# The relation between the Baum-Connes Conjecture and the Trace Conjecture


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**Abstract**

We prove a version of the $L^2$-index Theorem of Atiyah, which uses the universal center-valued trace instead of the standard trace. We construct for $G$-equivariant K-homology an equivariant Chern character, which is an isomorphism and lives over the ring $\mathbb{Z} \subset \Lambda^G \subset \mathbb{Q}$ obtained from the integers by inverting the orders of all finite subgroups of $G$. We use these two results to show that the Baum-Connes Conjecture implies the modified Trace Conjecture, which says that the image of the standard trace $K_0(C_r^*(G)) \to \mathbb{R}$ takes values in $\Lambda^G$. The original Trace Conjecture predicted that its image lies in the additive subgroup of $\mathbb{R}$ generated by the inverses of all the orders of the finite subgroups of $G$, and has been disproved by Roy [13].



Key words: Baum-Connes Conjecture, Trace Conjecture, equivariant Chern character, $L^2$-index theorem.
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## 0. Introduction and statements of results

Throughout this paper let $G$ be a discrete group. The *Baum-Connes Conjecture for $G$* says that the assembly map

$$\operatorname{asmb}^G : K_0^G(\underline{E}G) \to K_0(C_r^*(G))$$

from the equivariant $K$-homology of the classifying space for proper $G$-actions $\underline{E}G$ to the topological $K$-theory of the reduced $C^*$-algebra $C_r^*(G)$ is bijective [3, page 8], [5, Conjecture 3.1]. In connection with this conjecture Baum and Connes [3, page 21] also made the sometimes so called *Trace Conjecture.* It says that the image of the composition

$$K_0(C_r^*(G)) \xrightarrow{i} K_0(\mathcal{N}(G)) \xrightarrow{\operatorname{tr}_{\mathcal{N}(G)}} \mathbb{R}$$


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is the additive subgroup of $\mathbb{Q}$ generated by all numbers $\frac{1}{|H|}$, where $H \subset G$ runs though all finite subgroups of $G$. Here $\mathcal{N}(G)$ is the group von Neumann algebra, $i$ the change of rings homomorphism associated to the canonical inclusion $C_r^*(G) \to \mathcal{N}(G)$ and $\mathrm{tr}_{\mathcal{N}(G)}$ is the map induced by the standard von Neumann trace $\mathrm{tr}_{\mathcal{N}(G)} : \mathcal{N}(G) \to \mathbb{C}$. Roy has construced a counterexample to the Trace Conjecture in this form in [13] based on her article [14]. She constructs a group $\Gamma$, whose finite subgroups are all of order 1 or 3, together with an element in $K_0^G(\underline{E}G)$, whose image under $\mathrm{tr}_{\mathcal{N}(\Gamma)} \circ i \circ \mathrm{asmb}$ is $-\frac{1105}{9}$. The point is that $3 \cdot \frac{1105}{9}$ is not an integer. Notice that Roy's example does not imply that the Baum-Connes Conjecture does not hold for $\Gamma$. Since the group $\Gamma$ contains a torsionfree subgroup of index 9 and the Trace Conjecture for torsionfree groups does follow from the Baum-Connes Conjecture, the Baum-Connes Conjecture predicts that the image of $\mathrm{tr}_{\mathcal{N}(\Gamma)} \circ i : K_0(C_r^*(\Gamma)) \to \mathbb{R}$ is contained in $\{r \in \mathbb{R} \mid 9 \cdot r \in \mathbb{Z}\}$. So one could hope that the following version of the Trace Conjecture is still true. Denote by

$$\Lambda^G \quad := \quad \mathbb{Z}\left[\frac{1}{|\mathcal{F}in(G)|}\right] \tag{0.1}$$

the ring $\mathbb{Z} \subset \Lambda^G \subset \mathbb{Q}$ obtained from $\mathbb{Z}$ by inverting all the orders $|H|$ of finite subgroups of $G$. For Roy's group $\Gamma$ this is $\{m \cdot 3^{-n} \mid m, n \in \mathbb{Z}, n \geq 0\}$ and obviously contains $-\frac{1105}{9}$.

