# DEFINING RELATIONS FOR AUTOMORPHISM GROUPS OF FREE ALGEBRAS

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Abstract. We describe a set of defining relations for automorphism groups of finitely generated free algebras of Nielsen-Schreier varieties. In particular, this gives a representation of the automorphism groups of free Lie algebras by generators and defining relations.

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## 1. Introduction

It is well known [4, 6, 7, 10] that all automorphisms of polynomial algebras and free associative algebras in two variables are tame. Moreover, the groups of automorphisms of polynomial algebras and free associative algebras in two variables are isomorphic and have a nice representation as a free product of groups (see, for example [2, 5]).

It was recently proved that the automorphism groups of polynomial algebras [16, 17, 22] and free associative algebras [23, 25] in three variables over a field of characteristic 0 cannot be generated by all elementary automorphisms, i.e. there exist wild automorphisms. Defining relations of the tame automorphism group of polynomial algebra in three variables were described in [23, 24].

There are several well-known descriptions of the automorphism group of a free group by generators and defining relations (see, for example [26]). P.Cohn proved [1] that all automorphisms of finitely generated free Lie algebras are tame. Later this result was extended to free algebras of Nielsen-Schreier varieties [9]. Recall that a variety of universal algebras is called Nielsen-Schreier, if any subalgebra of a free algebra of this variety is free, i.e. an analog of the classical Nielsen-Schreier theorem is true. The varieties of all nonassociative algebras [8], commutative and anticommutative algebras [18], Lie algebras [18, 27] are Nielsen-Schreier. Other examples of Nielsen-Schreier varieties can be found in [11, 15, 19, 20].

So, the automorphism groups of free algebras of Nielsen-Schreier varieties are generated by all elementary automorphisms. In this paper we describe a set of defining relations of these groups. In fact, we show that the relations for elementary automorphisms studied in [23, 24] are defining relations in this case. Note that groups of automorphisms of free algebras of Nielson-Schreier varieties (tame automorphism groups of polynomial algebras and free associative algebras) of rank at least four over a field of characteristic 0 do not admit a faithful representation by matrices over any field [14].

The paper is organized as follows. In Section 2 we describe a set of relations for elementary automorphisms and repeat the proofs of two lemmas from [24] for completeness. In Section 3 we give some well-known definitions and theorems about free algebras. In Section 4 we prove the main result.

## 2. Defining relations

Let $F$ be an arbitrary field, and let $\mathfrak{M}$ be an arbitrary variety of linear algebras over $F$. By $A = F_{\mathfrak{M}} < x_1, x_2, \ldots, x_n >$ denote the free algebra of $\mathfrak{M}$ with a free set of generators $X = \{x_1, x_2, \ldots, x_n\}$, and by $Aut\, A$ denote the group of all automorphisms of this algebra. Let $\phi = (f_1, f_2, \ldots, f_n)$ denote an automorphism $\phi$ of $A$ such that $\phi(x_i) = f_i$, $1 \leq i \leq n$. An automorphism

$$\sigma(i, \alpha, f) = (x_1, \ldots, x_{i-1}, \alpha x_i + f, x_{i+1}, \ldots, x_n), \tag{1}$$

where $0 \neq \alpha \in F$, $f \in F_{\mathfrak{M}} < X \setminus \{x_i\} >$, is called *elementary*. The subgroup $TA(A)$ of $Aut\, A$ generated by all elementary automorphisms is called the *tame automorphism group*, and the elements of this subgroup are called *tame automorphisms* of $A$. Nontame automorphisms of $A$ are called *wild*.

Now we describe some relations for elementary automorphisms (1). It is easy to check that

$$\sigma(i, \alpha, f)\sigma(i, \beta, g) = \sigma(i, \alpha\beta, \beta f + g). \tag{2}$$

Note that from this we obtain trivial relations $\sigma(i, 1, 0) = id$, where $1 \leq i \leq n$.

If $i \neq j$ and $f \in F_{\mathfrak{M}} < X \setminus \{x_i, x_j\} >$, then we have also

$$\sigma(i, \alpha, f)^{-1}\sigma(j, \beta, g)\sigma(i, \alpha, f) = \sigma(j, \beta, \sigma(i, \alpha, f)^{-1}(g)). \tag{3}$$

Consequently, if $i \neq j$ and $f, g \in F_{\mathfrak{M}} < X \setminus \{x_i, x_j\} >$, then the automorphisms $\sigma(i, \alpha, f)$, $\sigma(j, \beta, g)$ commute.

For every pair of integers $k, s$, where $1 \leq k \neq s \leq n$, we define a tame automorphism $(ks)$ by putting

$$(ks) = \sigma(s, -1, x_k)\sigma(k, 1, -x_s)\sigma(s, 1, x_k).$$

Note that the automorphism $(ks)$ of the algebra $A$ just permutes the variables $x_k$ and $x_s$. Now it is easy to see that

$$\sigma(i, \alpha, f)^{(ks)} = \sigma(j, \alpha, (ks)(f)), \tag{4}$$

where $x_j = (ks)(x_i)$.

Let $G(A)$ be the abstract group with generators (1) and defining relations (2)–(4).

