p–adic families of motives, Galois representations, and L–functions

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Hida [Hi2] has constructed interesting families of Galois representations of the type

$$\rho_p : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{Z}_p[[T]]), \quad G_{\mathbf{Q}} = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}),$$

which are non ramified outside p. These representations have the following property: if we consider the homomorphisms

$$\mathbf{Z}_p[[T]] \xrightarrow{s_k} \mathbf{Z}_p, \quad 1+T \ \mapsto (1+p)^{k-1},$$

then we obtain a family of Galois representations

$$\rho_p^{(k)} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{Z}_p),$$

which is parametrized by $k \in \mathbf{Z}$, and for $k = 2, 3, \cdots$,these representations are equivalent over $\mathbf{Q}_p$ to the p - representations of Deligne, attached to modular forms of weight k. This means that the representations of Hida are obtained by the p - adic interpolation of Delign's representations. A geometric interpretation of Hida's representations was given by Mazur and Wiles [Maz-W3]. For example, for the modular form Δ of weight 12 Hida has constructed a representation

$$\rho_{p,\Delta} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{Z}_p[[T]]),$$

as an example of his general theory, where the prime number p have the property $\tau(p) \not\equiv (\mathrm{mod} p)$ (e.g. $p < 2041, p \neq 2, 3, 5$ and 7).

Also, Hida has generalized his construction to the case of Hilbert modular forms.

In this paper we would like to describe a conjectural generalization of his construction to arbitrary motives, and to formulate a certain conjecture. The important case, in which this conjecture can be verified, corresponds to Hecke characters of CM-type. We describe certain p - adic families, which are in general bigger than those obtained by the cyclotomic twist. We stress the fact that the corresponding p - adic L - functions depend analytically on the parameter of these p - adic Galois representations.

Summary.

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Throught the paper we fix embeddings

$$i_\infty : \overline{\mathbf{Q}} \to \mathbf{C}, \quad i_p : \overline{\mathbf{Q}} \to \mathbf{C}_p$$

and we shall often regard algebraic numbers (via these embeddings) as both complex and p - adic numbers, where $\mathbf{C}_p = \widehat{\overline{\mathbf{Q}}}_p$ is the Tate field (the completion of a fixed algebraic closure $\overline{\mathbf{Q}}_p$ of $\mathbf{Q}_p$), which is endowed with a unique norm $|\cdot|_p$ such that $|p|_p = p^{-1}$.

1. Motives and L - functions over a totally real field.

Let F be a totally real number field of degree $n = [F : \mathbf{Q}]$ and T is in another number field T which will be supposed a subfield of $\mathbf{C}$. By a motive M over F of with coefficients in T we shall mean a collection of the following objects:

$$M_{B,\sigma} = M_\sigma,\ M_{DR},\ M_\lambda,\ I_{\infty,\sigma},\ I_{\lambda,\sigma},$$

where σ runs over the set J_F of all complex embeddings of F,

M_σ is the Betti realization of M (with respect to the embedding $\sigma \in J_F$) which is a vector space over T of dimension d endowed for real $\sigma \in J_F$ with a T - rational involution ρ_σ ;

M_{DR} is the de Rham realization of M, a free $T \otimes F$ - module of rank d, endowed with a decreasing filtration $\{F^i_{DR}(M) \subset M_{DR} \mid i \in \mathbf{Z}\}$ of $T \otimes F$ - modules (which may not be free in some cases when $F \neq \mathbf{Q}$);

M_λ is the λ - adic realization of M at a finite place λ of the coefficient field T (a T_λ - vector space of degree d over T_λ, a completion of T at λ) which is a Galois module over $G_F = \mathrm{Gal}(\overline{F}/F)$ so that we a have a compatible system of λ - adic representations denoted by

$$r_{M,\lambda} = r_\lambda : G_F \to GL(M_\lambda).$$

Also,

$$I_{\infty,\sigma} : M_\sigma \otimes_T \mathbf{C} \to M_{DR} \otimes_{\sigma(F)\cdot T} \mathbf{C}$$

is the complex comparison isomorhism of complex vector spaces for each $\sigma \in J_F$,

$$I_{\lambda,\sigma} : M_\sigma \otimes_T T_\lambda \to M_\lambda$$

is the λ - adic comparison isomorphism of T_λ - vector spaces. It is assumed in the notation that the complex vector space $M_\sigma \otimes_{\mathbf{Q}} \mathbf{C}$ is decomposed in the Hodge bigraduation

$$M_\sigma \otimes_T \mathbf{C} = \oplus_{i,j} M_\sigma^{i,j}$$

in which $\rho_\sigma(M_\sigma^{i,j}) \subset M_\sigma^{j,i}$ for $\sigma \in J_F$ and the Hodge numbers

$$h(i,j,)_\sigma = h(i,j,M)_\sigma = \dim_{\mathbf{C}} M_\sigma^{i,j}$$

do not depend on σ. Moreover,

$$I_{\infty,\sigma}(\oplus_{i' \geq i} M_\sigma^{i',j}) = F^i_{DR}(M) \otimes_{F,\sigma} \mathbf{C}.$$

Also, $I_{\lambda,\sigma}$ takes ρ_σ to the r_λ - image of the Galois automorphism which is denoted by the same symbol $\rho_\sigma \in G_F$ and corresponds to the complex conjugation of $\mathbf{C}$ under an embedding of $\overline{F}$ to $\mathbf{C}$ extending σ. We assume that M is pure of weight w (i.e. $i+j=w$).

The L - function $L(M,s)$ of M is defined as the following Euler product:

$$L(M,s)=\prod_{\mathfrak{p}} L_{\mathfrak{p}}(M,\mathcal{N}\mathfrak{p}^{-s}),$$

extended over all maximal ideals $\mathfrak{p}$ of the maximal order $\mathcal{O}_F$ of F and where

$$L_{\mathfrak{p}}(M,X)^{-1}=\det(1-X\cdot r_{\lambda}(Fr_{\mathfrak{p}}^{-1})\mid M_{\lambda}^{I_{\mathfrak{p}}})=$$
$$(1-\alpha^{(1)}(\mathfrak{p})X)\cdot(1-\alpha^{(2)}(\mathfrak{p})X)\cdot\ldots\cdot(1-\alpha^{(d)}(\mathfrak{p})X)=$$
$$1+A_1(\mathfrak{p})X+\ldots+A_d(\mathfrak{p})X^d;$$

here $\mathcal{N}\mathfrak{p}$ is the norm of $\mathfrak{p}$ and $Fr_{\mathfrak{p}}\in G_F$ is the Frobenius element at $\mathfrak{p}$, defined modulo conjugation and modulo the inertia subgroup $I_{\mathfrak{p}}\subset G_{\mathfrak{p}}\subset G_F$ of the decomposition group $G_{\mathfrak{p}}$ (of any extension of $\mathfrak{p}$ to $\overline{F}$). We make the standard hypothesis that the coefficients of $L_{\mathfrak{p}}(M,\ X)^{-1}$ belong to T, and that they are independent of λ coprime with $\mathcal{N}\mathfrak{p}$. Therefore we can and we shall regard this polynomial both over $\mathbf{C}$ and over $\mathbf{C}_p$. We shall need the following twist operation: for an arbitrary motive M over F with coefficients in T an integer m and a Hecke character χ of finite order one can define the twist $N=M(m)(\chi)$ which is again a motive over F with the coefficient field $T(\chi)$ of the same rank d and weight w so that we haver

$$L(N,s)=\prod_{\mathfrak{p}} L_{\mathfrak{p}}(M,\chi(\mathfrak{p})\mathcal{N}\mathfrak{p}^{-s-n}).$$

§2. The group of Hida and the algebra of Iwasawa - Hida.

Now let us fix a motive M with coefficients in $T=\mathbf{Q}(\{a(\mathfrak{n})_{\mathfrak{n}}\})$ of rang d and of weight w, and let $\mathrm{End}_T M$ denote the endomorphism algebra of M (i.e. the algebra of T-linear endomorphisms of any M_σ, which commute with the Galois action under the comparison isomorphisms. Let

$$\mathrm{Gal}_p=\mathrm{Gal}(F^{ab}_{p,\infty}/F)$$

denotes the Galois group of the maximal abelian extension $F^{ab}_{p,\infty}$ of F unramified out side primes of F above p and ∞. Define $\mathcal{O}_{F,T,p}=\mathcal{O}_F\otimes\mathcal{O}_T\otimes\mathbf{Z}_p$.

Definition. The *group of Hida* $GH_M=GH_{M,p}$ is the following product

$$GH_M=\mathrm{End}_T^{\times}(\mathcal{O}_{F,T,p})\times\mathrm{Gal}_p,$$

where $\mathrm{End}_T^{\times}$ denotes the algebraic T-group of invertible elements of End_T and it is implicitely supposed that the group $\mathrm{End}_T^{\times}$ posesses an $\mathcal{O}_T$-integral structure given by an appropriate choice of an $\mathcal{O}_T$ - lattice. Consider next the $\mathbf{C}_p$ - analytic Lie group

$$\mathcal{X}_{M,p}=\mathrm{Hom}_{contin}(GH_M,\ \mathbf{C}_p^{\times})$$

consisting of all continuous characters of the Hida group GH_M, which contains the $\mathbf{C}_p$ - analytic Lie group

$$\mathcal{X}_p = \mathrm{Hom}_{contin}(\mathrm{Gal}_p,\ \mathbf{C}_p^\times)$$

consisting of all continuous characters of the Galois group Gal_p (via the projection of GH_M onto Gal_p.

The group $\mathcal{X}_{M,p}$ contains the discrete subgroup $\mathcal{A}$ of arithmetical characters of the type

$$\chi \cdot \eta \cdot \mathcal{N}cx_p^m = (\chi, \eta, m),$$

where

$$\chi \in \mathcal{X}_{M,p}^{tors}$$

is a character of finite order of GH_M, η is a T - algebraic character of $\mathrm{End}_T^\times(\mathcal{O}_{F,T,p})$, $m \in \mathbf{Z}$, and $\mathcal{N}x_p$ denotes the following natural norm homomorphism

$$\mathcal{N}x_p : \mathrm{Gal}_p \to \mathrm{Gal}(\mathbf{Q}_{p,\infty}^{ab}/\mathbf{Q}) \cong \mathbf{Z}_p^\times \to \mathbf{C}_p^\times,\ \mathcal{N}x_p \in \mathcal{X}_p.$$

Definition. The *algebra of Iwasawa - Hida* $I_M = I_{M,p}$ of M at p is the completed group ring $\mathcal{O}_p[[GH_M]]$, where $\mathcal{O}_p$ denotes the ring of integers of the Tate field $\mathbf{C}_p$.

Note that this definition is completely analogous to the usual definition of the Iwasawa algebra Λ as the completed group ring $\mathbf{Z}_p[[\mathbf{Z}_p]]$ if we take into account that $\mathbf{Z}_p$ coincides with the factor group of $\mathbf{Z}_p^\times$ modulo its torsion subgroup.

Now for each arithmetic point $P = (\chi, \eta, m) \in \mathcal{A}$ we have a homomorphism

$$\nu_P : GH_M \to \mathcal{O}_p$$

which is defined by the corresponding group homomorphism

$$P : GH_M \to \mathcal{O}_p^\times \subset \mathbf{C}_p^\times.$$

For a I_M-module N and $P \in \mathcal{A}$ we put

$$N_P = N \otimes_{I_M, \nu_P} \mathcal{O}_p$$

("reduction of N modulo P", or a fiber of N at P).

Therefore, for each Galois representation

$$r : G_F \to \mathrm{GL}(N)$$

its reduction $r \bmod P$ is defined as the natural composition:

$$G_F \to \mathrm{GL}(N) \to \mathrm{GL}(N_P).$$

§3. A conjecture on the existence of p - adic families of galois representations attached to motives.

Note first that the fixed embeddings $T \hookrightarrow \mathbf{C}$,

$$i_\infty : \overline{\mathbf{Q}} \to \mathbf{C},\ \ i_p : \overline{\mathbf{Q}} \to \mathbf{C}_p$$

define a place $\lambda(p)$ of T attached to the corresponding composition

$$T \hookrightarrow oe\mathbf{Q} \xrightarrow{i_p} \mathbf{C}_p.$$

Conjecture I. For every M over F of rang d with coefficients in T there esists a free I_M-module M_I of the same rang d, a Galois representation

$$r_I : G_F \to \mathrm{GL}(M_I),$$

an infinite subset $\mathcal{A}' \subset \mathcal{A}$ of "positive" characters, and a distinguished point $P_0 \in \mathcal{A}$ such that

(a) the reduced Galois representation

$$r_{I,P_0} : G_F \to \mathrm{GL}(M_{I,P_0})$$

is equivalent over $\mathbf{C}_p$ to the $\lambda(p)$ - adic representation $r_{M,\lambda(p)}$ of M at the distinguished place $\lambda(p)$;

(b) for every $P \in \mathcal{A}'$ there exists a motive M_P over F of the same rang d such that its Galois representation is equivalent over $\mathbf{C}_p$ to the reduction

$$r_{I,P} : G_F \to \mathrm{GL}(M_{I,P}).$$

We call the module M_P the *realization of Iwasawa* of M.

§4. A generalization of the Hasse invariant of a motive.

We define the *Hasse invariant* of a motive in terms of the *Newton polygons* and *the Hodge polygons* of a motive. Properties of these polygons are closely related to the notions of a p - ordinary and a p - admissible motive.

Now we are going to define the Newton polygon $P_{Newton,\sigma}(u) = P_{Newton,\sigma}(u, M)$ and the Hodge polygon $P_{Hodge,\sigma}(u) = P_{Hodge,\sigma}(u, M)$ attached to M, σ. First for $\mathfrak{p} = \mathfrak{p}(\sigma)$ we consider (using i_∞) the local $\mathfrak{p}$ - polynomial

$$L_\mathfrak{p}(M, X)^{-1} = 1 + A_1(\mathfrak{p})X + \cdots + A_d(\mathfrak{p})X^d$$

$$= (1 - \alpha^{(1)}(\mathfrak{p})X) \cdot (1 - \alpha^{(2)}(\mathfrak{p})X) \cdot \ldots \cdot (1 - \alpha^{(d)}(\mathfrak{p})X),$$

and we assume that its inverse roots are indexed in such a way that

$$\mathrm{ord}_p \alpha^{(1)}(\mathfrak{p}) \le \ \mathrm{ord}_p \alpha^{(2)}(\mathfrak{p}) \le \cdots \le \ \mathrm{ord}_p \alpha^{(d)}(\mathfrak{p})$$

Definition. The *Newton polygon* $P_{Newton,\sigma}(u)(0 \le u \le d)$ of M at $\mathfrak{p} = \mathfrak{p}(\sigma)$ is the convex hull of the points $(i, \mathrm{ord}_p A_i(\mathfrak{p}))$ $(i = 0, 1, \cdots, d)$.

The important property of the Newton polygon is that the length the horizontal segment of slope i is equal to the number of the inverse roots $\alpha^{(j)}(\mathfrak{p})$ such that $\mathrm{ord}_p \alpha^{(j)}(\mathfrak{p}) = i$ (note that this number may not necessarily be integer but this will be the case for the p - ordinary motives below).

The *Hodge polygon* $P_{Hodge,\sigma}(u)$ $(0 \le u \le d)$ of M at σ is defined using the Hodge decomposition of the d - dimensional $\mathbf{C}$ - vector space

$$M_\sigma = M_\sigma \otimes_T \mathbf{C} = \oplus_{i,j} M_\sigma^{i,j}$$

where $M_\sigma^{i,j}$ as a $\mathbf{C}$ - subspace. Note that the dimension $h_\sigma(i,j) = \dim_{\mathbf{C}} M_\sigma^{i,j}$ may depend on σ.

Definition. The Hodge polygon $P_{Hodge,\sigma}(u)$ is a function $[0,d] \to \mathbf{R}$ whose graph is a polygon which passes through the vertices

$$(0,0),\ \ldots,\ (\sum_{i' \le i} h_\sigma(i',j),\ \sum_{i' \le i} h_\sigma i' h_\sigma(i',j)),$$

so that the length of the horizontal segment of the slope i is equal to the dimension $h_\sigma(i,j)$.

Now we recall the definition of a p - ordinary motive in the simplest case $F = T = \mathbf{Q}$ (see [Co], [Co - PeRi]). We assume that M is pure of weight w and rank d. Let $G_p = \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ be the decomposition group (of a place in T over p) and

$$\psi_p : G_p \to \mathbf{Z}_p^\times$$

be the cyclotomic character of G_p. Then M is called p ordinary at p if the following conditions are satisfied:

(i) The inertia group $I_p \subset G_p$ acts trivially on each of the l - adic realizations M_l for $l \neq p$;

(ii) There exists a decreasing filtration $F_p^i V$ on $V = M_p = M_B \otimes \mathbf{Q}_p$ of $\mathbf{Q}_p$ - subspaces which are stable under the action of G_p such that for all $i \in \mathbf{Z}$ the group G_p acts on $F_p^i V / F_p^{i+1} V$ via some power of the cyclotomic character, say $\psi_p^{-e_i}$. Then

$$e_1(M) \geq \cdots \geq e_t(M)$$

and the following properties take place:

(a)

$$\dim_{\mathbf{Q}_p} F_p^i V / F_p^{i+1} V = h(e_i, w - e_i);$$

(b) The Hodge polygon and the Newton polygon of M coincide:

$$P_{Newton}(u) = P_{Hodge}(u).$$

If furthermore M is critical at $s = 0$ then it is easy to verify that the number d_p of the inverse roots $\alpha^{(j)}(p)$ with

$$\mathrm{ord}_p \alpha^{(j)}(p) < 0 \text{ is equal to } d^+ = d^+(M) \text{ of } M_\sigma^+.$$

In the general case (of a motive M over F with coefficients in T) the notion of a p - ordinary motive can be defined using the restriction of the ground field F to $\mathbf{Q}$ and the restriction of the coefficient field T to $\mathbf{Q}$ (the last operation corresponds to fogetting of the T - module structure on the realizations of M). In this way we get a motive M' over with coefficients in $\mathbf{Q}$ of the same weight w and the rank $\mathrm{rk}(M') = [F:\mathbf{Q}][T:\mathbf{Q}] \cdot d$.

However, it turns out that the notion of a p-ordinary motive is too restrictive, and we introduce the following weaker form of it.

Definition. The motive M over F with coefficients in T is called *admissible* at p if for all $\sigma \in J_F$ we have that

$$P_{Newton,\sigma}(d_\sigma^+) = P_{Hodge,\sigma}(d_\sigma^+)$$

here $d_\sigma^+ = d_\sigma^+(M)$ is the dimension of M_σ, $\sigma \in J_F$.

In the general case we use the following vector quantity $h = (h_\sigma)_\sigma$ which is defined in terms of the difference between the Newton polygon and the Hodge polygon of M:

$$h_\sigma = P_{Newton,\sigma}(d_\sigma^+) - P_{Hodge,\sigma}(d_\sigma^+.$$

We call the vector $h = h(M) = (h_\sigma)_\sigma$ the *Hasse invariant* of M at p. Note the following important properties of the quantity h:

(i) $h = h(M)$ does not change if we replace M by its Tate twist.

(ii) $h = h(M)$ does not change if we replace M by its twist $M = M(\chi)$ with a Hecke character χ of finite order whose conductor is prime to $\mathfrak{p}$.

(iii) $h = h(M)$ does not change if we replace M by its dual $M^\vee$

In the next section we state in terms of this quantity a general conjecture on p - adic L - functions.