**Conjecture 0.2 (Modified Trace Conjecture for a group $G$)** *The image of the composition*

$$K_0(C_r^*(G)) \xrightarrow{i} K_0(\mathcal{N}(G)) \xrightarrow{\mathrm{tr}_{\mathcal{N}(G)}} \mathbb{R}$$

*is contained in $\Lambda^G$.*

The motivation for this paper is to prove

**Theorem 0.3** *The image of the composition*

$$\Lambda^G \otimes_{\mathbb{Z}} K_0^G(\underline{E}G) \xrightarrow{\mathrm{id} \otimes \mathrm{asmb}^G} \Lambda^G \otimes_{\mathbb{Z}} K_0(C_r^*(G)) \xrightarrow{i} \Lambda^G \otimes_{\mathbb{Z}} K_0(\mathcal{N}(G)) \xrightarrow{\mathrm{tr}_{\mathcal{N}(G)}} \mathbb{R}$$

*is $\Lambda^G$.*

*In particular the modified Trace Conjecture 0.2 holds for $G$, if the assembly map $\mathrm{asmb}^G : K_0^G(\underline{E}G) \to K_0(C_r^*(G))$ appearing in the Baum-Connes Conjecture is surjective.*

In order to prove Theorem 0.3 (actually a generalization of it in Theorem 0.8), we will prove a slight generalization of Atiyah's $L^2$-Index Theorem and construct an equivariant Chern character for equivariant $K$-homology of proper $G$-$CW$-complexes, which is bijective and defined after applying $\Lambda^G \otimes_{\mathbb{Z}} -$.

Let $M$ be a closed Riemannian manifold and $D^* = (D^*, d^*)$ be an elliptic complex of differential operators of order 1 on $M$. Denote by $\mathrm{index}(D^*) \in \mathbb{Z}$ its index. Let $\overline{M} \to M$ be a $G$-covering. Then one can lift $D^*$ to an elliptic $G$-equivariant complex $\overline{D}^*$. Using the trace $\mathrm{tr}_{\mathcal{N}(G)} : \mathcal{N}(G) \to \mathbb{C}$ Atiyah [1] defines its *$L^2$-index* $\mathrm{index}_{\mathcal{N}(G)}(\overline{D}^*) \in \mathbb{R}$ and shows

$$\mathrm{index}(D^*) \quad = \quad \mathrm{index}_{\mathcal{N}(G)}(\overline{D}^*).$$

The $L^2$-index theorem of Atiyah implies that the composition

$$K_0^G(EG) \xrightarrow{\mathrm{asmb}^G} K_0(C_r^*(G)) \xrightarrow{i} K_0(\mathcal{N}(G)) \xrightarrow{\mathrm{tr}_{\mathcal{N}(G)}} \mathbb{R}$$

agrees with the composition

$$K_0^G(EG) \xrightarrow{\mathrm{ind}_{G \to \{1\}}} K_0(BG) \xrightarrow{K_0(\mathrm{pr})} K_0(*) \xrightarrow{\mathrm{asmb}^{\{1\}}} K_0(C_r^*(\{1\})) \xrightarrow{\dim_{\mathbb{C}}} \mathbb{Z} \hookrightarrow \mathbb{R}.$$

Since for a torsionfree group $G$ the spaces $EG$ and $\underline{E}G$ agree, the Baum-Connes Conjecture for a torsionfree group $G$ does imply that the image of $K_0(C_r^*(G)) \xrightarrow{i} K_0(\mathcal{N}(G)) \xrightarrow{\operatorname{tr}_{\mathcal{N}(G)}} \mathbb{R}$ is $\mathbb{Z}$ [3, Corollary 1 on page 21]. Instead of using the the standard von Neumann trace $\operatorname{tr}_{\mathcal{N}(G)} : \mathcal{N}(G) \to \mathbb{C}$, one can use the universal center-valued trace $\operatorname{tr}^u_{\mathcal{N}(G)} : \mathcal{N}(G) \to \mathcal{Z}(\mathcal{N}(G))$ to define an index

$$\operatorname{index}^u_{\mathcal{N}(G)}(\overline{D}^*) \in \mathcal{Z}(\mathcal{N}(G)),$$

which takes values in the center $\mathcal{Z}(\mathcal{N}(G))$ of the group von Neumann algebra $\mathcal{N}(G)$. Thus we get additional information, namely, for any element $g \in G$, whose conjugacy class $(g)$ is finite, we get a complex number. However, it turns out that the value at classes $(g)$ with $g \neq 1$ is zero and that the value at $(1)$ is the index of $D^*$. Namely, we will show in Section 1