**Lemma 1.** *The subgroup of $G(A)$ generated by all elements $(ks)$, where $1 \leq k \neq s \leq n$, is isomorphic to the symmetric group $S_n$.*

*Proof.* By (2) and (3), we have

$$\begin{aligned}(ks)^2 &= \sigma(s,-1,x_k)\sigma(k,1,-x_s)\sigma(s,1,x_k)\sigma(s,-1,x_k)\sigma(k,1,-x_s)\sigma(s,1,x_k)\\ &= \sigma(s,-1,x_k)\sigma(k,1,-x_s)\sigma(s,-1,0)\sigma(k,1,-x_s)\sigma(s,1,x_k)\\ &= \sigma(s,-1,x_k)\sigma(s,-1,0)\sigma(k,1,-x_s)^{\sigma(s,-1,0)}\sigma(k,1,-x_s)\sigma(s,1,x_k)\\ &= \sigma(s,1,-x_k)\sigma(k,1,x_s)\sigma(k,1,-x_s)\sigma(s,1,x_k) = \sigma(s,1,-x_k)\sigma(s,1,x_k) = id.\end{aligned}$$

Then (4) gives

$$\begin{aligned}(ks)^{(sk)} &= \sigma(s,-1,x_k)^{(sk)}\sigma(k,1,-x_s)^{(sk)}\sigma(s,1,x_k)^{(sk)}\\ &= \sigma(k,-1,x_s)\sigma(s,1,-x_k)\sigma(k,1,x_s) = (sk),\end{aligned}$$

i.e. $(ks)=(sk)$. Now it is not difficult to deduce from (2)–(4) that

$$[(ij),(ks)] = id,\ \ (ik)^{(is)} = (ks),$$

where $i,j,k,s$ are all distinct. It is immediate that the given relations imply the defining relations of the group $S_n$ with respect to the system of generators $(i\ i+1)$, where $1 \leq i \leq n-1$, which are indicated in [3]. □

By Lemma 1, the elements of the symmetric group $S_n$ can be identified with elements of $G(A)$. Note that (4) can be rewritten as

$$\sigma(i,\alpha,f)^{\pi} = \sigma(\pi^{-1}(i),\alpha,\pi^{-1}(f)),$$

where $\pi \in S_n$.

It is well known that the group of affine automorphisms $Af_n(F)$ of the algebra $A$ is generated by all affine elementary automorphisms.

**Lemma 2.** *The relations (2)–(4) for elementary affine automorphisms are defining relations of the group $Af_n(F)$.*

*Proof.* Let $\varphi$ be a product of elementary affine automorphisms. Suppose that $\varphi = id$. By (2) and (3), we can represent $\varphi$ in the form

$$\varphi = \sigma(1,1,\alpha_1)\sigma(2,1,\alpha_2)\ldots\sigma(n,1,\alpha_n)\varphi',$$

where $\varphi'$ is a product of elementary linear automorphisms. Obviously, $\alpha_1=\alpha_2=\ldots=\alpha_n=0$. Therefore we can assume that $\varphi$ is a product of elementary linear automorphisms. By (2) and (3), we can easily represent $\varphi$ in the form

$$\varphi = \sigma(1,\alpha_1,0)\sigma(2,\alpha_2,0)\ldots\sigma(n,\alpha_n,0)\varphi',$$

where $\varphi'$ is a product of elementary automorphisms of the type $\sigma(i,1,f)$. By (2)–(4), we have

$$\begin{aligned}&\sigma(k,\alpha,0) = \sigma(s,\alpha,0)^{(ks)}\\ &= \sigma(s,-1,x_k)\sigma(k,1,-x_s)\sigma(s,1,x_k)\sigma(s,\alpha,0)\sigma(s,-1,x_k)\sigma(k,1,-x_s)\sigma(s,1,x_k)\\ &= \sigma(s,-1,x_k)\sigma(k,1,-x_s)\sigma(s,-\alpha,0)\sigma(s,1,(1-\alpha)x_k)\sigma(k,1,-x_s)\sigma(s,1,x_k)\\ &= \sigma(s,-1,x_k)\sigma(s,-\alpha,0)\sigma(k,1,\alpha^{-1}x_s)\sigma(s,1,(1-\alpha)x_k)\sigma(k,1,-x_s)\sigma(s,1,x_k)\\ &= \sigma(s,\alpha,0)\sigma(s,1,-\alpha x_k)\sigma(k,1,\alpha^{-1}x_s)\sigma(s,1,(1-\alpha)x_k)\sigma(k,1,-x_s)\sigma(s,1,x_k).\end{aligned}$$

By using this relation, we can represent $\varphi$ in the form

$$\varphi = \sigma(n, \beta_n, 0)\varphi',$$

where $\varphi'$ is a product of elementary linear automorphisms of the form $\sigma(i, 1, f)$. Hence $\beta_n = 1$. Note that $\sigma(i, 1, f)$ can be represented as a product of automorphisms

$$X_{ij}(\lambda) = \sigma(j, 1, \lambda x_i), \quad \lambda \in F, \quad i \neq j. \tag{5}$$

Thus, we can assume that $\varphi$ is a product of automorphisms of the form (5).

Let $G$ be the subgroup of $TA(A)$ generated by all automorphisms of the form (5). We define a map

$$J : G \longrightarrow SL_n(F),$$

where $J(\psi)$ is the Jacobian matrix of $\psi \in G$. By $e_{ij}$ denote the standard matrix units and by $E_{ij}(\lambda)$ denote the elementary matrix $E + \lambda e_{ij}$, where $E$ is the unit matrix, $i \neq j$, and $\lambda \in F$. It is easy to check that

$$J(X_{ij}(\lambda)) = E_{ij}(\lambda),$$

and that $J$ is an isomorphism of groups.