§5. A conjecture on the existence of certain families of p - adic L - functions.

We are going to describe families of p - adic L - functions as certain analytic functions on the total analytic space, the $\mathbf{C}_p$ - analytic Lie group

$$\mathcal{X}_{M,p} = \mathrm{Hom}_{contin}(GH_M,\ \mathbf{C}_p^\times),$$

which contain the $\mathbf{C}_p$ - analytic Lie subgroup (the cyclotomic line) $\mathcal{X}_p \subset \mathcal{X}_{M,p}$:

$$\mathcal{X}_p = \mathrm{Hom}_{contin}(\mathrm{Gal}_p,\ \mathbf{C}_p^\times).$$

In order to do this we need a modified L - function of a motive over F. Following J.Coates this modified L - function has a form appropriate for further use in the p - adic construction. First we multiply $L(M,s)$ by an appropriate factor at infinity and define

$$\Lambda_{(\infty)}(M,\ s) = E_\infty(M,\ s)L(M,s)$$

as $\Lambda_{(\infty)}(\tau,\ R_{F/\mathbf{Q}}M,\ \rho, s)$ in the notation of J.Coates [Co] with $\rho = i$ so that $E_\infty(M,s) = E_\infty(\tau, R_{F/\mathbf{Q}}M, \rho, s)$ is the modified Γ - factor at infinity which actually does not depend on the fixed embedding τ of T into $\mathbf{C}$. Also we put

$$\Omega^\nu(M) = (\Omega^\nu(M)^{(\tau)}) = c^\nu(RM)(2\pi i)^{r(RM)} \in (T \otimes \mathbf{C})^\times$$

where

$$\nu = (-1)^m, r(RM) = \sum_{j<0} jh(i,j,R_{F/\mathbf{Q}}M) = \sum_{j<0} jh(i,j,M), \quad n = [F:\mathbf{Q}],$$

$c^\nu(RM) = c^\nu(R_{F/\mathbf{Q}}M)$ is the period of $R_{F/\mathbf{Q}}M$. Then the period conjecture of Deligne can be stated in the following convenient form: if $s = 0$ is critical for M then for any m such that $M(m)$ is critical at $s = 0$ we have that

$$\frac{\Lambda_\infty(M(m),0)}{\Omega^\nu(M)} \in T.$$

In order to deduce this statement from the original conjecture on critical values we can use the same arguments as in the J.Coates's work [Co], where it was shown that

$$E_\infty(M,\ 0) \sim (2\pi i)^{r(RM)} \bmod \mathbf{Q}^\times,$$

and it follows that

$$E_\infty(M(m),\ 0) \sim (2\pi i)^{r(RM)-md^\epsilon(RM)} = (2\pi i)^{n(r(M)-md^\epsilon(M))} \bmod \mathbf{Q}^\times,$$

where $\varepsilon = +$ if $j < 0$ and $\varepsilon = -$ if ≥ 0 for $j = w/2$. If we combine this fact with the equivalence

$$c^+(M(m)) \sim (2\pi i)^{d^\nu nm} c^\nu(M) \bmod T^\times$$

we deduce from the above form of the conjecture that

$$\Lambda_{(\infty)}(M(m),\ 0) \sim (2\pi i)^{n(r(M)-md^\epsilon(M)+md^\nu(M))} c^\nu(M)$$

Note that in our situation we have that $d^\epsilon(M) = d^\nu(M)$ because both M and $M(m)$ are critical at $s = 0$: we have that $\nu = +$ only for $j - m < 0$ because $M(m)$ is critical but according to Lemma 3 in [Co] the condition $j < 0$ is equivalent in this situation to $j - m < 0$.

Modified conjecture on the critical values. Assume that M is critical at $s = 0$. Then there exist constants $c^{\varepsilon_\sigma}(\sigma, M) \in (T \otimes \mathbf{C})^\times (\varepsilon_\sigma = \pm)$ defined modulo $T^\times$ such that if we put for a given sign $\varepsilon_0 = (\varepsilon_{0,\sigma}) \in \mathrm{Sgn}_F$

$$\Omega(\varepsilon_0, M) = (1 \otimes (2\pi i))^{nr(M)} \prod_\sigma c^{\varepsilon_{0,\sigma}}(\sigma, M)$$

with $r(M) = \sum_{j<0} jh(i,j,M)$ then for any integer m and Hecke character χ such that $M(\chi)(m)$ is critical at $s = 0$ and $\varepsilon_\sigma(\chi)\nu = \varepsilon_{0,\sigma}$ we have that

$$\Lambda_{(\infty)}(M(\chi)(m),\ 0)((G(\chi)^{-1}(1 \otimes D_F^{1/2}))^{d^{\varepsilon_0}(M)}\Omega(\varepsilon_0, M))^{-1} \in T(\chi)$$

where $\nu = sgn((-1)^m) = \pm$.

We recall that by definition

$$E_\infty(M, s) = E_\infty(\tau, R_{F/\mathbf{Q}}M, \rho, s) = E_\infty(U, \rho, s),$$

where U runs over direct summands of the Hodge decomposition, $\rho = i$ and $E_\infty(U, \rho, s)$ is given by:

(a) If $U = M^{j,k} \oplus M^{k,j}$ with $j < k$, then $E_\infty(U, \rho, s) = \Gamma_{\mathbf{C},\rho}(s - j)^{h(j,k)}$;

(b) If $U = M^{k,k}$ with $k \geq 0$, then $E_\infty(U, \rho, s) = 1$;

(c) If $U = M^{k,k}$ with $k < 0$, then $E_\infty(U,\rho,s) = R_\infty(U,\rho,s)$. Here $\rho^{-s} = \exp(-\rho\pi s/2)$, $\Gamma_{\mathbf{C},\rho}(s) = \rho^{-s}\Gamma_{\mathbf{C}}(s)$, $\Gamma_{\mathbf{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$, $\Gamma_{\mathbf{R}}(s) = \pi^{-s/2}\Gamma(s/2)$,

$$R_\infty(U,\rho,s) = L_\infty(\tau,U,\ s)/(\varepsilon_\infty(\tau,U,\rho,s)L_\infty(\tau,\ U^\vee(1),\ -s))$$

with L - and ε - factors described in [De3] on p.329, so that we have in case (c)

$$R_\infty(U,\rho,s) = \frac{\Gamma_{\mathbf{R}}(s-k+\delta)}{i^\delta\Gamma_{\mathbf{R}}(1-s+k-\delta)} = \frac{2\Gamma(s-k+\delta)\cos(\pi(s-k+\delta)/2}{i^\delta(2\pi)^{s-k+\delta}}$$

where $\delta = 0,1$ is chosen according with the sign of the scalar action of ρ_σ on $U = M_\sigma^{k,k}$ so that ρ_σ acts as $(-1)^{k+\delta}$.

We define

$$\Lambda_{p,(\infty)}(M(m)(\chi),s) =$$
$$\Lambda_{(\infty)}(M(m)(\chi),s)(G(\chi)^{-1}D_F^{1/2})^{d^{\epsilon_0}(M(m)(\chi))}\prod_{\mathfrak{p}\not|p}A_{\mathfrak{p}}(M(m)(\chi),s),$$

where

$$A_{\mathfrak{p}}(M(\chi),s) = \begin{cases} \prod_{i=d^++1}^{d}(1-\chi(\mathfrak{p})\alpha^{(i)}(\mathfrak{p})\mathcal{N}\mathfrak{p}^{-s})\prod_{i=1}^{d^+}(1-\chi^{-1}(\mathfrak{p})\alpha^{(i)}(\mathfrak{p})^{-1}\mathcal{N}\mathfrak{p}^{s-1}) \\ \qquad\qquad\qquad \text{for } \mathfrak{p}\not|\,\mathfrak{c}(\chi) \\ \prod_{i=1}^{d^+}\left(\frac{\mathcal{N}\mathfrak{p}^s}{\alpha^{(i)}(\mathfrak{p})}\right)^{\mathrm{ord}_{\mathfrak{p}}\mathfrak{c}(\chi)}, \text{otherwise.} \end{cases}$$

Let $\mathcal{A}$ be the discrete subgroup $\mathcal{A}$ of arithmetical characters,

$$\chi\cdot\eta\cdot Ncx_p^m = (\chi,\eta,m)\in\mathcal{A},$$

$\mathcal{A}'\subset\mathcal{A}$ the subset of "positive" $P_0\in\mathcal{A}$ characters, a distinguished point of conjecture I. Let $\mathcal{A}''\subset\mathcal{A}'$ be the subset of critical elements, which consists of those P, for which the corresponding motives M_P are critical (at $s=0$). Now we are ready to formulate the following

Conjecture II. For a canonical choice of periods $\Omega(P)\in\mathbf{C}^\times$ for $P\in\mathcal{A}''$ there exists a $\mathbf{C}_p$ - meromorphic function

$$\mathcal{L}_M : \mathcal{X}_{M,p}\to\mathbf{C}_p$$

with the properties:

(i)

$$\mathcal{L}_M(P) = \frac{\Lambda_{p,(\infty)}(M(m)(\chi),0)}{\Omega(P)}$$

for almost all $P\in\mathcal{A}''$;

(ii) For arithmetic points of type

$$P = (\chi,\eta,m)\in\mathcal{A}''$$

with η fixed there exists a finite set $\Xi \subset \mathcal{X}_{M,p}$ of p - adic characters and positive integers $n(\xi)$ (for $\xi \in \Xi$) such that for any $g_0 \in \mathrm{Gal}_p$ we have that the function

$$\prod_{\xi\in\Xi}(x(g_0) - \xi(g_0))^{n(\xi)}\mathcal{L}_M(x \cdot P)$$

is holomorphic on $\mathcal{X}_p$;

(iii) For arithmetic points of type

$$P = (\chi, \eta, m) \in \mathcal{A}''$$

with η fixed the function in (ii) is bounded if and only if the Hasse invariant $h(P) = h(M_P)$ vanishes;

(iv) In the general case the function $\mathcal{L}_M(P \cdot x)$ of $x \in \mathcal{X}_p$ is of logarithmic growth type $o(\log \mathcal{N}(\cdot)^h{}_0$ with

$$h_0 = [\max_\sigma h_\sigma] + 1.$$

§6. Examples.

6.1. Hecke characters of CM-type. Let $K \supset F$ be a totally imaginary quadratic extension, and $\eta : \mathbf{A}_K^\times/K^\times \to \mathbf{C}^\times$ be an algebraic Hecke's Grössencharakter such that

$$\eta((\alpha)) = \left(\frac{\alpha^{\phi_1}}{|\alpha^{\phi_1}|}\right)^{w_1} \cdots \left(\frac{\alpha^{\phi_n}}{|\alpha^{\phi_n}|}\right)^{w_n} \cdot \mathcal{N}(\alpha)^{w_0/2-1}$$

for $\alpha \in K, \alpha \equiv 1(\bmod \mathfrak{c}(\eta))$, where $\Sigma = bra\sigma_i : K \to \mathbf{C}$, is a fixed CM-type of K, w_i are positive integers, $w_0 = \max_i w_i$. Then there exists a Hilbert modular form $\mathbf{f}$ of weight $k = (w_1+1, \cdots, w_n+1)$ such that $L(s, \mathbf{f}) = L(s, \eta)$, and $M(\mathbf{f})$ coincides with the motive $M(\eta) = R_{K/F}[\eta]$ obtained by restriction of scalars from the motive $[\eta]$ (the last motive exists as an object of the category of motives of CM-type, see [Bl1]).

The Hodge structure of $M(\eta)_\sigma$ has the form

$$((w_0 - w_\sigma)/2, (w_0 + w_\sigma)/2) + ((w_0 + w_\sigma)/2, (w_0 - w_\sigma)/2),$$

Let

$$\mathfrak{p} = \mathfrak{p}_\sigma = \begin{cases} \mathfrak{P}\mathfrak{P}', & \text{if } \mathfrak{p} \text{ splits in } K, \\ \mathfrak{P}, & \text{if } \mathfrak{p} \text{ is inert in } K. \end{cases}$$

Then the local factor of $L(M(\eta), s)$ is given by

$$L_\mathfrak{p}(M(\eta), X)^{-1} = \begin{cases} (1 - \eta(\mathfrak{P}X))(1 - \eta(\mathfrak{P}')X), & \text{if } \mathfrak{p} \text{ splits in } K, \\ (1 - \eta(\mathfrak{P})^2, & \text{if } \mathfrak{p} \text{ is inert in } K. \end{cases}$$

Therefore the generalized Hasse invariant $h = (h_\sigma)_\sigma$ of $M(\eta)$ is given by

$$h_\sigma = \begin{cases} 0, & \text{if } \mathfrak{p} \text{ splits in } K, \\ w_0/2, & \text{if } \mathfrak{p} \text{ is inert in } K. \end{cases}$$

In the additive notation the type of η can be written in the following form:

$$\sum_\sigma \frac{w_\sigma}{2}(\sigma - \overline{\sigma}) + \frac{w_0}{2}\sum_\sigma(\sigma + \overline{\sigma}) = \sum_\sigma d_\sigma(\sigma - \overline{\sigma}) + m_0 \sum_\sigma \sigma,$$

where $m_0 = w_0, d_\sigma = (w_\sigma - w_0)/2$. Using a shift one sees that the point $s = m$ is critical for $L(s,\eta)$ iff $s = 0$ is critical for the character $\lambda(\mathfrak{a}) = \eta(\mathfrak{a})\mathcal{N}\mathfrak{a}^{-m}$.

Then one sees that the character η of the type $\sum_\sigma d_\sigma(\sigma - \overline{\sigma}) + m_0 \sum_\sigma \sigma$ is critical at 0 iff

$$m_0, d_\sigma \geq 0 \quad \text{or } m_0 \leq 1, d_\sigma \geq 1 - m_0.(\text{ for all } \sigma) \qquad (*)$$

In order to state the thorem of Katz on p-adic L-functions of CM-fields (in a simplified form) we let $\mathfrak{C} \subset \mathcal{O}_K$ denote an integer ideal of the maximal order of K, $G_\infty(\mathfrak{C})$ the ray class group of K of conductor $\mathfrak{C}p^\infty$.

For each CM-type Σ one can canonicaly choose a constants

$$\Omega_\infty = (\Omega_\infty(\sigma))_{\sigma\in\Sigma} \in (\mathbf{C}^\times)^n,$$

$$\Omega_p = (\Omega_p(\sigma))_{\sigma\in\Sigma} \in (\mathbf{C}_p^\times)^n$$

(complex and p-adic periods).

The theorem of Katz states that under the assumption $h = 0$ there exists a bounded p-adic measure μ on $G_\infty(\mathfrak{C})$ such that for all critical characters λ of conductor dividing $\mathfrak{C}p^\infty$ the value of the p - adic integal

$$\frac{\int_{G_\infty(\mathfrak{C})} \hat{\lambda}\, d\mu}{\Omega_p^{m_0\Sigma+2d}}$$

coinsides essentially with the normalized special value

$$frac\Lambda_{(p,\infty)}(\lambda,0),\Omega_\infty^{m_0\Sigma+2d},$$

where $\hat{\lambda}$ denotes the p - adic avatar of λ.

This theorem provides an example of a p -adic family of Conjectures I and II, because $\mathrm{End}_T(M)$ is essentially K, and by class field theory GH_M is related to $G_\infty(\mathfrak{C})$.

6.2. Families of Hida of Hilbert modular forms. In this case we start from a motive $M(\mathbf{f})$ attached to a (general) Hilbert modular form and obtain the group $GH_{\mathbf{f}} = \mathcal{O}_{F,T,p}^\times \times \mathrm{Gal}_p$w hose characters parametrize "the weights" of Hilbert modular forms in the corresponding family. **7. Hilbert modular forms and motives associated with them.**

We use the notation of Shimura [Shi6], [Shi10] and we regard the group $GL_2(F)$ as the group $G_\mathbf{Q}$ of all $\mathbf{Q}$ - rational points of a certain $\mathbf{Q}$ - subgroup $G \subset GL_{2n}$. Then Hilbert modular forms will be regarded as complex fuctions on the adelic group $G_\mathbf{A} = G(\mathbf{A})$ which is apparently identified with the product

$$GL_2(F_\mathbf{A}) = G_\infty \times G_{\widehat{\mathbf{Q}}}$$

where

$$G_\infty = GL_2(F_\infty) \cong GL_2(\mathbf{R})^n,\ G_{\widehat{\mathbf{Q}}} = GL_2(\widehat{F}),$$

$\mathbf{A}$, $F_{\mathbf{A}}$ denote the rings of finite adels of $\mathbf{Q}$ and F respectively.
The subgroup

$$G_\infty^+ = GL_2^+(F_\infty) \cong GL_2^+(\mathbf{R})^n$$

consists of all elements

$$\alpha = (\alpha_1, \cdots, \alpha_n), \alpha = \begin{pmatrix} a\ b \\ c\ d \end{pmatrix}$$

such that $\det\alpha_\nu > 0$, $\nu = 1, 2, \cdots, n$. Every element $\alpha \in G_\infty^+$ acts on the product $\mathfrak{H}^n$ of n copies of the upper half plane according to the formula

$$\alpha(z_1,\ \cdots,\ z_n) = (\alpha_1(z_1),\ \cdots,\ \alpha_n(z_n)),$$

where

$$\alpha_\nu(z_\nu) = (a_\nu z_\nu + b_\nu)/(c_\nu z_\nu + d_\nu)\quad (\nu = 1,\ 2,\ \cdots,\ n)$$

For $z = (z_1, z_2, \cdots, z_n) \in \mathfrak{H}^n$ we put $e_F(z) = e(\{z\}), \{z\} = z_1 + \ldots + z_n$ and $e(x) = \exp(2\pi i x)$ and we use the notations $\mathcal{N}z = z_1 \cdot \ldots \cdot z_n$, and $\mathbf{i} = (i,\ \ldots,\ i)$. For $\alpha \in G_\infty^+$, an integer n - tuple $k = (k_1, \cdots, k_n)$ and an arbitrary function $f{:}\mathfrak{H}^n \to \mathbf{C}$ we use the notation

$$(f\mid_k \alpha)(z) = \prod_\nu (c_\nu z_\nu + d_\nu)^{-k_\nu} f(\alpha(z)) \det(\alpha_\nu)^{k_\nu/2}$$

Let $\mathfrak{c}_\mathfrak{p} \subset \mathcal{O}_F$ be an integral ideal, $\mathfrak{c} = \mathfrak{c}\mathcal{O}_\mathfrak{p}$ its $\mathfrak{p}$ - part, $\mathfrak{d}_\mathfrak{p} = \mathfrak{d}\mathcal{O}_\mathfrak{p}$ the local different. We shall need the open compact subgroups $W = W_\mathfrak{c} \subset G_\mathbf{A}$ defined by

$$W_\mathfrak{c} = G_\infty^+ \times \quad W_\mathfrak{c}(\mathfrak{p}),$$

$$W_\mathfrak{c}(\mathfrak{p}) = \begin{pmatrix} a\ b \\ c\ d \end{pmatrix} \in GL_2(F_\mathfrak{p}) \mid b \in \mathfrak{d}_\mathfrak{p}^{-1},\ c \in \mathfrak{d}_\mathfrak{p}\mathfrak{c}_\mathfrak{p},\ a, d \in \mathcal{O}_\mathfrak{p}, ad - bc \in \mathcal{O}_\mathfrak{p}^\times$$

By a Hilbert automorphic form of the weight $k = (k_1, \cdots, k_n)$, the level $\mathfrak{c}$, and the Hecke character ψ we mean a function

$\mathbf{f} : G_\mathbf{A} \to \mathbf{C}$ satisfying the following conditions (7.1) – (7.3):

$$\mathbf{f}(s\alpha x) = \psi(s)\mathbf{f}(x) \text{ for all } x \in G_\mathbf{A},$$
$$s \in F_\mathbf{A}^\times \text{ (the center of } G_\mathbf{A}), \text{ and } \alpha \in G_\mathbf{Q} \tag{7.1}$$

We let $\psi_0{:}(\mathcal{O}/\mathfrak{c})^\times \to \mathbf{C}^\times$ denote the $\mathfrak{c}$ - part of the character ψ and the extend the definition of ψ over the group $W_\mathfrak{c}$ by the formula

$$\psi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \psi_0(a_\mathfrak{c} \bmod \mathfrak{c})$$

then for all $x \in G_\mathbf{A}$

$$\mathbf{f}(xw) = \psi(w^\iota)\mathbf{f}(x) \text{ for } w \in W_\mathfrak{c} \text{ with } w_\infty = 1, \tag{7.2}$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\iota = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

If

$$w = w(\theta) \text{ where } w(\theta) = (w_1(\theta_1), \ \cdots, \ w_n(\theta_n)),$$
$$w_\nu(\theta_\nu) = \begin{pmatrix} \cos\theta_\nu & -\sin\theta_\nu \\ \sin\theta_\nu & \cos\theta_\nu \end{pmatrix},$$

then

$$\mathbf{f}(xw(\theta)) = \mathbf{f}(x)\exp(-i(k_1\theta_1 + ... + k_n\theta_n)).$$

An automorphic form $\mathbf{f}$ is called a cusp form if

$$\int_{F_\mathbf{A}/F} \mathbf{f}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} g \, dt = 0 \text{ for all } g \in G_\mathbf{A}. \tag{7.3}$$

The vector space $\mathcal{M}_k(\mathfrak{c}, \psi)$ of Hilbert automorphic forms of holomorphic type is defined as the set functions satisfying (7.1)–(7.3) and the following holomorphy condition (7.4): for any $x \in G_\mathbf{A}$ with $x_\infty = 1$ there exists a holomorphic function $g_x{:}\mathfrak{H}^n \to \mathbf{C}$ such that for all $y = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_\infty^+$ we have that

$$\mathbf{f}(xw) = (g_x \mid_k w)(\mathbf{i}) \tag{7.4}$$

(in the case $F = \mathbf{Q}$ we must also require that the function g_x is holomorphic at the cusps). Let $\mathcal{S}_k(\mathfrak{c}, \psi) \subset \mathcal{M}_k(\mathfrak{c}, \psi)$ be the subspace of cusp forms.