**Theorem 0.4** *Under the conditions above we get in $\mathcal{Z}(\mathcal{N}(G))$*

$$\operatorname{index}^u_{\mathcal{N}(G)}(\overline{D}^*) \quad = \quad \operatorname{index}(D^*) \cdot 1_{\mathcal{N}(G)}.$$

As an illustration we discuss the special case, where $G$ is finite, $M$ is an oriented closed $4k$-dimensional manifold with free orientation preserving $G$-action and $D^*$ is the signature operator. Then Theorem 0.4 reduces to the well-known statement that the equivariant signature

$$\operatorname{sign}^G(M) := [H_{2k}(M)^+] - [H_{2k}(M)^-] \ \in \operatorname{Rep}_{\mathbb{C}}(G)$$

is equal to $\operatorname{sign}(G\backslash M) \cdot [\mathbb{C}G]$ for $\operatorname{sign}(G\backslash M) \in \mathbb{Z}$ the (ordinary) signature of $G\backslash M$. We mention that this implies $\operatorname{sign}(M) = |G| \cdot \operatorname{sign}(G\backslash M)$. Theorem 0.4 is a special case of Theorem 5.4 but we will need it in the proof of Theorem 5.4 and therefore will have to prove it first.

The second ingredient is a variation of the equivariant Chern character of [11] for equivariant $K$-homology of proper $G$-$CW$-complexes. Recall that proper means that all isotropy groups are finite. The construction in [11] works for equivariant homology theories with a Mackey structure on the coefficient system in general, but requires to invert all primes. The construction we will give here works after applying $\Lambda^G \otimes_{\mathbb{Z}} ?$ and has a different source.

Denote for a proper $G$-$CW$-complex $X$ by $\mathcal{F}(X)$ the set of all subgroups $H \subset G$, for which $X^H \neq \emptyset$, and by

$$\Lambda^G(X) \quad := \quad \mathbb{Z}\left[\frac{1}{\mathcal{F}(X)}\right] \tag{0.5}$$

the ring $\mathbb{Z} \subset \Lambda^G(X) \subset \Lambda^G$ obtained from $\mathbb{Z}$ by inverting the orders of all subgroups $H \in \mathcal{F}(X)$. Denote by

$$J^G \quad \text{resp. } J^G(X) \tag{0.6}$$

the set of conjugacy classes $(C)$ of finite cyclic subgroups $C \subset G$ resp. the subset $J^G(X) \subset J^G$ of conjugacy classes $(C)$ of finite cyclic subgroups $C \subset G$, for which $X^C$ is non-empty. Obviously $\Lambda^G = \Lambda^G(\underline{E}G)$ and $J^G = J^G(\underline{E}G)$ since $\underline{E}G$ is characterized up to $G$-homotopy by the property that $\underline{E}G^H$ is contractible (and hence non-empty) for finite $H \subset G$ and empty for infinite $H \subset G$. Let $C \subset G$ be a finite cyclic subgroup. Let $C_GC$ be the centralizer and $N_GC$ be the normalizer of $C \subset G$. Let $W_GC$ be the quotient $N_GC/C_GC$. We will construct an idempotent $\Theta_C \in \Lambda^C \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{Q}}(C)$ which acts on $\Lambda^C \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C)$. We will see in Lemma 3.4 (b) that the cokernel of

$$\oplus_{D \subset C, D \neq C} \operatorname{ind}_D^C : \oplus_{D \subset C, D \neq C} \mathbb{Z}\left[\frac{1}{|C|}\right] \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(D) \to \mathbb{Z}\left[\frac{1}{|C|}\right] \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C)$$

is isomorphic to the image of the idempotent endomorphism

$$\theta_C : \mathbb{Z}\left[\frac{1}{|C|}\right] \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C) \to \mathbb{Z}\left[\frac{1}{|C|}\right] \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C).$$

After introducing and proving some preliminary results about modules over a category and representation theory of finite groups in Sections 2 and 3, we will prove in Section 4