Now it is sufficient to prove that every relation of the group $SL_n(F)$ is a corollary of (2)–(4). Obviously, (2)–(3) cover the Steinberg relations (see, for example [13]). Besides, according to [13], we need to check the relation

$$\{u, v\} = id, \quad 0 \neq u, v \in F,$$

where

$$\begin{aligned}
\{u, v\} &= h_{ij}(uv)h_{ij}(u)^{-1}h_{ij}(v)^{-1},\\
h_{ij}(u) &= w_{ij}(u)w_{ij}(-1),\\
w_{ij}(u) &= X_{ij}(u)X_{ji}(-u^{-1})X_{ij}(u).
\end{aligned}$$

Applying (2)–(4) we have

$$\begin{aligned}
w_{ij}(u) &= \sigma(j, 1, ux_i)\sigma(i, 1, -u^{-1}x_j)\sigma(j, 1, ux_i)\\
&= \sigma(j, 1, ux_i)\sigma(i, u, 0)\sigma(i, 1, -x_j)\sigma(i, u^{-1}, 0)\sigma(j, 1, ux_i)\\
&= \sigma(i, u, 0)\sigma(j, 1, ux_i)^{\sigma(i,u,0)}\sigma(i, 1, -x_j)\sigma(j, 1, ux_i)^{\sigma(i,u,0)}\sigma(i, u^{-1}, 0)\\
&= \sigma(i, u, 0)\sigma(j, 1, x_i)\sigma(i, 1, -x_j)\sigma(j, 1, x_i)\sigma(i, u^{-1}, 0)\\
&= \sigma(i, u, 0)\sigma(j, -1, 0)\sigma(j, -1, x_i)\sigma(i, 1, -x_j)\sigma(j, 1, x_i)\sigma(i, u^{-1}, 0)\\
&= \sigma(i, u, 0)\sigma(j, -1, 0)(ij)\sigma(i, u^{-1}, 0) = (ij)\sigma(i, u, 0)^{(ij)}\sigma(j, -1, 0)^{(ij)}\sigma(i, u^{-1}, 0)\\
&= (ij)\sigma(j, u, 0)\sigma(i, -1, 0)\sigma(i, u^{-1}, 0) = (ij)\sigma(j, u, 0)\sigma(i, -u^{-1}, 0).
\end{aligned}$$

Consequently,

$$\begin{aligned}
h_{ij}(u) = w_{ij}(u)w_{ij}(-1) &= (ij)\sigma(j, u, 0)\sigma(i, -u^{-1}, 0)(ij)\sigma(j, -1, 0)\sigma(i, 1, 0)\\
&= \sigma(j, u, 0)^{(ij)}\sigma(i, -u^{-1}, 0)^{(ij)}\sigma(j, -1, 0)\\
&= \sigma(i, u, 0)\sigma(j, -u^{-1}, 0)\sigma(j, -1, 0) = \sigma(i, u, 0)\sigma(j, u^{-1}, 0).
\end{aligned}$$

Hence

$$\{u,v\} = h_{ij}(uv)h_{ij}(u)^{-1}h_{ij}(v)^{-1}$$
$$= \sigma(i,uv,0)\sigma(j,(uv)^{-1},0)\sigma(i,u,0)\sigma(j,u^{-1},0)\sigma(i,v,0)\sigma(j,v^{-1},0) = id.$$

Thus we can say that every relation of the group $SL_n(F)$ follows from (2)–(4). □

## 3. Reductions of automorphisms

Let $\mathfrak{M}$ be an arbitrary homogeneous variety of linear algebras over a field $F$. Recall that if $F$ is infinite, then any variety of linear algebras over $F$ is homogeneous [28]. Let $A = F_{\mathfrak{M}} < x_1, x_2, \ldots, x_n >$ be the free algebra of $\mathfrak{M}$ with free generators $x_1, x_2, \ldots, x_n$. The highest homogeneous part $\overline{f}$ and the degree $\deg f$ can be defined in the usual way. If $f_1, f_2, \ldots, f_k \in A$, then denote by $< f_1, f_2, \ldots, f_k >$ the subalgebra of $A$ generated by these elements.

Let $\theta = (f_1, f_2, \ldots, f_k)$ be an arbitrary $k$-tuple of elements of the algebra $A$. The number

$$\deg\theta = \deg f_1 + \deg f_2 + \ldots + \deg f_k$$

is called the *degree* of $\theta$.

Recall that an *elementary transformation* of a $k$-tuple $\theta = (f_1, f_2, \ldots, f_k)$ is, by definition, a transformation that changes only one element $f_i$ to an element of the form $\alpha f_i + g$, where $0 \neq \alpha \in F$ and $g \in \langle\{f_j | j \neq i\}\rangle$. The notation

$$\theta \to \tau$$

means that the $k$-tuple $\tau$ is obtained from $\theta$ by a single elementary transformation. A $k$-tuple $\theta$ is called *elementarily reducible* or *admits an elementary reduction* if there exists a $k$-tuple $\tau$ such that $\theta \to \tau$ and $\deg\tau < \deg\theta$. The element $f_i$ of the $k$-tuple $\theta$ which was changed in $\tau$ to an element of less degree is called *reducible* and we will say also that $f_i$ is *reduced in $\theta$ by the $k$-tuple $\tau$*.

Consider a finite number of elements

(6) $$f_1, f_2, \ldots, f_k$$

of the algebra $A$. The elements (6) are called *free* if the subalgebra $< f_1, f_2, \ldots, f_k >$ of $A$ is a free algebra of the variety $\mathfrak{M}$ with free generators (6). If

$$\overline{f_i} \notin < \overline{f_1}, \ldots, \overline{f_{i-1}}, \overline{f_{i+1}}, \ldots, \overline{f_n} >$$

for any $i$, then the elements (6) are called *reduced*.

From any $k$-tuple $(f_1, f_2, \ldots, f_k)$ by several elementary transformations we can get a $k$-tuple $(g_1, g_2, \ldots, g_s, 0, \ldots, 0)$, where $s \leq k$, such that the elements $g_1, g_2, \ldots, g_s$ are reduced. Note that $< f_1, f_2, \ldots, f_k > = < g_1, g_2, \ldots, g_s >$.

The statement of the next lemma is well known (see, for example [18]) and very useful in studying free algebras.

**Lemma 3.** *Assume that the elements $\overline{f_1}, \overline{f_2}, \ldots, \overline{f_k}$ are free. If $f \in < f_1, f_2, \ldots, f_k >$, then $\overline{f} \in < \overline{f_1}, \overline{f_2}, \ldots, \overline{f_k} >$.*

From now on we assume that $\mathfrak{M}$ is a homogeneous Nielsen-Schreier variety of linear algebras. The main property of Nielsen-Schreier varieties is given in the next lemma (see, for example [9]).