Hecke operators which act on $\mathcal{S}_k(\mathfrak{c}, \psi)$ and $\mathcal{M}_k(\mathfrak{c}, \psi)$ are introduced by means of the double cosets of the type WyW for y in the semigroup

$$Y_\mathfrak{c} = G_\mathbf{A} \cap (G_\infty^\times \times \ \dot{Y}_\mathfrak{c}(\mathfrak{p}),$$

where

$$Y_\mathfrak{c}(\mathfrak{p}) = \left\{ \begin{pmatrix} a\, b \\ c\, d \end{pmatrix} \in GL_2(F_\mathfrak{p}) \mid b \in \mathfrak{d}_\mathfrak{p}^{-1}, \ c \in \mathfrak{d}_\mathfrak{p}\mathfrak{c}_\mathfrak{p}, \ a, \ d \in \mathcal{O}_\mathfrak{p}, \ a\mathcal{O}_\mathfrak{p} + \mathfrak{c}_\mathfrak{p} = \mathcal{O}_\mathfrak{p} \right\}.$$

The Hecke algebra $\mathcal{H}_\mathfrak{c}$ consists of all formal finite sums of the type $\sum_y c_y WyW$, with the multiplication in $\mathcal{H}_\mathfrak{c}$ defined by a standard rule. By definition $T_\mathfrak{c}(m)$ is the element of $\mathcal{H}_\mathfrak{c}$ obtained by taking the sum of all different WwW with $w \in Y_\mathfrak{c}$ such that $\mathrm{div}(\det(y)) = \mathfrak{m}$. Let

$$T_\mathfrak{c}(\mathfrak{m})' = \mathcal{N}(\mathfrak{m})^{(k_0-2)/2} T_\mathfrak{c}(\mathfrak{m})$$

be the normalized Hecke operator, where k_0 denotes the maximal component of the weight k. Suppose that $\mathbf{f} \in \mathcal{S}_k(\mathfrak{c}, \psi)$ is an eigenform of all $T_{\mathfrak{c}}(\mathfrak{m})'$ with the eigenvalues $C(\mathfrak{m}, \mathbf{f})$. Then there is the following Euler product expansion:

$$L(s,\mathbf{f}) = \sum_{\mathfrak{n}} C(\mathfrak{n},\mathbf{f})\mathcal{N}\mathfrak{n}^{-s} = \prod_{\mathfrak{p}} (1 - C(\mathfrak{p},\mathbf{f})\mathcal{N}\mathfrak{p}^{-s} + \psi(\mathfrak{p})\mathcal{N}\mathfrak{p}^{k_0-1-2s})^{-1}$$

All of the numbers $C(\mathfrak{n}, \mathbf{f})$ are known to be algebraic integers.

Let $\mathbf{f} \in \mathcal{S}_k(\mathfrak{c}, \psi)$ be a primitive Hilbert cusp eigenform. In this case the numbers $C(\mathfrak{n}, \mathbf{f})$ can be regarded as the normalized Fourier coefficients of $\mathbf{f}$. The important analytic property of the corresponding L - function $L(\mathbf{f}, s)$ (see [Shi6], p.655) is that it admits a holomorphic analytic continuation onto the entire complex plane, and if we set

$$\Lambda(\mathbf{f},s) = \prod_{i=1}^{n} \Gamma_{\mathbf{C}}(s - (k_0 - k_i)/2) L(\mathbf{f},s)$$

then $\Lambda(\mathbf{f},s)$ satisfies a functional equation, which expresses $\Lambda(\mathbf{f},s)$ in terms of the function $\Lambda(\mathbf{f}^\rho, k_0 - s)$. According to the general conjecture on the analytic properties of the L - functions of motives we may suggest that $\mathbf{f}$ should correspond to a motive $M = M(\mathbf{f})$ over F of rank 2, and weight k_0 with coefficients in a field T containing all $C(\mathfrak{n}, \mathbf{f})$ such that

$$L(M,s) = L(s,\mathbf{f}),\ \Lambda(M,s) = \Lambda(s,\mathbf{f})$$

and for fixed embeddings $\tau \in J_T$ and $\sigma = \sigma_i \in J_F$ the Hodge decomposition of M_{σ_i} is given by

$$M_{\sigma_i}^{(\tau)} = M_{\sigma_i} \otimes_{T,\tau} \mathbf{C} = \tag{7.5}$$

$$M_{\sigma_i}^{(\tau)(k_0-k_i)/2,(k_0+k_i)/2-1} \oplus M_{\sigma_i}^{(\tau)(k_0+k_i)/2-1,(k_0-k_i)/2}$$

where k_i is the component of the weight k, attached to the fixed embedding σ_i (as was mentioned above this decomposition may depend on τ and σ_i). It is obvious from (7.5) that if such motive exists then the weight k must saisfy the condition $k_1 \equiv k_2 \equiv \ldots \equiv k_n \bmod 2$.

There are several confirmation of the conjecture. First of all it is known in the elliptic modular case $F = \mathbf{Q}$ due to U.Jannsen and A.J.Scholl [Ja], [Scho]; the existence of the Galois representations of $\mathrm{Gal}(\overline{F}/F)$ corresponding to λ - adic realizations of these motives was discovered earlier by Deligne [Del]. If we restrict such representation to the subgroup we obtain the L - function of certain Hilbert modular form of the same (scalar) weight which is the Doi - Naganuma lift (or "base change") of the original elliptic cusp form. In the general case the existence of Galois representations attached to Hilbert modular forms was established by Rogawski - Tunnell [Ro-Tu] and Ohta [Oh] (n odd) (under a local hypothesis) and by R.Taylor [Ta] in the general case. Also a number of results on special values of the fuction $L(s, \mathbf{f})$ is known, which math the above conjectures on the critical values and on the p - adic L - functions [Shi1], [Man], [Ka1]. As in the elliptic modular case there is a conjectural link between motives of the type $M(\mathbf{f})$ and the cohomology of certain Kuga - Shimura variety (fiber product of several copies of the universal Hilbert - Blumenthal abelian variety with a fixed level structure and and endomorphisms): namely, for the decomposition $R_{F/\mathbf{Q}}M =_i {}_1M^{\sigma_i}$ the tensor product $\otimes_{i=1}^n M^{\sigma_i}$ is a motive over $\mathbf{Q}$ of rank 2^n which conjecturally lies in the above cohomology, see the interesting discussion of this link in [Ha2], [Oda]. In case $k_1 = ... = k_n = 2$ the motives have the Hodge type $H^{0,1} \oplus H^{1,0}$. In some cases (e.g. when n is odd) the motives M^{σ_i} can be realized as factors of Jacobians of Shimura curves corresponding to quaternion algebras, which split at one fixed infinite place σ_i and ramified at all other infinite places $\sigma_j (j \neq i)$ ([Shi7]; see also forthcoming work of M. Harris).

8. Examples.

8.1. Periods of Hilbert cusp forms.

Let $\mathbf{f} \in \mathcal{S}_k(\mathfrak{c}, \psi)$ be a primitive Hilbert cusp eigenform which is supposed to be "motivic" in the sense of the previous section, and let

$$L(s, \mathbf{f}(\chi)) = \sum_{\mathfrak{n}} \chi(\mathfrak{n}) C(\mathfrak{n}, \mathbf{f}) \mathcal{N}\mathfrak{n}^{-s} = (1 - \chi(\mathfrak{p}) C(\mathfrak{p}, \mathbf{f}) \mathcal{N}\mathfrak{p}^{-s} + \chi^2(\mathfrak{p}) \psi(\mathfrak{p}) \mathcal{N}\mathfrak{p}^{k_0 - 1 - 2s})^{-1}.$$

Then the critical strip of $L(s, \mathbf{f}(\chi))$ is given by $m_* \le m \le m^*$,

$$m_* = \max\{(k_0 - k_i)/2\} + 1, \quad m^* = \min\{(k_0 + k_i)/2\} - 1.$$

Using the Rankin - Selberg method G.Shimura proved that there exist constants

$$c^{\pm}(\sigma, \mathbf{f}) \in (T \otimes \mathbf{C})^{\times}$$

defined modulo $T^{\times}$ such that if we put

$$c^{\varepsilon}(\chi, \mathbf{f}) = D_F^{-1/2} G(\chi) \prod_{\sigma \in J_F} c^{\varepsilon \cdot \varepsilon_\sigma(\chi)}(\sigma, \mathbf{f})$$

then for all $m \in \mathbf{Z}, m_* \le m \le m^*$ we have that

$$\frac{i^{nm} \Lambda(m, \ \mathbf{f}(\chi))}{c^{\nu}(\chi, \mathbf{f})} \in T(\chi),$$

where $\nu = (-1)^m$.

This statement coincides with that of the modified period conjecture if we take for $c^{\pm}(\sigma, M(\mathbf{f}))$ the quantities $c^{\pm}(\sigma, \mathbf{f})$.

In order to formulate the results on p - adic L - functions, put

$$1 - C(\mathfrak{p}, \mathbf{f})X + \psi(\mathfrak{p})\mathcal{N}\mathfrak{p}^{k_0 - 1}X^2 = (1 - \alpha(\mathfrak{p})X)(1 - \alpha'(\mathfrak{p})X) \in \mathbf{C}_p[X]$$

where $\alpha(\mathfrak{p})$, $\alpha'(\mathfrak{p})$ are the inverse roots of the Hecke polynomial assuming that

$$\mathrm{ord}_p \alpha(\mathfrak{p}) \le \mathrm{ord}_p \alpha'(\mathfrak{p}).$$

Note that in the p - ordinary case we should have

$$\mathrm{ord}_p\alpha(\mathfrak{p}) = (k_0 - k_i)/2, \mathrm{ord}_p\alpha'(\mathfrak{p}) = (k_0 + k_i)/2 - 1$$

for the prime $\mathfrak{p} = \mathfrak{p}_i = \mathfrak{p}(\sigma_i)$ attached to the embedding σ_i (see Section 5).

8.2. Theorem. Put $h = [\max(\mathrm{ord}_p(\alpha(\mathfrak{p}(\sigma_i)) - (k_0 - k_i)/2)] + 1$. Then for each sign $\varepsilon_0 = \{\varepsilon_{0,\sigma}\} \in \mathrm{Sgn}_F$ there exists a $\mathbf{C}_p$ - analytic function $L_{(p)}^{(\varepsilon_0)}$on $\mathcal{X}_p$ of the type $o(\log^h)$ with the properties:

(i) for all $m \in \mathbf{Z}$, $m_* \le m \le m^*$, and for all Hecke characters of finite order $\chi \in \mathcal{X}_p^{\mathrm{tors}}$ with $\nu\varepsilon_\sigma(\chi) = \varepsilon_{0,\sigma}$ $(\sigma \in J_F)$ the following equality holds

$$L_{(p)}^{(\varepsilon_0)}(\chi\mathcal{N}x_p^m) = \frac{D_F^m i^{mn}}{G(\chi)} \prod_{\mathfrak{p}|p} A_\mathfrak{p}(\mathbf{f}(\chi), m) \cdot \frac{\Lambda(\mathbf{f}(\chi), m)}{\Omega(\varepsilon_0,\ \mathbf{f})},$$

where

$$A_\mathfrak{p}(\mathbf{f}(\chi), m) = \begin{cases} (1 - \chi(\mathfrak{p})\alpha'(\mathfrak{p})\mathcal{N}\mathfrak{p}^{-m})(1 - \chi^{-1}(\mathfrak{p})\alpha(\mathfrak{p})^{-1}\mathcal{N}\mathfrak{p}^{m-1}), \\ \qquad \text{if } \mathfrak{p} \mid \mathfrak{c}(\chi) \\ \left(\frac{\mathcal{N}\mathfrak{p}^m}{\alpha(\mathfrak{p})}\right)^{\mathrm{ord}_\mathfrak{p}\mathfrak{c}(\chi)}, \\ \qquad \text{if } \mathfrak{p} \not| \mathfrak{c}(\chi) \end{cases}$$

and the constant $\Omega(\varepsilon_0,\ \mathbf{f})$ is given by

$$\Omega(\varepsilon_0,\ \mathbf{f}) = (2\pi i)^{-nm_*} \cdot D_F^{1/2} \cdot \prod_\sigma c^{\varepsilon_{0,\sigma}}(\sigma, \mathbf{f}),$$

(ii) If $h \le m^* - m_* + 1$ then the function $L_{(p)}^{(\varepsilon_0)}$ on $\mathcal{X}_p$ is uniquely determined by (i).

(iii) If

$$\max(\mathrm{ord}_p(\alpha(\mathfrak{p}(\sigma_i)) - (k_0 - k_i)/2) = 0$$

then the function $L_{(p)}^{(\varepsilon_0)}$ is bounded on $\mathcal{X}_p$.

In the p - ordinary case this theorem was established by Yu.I.Manin (in a less explicit form) using the theory of generalized modular symbols on Hilbert - Blumenthal modular varieties. The non p - ordinary case was treated only for $F = \mathbf{Q}$ by Višik [V1]. For an arbitrary totally real field F one can use the Rankin method and the technique of the Shimura's work [Shi6].

8.3. The Rankin convolution and the tensor product of motives. Let us consider the Rankin convolution

$$L(s, \mathbf{f}, \mathbf{g}) = \sum_\mathfrak{n} C(\mathfrak{n}, \mathbf{f})\, C(\mathfrak{n}, \mathbf{g})\mathcal{N}(\mathfrak{n})^{-s} \tag{8.1}$$

attached to two Hilbert modular forms $\mathbf{f}$, $\mathbf{g}$ over a totally real field F of degree $n = [F : \mathbf{Q}]$, where $C(\mathfrak{n}, \mathbf{f})$, $C(\mathfrak{n}, \mathbf{g})$ are normalized "Fourier coefficients" of $\mathbf{f}$ and $\mathbf{g}$, indexed by integral ideals $\mathfrak{n}$ of the maximal order $\mathcal{O}_F \subset F$ (see §7). We suppose that $\mathbf{f}$ is a primitive cusp form of vector weight $k = (k_1, \cdots, k_n)$, and $\mathbf{g}$ a primitive cusp form of weight $l = (l_1, \cdots, l_n)$ We assume that for a decomposition of J_F into a disjoint union $J_F = J \cup J'$ the following condition is satisfied

$$k_i > l_i \text{ (for } \sigma_i \in J), \quad \text{and} \quad l_i > k_i \text{ (for } \sigma_i \in J'). \tag{8.2}$$

Also, assume that

$$k_1 \equiv k_2 \equiv \cdots \equiv k_n \bmod 2, \tag{8.3}$$

and

$$l_1 \equiv l_2 \equiv \cdots \equiv l_n \bmod 2. \tag{8.4}$$

Let $\mathfrak{c}(\mathbf{f}) \subset \mathcal{O}_F$ denote the conductor and ψ the character of $\mathbf{f}$ and $\mathfrak{c}(\mathbf{g})$, ω denote the conductor and the character of $\mathbf{g}$ ($\psi, \omega : \mathbf{A}_F^\times / F^\times \to \mathbf{C}^\times$ being Hecke characters of finite order).

