{\mathbb{b}}(t)) = t^{(\mathbb{a},\mathbb{b})} \quad \text{for all } t \in \mathbb{G}_m \ (= \mathbb{C}^*).$$

(1.1) DEFINITION: A non-empty subset σ of N is called a <u>cone</u> if the following conditions are satisfied:

(a) If $\mathbb{b} \in N$ satisfies $\beta\mathbb{b} \in \sigma$ for some $\beta \in \mathbb{Z}_+$, then $\mathbb{b} \in \sigma$.

(b) If $0 \neq \mathbb{b} \in \sigma$, then $-\mathbb{b} \notin \sigma$.

(c) $0 \in \sigma$.

(d) In terms of the natural additive structure of N, σ is a semigroup generated by its finite subset.

For a cone σ, there exists a unique irredundant finite subset $\{ b^1, b^2, \ldots , b^m \}$ of σ such that $\sigma = \sum_{k=1}^{m} \mathbb{Z}_0 b^k$. These b^1, $b^2, \ldots , b^m$ are called the <u>fundamental</u> <u>generators</u> of the cone σ.

(1.2) DEFINITION: A non-empty subset τ of a cone σ is called a <u>face</u> of σ, denoted by $\tau \leqq \sigma$, if there exists an element a of M such that $(a,b) \geqq 0$ for all b in σ and that $\tau = \{ b \in \sigma \mid (a,b) = 0 \}$. A <u>finite</u> <u>polyhedral</u> <u>decomposition</u> of N is a finite set Δ of cones in N such that

(a) if $\tau \leqq \sigma \in \Delta$, then $\tau \in \Delta$;

(b) if $\sigma, \tau \in \Delta$, then $\sigma \cap \tau \leqq \sigma$ and $\sigma \cap \tau \leqq \tau$;

(c) $N = \bigcup_{\sigma \in \Delta} \sigma$.

For every finite polyhedral decomposition Δ of N, we put

$$\Delta(i) := \{ \sigma \in \Delta \mid \dim \sigma = i \}, \qquad 0 \leqq i \leqq n,$$

where $\dim \sigma$ denotes the dimension of the real vector space spanned by σ in $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$.

(1.3) DEFINITION: A finite polyhedral decomposition Δ of N is said to be <u>nonsingular</u> if for each $\sigma \in \Delta(n)$, the set of fundamental generators of σ consists of n elements and forms a $\mathbb{Z}$-basis for N. For every nonsingular Δ, the set of fundamental generators of each element of $\Delta(i)$ consists of exactly i elements and is completed to a $\mathbb{Z}$-basis for N.

We shall now quote the following fundamental results due to Demazure [6], Miyake and Oda [18], and Mumford et al. [19] :

(1.4) THEOREM: To every nonsingular finite polyhedral decomposition Δ of N, one can uniquely associate an n-dimensional irreducible nonsingular G-equivariant compactification G_Δ of G possessing the following two properties:

(a) To each $\sigma \in \Delta(i)$, $0 \leqq i \leqq n$, there corresponds a unique $(n-i)$-dimensional G-orbit, denoted by $\mathbb{O}^\sigma$, such that G_Δ is expressible as

$$G_\Delta = \bigcup_{\sigma \in \Delta} \mathbb{O}^\sigma \qquad \text{(disjoint union)}.$$

Furthermore, the closure $D(\sigma)$ of $\mathbb{O}^\sigma$ in G_Δ is an irreducible nonsingular $(n-i)$-dimensional G-stable subvariety of G_Δ written in the form

$$D(\sigma) = \bigcup_{\tau \geqq \sigma} \mathbb{O}^\tau \qquad \text{(disjoint union)}.$$

(b) For each $\sigma \in \Delta(n)$, $U_\sigma := \bigcup_{\tau \leqq \sigma} \mathbb{O}^\tau$ forms an affine open G-stable neighbourhood of $\mathbb{O}^\sigma$ in G_Δ satisfying the conditions

$$G \subsetneqq U_\sigma \cong \mathbb{A}^n(\mathbb{C})$$

and

$$G_\Delta = \bigcup_{\sigma \in \Delta(n)} U_\sigma .$$

Let $\{ \mathbb{b}(\sigma)^1, \mathbb{b}(\sigma)^2, \ldots, \mathbb{b}(\sigma)^n \}$ be the set of fundamental generators of σ (which forms a $\mathbb{Z}$-basis for N), and let $\{ \mathbb{a}(\sigma)^1, \mathbb{a}(\sigma)^2, \ldots, \mathbb{a}(\sigma)^n \}$ be the dual basis for M defined by the relation $(\mathbb{a}(\sigma)^i, \mathbb{b}(\sigma)^j) = \delta_{ij}$. Then the corresponding characters