**Lemma 4.** *Assume that $f_1, f_2, \ldots, f_k$ are homogeneous elements of $A$ and* $\deg f_1 \leq \deg f_2, \leq \ldots \leq \deg f_k$*. If the elements $f_1, f_2, \ldots, f_k$ are not free, then there exists $i$ such that $f_i \in < f_1, f_2, \ldots, f_{i-1} >$.*

**Corollary 1.** *Any finite reduced system of elements of the algebra $A$ is free.*

Note that the statement of this corollary for infinite systems of elements is also true [9]. Free systems of elements in free algebras were studied in [12, 21] via Fox derivatives.

**Corollary 2.** *Any automorphism of the algebra $A$ of degree more than $n$ is elementarily reducible.*

**Corollary 3.** *Automorphisms of the algebra $A$ are tame.*

Suppose that $\theta = (f_1, f_2, \ldots, f_n)$ and $\sigma(i, \alpha, f)$ is an elementary automorphism of the form (1). If

$$\tau = (f_1, \ldots, f_{i-1}, \alpha f_i + f(f_1, \ldots, f_n), f_{i+1}, \ldots, f_n),$$

then instead of $\theta \to \tau$ we often write

$$\theta \stackrel{\sigma(i,\alpha,f)}{\longrightarrow} \tau.$$

Assume that

$$\theta = \phi_1 \phi_2 \ldots \phi_r \in Aut\, A, \tag{7}$$

where $\phi_i$, $1 \leq i \leq r$, are elementary automorphisms. Put

$$\psi_i = \phi_1 \phi_2 \ldots \phi_i, \;\; 0 \leq i \leq r.$$

In particular, we have

$$\psi_r = \theta, \;\; \psi_0 = id.$$

To (7) corresponds the sequence of elementary transformations

$$id = \psi_0 \stackrel{\phi_1}{\to} \psi_1 \stackrel{\phi_2}{\to} \psi_2 \stackrel{\phi_3}{\to} \ldots \stackrel{\phi_r}{\to} \psi_r = \theta. \tag{8}$$

So, every tame automorphism $\theta$ has a sequence of elementary transformations of the form (8). If $\deg \theta > n$ and $\deg \psi_i < \deg \theta$ for any $i < r$, then the sequence (8) will be called a *minimal representation* of $\theta$. Note that the representations (7) and (8) of the automorphism $\theta$ are equivalent. If (8) is a minimal representation of $\theta$, then the representation (7) will be also called a *minimal representation* of $\theta$.

## 4. The main result

As above, $\mathfrak{M}$ is a homogeneous Nielsen-Schreier variety of linear algebras over a field $F$, and $A = F_{\mathfrak{M}} < x_1, x_2, \ldots, x_n >$ is a free algebra of $\mathfrak{M}$. We know that $TA(A) = Aut\, A$ and the elementary automorphisms (1) are generators of the group $Aut\, A$.

**Theorem 1.** *The relations (2)–(4) are defining relations of the group Aut A with respect to the generators (1).*

**Beginning of the proof.** Assume that

$$\varphi_1\varphi_2\dots\varphi_k = id = (x_1, x_2, \dots, x_n), \tag{9}$$

where $\varphi_i$, $1 \leq i \leq k$, are elementary automorphisms. Put

$$\theta_i = \varphi_1\varphi_2\dots\varphi_i,\ 0 \leq i \leq k.$$

In particular, we have $\theta_0 = \theta_k = (x_1, x_2, \dots, x_n)$. To (9) corresponds the sequence of elementary transformations

$$id = \theta_0 \xrightarrow{\varphi_1} \theta_1 \xrightarrow{\varphi_2} \dots \xrightarrow{\varphi_k} \theta_k = id. \tag{10}$$

Put $d = max\{\deg\ \theta_i | 0 \leq i \leq k\}$. Let $i_1$ be the minimal number and $i_2$ be the maximal number which satisfy the equations $\deg\ \theta_{i_1} = d$ and $\deg\ \theta_{i_2} = d$. Put $q = i_2 - i_1$. The pair $(d, q)$ will be called the *exponent* of the relation (9).

To prove the theorem, we show that (9) follows from (2)–(4). Assume that our theorem is not true. Call a relation of the form (9) *trivial* if it follows from (2)–(4). We choose a nontrivial relation (9) with the minimal exponent $(d, q)$ with respect to the lexicographic order. To arrive at a contradiction, we show that (9) is also trivial.

If $d = n$, then Lemma 2 gives the triviality of the relation (9). Therefore we can assume that $d > n$.

Our plan is to change the product (9) by using (2)–(4) and to obtain a new sequence (10) whose exponent is strictly less than $(d, q)$. Below we prove Lemmas 5–14 and then complete the proof of the theorem. □

Denote by $t = [\frac{q}{2}]$ the integral part of $\frac{q}{2}$. Put also

$$\phi = \theta_{i_1+t-1},\ \theta = \theta_{i_1+t},\ \tau = \theta_{i_1+t+1},\ \sigma_1 = \varphi_{i_1+t},\ \sigma_2 = \varphi_{i_1+t+1}.$$

Then we have

$$\phi \xrightarrow{\sigma_1} \theta \xrightarrow{\sigma_2} \tau. \tag{11}$$

**Lemma 5.** *The following statements are true:*

(1) $d = \deg\ \theta$, $t = 0$, *and*

$$\theta = \varphi_1\varphi_2\dots\varphi_{i_1+t} \tag{12}$$

*is a minimal representation of* $\theta$.

(2) *If* $q = 0$, *then*

$$\theta = \varphi_k^{-1}\varphi_{k-1}^{-1}\dots\varphi_{i_1+t+1}^{-1}$$

*is also a minimal representation of* $\theta$.