The essential property of the convolution

$$L(s, \mathbf{f}, \mathbf{g}(\chi)) = \sum_{\mathfrak{n}} \chi(\mathfrak{n}) C(\mathfrak{n}, \mathbf{f}) C(\mathfrak{n}, \mathbf{g}) \mathcal{N}(\mathfrak{n})^{-s}$$

(twisted with a Hecke characer χ of finite order) is the following Euler product decomposition

$$\begin{aligned} &L_{\mathfrak{c}}(2s + 2 - k_0 - l_0, \psi\omega\chi^2) L(s, \mathbf{f}, \mathbf{g}(\chi)) = \\ &\prod_{\mathfrak{q}} \left((1 - \chi(\mathfrak{q})\alpha(\mathfrak{q})\beta(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{-s})(1 - \chi(\mathfrak{q})\alpha(\mathfrak{q})\beta'(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{-s}) \times \right. \\ &\left. \times (1 - \chi(\mathfrak{q})\alpha'(\mathfrak{q})\beta(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{-s})(1 - \chi(\mathfrak{q})\alpha'(\mathfrak{q})\beta'(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{-s}) \right)^{-1}, \end{aligned} \tag{8.5}$$

where the numbers $\alpha(\mathfrak{q})$, $\alpha'(\mathfrak{q})$, $\beta(\mathfrak{q})$, and $\beta'(\mathfrak{q})$ are roots of the Hecke polynomials

$$X^2 - C(\mathfrak{q}, \mathbf{f})X + \psi(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{k_0 - 1} = (X - \alpha(\mathfrak{q}))(X - \alpha'(\mathfrak{q})),$$

and

$$X^2 - C(\mathfrak{q}, \mathbf{g})X + \omega(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{l_0 - 1} = (X - \beta(\mathfrak{q}))(X - \beta'(\mathfrak{q})).$$

The decomposition (8.5) is not difficult to deduce from the following elementary lemma on rational functions, applied to each of the Euler $\mathfrak{q}$-factors: if

$$\sum_{i=0}^{\infty} A_i X^i = \frac{1}{(1 - \alpha X)(1 - \alpha' X)}, \quad \sum_{i=0}^{\infty} B_i X^i = \frac{1}{(1 - \beta X)(1 - \beta' X)},$$

then

$$\sum_{i=0}^{\infty} A_i B_i X^i = \frac{1 - \alpha\alpha'\beta\beta' X^2}{(1 - \alpha\beta X)(1 - \alpha\beta' X)(1 - \alpha'\beta X)(1 - \alpha'\beta' X)}. \tag{8.6}$$

Assume that there exist motives $M(\mathbf{f})$ and $M(\mathbf{g})$ associated with $\mathbf{f}$ and $\mathbf{g}$. Then

$$L_{\mathfrak{c}}(2s + 2 - k - l, \psi\omega\chi^2) L(s, \mathbf{f}, \mathbf{g}(\chi)) = L(M[\chi], s)$$

where $M = M(\mathbf{f}) \otimes_F M(\mathbf{g})$ is the tensor product of motives over F with coefficients in some common number field T. Using the Hogde decompositions for $M(\mathbf{f})$ and $M(\mathbf{g})$ and the Künneth formula for $M = M(\mathbf{f}) \otimes_F M(\mathbf{g})$ we see that under our assumption the motive M has $d = 4$, $w = k_0 + l_0 - 2$, and the following Hodge type:

$$M_{\sigma_i} \otimes \mathbf{C} \cong$$
$$\oplus_{\tau \in J_T} \big(M_{\sigma_i}^{(k_0+l_0-k_i^\tau-l_i^\tau)/2,(k_0+l_0+k_i^\tau+l_i^\tau)/2-2)} \oplus M_{\sigma_i}^{(k_0+l_0+k_i^\tau+l_i^\tau)/2-2,(k_0+l_0-k_i^\tau-l_i^\tau)/2}$$
$$\oplus M_{\sigma_i}^{(k_0+l_0-|k_i^\tau-l_i^\tau|)/2-1,(k_0+l_0+|k_i^\tau-l_i^\tau|)/2-1} \oplus M_{\sigma_i}^{(k_0+l_0+|k_i^\tau-l_i^\tau|)/2-1,(k_0+l_0-|k_i^\tau-l_i^\tau|)/2-1}\big).$$

Moreover,

$$\Lambda(M[\chi], s) = \Lambda(s, \mathbf{f}, \mathbf{g}(\chi)) =$$
$$\prod_{i=1}^{n} \left(\Gamma_{\mathbf{C}}(s - (k_0 + l_0 - k_i - l_i)/2)\Gamma_{\mathbf{C}}(s - (k_0 + l_0 - |k_i - l_i|)/2 + 1)\right)\times$$
$$\times L_c(2s + 2 - k_0 - l_0, \psi\omega\chi^2)L(s, \mathbf{f}, \mathbf{g}(\chi)),$$

and this function satisfies a functional equation of the type $s \mapsto k_0 + l_0 - 2 - s$.

8.4. The critical values of the Rankin convolution. Let us now set

$$m_* = \max_i((k_0 + l_0 - |k_i - l_i|)/2 - 1), +1, \quad m^* = k_0 + l_0 - 2 - m_*.$$

The periods $c^{\pm}(\sigma, M)$ can be easily computed in terms of $c^{\pm}(\sigma, M)$ (as in the elliptic modular case; see a more general calculation in [Bl2]). As a result one gets that $c^{\pm}(\sigma, M) = c(\sigma, M)$ does not depend on the sign $\pm$, and is given by

$$c^{\pm}(\sigma, M) = \begin{cases} c^+(\sigma, \mathbf{f})c^-(\sigma, \mathbf{f})\delta(\sigma, \mathbf{g}), & \text{if } \sigma \in J \\ c^+(\sigma, \mathbf{g})c^-(\sigma, \mathbf{g})\delta(\sigma, \mathbf{f}), & \text{if } \sigma \in J'. \end{cases}$$

Moreover,

$$c^{\pm}(M[\chi]) = G(\chi)^{-2} \prod_{\sigma \in J} c^{\pm}(\sigma, M).$$

Let us apply the modified conjecture on special values to the L-function

$$\Lambda(M[\chi], s) = \Lambda(s, \mathbf{f}, \mathbf{g}(\chi)),$$

and set $c(\mathbf{f}, \mathbf{g}) = \prod_\sigma c^+(\sigma, M)$,

$$c(J, \mathbf{f}) = \prod_{\sigma \in J} c^+(\sigma, \mathbf{f})c^-(\sigma, \mathbf{f}), \ c(J', \mathbf{g}) = \prod_{\sigma \in J'} c^+(\sigma, \mathbf{g})c^-(\sigma, \mathbf{g}),$$

and

$$\delta(J, \mathbf{f}) = \prod_{\sigma \in J} \delta(\sigma, \mathbf{f}), \ \delta(J', \mathbf{g}) = \prod_{\sigma \in J'} \delta(\sigma, \mathbf{g}).$$

Then we see that

$$c(J, \mathbf{f})c(J', \mathbf{f}) = \langle \mathbf{f}, \mathbf{f} \rangle, \quad \delta(J, \mathbf{f})\delta(J', \mathbf{f}) = G(\psi)^{-1}(2\pi i)^{n(k_0-1)},$$
$$c(J, \mathbf{g})c(J', \mathbf{g}) = \langle \mathbf{g}, \mathbf{g} \rangle, \quad \delta(J, \mathbf{f})\delta(J', \mathbf{g}) = G(\omega)^{-1}(2\pi i)^{n(l_0-1)},$$

and
$$c(M[\chi]) = c^{\pm}(M[\chi]) = G(\chi)^{-2}c(J,\mathbf{f})\delta(J,\mathbf{g})c(J',\mathbf{g})\delta(J',\mathbf{f}).$$

With this notation the conjecture 1.8 takes the following form: for all Hecke characters χ of finite order and $r \in \mathbf{Z}, m_* \geq r \leq m^*$ we have that
$$\frac{\Lambda(r,\mathbf{f},\mathbf{g}(\chi))}{G(\chi)^{-2}c(J,\mathbf{f})\delta(J,\mathbf{g})c(J',\mathbf{g})\delta(J',\mathbf{f})} = \frac{\Lambda(M[\chi],r)}{G(\chi)^{-2}c(M)} \in \mathbf{Q}(\mathbf{f},\mathbf{g},\chi).$$

8.5. Let us consider the special case when $J' = \emptyset$, i.e. $k_i > l_i$ for all $\sigma_i \in J_F$. Then
$$c(J,\mathbf{f}) = c(J_F,\mathbf{f}) = \langle \mathbf{f},\mathbf{f}\rangle, \quad \delta(J,\mathbf{g}) = \delta(J_F,\mathbf{g}) = G(\omega)^{-1}(2\pi i)^{n(l_0-1)},$$
and the above property transforms to the following:
$$\frac{\Lambda(r,\mathbf{f},\mathbf{g}(\chi))}{G(\chi)^{-2}\langle \mathbf{f},\mathbf{f}\rangle, G(\omega)^{-1}(2\pi i)^{n(l_0-1)}} \in \mathbf{Q}(\mathbf{f},\mathbf{g},\chi),$$
where $\mathbf{Q}(\mathbf{f},\mathbf{g},\chi)$ denotes the subfield of $\mathbf{C}$ generated by the Fourier coefficients of $\mathbf{f}$ and $\mathbf{g}$, and the values of χ. This algebraicity property was established by G.Shimura [Sh1] by means of a version of the Rankin –Selberg method.

In the general case the above algebraicity property was also studies by G.Shimura [Sh2], [Sh3] (for some special Hilbert modular forms, coming from quaternion algebras) and by M.Harris [Ha3] using the theory of arithmetical vector bundles on Shimura varieties. The idea of the proof was to replace the original automorphic cusp form $\mathbf{f} : G(\mathbf{A}) \to \mathbf{C}$ of holomorphic type by another cusp form $\mathbf{f}^J : G(\mathbf{A}) \to \mathbf{C}$ such that
$$\mathbf{f}^J(g_1,\cdots,g_n) = \mathbf{f}(g_1j_1,\cdots,g_nj_n),$$
where $g_i \in \mathrm{GL}_2(\mathbf{R})$,
$$j_i = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \text{if } i \in J \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & \text{if } i \in J'. \end{cases}$$

Then $\mathbf{f}^J$ can be described by functions $\mathbf{f}^J_\chi$ on $\mathfrak{H}^n$, which are holomorphic in z_i $(i \in J)$ and antiholomorphic in z_i $(i \in J')$. Then the differential forms
$$\mathbf{f}^J_\chi \wedge_{i\in J} d\bar{z}_i$$
define a certain class $cl(\mathbf{f}^J)$ of the degree $|J|$ in the coherent cohomology of the Hilbert – Blumenthal modular variety, or rather its toroidal compactification ([Ha1], [Ha2]). This space of coherent cohomology has a natural rational structure over a certain number field F^J, defined in terms of canonical models. From the theory of new forms it follows that there exist a constant $\nu(J,\mathbf{f}) \in \mathbf{C}^\times$ such that the differential form attached to $\nu(J,\mathbf{f})^{-1}\mathbf{f}^J$ is rational over the extension of F^J obtained by adjoining the Hecke eigenvalues of $\mathbf{f}$. Then the critical values of the type $\Lambda(r,\mathbf{f},\mathbf{g})$ can be expressed in terms of a cup product of the form
$$cl(\mathbf{f}^J) \cup cl(\mathbf{g}^{J'}) \cup E,$$
where E is a (nearly) holomorphic Eisenstein series. Then the above algebraicity property can be deduced from the fact that the cup product preserves the rational structure

in the coherent cohomology. However, the technical details of the proof are quite difficult.

8.6. p-adic convolutions of Hilbert cusp forms. Now we give a precise description of the p-adic convolution of $\mathbf{f}$ and $\mathbf{g}$ assuming that both $\mathbf{f}$ and $\mathbf{g}$ are p-ordinary, i.e.

$$\begin{aligned} \mathrm{ord}_p\alpha(\mathfrak{p}_i) &= (k_0 - k_i)/2, \quad \mathrm{ord}_p\alpha'(\mathfrak{p}_i) = (k_0 + k_i)/2 - 1, \\ \mathrm{ord}_p\beta(\mathfrak{p}_i) &= (l_0 - l_i)/2, \quad \mathrm{ord}_p\beta'(\mathfrak{p}_i) = (l_0 + l_i)/2 - 1, \end{aligned}$$

or equivalently, $\mathrm{ord}_p C(\mathfrak{p}_i, \mathbf{f}) = (k_0 - k_i)/2$, and $\mathrm{ord}_p C(\mathfrak{p}_i, \mathbf{g}) = (l_0 - l_i)/2$. We assume also that the conductors of $\mathbf{f}$ and $\mathbf{g}$ are coprime to p and we set

$$\begin{aligned} &Ai_p(s, \mathbf{f}, \mathbf{g}(\chi)) = \\ &\prod_{\sigma_i \in J \setminus S(\chi)} (1 - \chi(\mathfrak{p}_i)\alpha'(\mathfrak{p}_i)\beta(\mathfrak{p}_i)\mathcal{N}\mathfrak{p}_i^{-s})(1 - \chi(\mathfrak{p}_i)\alpha'(\mathfrak{p}_i)\beta'(\mathfrak{p}_i)\mathcal{N}\mathfrak{p}_i^{-s}) \times \\ &\times (1 - \chi(\mathfrak{p}_i)^{-1}\alpha(\mathfrak{p}_i)^{-1}\beta(\mathfrak{p}_i)^{-1}\mathcal{N}\mathfrak{p}_i^{s-1})(1 - \chi(\mathfrak{p}_i)^{-1}\alpha(\mathfrak{p}_i)^{-1}\beta'(\mathfrak{p}_i)^{-1}\mathcal{N}\mathfrak{p}_i^{s-1}) \times \\ &\times \prod_{\sigma_i \in J' \setminus S(\chi)} (1 - \chi(\mathfrak{p}_i)\alpha(\mathfrak{p}_i)\beta'(\mathfrak{p}_i)\mathcal{N}\mathfrak{p}_i^{-s})(1 - \chi(\mathfrak{p}_i)\alpha'(\mathfrak{p}_i)\beta'(\mathfrak{p}_i)\mathcal{N}\mathfrak{p}_i^{-s}) \times \\ &\times (1 - \chi(\mathfrak{p}_i)^{-1}\alpha(\mathfrak{p}_i)^{-1}\beta(\mathfrak{p}_i)^{-1}\mathcal{N}\mathfrak{p}_i^{s-1})(1 - \chi(\mathfrak{p}_i)^{-1}\alpha'(\mathfrak{p}_i)^{-1}\beta(\mathfrak{p}_i)^{-1}\mathcal{N}\mathfrak{p}_i^{s-1}). \end{aligned}$$

Then we introduce the following constant:

$$\begin{aligned} &\Omega(\mathbf{f}, \mathbf{g}) = c(J, \mathbf{f})\delta(J, \mathbf{g})c(J', \mathbf{g})\delta(J', \mathbf{f}) = \\ &\prod_{\sigma \in J} c^+(\sigma, \mathbf{f})c^-(\sigma, \mathbf{f})\delta(\sigma, \mathbf{g}) \prod_{\sigma \in J'} c^+(\sigma, \mathbf{g})c^-(\sigma, \mathbf{g})\delta(\sigma, \mathbf{f}) \end{aligned}$$

8.7. Description of the p-adic convolution. *Under the conventions and notation as above there exists a bounded $\mathbf{C}_p$-valued measure $\mu = \mu_{\mathbf{f},\mathbf{g}}$ on Gal_p, which is uniquely determined by the following condition: for all Hecke characters $\chi \in \mathcal{X}_p^{\mathrm{tors}}$ and all $r \in \mathbf{Z}$ satisfying $m_* \geq r \leq m^*$ the following equality holds:*

$$\begin{aligned} &\int_{\mathrm{Gal}_p} \chi^{-1}\mathcal{N}x_p^r d\mu_{\mathbf{f},\mathbf{g}} = \\ &\quad i_p\Bigg(\frac{D_F^{2r}(-1)^r}{G(\chi)^2}\frac{\Lambda(r, \mathbf{f}, \mathbf{g}(\chi))}{\Omega(\mathbf{f}, \mathbf{g})\Phi_p(r, \mathbf{f}, \mathbf{g}(\chi))} \times \\ &\quad \times \prod_{\sigma_i \in J}\left(\frac{\mathcal{N}\mathfrak{p}_i^{r-1}}{\alpha(\mathfrak{p}_i)^2\beta(\mathfrak{p}_i)\beta'(\mathfrak{p}_i)}\right)^{\mathrm{ord}_{\mathfrak{p}_i} c(\chi)} \prod_{\sigma_i \in J'}\left(\frac{\mathcal{N}\mathfrak{p}_i^{r-1}}{\beta(\mathfrak{p}_i)^2\alpha(\mathfrak{p}_i)\alpha'(\mathfrak{p}_i)}\right)^{\mathrm{ord}_{\mathfrak{p}_i} c(\chi)}\Bigg), \end{aligned}$$

and the measure $\mu_{\mathbf{f},\mathbf{g}}$ defines a bounded $\mathbf{C}_p$-analytic function

$$L_{\mathbf{f},\mathbf{g}} : \mathcal{X}_p \to \mathbf{C}_p, \quad \mathcal{X}_p \ni x \mapsto \int_{\mathrm{Gal}_p} x \, d\mu_{\mathbf{f},\mathbf{g}}$$

(the p-adic Mellin transform of $\mu_{\mathbf{f},\mathbf{g}}$), which is uniquely determined by its values on the characters $x = \chi^{-1}\mathcal{N}x_p^r \in \mathcal{X}_p$.

(Note that the above expression could be written in a slightly simplier form if we take into account the equalities:

$$\alpha(\mathfrak{p})^2\beta(\mathfrak{p})\beta'(\mathfrak{p}) = \alpha(\mathfrak{p})^2\omega(\mathfrak{p})\mathcal{N}\mathfrak{p}^{l_0-1}, \quad \beta(\mathfrak{p})^2\alpha(\mathfrak{p})\alpha'(\mathfrak{p}) = \beta(\mathfrak{p})^2\psi(\mathfrak{p})\mathcal{N}\mathfrak{p}^{k_0-1}.)$$

8.8. Concluding remarks. The existence of the p-adic measure in 8.7 is known in the special case, and $J = \emptyset$ (see [Pa2]), where $\mathbf{f}$ and $\mathbf{g}$ are assumed to be automorphic forms of scalar weights k and l, $k > l$. One verifies easily that the description 8.7 perfectly matches with the modified period conjecture and with the general conjecture on the p - adic L - functions of Section 6. Also, this construction was recently extended by My Vinh Quang (Moscow University) to Hilbert automorphic forms $\mathbf{f}$ and $\mathbf{g}$ of arbitrary vector weights $k = (k_1, \cdots, k_n)$, and $l = (l_1, \cdots, l_n)$ such that $k_i > l_i$ for all $i = 1, \cdots, n$, and to the *non-p-ordinary*, i.e. *supersingular* case, when $| i_p(\alpha(\mathfrak{p}) |_p < 1$ for all $\mathfrak{p} \mid p$. In this situation the p-adic convolution of $L_{\mathbf{f},\mathbf{g}}$ is also uniquely determined by the above condition provided that it has the prescribed logarithmic growth on $\mathcal{X}_p$ (see [V1]).

In the general case the proof of the algebraic properties of the Rankin convolution in [Ha3] can be used also in order to carry out a p-adic construction. First of all, one obtains an expression for complex-valued distributions attached to $\Lambda(r, \mathbf{f}, \mathbf{g}(\chi))$ in terms of the cup product of certain coherent cohomology classes, and one verifies that these distributions take algebraic values. Then, integrality properties of the arithmetic vector bundles can be used for proving some generalized Kummer congruences for the values of these distributions, which is equivalent to the existence of p-adic L-functions in 8.7 However, some essential technical difficulties remain in the general case, and 8.7 can not be regarded yet as a theorem proven in full generality.

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Introduction

Hida [Hi2] has constructed interesting families of Galois representations of the type

$$\rho_p : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{Z}_p[[T]]), \quad G_{\mathbf{Q}} = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}),$$

which are non ramified outside p. These representations have the following property: if we consider the homomorphisms

$$\mathbf{Z}_p[[T]] \xrightarrow{s_k} \mathbf{Z}_p, \quad 1+T \mapsto (1+p)^{k-1},$$

then we obtain a family of Galois representations

$$\rho_p^{(k)} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{Z}_p),$$

which is parametrized by $k \in \mathbf{Z}$, and for $k = 2, 3, \cdots$,these representations are equivalent over $\mathbf{Q}_p$ to the p - adic representations of Deligne, attached to modular forms of weight k. This means that the representations of Hida are obtained by the p - adic interpolation of Deligne's representations. A geometric interpretation of Hida's representations was given by Mazur and Wiles [Maz-W3]. For example, for the modular form Δ of weight 12 Hida has constructed a representation

$$\rho_{p,\Delta} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{Z}_p[[T]]),$$

as an example of his general theory, where the prime number p have the property $\tau(p) \not\equiv (\mathrm{mod} p)$ (e.g. $p < 2041, p \neq 2, 3, 5$ and 7).

Also, Hida has generalized his construction to the case of Hilbert modular forms.