**Theorem 0.7** *Let $X$ be a proper $G$-CW-complex. Put $\Lambda = \Lambda^G(X)$ and $J = J^G(X)$. Then there is for $p = 0, 1$ a natural isomorphism called* equivariant Chern character

$$\begin{aligned}\operatorname{ch}_p^G(X) : \oplus_{(C)\in J}\ \Lambda \otimes_{\mathbb{Z}} K_p(C_GC\backslash X^C) \otimes_{\Lambda[W_GC]} \operatorname{im}\left(\theta_C : \Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C) \to \Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C)\right) \\ \xrightarrow{\cong} \Lambda \otimes_{\mathbb{Z}} K_p^G(X).\end{aligned}$$

Notice that the equivariant Chern character of Theorem 0.7 reduces to the obvious isomorphism $K_0(G\backslash X) \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(\{1\}) \xrightarrow{\cong} K_0^G(X)$, if $G$ acts freely on $X$. In the special case, where $G$ is finite, $X$ is the one-point-space $\{*\}$ and $p = 0$, the equivariant Chern character reduces to an isomorphism

$$\begin{aligned}\oplus_{(C)\in J^G}\ \mathbb{Z}\left[\frac{1}{|G|}\right] \otimes_{\mathbb{Z}[\frac{1}{|G|}][W_GC]} \operatorname{im}\left(\theta_C : \mathbb{Z}\left[\frac{1}{|G|}\right] \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C) \to \mathbb{Z}\left[\frac{1}{|G|}\right] \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C)\right) \\ \xrightarrow{\cong}\ \mathbb{Z}\left[\frac{1}{|G|}\right] \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(G).\end{aligned}$$

This is a strong version of the well-known theorem of Artin that the map induced by induction

$$\oplus_{(C)\in J^G}\ \mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C) \to \mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(G)$$

is surjective for any finite group $G$. Theorem 0.7 gives a computation of $\Lambda^G \otimes K_0^G(\underline{E}G)$, namely

$$\begin{aligned}\oplus_{(C)\in J^G}\ \Lambda^G \otimes_{\mathbb{Z}} K_p(B(C_GC)) \otimes_{\Lambda^G[W_GC]} \operatorname{im}\left(\theta_C : \Lambda^G \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C) \to \Lambda^G \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C)\right) \\ \xrightarrow{\cong}\ \Lambda^G \otimes_{\mathbb{Z}} K_p^G(\underline{E}G).\end{aligned}$$

Another construction of an equivariant Chern character using completely different methods can be found in [4]. However, it works only after applying $\mathbb{C} \otimes_{\mathbb{Z}} -$ and therefore cannot be used for our purposes here.

In Theorem 5.4 we will identify the composition of the Chern character of Theorem 0.7 with the map

$$\Lambda^G \otimes_{\mathbb{Z}} K_0^G(\underline{E}G) \xrightarrow{\operatorname{id}\otimes \operatorname{asmb}^G} \Lambda^G \otimes_{\mathbb{Z}} K_0(C_r^*(G)) \xrightarrow{\operatorname{id}\otimes i} \Lambda^G \otimes_{\mathbb{Z}} K_0(\mathcal{N}(G))$$

with an easier to understand and to calculate homomorphism, whose image is obvious from its definition. This will immediately imply

**Theorem 0.8** *Let $\Lambda^G$ resp. $J^G$ be the ring resp. set introduced in* (0.1) *resp.* (0.6)*. Then the image of the composition*

$$\Lambda^G \otimes_{\mathbb{Z}} K_0^G(\underline{E}G) \xrightarrow{\operatorname{id}\otimes_{\mathbb{Z}} \operatorname{asmb}^G} \Lambda^G \otimes_{\mathbb{Z}} K_0(C_r^*(G)) \xrightarrow{\operatorname{id}\otimes_{\mathbb{Z}} i} \Lambda^G \otimes_{\mathbb{Z}} K_0(\mathcal{N}(G))$$

*is the image of the map given by induction*

$$\oplus_{(C)\in J^G}\ \operatorname{id}\otimes \operatorname{ind}_C^G : \oplus_{(C)\in J^G}\ \Lambda^G \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C) \to \Lambda^G \otimes_{\mathbb{Z}} K_0(\mathcal{N}(G)).$$

Now Theorem 0.3 follows from Theorem 0.8.