(3) *If* $q = 1$, *then* $(d(\tau), t(\tau)) = (d, t)$ *and*

$$\tau = \varphi_k^{-1}\varphi_{k-1}^{-1}\dots\varphi_{i_1+t+2}^{-1}$$

*is a minimal representation of* $\tau$. *Moreover, in (9) the product (12) can be replaced by an arbitrary minimal representation of* $\theta$.

*Proof.* Assume that $(d(\theta), t(\theta)) < (d, t)$ and let (7) be a minimal representation of $\theta$. Then (9) is a consequence of the equalities

$$\varphi_1\varphi_2 \dots \varphi_{i_1+t}\phi_r^{-1} \dots \phi_2^{-1}\phi_1^{-1} = id, \tag{13}$$

$$\phi_1\phi_2 \dots \phi_r\varphi_{i_1+t+1} \dots \varphi_{k-1}\varphi_k = id. \tag{14}$$

To (13) corresponds the sequence of elementary transformations

$$(x_1, x_2, \dots, x_n) \to \theta_1 \to \dots \to \theta_{i_1+t} = \theta = \psi_r \to \psi_{r-1} \dots \to \psi_1 \to (x_1, x_2, \dots, x_n),$$

and to (14) corresponds

$$(x_1, x_2, \dots, x_n) \to \psi_1 \to \dots \to \psi_r = \theta = \theta_{i_1+t} \to \theta_{i_1+t+1} \to \dots \to \theta_{k-1} \to (x_1, x_2, \dots, x_n).$$

Since $(d(\theta), t(\theta)) < (d, t)$, it follows that (13) and (14) have exponents strictly less than $(d, q)$. This gives the first statement of the lemma.

It is obvious that the relation

$$\varphi_k^{-1}\varphi_{k-1}^{-1} \dots \varphi_1^{-1} = id$$

has the same exponent $(d, q)$. Applying the first statement of the lemma to this relation, we get the second statement of the lemma, as well as the minimality of the representation of $\tau$ if $q = 1$. If $q = 1$, then (13) has exponent strictly less than $(d, q)$, and (14) has the exponent $(d, q)$. Consequently, (9) and (14) are equivalent modulo (2)–(4). Thus $\theta$ can be changed by an arbitrary minimal representation in (14). □

Put $\theta = (f_1, f_2, \dots, f_n)$. According to Lemma 5, $t = 0$, $q = 0, 1$, and

$$\deg\, \phi < \deg\, \theta = d \geq \deg\, \tau.$$

Without loss of generality, we can assume that

$$\tau = (f_1, f_2, \dots, f_{n-1}, f), \tag{15}$$

where

$$f = \beta f_n + B, \quad B = b(f_1, f_2 \dots, f_{n-1}), \quad \deg\, B \leq \deg f_n.$$

**Lemma 6.** *If $\phi$ reduces the element $f_n$ of $\theta$, then the relation (9) is trivial.*

*Proof.* Applying (2) we can replace $\sigma_1\sigma_2$ by an elementary automorphism. Obviously, this replacement also decreases the exponent of (10). □

By Lemma 6, we can assume that $\phi$ reduces one of the elements $f_1, f_2, \dots, f_{n-1}$ of $\theta$.

**Lemma 7.** *Assume that $\phi$ reduces the element $f_i$ of $\theta$, where $1 \leq i \leq n-1$. If $\phi'$ also reduces the element $f_i$ of $\theta$, then in (11) the automorphism $\phi$ can be replaced by $\phi'$.*

*Proof.* According to (2), in this case the elementary transformation $\phi \to \theta$ can be changed to $\phi \to \phi' \to \theta$. Since $\deg\, \phi' < \deg\, \theta = d$, the exponent $(d, q)$ of the sequence (10) does not change after this replacement. But in the new sequence (10) we have $\phi'$ instead of $\phi$. □

From now on we assume that $\phi$ reduces the element $f_i$ of $\theta$, where $1 \leq i \leq n-1$. Taking Lemma 7 into account, we can assume that

$$\phi = (f_1, \dots, f_{i-1}, g_i, f_{i+1}, \dots, f_n), \tag{16}$$

where

$$g_i = f_i - C, \quad C = c(f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_n), \quad \deg g_i < \deg f_i.$$

Thus, we defined the members of the sequence (11) and we have

$$\sigma_1 = \sigma(i, 1, c(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)), \quad \sigma_2 = \sigma(n, \beta, b(x_1 \dots, x_{n-1})).$$

**Lemma 8.** *If the elements $\overline{f_1}, \dots, \overline{f_{i-1}}, \overline{f_{i+1}}, \dots, \overline{f_{n-1}}$ are not free, then the relation (9) is trivial.*

*Proof.* If the elements $\overline{f_1}, \dots, \overline{f_{i-1}}, \overline{f_{i+1}}, \dots, \overline{f_{n-1}}$ are not free, then according to Lemma 4, there exists $r$ such that

$$\overline{f_r} \in < \overline{f_1}, \dots, \overline{f_{r-1}}, \overline{f_{r+1}}, \dots, \overline{f_{i-1}}, \overline{f_{i+1}}, \dots, \overline{f_{n-1}} > .$$

Suppose that

$$\overline{f_r} = T(\overline{f_1}, \dots, \overline{f_{r-1}}, \overline{f_{r+1}}, \dots, \overline{f_{i-1}}, \overline{f_{i+1}}, \dots, \overline{f_{n-1}})$$

and put

$$g_r = f_r - T(f_1, \dots, f_{r-1}, f_{r+1}, \dots, f_{i-1}, f_{i+1}, \dots, f_{n-1}).$$