In this paper we would like to describe a conjectural generalization of his construction to arbitrary motives, and to formulate a certain conjecture. The important case, in which this conjecture can be verified, corresponds to Hecke characters of CM-type. We describe certain p - adic families, which are in general bigger than those obtained by the cyclotomic twist. We stress the fact that the corresponding p - adic L - functions depend analytically on the parameter of these p - adic Galois representations.

Summary.

§1. Motives and L - functions over a totally real field.

§2. The group of Hida and the algebra of Iwasawa - Hida.

§3. A conjecture on the existence of p - adic families of Galois representations attached to motives.

§4. A generalization of the Hasse invariant of a motive.

§5. A conjecture on the existence of certain families of p - adic L - functions.

§6. Hilbert modular forms and motives associated with them.

§7. Example: Hecke characters of CM-type.

§8. p-adic L-functions of Hilbert modular forms and their convolutions.

Throught the paper we fix embeddings

$$i_\infty : \overline{\mathbf{Q}} \to \mathbf{C}, \quad i_p : \overline{\mathbf{Q}} \to \mathbf{C}_p$$

and we shall often regard algebraic numbers (via these embeddings) as both complex and p - adic numbers, where $\mathbf{C}_p = \widehat{\overline{\mathbf{Q}}}_p$ is the Tate field (the completion of a fixed algebraic closure $\overline{\mathbf{Q}}_p$ of $\mathbf{Q}_p$), which is endowed with a unique norm $|\cdot|_p$ such that $|p|_p = p^{-1}$.

§1. Motives and L - functions over a totally real field.

Let F be a totally real number field of degree $n = [F : \mathbf{Q}]$ and T is another number field T which will be supposed a subfield of $\mathbf{C}$. By a motive M over F of with coefficients in T we shall mean a collection of the following objects:

$$M_{B,\sigma} = M_\sigma, \ M_{DR}, \ M_\lambda, \ I_{\infty,\sigma}, \ I_{\lambda,\sigma},$$

where σ runs over the set J_F of all complex embeddings of F,

M_σ is the Betti realization of M (with respect to the embedding $\sigma \in J_F$) which is a vector space over T of dimension d endowed for real $\sigma \in J_F$ with a T - rational involution ρ_σ ;

M_{DR} is the de Rham realization of M, a free $T \otimes F$ - module of rank d, endowed with a decreasing filtration $\{F^i_{DR}(M) \subset M_{DR} \mid i \in \mathbf{Z}\}$ of $T \otimes F$ - modules (which may not be free in some cases when $F \neq \mathbf{Q}$);

M_λ is the λ - adic realization of M at a finite place λ of the coefficient field T (a T_λ - vector space of degree d over T_λ, a completion of T at λ) which is a Galois module over $G_F = \mathrm{Gal}(\overline{F}/F)$ so that we a have a compatible system of λ - adic representations denoted by

$$r_{M,\lambda} = r_\lambda : G_F \to GL(M_\lambda).$$

Also,

$$I_{\infty,\sigma} : M_\sigma \otimes_T \mathbf{C} \to M_{DR} \otimes_{\sigma(F)\cdot T} \mathbf{C}$$

is the complex comparison isomorhism of complex vector spaces for each $\sigma \in J_F$,

$$I_{\lambda,\sigma} : M_\sigma \otimes_T T_\lambda \to M_\lambda$$

is the λ - adic comparison isomorphism of T_λ - vector spaces. It is assumed in the notation that the complex vector space $M_\sigma \otimes_{\mathbf{Q}} \mathbf{C}$ is decomposed in the Hodge bigraduation

$$M_\sigma \otimes_T \mathbf{C} = \oplus_{i,j} M_\sigma^{i,j}$$

in which $\rho_\sigma(M_\sigma^{i,j}) \subset M_\sigma^{j,i}$ for $\sigma \in J_F$ and the Hodge numbers

$$h_\sigma(i,j) = h_\sigma(i,j,M) = \dim_{\mathbf{C}} M_\sigma^{i,j}$$

do not depend on σ. Moreover,

$$I_{\infty,\sigma}(\oplus_{i'\geq i}M_\sigma^{i',j}) = F_{DR}^i(M)\otimes_{F,\sigma}\mathbf{C}.$$

Also, $I_{\lambda,\sigma}$ takes ρ_σ to the r_λ - image of the Galois automorphism which is denoted by the same symbol $\rho_\sigma \in G_F$ and corresponds to the complex conjugation of $\mathbf{C}$ under an embedding of $\overline{F}$ to $\mathbf{C}$ extending σ. We assume that M is pure of weight w (i.e. $i+j=w$).

The L - function $L(M,s)$ of M is defined as the following Euler product:

$$L(M,s) = \prod_{\mathfrak{p}} L_{\mathfrak{p}}(M, \mathcal{N}\mathfrak{p}^{-s}),$$

extended over all maximal ideals $\mathfrak{p}$ of the maximal order $\mathcal{O}_F$ of F and where

$$L_{\mathfrak{p}}(M,X)^{-1} = \det(1 - X\cdot r_\lambda(Fr_{\mathfrak{p}}^{-1}) \mid M_\lambda^{I_{\mathfrak{p}}}) =$$
$$(1-\alpha^{(1)}(\mathfrak{p})X)\cdot(1-\alpha^{(2)}(\mathfrak{p})X)\cdot\ldots\cdot(1-\alpha^{(d)}(\mathfrak{p})X) =$$
$$1 + A_1(\mathfrak{p})X + \ldots + A_d(\mathfrak{p})X^d;$$

here $\mathcal{N}\mathfrak{p}$ is the norm of $\mathfrak{p}$ and $Fr_{\mathfrak{p}} \in G_F$ is the Frobenius element at $\mathfrak{p}$, defined modulo conjugation and modulo the inertia subgroup $I_{\mathfrak{p}} \subset G_{\mathfrak{p}} \subset G_F$ of the decomposition group $G_{\mathfrak{p}}$ (of any extension of $\mathfrak{p}$ to $\overline{F}$). We make the standard hypothesis that the coefficients of $L_{\mathfrak{p}}(M, X)^{-1}$ belong to T, and that they are independent of λ coprime to $\mathcal{N}\mathfrak{p}$. Therefore we can and we shall regard this polynomial both over $\mathbf{C}$ and over $\mathbf{C}_p$. We shall need the following twist operation: for an arbitrary motive M over F with coefficients in T an integer m and a Hecke character χ of finite order one can define the twist $N = M(m)(\chi)$ which is again a motive over F with the coefficient field $T(\chi)$ of the same rank d and weight w so that we haver

$$L(N,s) = \prod_{\mathfrak{p}} L_{\mathfrak{p}}(M, \chi(\mathfrak{p})\mathcal{N}\mathfrak{p}^{-s-n}).$$

§2. The group of Hida and the algebra of Iwasawa - Hida.

Now let us fix a motive M with coefficients in $T = \mathbf{Q}(\langle a(\mathfrak{n})_{\mathfrak{n}}\rangle)$ of rang d and of weight w, and let $\mathrm{End}_T M$ denote the endomorphism algebra of M (i.e. the algebra of T-linear endomorphisms of any M_σ, which commute with the Galois action under the comparison isomorphisms). Let

$$\mathrm{Gal}_p = \mathrm{Gal}(F_{p,\infty}^{ab}/F)$$

denotes the Galois group of the maximal abelian extension $F_{p,\infty}^{ab}$ of F unramified outside primes of F above p and ∞. Define $\mathcal{O}_{F,T,p} = \mathcal{O}_F\otimes\mathcal{O}_T\otimes\mathbf{Z}_p$.

Definition. The *group of Hida* $GH_M = GH_{M,p}$ is the following product

$$GH_M = \mathrm{End}_T^\times(\mathcal{O}_{F,T,p})\times\mathrm{Gal}_p,$$

where $\mathrm{End}_T^\times$ denotes the algebraic T-group of invertible elements of End_T (it is implicitely supposed that the group $\mathrm{End}_T^\times$ posesses an $\mathcal{O}_T$-integral structure given by an appropriate choice of an $\mathcal{O}_T$ - lattice). Consider next the $\mathbf{C}_p$ - analytic Lie group

$$\mathcal{X}_{M,p} = \mathrm{Hom}_{contin}(GH_M,\ \mathbf{C}_p^\times)$$

consisting of all continuous characters of the Hida group GH_M, which contains the $\mathbf{C}_p$ - analytic Lie group

$$\mathcal{X}_p = \mathrm{Hom}_{contin}(\mathrm{Gal}_p,\ \mathbf{C}_p^\times)$$

consisting of all continuous characters of the Galois group Gal_p (via the projection of GH_M onto Gal_p.

The group $\mathcal{X}_{M,p}$ contains the discrete subgroup $\mathcal{A}$ of arithmetical characters of the type

$$\chi \cdot \eta \cdot Ncx_p^m = (\chi, \eta, m),$$

where

$$\chi \in \mathcal{X}_{M,p}^{tors}$$

is a character of finite order of GH_M, η is a T - algebraic character of $\mathrm{End}_T^\times(\mathcal{O}_{F,T,p})$, $m \in \mathbf{Z}$, and $\mathcal{N}x_p$ denotes the following natural norm homomorphism

$$\mathcal{N}x_p : \mathrm{Gal}_p \to \mathrm{Gal}(\mathbf{Q}_{p,\infty}^{ab}/\mathbf{Q}) \cong \mathbf{Z}_p^\times \to \mathbf{C}_p^\times,\ \mathcal{N}x_p \in \mathcal{X}_p.$$

Definition. The *algebra of Iwasawa - Hida* $I_M = I_{M,p}$ of M at p is the completed group ring $\mathcal{O}_p[[GH_M]]$, where $\mathcal{O}_p$ denotes the ring of integers of the Tate field $\mathbf{C}_p$.

Note that this definition is completely analogous to the usual definition of the Iwasawa algebra Λ as the completed group ring $\mathbf{Z}_p[[\mathbf{Z}_p]]$ if we take into account that $\mathbf{Z}_p$ coincides with the factor group of $\mathbf{Z}_p^\times$ modulo its torsion subgroup.

Now for each arithmetic point $P = (\chi, \eta, m) \in \mathcal{A}$ we have a homomorphism

$$\nu_P : GH_M \to \mathcal{O}_p$$

which is defined by the corresponding group homomorphism

$$P : GH_M \to \mathcal{O}_p^\times \subset \mathbf{C}_p^\times.$$

For a I_M-module N and $P \in \mathcal{A}$ we put

$$N_P = N \otimes_{I_M,\nu_P} \mathcal{O}_p$$

("reduction of N modulo P", or a fiber of N at P).

Therefore, for each Galois representation

$$r_N : G_F \to \mathrm{GL}(N)$$

its reduction $r_{N_P} = r \bmod P$ is defined as the natural composition:

$$G_F \to \mathrm{GL}(N) \to \mathrm{GL}(N_P).$$

§3. A conjecture on the existence of p - adic families of Galois representations attached to motives.

Note first that the fixed embeddings $T \hookrightarrow \mathbf{C}$,

$$i_\infty : \overline{\mathbf{Q}} \to \mathbf{C}, \quad i_p : \overline{\mathbf{Q}} \to \mathbf{C}_p$$

define a place $\lambda(p)$ of T attached to the corresponding composition

$$T \hookrightarrow \overline{\mathbf{Q}} \xrightarrow{i_p} \mathbf{C}_p.$$

Conjecture I. *For every M over F of rang d with coefficients in T there esists a free I_M-module M_I of the same rang d, a Galois representation*

$$r_I : G_F \to \mathrm{GL}(M_I),$$

an infinite subset $\mathcal{A}' \subset \mathcal{A}$ of "positive" characters, and a distinguished point $P_0 \in \mathcal{A}$ such that

(a) the reduced Galois representation

$$r_{I,P_0} : G_F \to \mathrm{GL}(M_{I,P_0})$$

is equivalent over $\mathbf{C}_p$ to the $\lambda(p)$ - adic representation $r_{M,\lambda(p)}$ of M at the distinguished place $\lambda(p)$;

(b) for every $P \in \mathcal{A}'$ there exists a motive M_P over F of the same rang d such that its Galois representation is equivalent over $\mathbf{C}_p$ to the reduction

$$r_{I,P} : G_F \to \mathrm{GL}(M_{I,P}).$$

We call the module M_P the *realization of Iwasawa of M*.

§4. A generalization of the Hasse invariant of a motive.

We define the *Hasse invariant* of a motive in terms of the *Newton polygons* and *the Hodge polygons* of a motive. Properties of these polygons are closely related to the notions of a p - ordinary and a p - admissible motive.

Now we are going to define the Newton polygon $P_{Newton,\sigma}(u) = P_{Newton,\sigma}(u, M)$ and the Hodge polygon $P_{Hodge,\sigma}(u) = P_{Hodge,\sigma}(u, M)$ attached to M, σ. First for $\mathfrak{p} = \mathfrak{p}(\sigma)$ we consider (using i_∞) the local $\mathfrak{p}$ - polynomial

$$L_\mathfrak{p}(M, X)^{-1} = 1 + A_1(\mathfrak{p})X + \cdots + A_d(\mathfrak{p})X^d$$
$$= (1 - \alpha^{(1)}(\mathfrak{p})X) \cdot (1 - \alpha^{(2)}(\mathfrak{p})X) \cdot \ldots \cdot (1 - \alpha^{(d)}(\mathfrak{p})X),$$

and we assume that its inverse roots are indexed in such a way that

$$\mathrm{ord}_p \alpha^{(1)}(\mathfrak{p}) \le \mathrm{ord}_p \alpha^{(2)}(\mathfrak{p}) \le \cdots \le \mathrm{ord}_p \alpha^{(d)}(\mathfrak{p})$$

Definition. The *Newton polygon* $P_{Newton,\sigma}(u) (0 \le u \le d)$ of M at $\mathfrak{p} = \mathfrak{p}(\sigma)$ is the convex hull of the points $(i, \mathrm{ord}_p A_i(\mathfrak{p}))$ $(i = 0, 1, \cdots, d)$.

The important property of the Newton polygon is that the length the horizontal segment of slope $i \in \mathbf{Q}$ is equal to the number of the inverse roots $\alpha^{(j)}(\mathfrak{p})$ such that

$\mathrm{ord}_p\alpha^{(j)}(\mathfrak{p}) = i$ (note that i may not necessarily be integer but this will be the case for the p - ordinary motives below).

The *Hodge polygon* $P_{Hodge,\sigma}(u)$ $(0 \le u \le d)$ of M at σ is defined using the Hodge decomposition of the d - dimensional $\mathbf{C}$ - vector space

$$M_\sigma = M_\sigma \otimes_T \mathbf{C} = \oplus_{i,j} M_\sigma^{i,j}$$

where $M_\sigma^{i,j}$ as a $\mathbf{C}$ - subspace. Note that the dimension $h_\sigma(i,j) = \dim_{\mathbf{C}} M_\sigma^{i,j}$ may depend on σ.

Definition. The Hodge polygon $P_{Hodge,\sigma}(u)$ is a function $[0,d] \to \mathbf{R}$ whose graph consists of segments passing through the points

$$(0,0), \; ..., \; (\sum_{i' \le i} h_\sigma(i',j), \; \sum_{i' \le i} h_\sigma i' h_\sigma(i',j)),$$

so that the length of the horizontal segment of the slope $i \in \mathbf{Z}$ is equal to the dimension $h_\sigma(i,j)$.

Now we recall the definition of a p - ordinary motive in the simplest case $F = T = \mathbf{Q}$ (see [Co], [Co - PeRi]). We assume that M is pure of weight w and rank d. Let $G_p = \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ be the decomposition group (of a place in T over p) and

$$\psi_p : G_p \to \mathbf{Z}_p^\times$$

be the cyclotomic character of G_p. Then M is called p ordinary at p if the following conditions are satisfied:

(i) The inertia group $I_p \subset G_p$ acts trivially on each of the l - adic realizations M_l for $l \ne p$;

(ii) There exists a decreasing filtration $F_p^i V$ on $V = M_p = M_B \otimes \mathbf{Q}_p$ of $\mathbf{Q}_p$ - subspaces which are stable under the action of G_p such that for all $i \in \mathbf{Z}$ the group G_p acts on $F_p^i V / F_p^{i+1} V$ via some power of the cyclotomic character, say $\psi_p^{-e_i}$. Then

$$e_1(M) \ge \cdots \ge e_t(M)$$

and the following properties take place:

(a)

$$\dim_{\mathbf{Q}_p} F_p^i V / F_p^{i+1} V = h(e_i, w - e_i);$$

(b) The Hodge polygon and the Newton polygon of M coincide:

$$P_{Newton}(u) = P_{Hodge}(u).$$

If furthermore M is critical at $s = 0$ then it is easy to verify that the number d_p of the inverse roots $\alpha^{(j)}(p)$ with

$$\mathrm{ord}_p\alpha^{(j)}(p) < 0 \text{ is equal to } d^+ = d^+(M) \text{ of } M_\sigma^+.$$

In the general case (of a motive M over F with coefficients in T) the notion of a p - ordinary motive can be defined using the restriction of the ground field F to $\mathbf{Q}$ and the restriction of the coefficient field T to $\mathbf{Q}$ (the last operation corresponds to fogetting of the T - module structure on the realizations of M). In this way we get a motive M' over with coefficients in $\mathbf{Q}$ of the same weight w and the rank $\mathrm{rk}(M') = [F:\mathbf{Q}][T:\mathbf{Q}]\cdot d$.

However, it turns out that the notion of a p - ordinary motive is too restrictive, and we introduce the following weaker version of it.

Definition. The motive M over F with coefficients in T is called *admissible* at p if for all $\sigma \in J_F$ we have that

$$P_{Newton,\sigma}(d_\sigma^+) = P_{Hodge,\sigma}(d_\sigma^+)$$

here $d_\sigma^+ = d_\sigma^+(M)$ is the dimension of M_σ, $\sigma \in J_F$.

In the general case we use the following vector quantity $h = (h_\sigma)_\sigma$ which is defined in terms of the difference between the Newton polygon and the Hodge polygon of M:

$$h_\sigma = P_{Newton,\sigma}(d_\sigma^+) - P_{Hodge,\sigma}(d_\sigma^+).$$

We call the vector $h = h(M) = (h_\sigma)_\sigma$ the *Hasse invariant* of M at p. Note the following important properties of the quantity h:

(i) $h = h(M)$ does not change if we replace M by its Tate twist.

(ii) $h = h(M)$ does not change if we replace M by its twist $M = M(\chi)$ with a Hecke character χ of finite order whose conductor is prime to $\mathfrak{p}$.

(iii) $h = h(M)$ does not change if we replace M by its dual $M^\vee$

In the next section we state in terms of this quantity a general conjecture on p - adic L - functions.

§5. A conjecture on the existence of certain families of p - adic L - functions.