$$\chi_{\sigma;i} := \chi^{\mathbb{a}(\sigma)^i} \in \mathrm{Hom}_{\mathrm{alg\ gp}}(G, \mathbb{G}_m), \qquad 1 \leqq i \leqq n,$$

extend to rational functions on G_Δ, which are all regular

2. DEMAZURE'S RESULTS ON TORIC VARIETIES.

Throughout this section, we fix a nonsingular finite polyhedral decomposition Δ of N. Put $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$. Furthermore, for each $\rho \in \Delta(1)$, let b_ρ denote the unique fundamental generator of ρ. We now consider the divisor

$$K := - \sum_{\rho \in \Delta(1)} D(\rho)$$

on G_Δ . Recall the following fact due to Demazure [6]:

(2.1) THEOREM: K is a canonical divisor of G_Δ . Moreover, the following are equivalent:

(a) G_Δ is a toric Fano variety.

(b) -K is ample.

(c) -K is very ample.

(d) $\Sigma_{-K} := \{ a \in M_{\mathbb{R}} \mid (a, b_\rho) \leqq 1 \text{ for all } \rho \in \Delta(1) \}$ is an n-dimensional compact convex polyhedron whose vertices are exactly $\{ a_\tau \mid \tau \in \Delta(n) \}$, where each a_τ denotes the unique element of M such that $(a_\tau, b) = 1$ for all fundamental generators b of τ .

(2.2) REMARK: It is easily seen that $\mathbb{P}^2(\mathbb{C})$, $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$, S_i $(1 \leqq i \leqq 3)$ are the only possible 2-dimensional nonsingular toric Fano varieties. Recently, for dimension three also, all nonsingular toric Fano varieties are completely classified (cf. Batyrev [4] , K. Watanabe and M. Watanabe [23]).

(2.3) DEFINITION (Demazure [6 ; p.571]): An element a of M is called a root if there exists $\rho \in \Delta(1)$ such that $(a, b_\rho) = 1$ and that $(a, b_\sigma) \leqq 0$ for all $\sigma \in \Delta(1)$ with $\sigma \neq \rho$. Let $R(\Delta)$ be the set of all roots in M.

Now, as an immediate consequence of a result of Demazure [6; p. 581], one obtains:

(2.4) THEOREM: Let $\mathrm{Aut}(G_\Delta)$ be the group of all holomorphic automorphisms of G_Δ. Then $\mathrm{Aut}(G_\Delta)$ is a reductive algebraic group if and only if $-R(\Delta) := \{-a \mid a \in R(\Delta)\}$ coincides with $R(\Delta)$.

(2.5) REMARK: In view of this theorem and (2.2), it is now possible to determine all 3-dimensional nonsingular toric Fano varieties G_Δ with reductive $\mathrm{Aut}(G_\Delta)$. Such a G_Δ is, actually, isomorphic to one of the following (we owe the computation to Dr. T. Ashikaga):

$$\mathbb{P}^3(\mathbb{C}),\ \mathbb{P}^2(\mathbb{C})\times\mathbb{P}^1(\mathbb{C}),\ \mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C}),$$
$$\mathbb{P}^1(\mathbb{C})\times S_3,\ \mathbb{P}(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,-1)),\ F_1^5,$$

where we used the notation of K. Watanabe and M. Watanabe [23]. Obviously, the first three varieties admit an Einstein-Kähler form. Note that, for the last three varieties, $\mathrm{Aut}(G_\Delta)$ cannot act transitively on G_Δ. However, $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,-1))$ still admits an Einstein-Kähler form by virtue of a result of Sakane [22], partly because in this case, every maximal compact subgroup of $\mathrm{Aut}(G_\Delta)$ acts on G_Δ with principal orbits of real codimension one (cf. Appendix D).

The importance of (2.4) comes from the following theorem in differential geometry due to Matsushima [17]:

(2.6) THEOREM: Let Y be a compact complex connected manifold with $\dim_{\mathbb{C}} \mathrm{Aut}^0(Y) > 0$ (where $\mathrm{Aut}^0(Y)$ denotes the identity component of the group $\mathrm{Aut}(Y)$ of holomorphic automorphisms of Y). If Y admits an Einstein-Kähler form, then $\mathrm{Aut}(Y)$ is a reductive alge-