Then $\deg\, g_r < \deg\, f_r$. Put also

$$\begin{aligned}
\psi_1 &= (f_1, \dots, f_{r-1}, g_r, f_{r+1}, \dots, f_{i-1}, g_i, f_{i+1}, \dots, f_{n-1}, f_n),\\
\psi_2 &= (f_1, \dots, f_{r-1}, g_r, f_{r+1}, \dots, f_{i-1}, f_i, f_{i+1}, \dots, f_{n-1}, f_n),\\
\psi_3 &= (f_1, \dots, f_{r-1}, g_r, f_{r+1}, \dots, f_{i-1}, f_i, f_{i+1}, \dots, f_{n-1}, f).
\end{aligned}$$

Then we have the sequence of elementary transformations

$$\phi \xrightarrow{\delta_1} \psi_1 \xrightarrow{\delta_2} \psi_2 \xrightarrow{\delta_3} \psi_3 \xrightarrow{\delta_4} \tau, \tag{17}$$

where

$$\begin{aligned}
\delta_4 = \sigma(r, 1, T(x_1, \dots, x_{r-1}, x_{r+1}, \dots, x_{i-1}, x_{i+1}, \dots, x_{n-1})), \quad \delta_1 = \delta_4^{-1},\\
\delta_2 = \sigma(i, 1, c(x_1, \dots, x_{r-1}, \delta_4(x_r), x_{r+1}, \dots, x_{i-1}, x_{i+1}, \dots, x_n)),\\
\delta_3 = \sigma(n, \beta, b(x_1, \dots, x_{r-1}, \delta_4(x_r), x_{r+1}, \dots, x_{n-1})).
\end{aligned}$$

By (3) we have

$$\delta_1\delta_2\delta_3\delta_4 = \delta_2^{\delta_4}\delta_3^{\delta_4} = \sigma_1\sigma_2.$$

Then, we can replace the subsequence (11) of (10) by (17). Since $\deg\, \psi_1, \deg\, \psi_2, \deg\, \psi_3 < d$, the new sequence (10) has the exponent less than $(d, q)$. Consequently, the relation (9) is trivial. $\square$

**Lemma 9.** *If $\overline{f_i} \in \langle \overline{f_1}, \dots, \overline{f_{i-1}}, \overline{f_{i+1}}, \dots, \overline{f_{n-1}} \rangle$, then (9) is trivial.*

*Proof.* Assume that

$$\overline{f_i} = T(\overline{f_1}, \dots, \overline{f_{i-1}}, \overline{f_{i+1}}, \dots, \overline{f_{n-1}}).$$

According to Lemma 7, we can suppose that

$$g_i = f_i - T(f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_{n-1}).$$

Then $\sigma_1 = \sigma(i, 1, T)$. By (3), we have $\sigma_1\sigma_2 = \sigma(n, \beta, b_1)\sigma_1$, where $b_1 = \sigma_1(b) \in F_{\mathfrak{M}} < x_1, \ldots, x_{n-1} >$. After the corresponding replacement in (9), $\theta$ is replaced by

$$\theta' = (f_1, \ldots, f_{i-1}, g_i, f_{i+1}, \ldots, f_{n-1}, f)$$

in (10). Since deg $\theta' < d$, the exponent of (9) is decreased. □

**Lemma 10.** *If $\overline{f_n} \in \langle \overline{f_1}, \ldots, \overline{f_{i-1}}, \overline{f_{i+1}}, \ldots, \overline{f_{n-1}} \rangle$, then the relation (9) is trivial.*

*Proof.* Assume that

$$\overline{f_n} = T(\overline{f_1}, \ldots, \overline{f_{i-1}}, \overline{f_{i+1}}, \ldots, \overline{f_{n-1}})$$

and put

$$g_n = f_n - T(f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_{n-1}).$$

According to (3), we have

$$\sigma_1 = \sigma(i, 1, c(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)) = \delta_1\delta_2\delta_3,$$

where

$$\delta_1 = \sigma(n, 1, -T(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n-1})),$$
$$\delta_2 = \sigma(i, 1, c_1(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)),$$
$$\delta_3 = \sigma(n, 1, T(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n-1})).$$

After the corresponding replacement in (9), the elementary transformation $\phi \to \theta$ is replaced by the sequence of elementary transformations

$$\phi \to \psi_1 \to \psi_2 \to \theta,$$

where

$$\psi_1 = (f_1, \ldots, f_{i-1}, g_i, f_{i+1}, \ldots, f_{n-1}, g_n),$$
$$\psi_2 = (f_1, \ldots, f_{i-1}, f_i, f_{i+1}, \ldots, f_{n-1}, g_n).$$

Since deg $\psi_1$, deg $\psi_2 < d =$ deg $\theta$, the new sequence (10) has the same exponent $(d, q)$. However, instead of $\phi$ we have $\psi_2$, which reduces the element $f_n$ of $\theta$. By Lemma 6, we obtain the triviality of (9). □

**Lemma 11.** *If $B$ does not depend on $f_i$, then (9) is trivial.*

*Proof.* It means that $b$ does not depend on $x_i$. By (3) we have

$$\sigma_1\sigma_2 = \sigma(i, 1, c)\sigma(n, \beta, b) = \sigma(n, \beta, b)\sigma(2, 1, c_1),$$

where $c_1 = \sigma(n, \beta, b)^{-1}(c) \in F_{\mathfrak{M}} < X \setminus \{x_i\} >$. After the corresponding replacement in (9), instead of $\theta$ we obtain

$$\psi = (f_1, \ldots, f_{i-1}, g_i, f_{i+1}, \ldots, f_{n-1}, f).$$

Since deg $\psi < d$, this replacement also decreases the exponent of (9). □

**Lemma 12.** *If $\overline{f_i} = \gamma\overline{f_n} + T(\overline{f_1}, \ldots, \overline{f_{i-1}}, \overline{f_{i+1}}, \ldots, \overline{f_{n-1}})$, then (9) is trivial.*