We are going to describe families of p - adic L - functions as certain analytic functions on the total analytic space, the $\mathbf{C}_p$ - analytic Lie group

$$\mathcal{X}_{M,p} = \mathrm{Hom}_{contin}(GH_M,\ \mathbf{C}_p^\times),$$

which contain the $\mathbf{C}_p$ - analytic Lie subgroup (the cyclotomic line) $\mathcal{X}_p \subset \mathcal{X}_{M,p}$:

$$\mathcal{X}_p = \mathrm{Hom}_{contin}(\mathrm{Gal}_p,\ \mathbf{C}_p^\times).$$

In order to do this we need a modified L - function of a motive over F. Following J.Coates this modified L - function has a form appropriate for further use in the p - adic construction. First we multiply $L(M,s)$ by an appropriate factor at infinity and define

$$\Lambda_{(\infty)}(M,\ s) = E_\infty(M,\ s)L(M,s)$$

as $\Lambda_{(\infty)}(\tau,\ R_{F/\mathbf{Q}}M,\ \rho, s)$ in the notation of J.Coates [Co] with $\rho = i$ so that $E_\infty(M,s) = E_\infty(\tau, R_{F/\mathbf{Q}}M, \rho, s)$ is the modified Γ - factor at infinity which actually does not depend on the fixed embedding τ of T into $\mathbf{C}$. Also we put

$$\Omega^\nu(M) = (\Omega^\nu(M)^{(\tau)}) = c^\nu(RM)(2\pi i)^{r(RM)} \in (T\otimes\mathbf{C})^\times$$

where

$$\nu = (-1)^m, r(RM) = \sum_{j<0} jh(i,j,R_{F/\mathbf{Q}}M) = \sum_{j<0} jh(i,j,M), \quad n = [F:\mathbf{Q}],$$

$c^\nu(RM) = c^\nu(R_{F/\mathbf{Q}}M)$ is the period of $R_{F/\mathbf{Q}}M$. Note that the quantity $r(M)$ has a natural geometric interpretation as the minimum of the Hodge polygon $P_{Hodge}(M)$.

Then the period conjecture of Deligne can be stated in the following convenient form: if $s = 0$ is critical for M then for any m such that $M(m)$ is critical at $s = 0$ we have that

$$\frac{\Lambda_\infty(M(m),0)}{\Omega^\nu(M)} \in T.$$

In order to deduce this statement from the original conjecture on critical values we can use the same arguments as in the J.Coates's work [Co], where it was shown that

$$E_\infty(M,\ 0) \sim (2\pi i)^{r(RM)} \bmod \mathbf{Q}^\times,$$

and it follows that

$$E_\infty(M(m),\ 0) \sim (2\pi i)^{r(RM)-md^\varepsilon(RM)} = (2\pi i)^{n(r(M)-md^\varepsilon(M))} \bmod \mathbf{Q}^\times,$$

where $\varepsilon = +$ if $j < 0$ and $\varepsilon = -$ if ≥ 0 for $j = w/2$. If we combine this fact with the equivalence

$$c^+(M(m)) \sim (2\pi i)^{d^\nu nm} c^\nu(M) \bmod T^\times$$

we deduce from the above form of the conjecture that

$$\Lambda_{(\infty)}(M(m),\ 0) \sim (2\pi i)^{n(r(M)-md^\varepsilon(M)+md^\nu(M))} c^\nu(M)$$

Note that in our situation we have that $d^\varepsilon(M) = d^\nu(M)$ because both M and $M(m)$ are critical at $s = 0$: we have that $\nu = +$ only for $j - m < 0$ because $M(m)$ is critical but according to Lemma 3 in [Co] the condition $j < 0$ is equivalent in this situation to $j - m < 0$.

Modified conjecture on the critical values. Assume that M is critical at $s = 0$. Then there exist constants $c^{\varepsilon_\sigma}(\sigma, M) \in (T \otimes \mathbf{C})^\times (\varepsilon_\sigma = \pm)$ defined modulo $T^\times$ such that if we put for a given sign $\varepsilon_0 = (\varepsilon_{0,\sigma}) \in \mathrm{Sgn}_F$

$$\Omega(\varepsilon_0, M) = (1 \otimes (2\pi i))^{nr(M)} \prod_\sigma c^{\varepsilon_{0,\sigma}}(\sigma, M)$$

with $r(M) = \sum_{j<0} j h(i,j,M)$ then for any integer m and Hecke character χ such that $M(\chi)(m)$ is critical at $s = 0$ and $\varepsilon_\sigma(\chi)\nu = \varepsilon_{0,\sigma}$ we have that

$$\Lambda_{(\infty)}(M(\chi)(m),\ 0)((G(\chi)^{-1}(1 \otimes D_F^{1/2}))^{d^{\varepsilon_0}(M)}\Omega(\varepsilon_0, M))^{-1} \in T(\chi)$$

where $\nu = \mathrm{sgn}((-1)^m) = \pm$.

We recall that by definition

$$E_\infty(M,s) = E_\infty(\tau, R_{F/\mathbf{Q}}M, \rho, s) = E_\infty(U, \rho, s),$$

where U runs over direct summands of the Hodge decomposition, $\rho = i$ and $E_\infty(U, \rho, s)$ is given by:

(a) If $U = M^{j,k} \oplus M^{k,j}$ with $j < k$, then $E_\infty(U, \rho, s) = \Gamma_{\mathbf{C},\rho}(s-j)^{h(j,k)}$;

(b) If $U = M^{k,k}$ with $k \geq 0$, then $E_\infty(U, \rho, s) = 1$;

(c) If $U = M^{k,k}$ with $k < 0$, then $E_\infty(U,\rho,s) = R_\infty(U,\rho,s)$. Here $\rho^{-s} = \exp(-\rho\pi s/2)$, $\Gamma_{\mathbf{C},\rho}(s) = \rho^{-s}\Gamma_{\mathbf{C}}(s)$, $\Gamma_{\mathbf{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$, $\Gamma_{\mathbf{R}}(s) = \pi^{-s/2}\Gamma(s/2)$,

$$R_\infty(U,\rho,s) = L_\infty(\tau,U,\ s)/(\varepsilon_\infty(\tau,U,\rho,s)L_\infty(\tau,\ U^\vee(1),\ -s))$$

with L - and ε - factors described in [De3] on p.329, so that we have in case (c)

$$R_\infty(U,\rho,s) = \frac{\Gamma_{\mathbf{R}}(s-k+\delta)}{i^\delta\Gamma_{\mathbf{R}}(1-s+k-\delta)} = \frac{2\Gamma(s-k+\delta)\cos(\pi(s-k+\delta)/2}{i^\delta(2\pi)^{s-k+\delta}}$$

where $\delta = 0,1$ is chosen according with the sign of the scalar action of ρ_σ on $U = M_\sigma^{k,k}$ so that ρ_σ acts as $(-1)^{k+\delta}$.

We define

$$\Lambda_{(p,\infty)}(M(m)(\chi),s) =$$
$$(G(\chi)^{-1}D_F^{1/2})^{d^{+0}(M(m)(\chi))}\prod_{\mathfrak{p}|p} A_\mathfrak{p}(M(m)(\chi),s)\cdot\Lambda_{(\infty)}(M(m)(\chi),s),$$

where

$$A_\mathfrak{p}(M(\chi),s) = \begin{cases} \prod_{i=d^++1}^{d}(1-\chi(\mathfrak{p})\alpha^{(i)}(\mathfrak{p})\mathcal{N}\mathfrak{p}^{-s})\prod_{i=1}^{d^+}(1-\chi^{-1}(\mathfrak{p})\alpha^{(i)}(\mathfrak{p})^{-1}\mathcal{N}\mathfrak{p}^{s-1}) \\ \qquad\qquad\qquad \text{for } \mathfrak{p} \not| \mathfrak{c}(\chi) \\ \prod_{i=1}^{d^+}\left(\frac{\mathcal{N}\mathfrak{p}^s}{\alpha^{(i)}(\mathfrak{p})}\right)^{\mathrm{ord}_\mathfrak{p}\mathfrak{c}(\chi)}, \text{otherwise.} \end{cases}$$

Let $\mathcal{A}$ be the discrete subgroup $\mathcal{A}$ of arithmetical characters,

$$\chi\cdot\eta\cdot Ncx_p^m = (\chi,\eta,m)\in\mathcal{A},$$

$\mathcal{A}'\subset\mathcal{A}$ the subset of "positive" $P_0\in\mathcal{A}$ characters, a distinguished point of conjecture I. Let $\mathcal{A}''\subset\mathcal{A}'$ be the subset of critical elements, which consists of those P, for which the corresponding motives M_P are critical (at $s=0$). Now we are ready to formulate the following

Conjecture II. For a canonical choice of periods $\Omega(P)\in\mathbf{C}^\times$ for $P\in\mathcal{A}''$ there exists a $\mathbf{C}_p$ - meromorphic function

$$\mathcal{L}_M : \mathcal{X}_{M,p}\to\mathbf{C}_p$$

with the properties:

(i)

$$\mathcal{L}_M(P) = \frac{\Lambda_{p,(\infty)}(M(m)(\chi),0)}{\Omega(P)}$$

for almost all $P\in\mathcal{A}''$;

(ii) For arithmetic points of type

$$P = (\chi,\eta,m)\in\mathcal{A}''$$

with η fixed there exists a finite set $\Xi \subset \mathcal{X}_{M,p}$ of p - adic characters and positive integers $n(\xi)$ (for $\xi \in \Xi$) such that for any $g_0 \in \mathrm{Gal}_p$ we have that the function

$$\prod_{\xi\in\Xi}(x(g_0)-\xi(g_0))^{n(\xi)}\mathcal{L}_M(x\cdot P)$$

is holomorphic on $\mathcal{X}_p$;

(iii) For arithmetic points of type

$$P=(\chi,\eta,m)\in\mathcal{A}''$$

with η fixed the function in (ii) is bounded if and only if the Hasse invariant $h(P)=h(M_P)$ vanishes;

(iv) In the general case the function $\mathcal{L}_M(P\cdot x)$ of $x\in\mathcal{X}_p$ is of logarithmic growth type $o(\log\mathcal{N}(\cdot)^{h}{}_0$ with

$$h_0=[\max_\sigma h_\sigma]+1.$$

Don Blasius has suggested us the following modification of the Conjecture II, based on the theorem of Katz on p-adic L-functions of CM-type, and on the theory of p-adic periods [Bl3]:

Conjecture II′. There exists a certain choice of complex periods $\Omega_\infty(P)\in\mathbf{C}^\times$ and p-adic periods $\Omega_p(P)\in\mathbf{C}_p^\times$ for all $P\in\mathcal{A}''$ such that "the ratio" $\Omega_\infty(P)/\Omega_p(P)$ is canonicaly defined, and there exists a $\mathbf{C}_p$ - meromorphic function

$$\mathcal{L}_M:\mathcal{X}_{M,p}\to\mathbf{C}_p$$

with the properties:

(i)

$$\frac{L_M(P)}{\Omega_p(P)}=\frac{\Lambda_{p,(\infty)}(M(m)(\chi),0)}{\Omega(P)}$$

for almost all $P\in\mathcal{A}''$;

(ii) For arithmetic points of type

$$P=(\chi,\eta,m)\in\mathcal{A}''$$

with η fixed there exists a finite set $\Xi\subset\mathcal{X}_{M,p}$ of p - adic characters and positive integers $n(\xi)$ (for $\xi\in\Xi$) such that for any $g_0\in\mathrm{Gal}_p$ we have that the function

$$\prod_{\xi\in\Xi}(x(g_0)-\xi(g_0))^{n(\xi)}\mathcal{L}_M(x\cdot P)$$

is holomorphic on $\mathcal{X}_p$;

(iii) For arithmetic points of type

$$P=(\chi,\eta,m)\in\mathcal{A}''$$

with η fixed the function in (ii) is bounded if and only if the Hasse invariant $h(P)=h(M_P)$ vanishes;

(iv) In the general case the function $\mathcal{L}_M(P\cdot x)$ of $x \in \mathcal{X}_p$ is of logarithmic growth type $o(\log \mathcal{N}(\cdot)^{h}{}_0$ with

$$h_0 = [\max_\sigma h_\sigma] + 1.$$

§6. Hilbert modular forms and motives associated with them.

We use the notation of Shimura [Shi6], [Shi10] and we regard the group $GL_2(F)$ as the group $G_{\mathbf{Q}}$ of all $\mathbf{Q}$ - rational points of a certain $\mathbf{Q}$ - subgroup $G \subset GL_{2n}$. Then Hilbert modular forms will be regarded as complex fuctions on the adelic group $G_{\mathbf{A}} = G(\mathbf{A})$ which is apparently identified with the product

$$GL_2(F_{\mathbf{A}}) = G_\infty \times G_{\widehat{\mathbf{Q}}}$$

where

$$G_\infty = GL_2(F_\infty) \cong GL_2(\mathbf{R})^n,\ G_{\widehat{\mathbf{Q}}} = GL_2(\widehat{F}),$$

$\mathbf{A}$, $F_{\mathbf{A}}$ denote the rings of finite adels of $\mathbf{Q}$ and F respectively.
The subgroup

$$G_\infty^+ = GL_2^+(F_\infty) \cong GL_2^+(\mathbf{R})^n$$

consists of all elements

$$\alpha = (\alpha_1, \cdots, \alpha_n), \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

such that $\det\alpha_\nu > 0$, $\nu = 1, 2, \cdots, n$. Every element $\alpha \in G_\infty^+$ acts on the product $\mathfrak{H}^n$ of n copies of the upper half plane according to the formula

$$\alpha(z_1,\ \cdots,\ z_n) = (\alpha_1(z_1),\ \cdots,\ \alpha_n(z_n)),$$

where

$$\alpha_\nu(z_\nu) = (a_\nu z_\nu + b_\nu)/(c_\nu z_\nu + d_\nu)\quad (\nu = 1,\ 2,\ \cdots,\ n)$$

For $z = (z_1, z_2, \cdots, z_n) \in \mathfrak{H}^n$ we put $e_F(z) = e(\{z\}), \{z\} = z_1 + ... + z_n$ and $e(x) = \exp(2\pi i x)$ and we use the notations $\mathcal{N}z = z_1 \cdot ... \cdot z_n$, and $\mathbf{i} = (i,\ ...,\ i)$. For $\alpha \in G_\infty^+$, an integer n - tuple $k = (k_1, \cdots, k_n)$ and an arbitrary function $f: \mathfrak{H}^n \to \mathbf{C}$ we use the notation

$$(f\mid_k \alpha)(z) = \prod_\nu (c_\nu z_\nu + d_\nu)^{-k_\nu} f(\alpha(z)) \det(\alpha_\nu)^{k_\nu/2}$$

Let $\mathfrak{c}_\mathfrak{p} \subset \mathcal{O}_F$ be an integral ideal, $\mathfrak{c} = \mathfrak{c}\mathcal{O}_\mathfrak{p}$ its $\mathfrak{p}$ - part, $\mathfrak{d}_\mathfrak{p} = \mathfrak{d}\mathcal{O}_\mathfrak{p}$ the local different. We shall need the open compact subgroups $W = W_\mathfrak{c} \subset G_{\mathbf{A}}$ defined by

$$W_\mathfrak{c} = G_\infty^+ \times\quad W_\mathfrak{c}(\mathfrak{p}),$$

$$W_\mathfrak{c}(\mathfrak{p}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(F_\mathfrak{p}) \mid b \in \mathfrak{d}_\mathfrak{p}^{-1},\ c \in \mathfrak{d}_\mathfrak{p}\mathfrak{c}_\mathfrak{p},\ a, d \in \mathcal{O}_\mathfrak{p}, ad - bc \in \mathcal{O}_\mathfrak{p}^\times$$

By a Hilbert automorphic form of the weight $k = (k_1, \cdots, k_n)$, the level $\mathfrak{c}$, and the Hecke character ψ we mean a function

$\mathbf{f} : G_{\mathbf{A}} \to \mathbf{C}$ satisfying the following conditions (6.1) – (6.3):

$$\mathbf{f}(s\alpha x) = \psi(s)\mathbf{f}(x) \text{ for all } x \in G_{\mathbf{A}},$$
$$s \in F_{\mathbf{A}}^{\times} \text{ (the center of } G_{\mathbf{A}}), \text{ and } \alpha \in G_{\mathbf{Q}} \tag{6.1}$$

We let $\psi_0:(\mathcal{O}/\mathfrak{c})^{\times} \to \mathbf{C}^{\times}$ denote the $\mathfrak{c}$ - part of the character ψ and the extend the definition of ψ over the group $W_{\mathfrak{c}}$ by the formula

$$\psi\left(\left(\begin{array}{cc} a & b \\ c & d \end{array}\right)\right) = \psi_0(a_{\mathfrak{c}} \bmod \mathfrak{c})$$

then for all $x \in G_{\mathbf{A}}$

$$\mathbf{f}(xw) = \psi(w^{\iota})\mathbf{f}(x) \text{ for } w \in W_{\mathfrak{c}} \text{ with } w_{\infty} = 1, \tag{6.2}$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\iota} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

If

$$w = w(\theta) \text{ where } w(\theta) = (w_1(\theta_1), \ \cdots, \ w_n(\theta_n)),$$
$$w_{\nu}(\theta_{\nu}) = \begin{pmatrix} \cos\theta_{\nu} & -\sin\theta_{\nu} \\ \sin\theta_{\nu} & \cos\theta_{\nu} \end{pmatrix},$$

then

$$\mathbf{f}(xw(\theta)) = \mathbf{f}(x)\exp(-i(k_1\theta_1 + ... + k_n\theta_n)).$$

An automorphic form $\mathbf{f}$ is called a cusp form if

$$\int_{F_{\mathbf{A}}/F} \mathbf{f}\left(\begin{array}{cc} 1 & t \\ 0 & 1 \end{array}\right) g \, dt = 0 \text{ for all } g \in G_{\mathbf{A}}. \tag{6.3}$$

The vector space $\mathcal{M}_k(\mathfrak{c}, \psi)$ of Hilbert automorphic forms of holomorphic type is defined as the set functions satisfying (6.1) – (6.3) and the following holomorphy condition (6.4): for any $x \in G_{\mathbf{A}}$ with $x_{\infty} = 1$ there exists a holomorphic function $g_x:\mathfrak{H}^n \to \mathbf{C}$ such that for all $y = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_{\infty}^{+}$ we have that

$$\mathbf{f}(xw) = (g_x \mid_k w)(\mathbf{i}) \tag{6.4}$$

(in the case $F = \mathbf{Q}$ we must also require that the function g_x is holomorphic at the cusps). Let $\mathcal{S}_k(\mathfrak{c}, \psi) \subset \mathcal{M}_k(\mathfrak{c}, \psi)$ be the subspace of cusp forms.

Hecke operators which act on $\mathcal{S}_k(\mathfrak{c}, \psi)$ and $\mathcal{M}_k(\mathfrak{c}, \psi)$ are introduced by means of the double cosets of the type WyW for y in the semigroup

$$Y_{\mathfrak{c}} = G_{\mathbf{A}} \cap (G_{\infty}^{\times} \times \ \ Y_{\mathfrak{c}}(\mathfrak{p}),$$

where

$$Y_{\mathfrak{c}}(\mathfrak{p}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(F_{\mathfrak{p}}) \mid b \in \mathfrak{d}_{\mathfrak{p}}^{-1},\ c \in \mathfrak{d}_{\mathfrak{p}}\mathfrak{c}_{\mathfrak{p}},\ a,\ d \in \mathcal{O}_{\mathfrak{p}},\ a\mathcal{O}_{\mathfrak{p}} + \mathfrak{c}_{\mathfrak{p}} = \mathcal{O}_{\mathfrak{p}} \right\}.$$

The Hecke algebra $\mathcal{H}_{\mathfrak{c}}$ consists of all formal finite sums of the type $\sum_y c_y WyW$, with the multiplication in $\mathcal{H}_{\mathfrak{c}}$ defined by a standard rule. By definition $T_{\mathfrak{c}}(m)$ is the element of $\mathcal{H}_{\mathfrak{c}}$ obtained by taking the sum of all different WwW with $w \in Y_{\mathfrak{c}}$ such that $\mathrm{div}(\det(y)) = \mathfrak{m}$. Let

$$T_{\mathfrak{c}}(\mathfrak{m})' = \mathcal{N}(\mathfrak{m})^{(k_0-2)/2} T_{\mathfrak{c}}(\mathfrak{m})$$

be the normalized Hecke operator, where k_0 denotes the maximal component of the weight k. Suppose that $\mathbf{f} \in \mathcal{S}_k(\mathfrak{c},\ \psi)$ is an eigenform of all $T_{\mathfrak{c}}(\mathfrak{m})'$ with the eigenvalues $C(\mathfrak{m},\ \mathbf{f})$. Then there is the following Euler product expansion:

$$L(s,\mathbf{f}) = \sum_{\mathfrak{n}} C(\mathfrak{n},\mathbf{f})\mathcal{N}\mathfrak{n}^{-s} = \prod_{\mathfrak{p}} (1 - C(\mathfrak{p},\mathbf{f})\mathcal{N}\mathfrak{p}^{-s} + \psi(\mathfrak{p})\mathcal{N}\mathfrak{p}^{k_0-1-2s})^{-1}$$

All of the numbers $C(\mathfrak{n},\ \mathbf{f})$ are known to be algebraic integers.

Let $\mathbf{f} \in \mathcal{S}_k(\mathfrak{c},\ \psi)$ be a primitive Hilbert cusp eigenform. In this case the numbers $C(\mathfrak{n},\ \mathbf{f})$ can be regarded as the normalized Fourier coefficients of $\mathbf{f}$. The important analytic property of the corresponding L - function $L(\mathbf{f},\ s)$ (see [Shi6], p.655) is that it admits a holomorphic analytic continuation onto the entire complex plane, and if we set

$$\Lambda(\mathbf{f},s) = \prod_{i=1}^{n} \Gamma_{\mathbf{C}}(s - (k_0 - k_i)/2) L(\mathbf{f},s)$$

then $\Lambda(\mathbf{f},s)$ satisfies a functional equation, which expresses $\Lambda(\mathbf{f},s)$ in terms of the function $\Lambda(\mathbf{f}^{\rho},\ k_0 - s)$. According to the general conjecture on the analytic properties of the L - functions of motives we may suggest that $\mathbf{f}$ should correspond to a motive $M = M(\mathbf{f})$ over F of rank 2, and weight k_0 with coefficients in a field T containing all $C(\mathfrak{n},\ \mathbf{f})$ such that

$$L(M,s) = L(s,\mathbf{f}),\ \ \Lambda(M,s) = \Lambda(s,\mathbf{f})$$

and for fixed embeddings $\tau \in J_T$ and $\sigma = \sigma_i \in J_F$ the Hodge decomposition of M_{σ_i} is given by

$$M_{\sigma_i}^{(\tau)} = M_{\sigma_i} \otimes_{T,\tau} \mathbf{C} = \tag{6.5}$$
$$M_{\sigma_i}^{(\tau)(k_0-k_i)/2,(k_0+k_i)/2-1} \oplus M_{\sigma_i}^{(\tau)(k_0+k_i)/2-1,(k_0-k_i)/2}$$

where k_i is the component of the weight k, attached to the fixed embedding σ_i (as was mentioned above this decomposition may depend on τ and σ_i). It is obvious from (6.5) that if such motive exists then the weight k must saisfy the condition $k_1 \equiv k_2 \equiv \ldots \equiv k_n \bmod 2$.

There are several confirmation of the conjecture. First of all it is known in the elliptic modular case $F = \mathbf{Q}$ due to U.Jannsen and A.J.Scholl [Ja], [Scho]; the existence of the Galois representations of $\mathrm{Gal}(\overline{F}/F)$ corresponding to λ - adic realizations of these motives was discovered earlier by Deligne [Del]. If we restrict such representation to the subgroup we obtain the L - function of certain Hilbert modular form of the same (scalar) weight which is the Doi - Naganuma lift (or "base change") of the original elliptic cusp form. In the general case the existence of Galois representations attached to Hilbert modular forms was established by Rogawski - Tunnell [Ro-Tu] and Ohta [Oh] (n odd) (under a local hypothesis) and by R.Taylor [Ta] in the general case. Also a number of results on special values of the fuction $L(s, \mathbf{f})$ is known, which math the above conjectures on the critical values and on the p - adic L - functions [Shi1], [Man], [Ka1]. As in the elliptic modular case there is a conjectural link between motives of the type $M(\mathbf{f})$ and the cohomology of certain Kuga - Shimura variety (fiber product of several copies of the universal Hilbert - Blumenthal abelian variety with a fixed level structure and and endomorphisms): namely, for the decomposition $R_{F/\mathbf{Q}}M =_i {}_1M^{\sigma_i}$ the tensor product $\otimes_{i=1}^{n} M^{\sigma_i}$ is a motive over $\mathbf{Q}$ of rank 2^n which conjecturally lies in the above cohomology, see the interesting discussion of this link in [Ha2], [Oda]. In case $k_1 = ... = k_n = 2$ the motives have the Hodge type $H^{0,1} \oplus H^{1,0}$. In some cases (e.g. when n is odd) the motives M^{σ_i} can be realized as factors of Jacobians of Shimura curves corresponding to quaternion algebras, which split at one fixed infinite place σ_i and ramified at all other infinite places $\sigma_j (j \neq i)$ ([Shi7]; see also forthcoming work of M. Harris [Ha3], and [Bl-Ro]).

§7. Example: Hecke characters of CM-type.

7.1. Let $K \supset F$ be a totally imaginary quadratic extension, and $\eta : \mathbf{A}_K^\times / K^\times \to \mathbf{C}^\times$ be an algebraic Hecke's Grössencharakter such that

$$\eta((\alpha)) = \left(\frac{\alpha^{\phi_1}}{|\alpha^{\phi_1}|}\right)^{w_1} \cdots \left(\frac{\alpha^{\phi_n}}{|\alpha^{\phi_n}|}\right)^{w_n} \cdot \mathcal{N}(\alpha)^{w_0/2-1}$$

for $\alpha \in K, \alpha \equiv 1(\bmod \mathfrak{c}(\eta))$, where $\Sigma = bra\sigma_i : K \to \mathbf{C}$, is a fixed CM-type of K, w_i are positive integers, $w_0 = \max_i w_i$. Then there exists a Hilbert modular form $\mathbf{f}$ of weight $k = (w_1+1, \cdots, w_n+1)$ such that $L(s, \mathbf{f}) = L(s, \eta)$, and $M(\mathbf{f})$ coincides with the motive $M(\eta) = R_{K/F}[\eta]$ obtained by restriction of scalars from the motive $[\eta]$ (the last motive exists as an object of the category of motives of CM-type, see [Bl1]).

The Hodge structure of $M(\eta)_\sigma$ has the form

$$((w_0 - w_\sigma)/2, (w_0 + w_\sigma)/2) + ((w_0 + w_\sigma)/2, (w_0 - w_\sigma)/2),$$

Let

$$\mathfrak{p} = \mathfrak{p}_\sigma = \begin{cases} \mathfrak{P}\mathfrak{P}', & \text{if } \mathfrak{p} \text{ splits in } K, \\ \mathfrak{P}, & \text{if } \mathfrak{p} \text{ is inert in } K. \end{cases}$$

Then the local factor of $L(M(\eta), s)$ is given by

$$L_\mathfrak{p}(M(\eta), X)^{-1} = \begin{cases} (1 - \eta(\mathfrak{P})X))(1 - \eta(\mathfrak{P}')X), & \text{if } \mathfrak{p} \text{ splits in } K, \\ (1 - \eta(\mathfrak{P})^2, & \text{if } \mathfrak{p} \text{ is inert in } K. \end{cases}$$

Therefore the generalized Hasse invariant $h = (h_\sigma)_\sigma$ of $M(\eta)$ is given by

$$h_\sigma = \begin{cases} 0, & \text{if } \mathfrak{p} \text{ splits in } K, \\ w_0/2, & \text{if } \mathfrak{p} \text{ is inert in } K. \end{cases}$$

In the additive notation the type of η can be written in the following form:

$$\sum_\sigma \frac{w_\sigma}{2}(\sigma - \overline{\sigma}) + \frac{w_0}{2}\sum_\sigma(\sigma + \overline{\sigma}) = \sum_\sigma d_\sigma(\sigma - \overline{\sigma}) + m_0 \sum_\sigma \sigma,$$

where $m_0 = w_0, d_\sigma = (w_\sigma - w_0)/2$. Using a shift one sees that the point $s = m$ is critical for $L(s, \eta)$ iff $s = 0$ is critical for the character $\lambda(\mathfrak{a}) = \eta(\mathfrak{a})\mathcal{N}\mathfrak{a}^{-m}$.

Then one sees that the character η of the type $\sum_\sigma d_\sigma(\sigma - \overline{\sigma}) + m_0 \sum_\sigma \sigma$ is critical at 0 iff

$$m_0, d_\sigma \geq 0 \quad \text{or} \quad m_0 \leq 1, d_\sigma \geq 1 - m_0 (\text{ for all } \sigma) \qquad (*)$$

In order to state the thorem of Katz on p-adic L-functions of CM-fields (in a simplified form) we let $\mathfrak{C} \subset \mathcal{O}_K$ denote an integer ideal of the maximal order of K, $G_\infty(\mathfrak{C})$ the ray class group of K of conductor $\mathfrak{C}p^\infty$.

For each CM-type Σ one can canonicaly choose a constants

$$\Omega_\infty = (\Omega_\infty(\sigma))_{\sigma \in \Sigma} \in (\mathbf{C}^\times)^n,$$

$$\Omega_p = (\Omega_p(\sigma))_{\sigma \in \Sigma} \in (\mathbf{C}_p^\times)^n$$

(complex and p-adic periods).

The theorem of Katz states that under the assumption $h = 0$ there exists a bounded p-adic measure μ on $G_\infty(\mathfrak{C})$ such that for all critical characters λ of conductor dividing $\mathfrak{C}p^\infty$ the value of the p - adic integal

$$\frac{\int_{G_\infty(\mathfrak{C})} \hat{\lambda}\, d\mu}{\Omega_p^{m_0\Sigma + 2d}}$$

essentially coinsides with the normalized special value

$$\frac{\Lambda_{(p,\infty)}(\lambda, 0)}{\Omega_\infty^{m_0\Sigma + 2d}}$$

where $\hat{\lambda}$ denotes the p - adic avatar of λ.

This theorem provides an example of a p -adic family of Conjectures I and II, because $\mathrm{End}_T(M)$ is essentially K, and by class field theory GH_M is related to $G_\infty(\mathfrak{C})$.

7.2. Families of Hida of Hilbert modular forms. In this case we start from a motive $M(\mathbf{f})$ attached to a (general) Hilbert modular form and obtain the group $GH_{\mathbf{f}} = \mathcal{O}_{F,T,p}^\times \times \mathrm{Gal}_p$ whose characters parametrize "the weights" of Hilbert modular forms in the corresponding family.

§8. p-adic L-functions of Hilbert modular forms and their convolutions.

8.1. Periods of Hilbert modular forms.
Let $\mathbf{f} \in \mathcal{S}_k(\mathfrak{c}, \psi)$ be a primitive Hilbert cusp eigenform which is supposed to be "motivic" in the sense of Section 6, and let

$$L(s, \mathbf{f}(\chi)) = \sum_{\mathfrak{n}} \chi(\mathfrak{n}) C(\mathfrak{n}, \mathbf{f}) \mathcal{N}\mathfrak{n}^{-s} = (1 - \chi(\mathfrak{p}) C(\mathfrak{p}, \mathbf{f}) \mathcal{N}\mathfrak{p}^{-s} + \chi^2(\mathfrak{p}) \psi(\mathfrak{p}) \mathcal{N}\mathfrak{p}^{k_0 - 1 - 2s})^{-1}.$$

Then the critical strip of $L(s, \mathbf{f}(\chi))$ is given by $m_* \le m \le m^*$,

$$m_* = \max\{(k_0 - k_i)/2\} + 1, \quad m^* = \min\{(k_0 + k_i)/2\} - 1.$$

Using the Rankin - Selberg method G.Shimura proved that there exist constants

$$c^{\pm}(\sigma, \mathbf{f}) \in (T \otimes \mathbf{C})^{\times}$$

defined modulo $T^{\times}$ such that if we put

$$c^{\varepsilon}(\chi, \mathbf{f}) = D_F^{-1/2} G(\chi) \prod_{\sigma \in J_F} c^{\varepsilon \cdot \varepsilon_\sigma(\chi)}(\sigma, \mathbf{f})$$

then for all $m \in \mathbf{Z}, m_* \le m \le m^*$ we have that

$$\frac{i^{nm} \Lambda(m,\ \mathbf{f}(\chi))}{c^{\nu}(\chi, \mathbf{f})} \in T(\chi),$$

where $\nu = (-1)^m$.

This statement coincides with that of the modified period conjecture if we take for $c^{\pm}(\sigma, M(\mathbf{f}))$ the quantities $c^{\pm}(\sigma, \mathbf{f})$.

In order to formulate the results on p - adic L - functions, put

$$1 - C(\mathfrak{p}, \mathbf{f}) X + \psi(\mathfrak{p}) \mathcal{N}\mathfrak{p}^{k_0 - 1} X^2 = (1 - \alpha(\mathfrak{p}) X)(1 - \alpha'(\mathfrak{p}) X) \in \mathbf{C}_p[X]$$

where $\alpha(\mathfrak{p})$, $\alpha'(\mathfrak{p})$ are the inverse roots of the Hecke polynomial assuming that

$$\mathrm{ord}_p \alpha(\mathfrak{p}) \le \mathrm{ord}_p \alpha'(\mathfrak{p}).$$

Note that in the p - ordinary case we should have

$$\mathrm{ord}_p \alpha(\mathfrak{p}) = (k_0 - k_i)/2, \quad \mathrm{ord}_p \alpha'(\mathfrak{p}) = (k_0 + k_i)/2 - 1$$

for the prime $\mathfrak{p} = \mathfrak{p}_i = \mathfrak{p}(\sigma_i)$ attached to the embedding σ_i (see Section 5).

8.2. Theorem. *Put $h = [\max(\mathrm{ord}_p(\alpha(\mathfrak{p}(\sigma_i)) - (k_0 - k_i)/2)] + 1$. Then for each sign $\varepsilon_0 = \{\varepsilon_{0,\sigma}\} \in \mathrm{Sgn}_F$ there exists a $\mathbf{C}_p$ - analytic function $L_{(p)}^{(\varepsilon_0)}$ on $\mathcal{X}_p$ of the type $o(\log^h)$ with the properties:*

(i) for all $m \in \mathbf{Z}$, $m_* \le m \le m^*$, and for all Hecke characters of finite order $\chi \in \mathcal{X}_p^{tors}$ with $\nu\varepsilon_\sigma(\chi) = \varepsilon_{0,\sigma}$ ($\sigma \in J_F$) the following equality holds

$$L_{(p)}^{(\varepsilon_0)}(\chi \mathcal{N} x_p^m) = \frac{D_F^m i^{mn}}{G(\chi)} \prod_{\mathfrak{p} | p} A_{\mathfrak{p}}(\mathbf{f}(\chi), m) \cdot \frac{\Lambda(\mathbf{f}(\chi), m)}{\Omega(\varepsilon_0,\ \mathbf{f})},$$

where

$$A_{\mathfrak{p}}(\mathbf{f}(\chi), m) = \begin{cases} (1-\chi(\mathfrak{p})\alpha'(\mathfrak{p})\mathcal{N}\mathfrak{p}^{-m})(1-\chi^{-1}(\mathfrak{p})\alpha(\mathfrak{p})^{-1}\mathcal{N}\mathfrak{p}^{m-1}), \\ \qquad \text{if } \mathfrak{p} \mid \mathfrak{c}(\chi) \\ \left(\frac{\mathcal{N}\mathfrak{p}^m}{\alpha(\mathfrak{p})}\right)^{\mathrm{ord}_{\mathfrak{p}}\mathfrak{c}(\chi)}, \\ \qquad \text{if } \mathfrak{p} \not| \, \mathfrak{c}(\chi) \end{cases}$$

and the constant $\Omega(\varepsilon_0,\ \mathbf{f})$ *is given by*

$$\Omega(\varepsilon_0,\ \mathbf{f}) = (2\pi i)^{-nm_*} \cdot D_F^{1/2} \cdot \prod_{\sigma} c^{\varepsilon_{0,\sigma}}(\sigma, \mathbf{f}),$$

(ii) If $h \le m^* - m_* + 1$ *then the function* $L_{(p)}^{(\varepsilon_0)}$ *on* $\mathcal{X}_p$ *is uniquely determined by (i).*
(iii) If

$$\max(\mathrm{ord}_p(\alpha(\mathfrak{p}(\sigma_i)) - (k_0 - k_i)/2) = 0$$

then the function $L_{(p)}^{(\varepsilon_0)}$ *is bounded on* $\mathcal{X}_p$.

In the p - ordinary case this theorem was established by Yu.I.Manin (in a less explicit form) using the theory of generalized modular symbols on Hilbert - Blumenthal modular varieties. The non - p - ordinary case was treated only for $F = \mathbf{Q}$ by Višik [V1]. For an arbitrary totally real field F one can use the Rankin method and the technique of the Shimura's work [Shi6].

8.3. The Rankin convolution and the tensor product of motives. Let us consider the Rankin convolution

$$L(s, \mathbf{f}, \mathbf{g}) = \sum_{\mathfrak{n}} C(\mathfrak{n}, \mathbf{f})\, C(\mathfrak{n}, \mathbf{g}) \mathcal{N}(\mathfrak{n})^{-s} \tag{8.1}$$

attached to two Hilbert modular forms $\mathbf{f}$, $\mathbf{g}$ over a totally real field F of degree $n = [F : \mathbf{Q}]$, where $C(\mathfrak{n}, \mathbf{f})$, $C(\mathfrak{n}, \mathbf{g})$ are normalized "Fourier coefficients" of $\mathbf{f}$ and $\mathbf{g}$, indexed by integral ideals $\mathfrak{n}$ of the maximal order $\mathcal{O}_F \subset F$ (see §6). We suppose that $\mathbf{f}$ is a primitive cusp form of vector weight $k = (k_1, \cdots, k_n)$, and $\mathbf{g}$ a primitive cusp form of weight $l = (l_1, \cdots, l_n)$ We assume that for a decomposition of J_F into a disjoint union $J_F = J \cup J'$ the following condition is satisfied

$$k_i > l_i \text{ (for } \sigma_i \in J), \quad \text{and} \quad l_i > k_i \text{ (for } \sigma_i \in J'). \tag{8.2}$$

Also, assume that

$$k_1 \equiv k_2 \equiv \cdots \equiv k_n \bmod 2, \tag{8.3}$$

and

$$l_1 \equiv l_2 \equiv \cdots \equiv l_n \bmod 2. \tag{8.4}$$

Let $\mathfrak{c}(\mathbf{f}) \subset \mathcal{O}_F$ denote the conductor and ψ the character of $\mathbf{f}$ and $\mathfrak{c}(\mathbf{g})$, ω denote the conductor and the character of $\mathbf{g}$ ($\psi, \omega : \mathbf{A}_F^\times/F^\times \to \mathbf{C}^\times$ being Hecke characters of finite order).