*Proof.* By Lemma 9, we can assume that $\gamma \neq 0$. By Lemma 7, we can also assume that $C = \gamma f_n + T(f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_{n-1})$. Consequently,

$$\begin{aligned} g_i &= f_i - \gamma f_n - T(f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_{n-1}), \\ f_i &= g_i + \gamma f_n + T(f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_{n-1}), \\ f_n &= -\frac{1}{\gamma} g_i + \frac{1}{\gamma} f_i - \frac{1}{\gamma} T(f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_{n-1}). \end{aligned}$$

These equalities give rise to the sequence of elementary transformations

$$\psi_1 \to \psi_2 \to \theta,$$

where

$$\psi_1 = (f_1, \ldots, f_{i-1}, f_n, f_{i+1}, \ldots, f_{n-1}, g_i),$$
$$\psi_2 = (f_1, \ldots, f_{i-1}, f_i, f_{i+1}, \ldots, f_{n-1}, g_i).$$

We have

$$\sigma_1 = \sigma(i, 1, \gamma x_n + T) = \sigma(i, 1, \gamma x_n + T(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n-1})).$$

Applying (2) and (3) we get

$$\begin{aligned} \sigma_1 = \sigma(i,1,\gamma x_n)\sigma(i,1,T) &= \sigma(i,1,\gamma x_n)\sigma(n,-\gamma,x_i)\sigma(n,-\frac{1}{\gamma},\frac{1}{\gamma}x_i)\sigma(i,1,T) \\ &= \sigma(i,1,\gamma x_n)\sigma(n,-\gamma,x_i)\sigma(i,1,T)\sigma(n,-\frac{1}{\gamma},\frac{1}{\gamma}x_i)^{\sigma(i,1,T)} \\ &= \sigma(i,1,\gamma x_n)\sigma(n,-\gamma,x_i)\sigma(i,1,T)\sigma(n,-\frac{1}{\gamma},\frac{1}{\gamma}(x_i - T)) \\ = \sigma(i,1,\gamma x_n)\sigma(n,-\gamma,x_i)\sigma(i,\frac{1}{\gamma},-\frac{1}{\gamma}x_n)&\sigma(i,\gamma,x_n+T)\sigma(n,-\frac{1}{\gamma},\frac{1}{\gamma}(x_i - T)). \end{aligned}$$

Since the transposition $(in) \in S_n$ can be factored as a product of linear elementary automorphisms

$$(in) = \sigma(i,1,\gamma x_n)\sigma(n,-\gamma,x_i)\sigma(i,\frac{1}{\gamma},-\frac{1}{\gamma}x_n),$$

we obtain

$$\sigma_1 = (in)\sigma(i,\gamma,x_n+T)\sigma(n,-\frac{1}{\gamma},\frac{1}{\gamma}(x_i - T)).$$

Then

$$\theta = (in)\varphi_1^{(in)}\varphi_2^{(in)} \ldots \varphi_{i_1+t-1}^{(in)}\sigma(i,\gamma,x_n+T)\sigma(n,-\frac{1}{\gamma},\frac{1}{\gamma}(x_i - T)), \tag{18}$$

where $\varphi_j^{(in)}$ are elementary automorphisms, according to (4). To (18) corresponds the sequence of elementary transformations

$$\begin{aligned} (x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_{n-1}, x_n) &\mapsto (x_1, \ldots, x_{i-1}, x_n, x_{i+1}, \ldots, x_{n-1}, x_i) \to \\ &\theta_1' \to \theta_2' \to \ldots \to \theta_{i_1+t-1}' = \psi_1 \to \psi_2 \to \theta, \end{aligned}$$

where $\theta_i'$ is obtained from $\theta_i$ only by the permutation of the coordinates with numbers $i$ and $n$, and the transformation

$$(x_1,\dots,x_{i-1},x_i,x_{i+1},\dots,x_{n-1},x_n)\mapsto(x_1,\dots,x_{i-1},x_n,x_{i+1},\dots,x_{n-1},x_i)$$

is a composition of three elementary linear transformations.

If in (9) we replace $\theta$ by (18), then the exponent of (10) remains the same. But instead of $\phi$ we have $\psi_2$, which reduces the element $f_n$ of $\theta$, and Lemma 6 gives the triviality of (9). □

**Lemma 13.** *If the elements $\overline{f_1},\dots,\overline{f_{i-1}},\overline{f_{i+1}},\dots,\overline{f_n}$ are free, then (9) is trivial.*

*Proof.* By Lemma 3 and (16), we have

$$\overline{f_i}=\overline{C}=c(\overline{f_1},\dots,\overline{f_{i-1}},\overline{f_{i+1}},\dots,\overline{f_n}).$$

By Lemma 9 we can assume that $\overline{f_i}$ depends on $\overline{f_n}$. Consequently, $\deg f_n\le\deg f_i$.

Note that if the elements $\overline{f_1},\dots,\overline{f_{i-1}},\overline{f_i},\overline{f_{i+1}},\dots,\overline{f_{n-1}}$ are not free, then it follows that the elements $\overline{f_1},\dots,\overline{f_{i-1}},\overline{f_{i+1}},\dots,\overline{f_n}$ are also not free, which contradicts the condition of the lemma. Consequently, the elements $\overline{f_1},\dots,\overline{f_{i-1}},\overline{f_i},\overline{f_{i+1}},\dots,\overline{f_{n-1}}$ are free. Then $\overline{B}\in\langle\overline{f_1},\dots,\overline{f_{i-1}},\overline{f_i},\overline{f_{i+1}},\dots,\overline{f_{n-1}}\rangle$. By Lemma 11 we can assume that $B$ contains $f_i$. Then $\deg f_i\le\deg B\le\deg f_n$, i.e. $\deg f_i=\deg f_n$. Hence

$$\overline{C}=\overline{f_i}=\gamma\overline{f_n}+T(\overline{f_1},\dots,\overline{f_{i-1}},\overline{f_{i+1}},\dots,\overline{f_{n-1}}).$$