The essential property of the convolution

$$L(s,\mathbf{f},\mathbf{g}(\chi)) = \sum_{\mathfrak{n}} \chi(\mathfrak{n})C(\mathfrak{n},\mathbf{f})C(\mathfrak{n},\mathbf{g})\mathcal{N}(\mathfrak{n})^{-s}$$

(twisted with a Hecke characer χ of finite order) is the following Euler product decomposition

$$\begin{aligned} &L_c(2s+2-k_0-l_0,\psi\omega\chi^2)L(s,\mathbf{f},\mathbf{g}(\chi)) = \\ &\prod_{\mathfrak{q}} ((1-\chi(\mathfrak{q})\alpha(\mathfrak{q})\beta(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{-s})(1-\chi(\mathfrak{q})\alpha(\mathfrak{q})\beta'(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{-s})\times \\ &\times (1-\chi(\mathfrak{q})\alpha'(\mathfrak{q})\beta(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{-s})(1-\chi(\mathfrak{q})\alpha'(\mathfrak{q})\beta'(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{-s}))^{-1}, \end{aligned} \tag{8.5}$$

where the numbers $\alpha(\mathfrak{q})$, $\alpha'(\mathfrak{q})$, $\beta(\mathfrak{q})$, and $\beta'(\mathfrak{q})$ are roots of the Hecke polynomials

$$X^2 - C(\mathfrak{q},\mathbf{f})X + \psi(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{k_0-1} = (X-\alpha(\mathfrak{q}))(X-\alpha'(\mathfrak{q})),$$

and

$$X^2 - C(\mathfrak{q},\mathbf{g})X + \omega(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{l_0-1} = (X-\beta(\mathfrak{q}))(X-\beta'(\mathfrak{q})).$$

The decomposition (8.5) is not difficult to deduce from the following elementary lemma on rational functions, applied to each of the Euler $\mathfrak{q}$-factors: if

$$\sum_{i=0}^{\infty} A_i X^i = \frac{1}{(1-\alpha X)(1-\alpha' X)}, \quad \sum_{i=0}^{\infty} B_i X^i = \frac{1}{(1-\beta X)(1-\beta' X)},$$

then

$$\sum_{i=0}^{\infty} A_i B_i X^i = \frac{1-\alpha\alpha'\beta\beta' X^2}{(1-\alpha\beta X)(1-\alpha\beta' X)(1-\alpha'\beta X)(1-\alpha'\beta' X)}. \tag{8.6}$$

Assume that there exist motives $M(\mathbf{f})$ and $M(\mathbf{g})$ associated with $\mathbf{f}$ and $\mathbf{g}$. Then

$$L_c(2s+2-k-l,\psi\omega\chi^2)L(s,\mathbf{f},\mathbf{g}(\chi)) = L(M(\chi),s)$$

where $M = M(\mathbf{f}) \otimes_F M(\mathbf{g})$ is the tensor product of motives over F with coefficients in some common number field T. Using the Hogde decompositions for $M(\mathbf{f})$ and $M(\mathbf{g})$ and the Künneth formula for $M = M(\mathbf{f}) \otimes_F M(\mathbf{g})$ we see that under our assumption the motive M has $d = 4$, $w = k_0 + l_0 - 2$, and the following Hodge type:

$$\begin{aligned} &M_{\sigma_i} \otimes \mathbf{C} \cong \\ &\oplus_{\tau\in J_T} \big(M_{\sigma_i}^{(k_0+l_0-k_i^\tau-l_i^\tau)/2,(k_0+l_0+k_i^\tau+l_i^\tau)/2-2} \oplus M_{\sigma_i}^{(k_0+l_0+k_i^\tau+l_i^\tau)/2-2,(k_0+l_0-k_i^\tau-l_i^\tau)/2} \\ &\oplus M_{\sigma_i}^{(k_0+l_0-|k_i^\tau-l_i^\tau|)/2-1,(k_0+l_0+|k_i^\tau-l_i^\tau|)/2-1} \oplus M_{\sigma_i}^{(k_0+l_0+|k_i^\tau-l_i^\tau|)/2-1,(k_0+l_0-|k_i^\tau-l_i^\tau|)/2-1}\big). \end{aligned}$$

Moreover,

$$\Lambda(M(\chi),s) = \Lambda(s,\mathbf{f},\mathbf{g}(\chi)) =$$
$$\prod_{i=1}^{n} \left(\Gamma_{\mathbf{C}}(s-(k_0+l_0-k_i-l_i)/2)\Gamma_{\mathbf{C}}(s-(k_0+l_0-|k_i-l_i|)/2+1)\right)\times$$
$$\times L_c(2s+2-k_0-l_0,\psi\omega\chi^2)L(s,\mathbf{f},\mathbf{g}(\chi)),$$

and this function satisfies a functional equation of the type $s \mapsto k_0+l_0-2-s$.

8.4. The critical values of the Rankin convolution. Let us now set

$$m_* = \max_i((k_0+l_0-|k_i-l_i|)/2-1),+1, \quad m^* = k_0+l_0-2-m_*.$$

The periods $c^{\pm}(\sigma,M)$ can be easily computed in terms of $c^{\pm}(\sigma,M)$ (as in the elliptic modular case; see a more general calculation in [Bl2]). As a result one gets that $c^{\pm}(\sigma,M)=c(\sigma,M)$ does not depend on the sign $\pm$, and is given by

$$c^{\pm}(\sigma,M) = \begin{cases} c^+(\sigma,\mathbf{f})c^-(\sigma,\mathbf{f})\delta(\sigma,\mathbf{g}), & \text{if } \sigma\in J \\ c^+(\sigma,\mathbf{g})c^-(\sigma,\mathbf{g})\delta(\sigma,\mathbf{f}), & \text{if } \sigma\in J'. \end{cases}$$

Moreover,

$$c^{\pm}(M(\chi)) = G(\chi)^{-2}\prod_{\sigma\in J} c^{\pm}(\sigma,M).$$

Let us apply the modified conjecture on special values to the L-function

$$\Lambda(M(\chi),s) = \Lambda(s,\mathbf{f},\mathbf{g}(\chi)),$$

and set $c(\mathbf{f},\mathbf{g}) = \prod_\sigma c^+(\sigma,M)$,

$$c(J,\mathbf{f}) = \prod_{\sigma\in J} c^+(\sigma,\mathbf{f})c^-(\sigma,\mathbf{f}),\ c(J',\mathbf{g}) = \prod_{\sigma\in J'} c^+(\sigma,\mathbf{g})c^-(\sigma,\mathbf{g}),$$

and

$$\delta(J,\mathbf{f}) = \prod_{\sigma\in J}\delta(\sigma,\mathbf{f}),\ \delta(J',\mathbf{g}) = \prod_{\sigma\in J'}\delta(\sigma,\mathbf{g}).$$

Then we see that

$$c(J,\mathbf{f})c(J',\mathbf{f}) = \langle\mathbf{f},\mathbf{f}\rangle,\quad \delta(J,\mathbf{f})\delta(J',\mathbf{f}) = G(\psi)^{-1}(2\pi i)^{n(k_0-1)},$$
$$c(J,\mathbf{g})c(J',\mathbf{g}) = \langle\mathbf{g},\mathbf{g}\rangle,\quad \delta(J,\mathbf{f})\delta(J',\mathbf{g}) = G(\omega)^{-1}(2\pi i)^{n(l_0-1)},$$

and

$$c(M(\chi)) = c^{\pm}(M(\chi)) = G(\chi)^{-2}c(J,\mathbf{f})\delta(J,\mathbf{g})c(J',\mathbf{g})\delta(J',\mathbf{f}).$$

With this notation the modified conjecture on the critical values takes the following form: for all Hecke characters χ of finite order and $r\in\mathbf{Z}, m_*\geq r\leq m^*$ we have that

$$\frac{\Lambda(r,\mathbf{f},\mathbf{g}(\chi))}{G(\chi)^{-2}c(J,\mathbf{f})\delta(J,\mathbf{g})c(J',\mathbf{g})\delta(J',\mathbf{f})} = \frac{\Lambda(M(\chi),r)}{G(\chi)^{-2}c(M)} \in \mathbf{Q}(\mathbf{f},\mathbf{g},\chi).$$

8.5. Let us consider the special case when $J' = \emptyset$, i.e. $k_i > l_i$ for all $\sigma_i \in J_F$. Then

$$c(J, \mathbf{f}) = c(J_F, \mathbf{f}) = \langle \mathbf{f}, \mathbf{f} \rangle, \quad \delta(J, \mathbf{g}) = \delta(J_F, \mathbf{g}) = G(\omega)^{-1}(2\pi i)^{n(l_0-1)},$$

and the above property transforms to the following:

$$\frac{\Lambda(r, \mathbf{f}, \mathbf{g}(\chi))}{G(\chi)^{-2}\langle \mathbf{f}, \mathbf{f} \rangle, G(\omega)^{-1}(2\pi i)^{n(l_0-1)}} \in \mathbf{Q}(\mathbf{f}, \mathbf{g}, \chi),$$

where $\mathbf{Q}(\mathbf{f}, \mathbf{g}, \chi)$ denotes the subfield of $\mathbf{C}$ generated by the Fourier coefficients of $\mathbf{f}$ and $\mathbf{g}$, and the values of χ. This algebraicity property was established by G.Shimura [Sh1] by means of a version of the Rankin - Selberg method.

In the general case the above algebraicity property was also studies by G.Shimura [Sh2], [Sh3] (for some special Hilbert modular forms, coming from quaternion algebras) and by M.Harris [Ha3] using the theory of arithmetical vector bundles on Shimura varieties. The idea of the proof was to replace the original automorphic cusp form $\mathbf{f} : G(\mathbf{A}) \to \mathbf{C}$ of holomorphic type by another cusp form $\mathbf{f}^J : G(\mathbf{A}) \to \mathbf{C}$ such that

$$\mathbf{f}^J(g_1, \cdots, g_n) = \mathbf{f}(g_1 j_1, \cdots, g_n j_n),$$

where $g_i \in \mathrm{GL}_2(\mathbf{R})$,

$$j_i = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \text{if } i \in J \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & \text{if } i \in J'. \end{cases}$$

Then $\mathbf{f}^J$ can be described by functions $\mathbf{f}^J_\lambda$ on $\mathfrak{H}^n$, which are holomorphic in z_i ($i \in J$) and antiholomorphic in z_i ($i \in J'$). Then the differential forms

$$\mathbf{f}^J_\lambda \wedge_{i \in J} d\overline{z}_i$$

define a certain class $cl(\mathbf{f}^J)$ of the degree $|J|$ in the coherent cohomology of the Hilbert - Blumenthal modular variety, or rather its toroidal compactification ([Ha1], [Ha2]). This space of coherent cohomology has a natural rational structure over a certain number field F^J, defined in terms of canonical models. From the theory of new forms it follows that there exist a constant $\nu(J, \mathbf{f}) \in \mathbf{C}^\times$ such that the differential form attached to $\nu(J, \mathbf{f})^{-1}\mathbf{f}^J$ is rational over the extension of F^J obtained by adjoining the Hecke eigenvalues of $\mathbf{f}$. Then the critical values of the type $\Lambda(r, \mathbf{f}, \mathbf{g})$ can be expressed in terms of a cup product of the form

$$cl(\mathbf{f}^J) \cup cl(\mathbf{g}^{J'}) \cup E,$$

where E is a (nearly) holomorphic Eisenstein series. Then the above algebraicity property can be deduced from the fact that the cup product preserves the rational structure in the coherent cohomology. However, the technical details of the proof are quite difficult.

8.6. p-adic convolutions of Hilbert cusp forms. Now we give a precise description of the p-adic convolution of $\mathbf{f}$ and $\mathbf{g}$ assuming that both $\mathbf{f}$ and $\mathbf{g}$ are p-ordinary, i.e. for $\mathfrak{p}_i = \mathfrak{p}(\sigma_i)$ one has

$$\mathrm{ord}_p \alpha(\mathfrak{p}_i) = (k_0 - k_i)/2, \quad \mathrm{ord}_p \alpha'(\mathfrak{p}_i) = (k_0 + k_i)/2 - 1,$$
$$\mathrm{ord}_p \beta(\mathfrak{p}_i) = (l_0 - l_i)/2, \quad \mathrm{ord}_p \beta'(\mathfrak{p}_i) = (l_0 + l_i)/2 - 1,$$

or equivalently, $\mathrm{ord}_p C(\mathfrak{p}_i, \mathbf{f}) = (k_0 - k_i)/2$, and $\mathrm{ord}_p C(\mathfrak{p}_i, \mathbf{g}) = (l_0 - l_i)/2$. We assume also that the conductors of $\mathbf{f}$ and $\mathbf{g}$ are coprime to p and we set

$$\begin{aligned}
&A_p(s, \mathbf{f}, \mathbf{g}(\chi)) = \\
&\prod_{\sigma_i \in J \setminus S(\chi)} (1 - \chi(\mathfrak{p}_i)\alpha'(\mathfrak{p}_i)\beta(\mathfrak{p}_i)\mathcal{N}\mathfrak{p}_i^{-s})(1 - \chi(\mathfrak{p}_i)\alpha'(\mathfrak{p}_i)\beta'(\mathfrak{p}_i)\mathcal{N}\mathfrak{p}_i^{-s}) \times \\
&\times (1 - \chi(\mathfrak{p}_i)^{-1}\alpha(\mathfrak{p}_i)^{-1}\beta(\mathfrak{p}_i)^{-1}\mathcal{N}\mathfrak{p}_i^{s-1})(1 - \chi(\mathfrak{p}_i)^{-1}\alpha(\mathfrak{p}_i)^{-1}\beta'(\mathfrak{p}_i)^{-1}\mathcal{N}\mathfrak{p}_i^{s-1}) \times \\
&\times \prod_{\sigma_i \in J' \setminus S(\chi)} (1 - \chi(\mathfrak{p}_i)\alpha(\mathfrak{p}_i)\beta'(\mathfrak{p}_i)\mathcal{N}\mathfrak{p}_i^{-s})(1 - \chi(\mathfrak{p}_i)\alpha'(\mathfrak{p}_i)\beta'(\mathfrak{p}_i)\mathcal{N}\mathfrak{p}_i^{-s}) \times \\
&\times (1 - \chi(\mathfrak{p}_i)^{-1}\alpha(\mathfrak{p}_i)^{-1}\beta(\mathfrak{p}_i)^{-1}\mathcal{N}\mathfrak{p}_i^{s-1})(1 - \chi(\mathfrak{p}_i)^{-1}\alpha'(\mathfrak{p}_i)^{-1}\beta(\mathfrak{p}_i)^{-1}\mathcal{N}\mathfrak{p}_i^{s-1}).
\end{aligned}$$

Then we introduce the following constant:

$$\begin{aligned}
&\Omega(\mathbf{f}, \mathbf{g}) = c(J, \mathbf{f})\delta(J, \mathbf{g})c(J', \mathbf{g})\delta(J', \mathbf{f}) = \\
&\prod_{\sigma \in J} c^+(\sigma, \mathbf{f})c^-(\sigma, \mathbf{f})\delta(\sigma, \mathbf{g}) \prod_{\sigma \in J'} c^+(\sigma, \mathbf{g})c^-(\sigma, \mathbf{g})\delta(\sigma, \mathbf{f})
\end{aligned}$$

8.7. Description of the p-adic convolution. *Under the conventions and notation as above there exists a bounded $\mathbf{C}_p$-valued measure $\mu = \mu_{\mathbf{f},\mathbf{g}}$ on Gal_p, which is uniquely determined by the following condition: for all Hecke characters $\chi \in \mathcal{X}_p^{\mathrm{tors}}$ and all $r \in \mathbf{Z}$ satisfying $m_* \le r \le m^*$ the following equality holds:*

$$\begin{aligned}
&\int_{\mathrm{Gal}_p} \chi^{-1}\mathcal{N}x_p^r d\mu_{\mathbf{f},\mathbf{g}} = \\
&\qquad i_p\Big(\frac{D_F^{2r}(-1)^r}{G(\chi)^2}\frac{\Lambda(r, \mathbf{f}, \mathbf{g}(\chi))}{\Omega(\mathbf{f}, \mathbf{g})}\prod_{\mathfrak{p}|p} A_p(r, \mathbf{f}, \mathbf{g}(\chi)) \times \\
&\qquad \times \prod_{\sigma_i \in J}\Big(\frac{\mathcal{N}\mathfrak{p}_i^{r-1}}{\alpha(\mathfrak{p}_i)^2\beta(\mathfrak{p}_i)\beta'(\mathfrak{p}_i)}\Big)^{\mathrm{ord}_{\mathfrak{p}_i} c(\chi)} \prod_{\sigma_i \in J'}\Big(\frac{\mathcal{N}\mathfrak{p}_i^{r-1}}{\beta(\mathfrak{p}_i)^2\alpha(\mathfrak{p}_i)\alpha'(\mathfrak{p}_i)}\Big)^{\mathrm{ord}_{\mathfrak{p}_i} c(\chi)}\Big),
\end{aligned}$$

and the measure $\mu_{\mathbf{f},\mathbf{g}}$ defines a bounded $\mathbf{C}_p$-analytic function

$$L_{\mathbf{f},\mathbf{g}} : \mathcal{X}_p \to \mathbf{C}_p, \quad \mathcal{X}_p \ni x \mapsto \int_{\mathrm{Gal}_p} x \, d\mu_{\mathbf{f},\mathbf{g}}$$

(the p-adic Mellin transform of $\mu_{\mathbf{f},\mathbf{g}}$), which is uniquely determined by its values on the characters $x = \chi^{-1}\mathcal{N}x_p^r \in \mathcal{X}_p$.

(Note that the above expression could be written in a slightly simplier form if we take into account the equalities:

$$\alpha(\mathfrak{p})^2\beta(\mathfrak{p})\beta'(\mathfrak{p}) = \alpha(\mathfrak{p})^2\omega(\mathfrak{p})\mathcal{N}\mathfrak{p}^{l_0-1}, \quad \beta(\mathfrak{p})^2\alpha(\mathfrak{p})\alpha'(\mathfrak{p}) = \beta(\mathfrak{p})^2\psi(\mathfrak{p})\mathcal{N}\mathfrak{p}^{k_0-1}.)$$

8.8. Concluding remarks. The existence of the p-adic measure in 8.7 is known in the special case, and $J = \emptyset$ (see [Pa2]), where $\mathbf{f}$ and $\mathbf{g}$ are assumed to be automorphic forms of scalar weights k and l, $k > l$. One verifies easily that the description 8.7 perfectly matches with the modified period conjecture and with the general conjecture on the p - adic L - functions of Section 6. Also, this construction was recently extended by My Vinh Quang (Moscow University) to Hilbert automorphic forms $\mathbf{f}$ and $\mathbf{g}$ of arbitrary vector weights $k = (k_1, \cdots, k_n)$, and $l = (l_1, \cdots, l_n)$ such that $k_i > l_i$ for all $i = 1, \cdots, n$, and to the non-p-ordinary case. In this situation the p-adic convolution of $L_{\mathbf{f},\mathbf{g}}$ is also uniquely determined by the above condition provided that it has the prescribed logarithmic growth on $\mathcal{X}_p$ (see [V1]).

In the general case the proof of the algebraic properties of the Rankin convolution in [Ha3] can be used also in order to carry out a p-adic construction. First of all, one obtains an expression for complex-valued distributions attached to $\Lambda(r, \mathbf{f}, \mathbf{g}(\chi))$ in terms of the cup product of certain coherent cohomology classes, and one verifies that these distributions take algebraic values. Then, integrality properties of the arithmetic vector bundles can be used for proving some generalized Kummer congruences for the values of these distributions, which is equivalent to the existence of p-adic L-functions in 8.7. However, some essential technical difficulties remain in the general case, and 8.7 can not be regarded yet as a theorem proven in full generality.

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