Lemma 12 gives the triviality of (9). □

**Lemma 14.** *Assume that there exists $r$ such that $r\neq i$, $1\le i\le n-1$, and*

$$\overline{f_r}\in<\overline{f_1},\dots,\overline{f_{r-1}},\overline{f_{r+1}},\dots,\overline{f_{i-1}},\overline{f_{i+1}},\dots,\overline{f_n}>.$$

*Then in (11) the automorphism $\phi$ can be replaced by an automorphism which reduces the element $f_r$ of $\theta$.*

*Proof.* Assume that

$$\overline{f_r}=T(\overline{f_1},\dots,\overline{f_{r-1}},\overline{f_{r+1}},\dots,\overline{f_{i-1}},\overline{f_{i+1}},\dots,\overline{f_n})$$

where $T\in F_{\mathfrak{M}}<X\setminus\{x_r,x_i\}>$, and put

$$g_r=f_r-T(f_1,\dots,f_{r-1},f_{r+1},\dots,f_{i-1},f_{i+1},\dots,f_n).$$

By (3) we have

$$\sigma_1=\sigma(r,1,-T)\sigma(i,1,c_1)\sigma(r,1,T)$$

for some $c_1\in F_{\mathfrak{M}}<X\setminus\{x_i\}>$. After such replacement, instead of $\phi\to\theta$ we obtain

$$\phi\to\psi_1\to\psi_2\to\theta,$$

where

$$\psi_1=(f_1,\dots,f_{r-1},g_r,f_{r+1},\dots,f_{i-1},g_i,f_{i+1},\dots,f_n),$$
$$\psi_2=(f_1,\dots,f_{r-1},g_r,f_{r+1},\dots,f_{i-1},f_i,f_{i+1},\dots,f_n).$$

Since $\deg\psi_1,\deg\psi_2<d=\deg\theta$, the new sequence (10) has the same exponent. Then instead of $\phi$ in (11) we have $\psi_2$, which reduces the element $f_r$ of $\theta$. □

**Completion of the proof of Theorem 1.** By Lemma 13, we can assume that the elements

$$\overline{f_1}, \dots, \overline{f_{i-1}}, \overline{f_{i+1}}, \dots, \overline{f_n}$$

are not free. Then, according to Lemma 4, there exists $j \neq i$ such that

$$\overline{f_j} \in < \overline{f_1}, \dots, \overline{f_{j-1}}, \overline{f_{j+1}}, \dots, \overline{f_{i-1}}, \overline{f_{i+1}}, \dots, \overline{f_n} > .$$

By Lemma 10, we can assume that $j \neq n$, i.e. $j \leq n-1$. According to Lemma 8, we can also assume that $\overline{f_j}$ depends on $\overline{f_n}$. Consequently, $\deg f_j \geq \deg f_n$. If $\deg f_j = \deg f_n$, then from this we can easily obtain that $\overline{f_n} \in < \overline{f_1}, \dots, \overline{f_{i-1}}, \overline{f_{i+1}}, \dots, \overline{f_{n-1}} >$, and Lemma 10 gives the triviality of (9). Thus, it can be assumed that $\deg f_j > \deg f_n$. Moreover, by Lemma 14, we may assume that $\phi$ reduces the element $f_j$ of $\theta$. Interchanging $f_i$ and $f_j$, from now we can assume without of generality that $\deg f_i > \deg f_n$ and

$$\overline{f_i} \in < \overline{f_1}, \dots, \overline{f_{i-1}}, \overline{f_{i+1}}, \dots, \overline{f_n} > . \tag{19}$$

Suppose that the elements $\overline{f_1}, \overline{f_2}, \dots, \overline{f_{n-1}}$ are free. By Lemma 3, $\overline{B} \in < \overline{f_1}, \overline{f_2}, \dots, \overline{f_{n-1}} >$. If $B$ depends on $f_i$, then $\deg B \geq \deg f_i > \deg f_n$, which contradicts (15). If $B$ does not depend on $f_i$, then Lemma 11 gives the triviality of (9).

If the elements $\overline{f_1}, \overline{f_2}, \dots, \overline{f_{n-1}}$ are not free, then there exists $r$ such that

$$\overline{f_r} \in < \overline{f_1}, \dots, \overline{f_{r-1}}, \overline{f_{r+1}}, \dots, \overline{f_{n-1}} > . \tag{20}$$

By Lemma 9, we can take $r \neq i$. If $\overline{f_r}$ does not depend on $\overline{f_i}$, then Lemma 8 gives the triviality of (9). Assume that $\overline{f_r}$ depends on $\overline{f_i}$. Then, $\deg f_r \geq \deg f_i$. If $\deg f_r = \deg f_i$, then from this we can obtain that $\overline{f_i} \in < \overline{f_1}, \dots, \overline{f_{i-1}}, \overline{f_{i+1}}, \dots, \overline{f_{n-1}} >$, and Lemma 9 gives the triviality of (9). So, we can assume that $\deg f_r > \deg f_i$. Then (19) gives that

$$\overline{f_i} \in < \overline{f_1}, \dots, \overline{f_{r-1}}, \overline{f_{r+1}}, \dots, \overline{f_{i-1}}, \overline{f_{i+1}}, \dots, \overline{f_n} > .$$

Consequently,

$$\overline{f_r} \in < \overline{f_1}, \dots, \overline{f_{r-1}}, \overline{f_{r+1}}, \dots, \overline{f_{i-1}}, \overline{f_{i+1}}, \dots, \overline{f_n} > .$$

By Lemma 14, we can assume that $\phi$ reduces the element $f_r$ of $\theta$. Then, (20) and Lemma 9 gives the triviality of (9).

This completes the proof of Theorem 1. $\square$

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