The change of rings and K-theory map $l : K_0(\mathbb{C}G) \to K_0(C_r^*(G))$ from the algebraic $K_0$-group of the complex group ring $\mathbb{C}G$ to the topological $K_0$-group of $C_r^*(G)$ is in general far from being surjective. There is some evidence that it is injective after applying $\Lambda \otimes_{\mathbb{Z}}$? (see [11, Theorem 0.1]). Theorem 0.8 gives some evidence for the conjecture that the image of $\Lambda^G \otimes_{\mathbb{Z}} K_0(C_r^*(G)) \xrightarrow{\mathrm{id} \otimes_{\mathbb{Z}} i} \Lambda^G \otimes_{\mathbb{Z}} K_0(\mathcal{N}(G))$ agrees with the image of the composition $\Lambda^G \otimes_{\mathbb{Z}} K_0(\mathbb{C}G) \xrightarrow{l} \Lambda^G \otimes_{\mathbb{Z}} K_0(C_r^*(G)) \xrightarrow{\mathrm{id} \otimes_{\mathbb{Z}} i} \Lambda^G \otimes_{\mathbb{Z}} K_0(\mathcal{N}(G))$. The paper is organized as follows



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# 1. The $L^2$-index theorem

In this section we prove a slight generalization of the $L^2$-index theorem of Atiyah [1]. Let $\overline{M}$ be a Riemannian manifold (without boundary) together with a cocompact free proper action of $G$ by isometries. In other words, $M = G\backslash\overline{M}$ is a closed Riemannian manifold, the projection $p : \overline{M} \to M$ is a $G$-covering and $\overline{M}$ is equipped with the Riemannian metric induced by the one of $M$. Let $D^* = (D^*, d^*)$ be an elliptic complex of differential operators $d^p : D^p \to D^{p+1}$ of order 1 acting on the space of sections $D^p = C^\infty(E^p)$ of vector bundles $E^p \to M$. Define $\overline{E}^p$ by $p^*E^p$ and $\overline{D}^p$ by $L^2C^\infty(\overline{E}_p)$. Then $G$-acts on $\overline{E}^p$ and $\overline{D}^p$. Since differential operators are local operators, there is a unique lift of each operator $d^p$ to a $G$-equivariant differential operator $\widehat{d}^p : C^\infty(\overline{E}^p) \to C^\infty(\overline{E}^{p+1})$. We obtain an elliptic $G$-complex $(C^\infty(\overline{E}^*), \widehat{d}^*)$. Let $\overline{d}^p : \overline{D}^p \to \overline{D}^{p+1}$ be the minimal closure of $\widehat{d}^p$ which is the same as its maximal closure [1, Proposition 3.1].

Since $D^*$ is elliptic, each cohomology module $H^p(D^*) := \ker(d^p)/\operatorname{im}(d^{p-1})$ is a finitely generated $\mathbb{C}$-module. Hence we can define the *index* of the elliptic complex $D^*$ by

$$\operatorname{index}(D^*) \quad := \quad \sum_{p \geq 0} \dim_{\mathbb{C}}(H^p(D^*)) \ \in \mathbb{Z}. \tag{1.1}$$

Next we want to define an analogous invariant for the lifted complex $\overline{D}^*$. The group von Neumann algebra $\mathcal{N}(G)$ of $G$ is the $*$-algebra $\mathcal{B}(l^2(G))^G$ of all bounded $G$-equivariant operators $l^2(G) \to l^2(G)$, where we equip $l^2(G)$ with the obvious left $G$-action. Let

$$\operatorname{tr}_{\mathcal{N}(G)} : \mathcal{N}(G) \to \quad \mathbb{C} \tag{1.2}$$

be the *standard von Neumann trace*, which sends $f \in \mathcal{N}(G) = \mathcal{B}(l^2(G))^G$ to $\langle f(e), e\rangle_{l^2(G)}$, where $e$ denotes the element in $l^2(G)$ given by the unit element in $G \subset l^2(G)$. Denote by $\mathcal{Z}(\mathcal{N}(G))$ the center of $\mathcal{N}(G)$. There is the universal center-valued trace [7, Theorem 7.1.12 on page 462, Proposition 7.4.5 on page 483, Theorem 8.2.8 on page 517, Proposition 8.3.10 on page 525, Theorem 8.4.3 on page 532]

$$\operatorname{tr}^u_{\mathcal{N}(G)} : \mathcal{N}(G) \to \mathcal{Z}(\mathcal{N}(G)) \tag{1.3}$$

which is uniquely determined by the following two properties:

(a) $\mathrm{tr}^u$ is a trace with values in the center, i.e. $\mathrm{tr}^u$ is $\mathbb{C}$-linear, for $a \in \mathcal{N}(G)$ with $a \geq 0$ we have $\mathrm{tr}^u(a) \geq 0$ and $\mathrm{tr}^u(ab) = \mathrm{tr}^u(ba)$ for all $a, b \in \mathcal{N}(G)$;

(b) $\mathrm{tr}^u(a) = a$ for all $a \in Z(\mathcal{N}(G))$.

The map $\mathrm{tr}^u$ has the following further properties:

(c) $\mathrm{tr}^u$ is faithful;

(d) $\mathrm{tr}^u$ is normal. Equivalently, $\mathrm{tr}^u$ is continuous with respect to the ultraweak topology on $\mathcal{N}(G)$;

(e) $||\,\mathrm{tr}^u(a)|| \ \leq \ ||a||$ for $a \in \mathcal{N}(G)$;

(f) $\mathrm{tr}^u(ab) = a\,\mathrm{tr}^u(b)$ for all $a \in Z(\mathcal{N}(G))$ and $b \in \mathcal{N}(G)$;

(g) Let $p$ and $q$ be projections in $\mathcal{N}(G)$. Then $p$ and $q$ are equivalent, i.e. $p = vv^*$ and $q = v^*v$, if and only if $\mathrm{tr}^u(p) = \mathrm{tr}^u(q)$;

(h) Any linear functional $f : \mathcal{N}(G) \to \mathbb{C}$, which is continuous with respect to the norm topology on $\mathcal{N}(G)$ and which is central, i.e. $f(ab) = f(ba)$ for all $a, b \in \mathcal{N}(G)$, factorizes as

$$\mathcal{N}(G) \xrightarrow{\mathrm{tr}^u} Z(\mathcal{N}(G)) \xrightarrow{f|_{Z(\mathcal{N}(G))}} \mathbb{C}.$$

In particular $\mathrm{tr}_{\mathcal{N}(G)} \circ \mathrm{tr}^u_{\mathcal{N}(G)} = \mathrm{tr}_{\mathcal{N}(G)}$.

A *Hilbert $\mathcal{N}(G)$-module* $V$ is a Hilbert space $V$ together with a $G$-action by isometries such that there exists a Hilbert space $H$ and a $G$-equivariant projection $p : H \otimes l^2(G) \to H \otimes l^2(G)$ with the property that $V$ and $\mathrm{im}(p)$ are isometrically $G$-linearly isomorphic. Here $H \otimes l^2(G)$ is the tensor product of Hilbert spaces and $G$ acts trivially on $H$ and on $l^2(G)$ by the obvious left multiplication. Notice that $p$ is not part of the structure, only its existence is required. We call $V$ *finitely generated* if $H$ can be choosen to be finite-dimensional.

Our main examples of Hilbert $\mathcal{N}(G)$-modules are the Hilbert spaces $\overline{D}^p$ which are isometrically $G$-isomorphic to $L^2(C^\infty(E^p)) \otimes l^2(G)$. This can be seen using a fundamental domain $\mathcal{F}$ for the $G$-action on $\overline{M}$ which is from a measure theory point of view the same as $M$. A morphism $f : V \to W$ of Hilbert $\mathcal{N}(G)$-modules is a densely defined closed $G$-equivariant operator. The differentials $\overline{d}^p$ are morphisms of Hilbert $\mathcal{N}(G)$-modules.

Let $f : V \to V$ be a *morphism of Hilbert $\mathcal{N}(G)$-modules* which is positive. Choose a $G$-projection $p : H \otimes l^2(G) \to H \otimes l^2(G)$ and an isometric invertible $G$-equivariant operator $u : \mathrm{im}(p) \to V$. Let $\{b_i \mid i \in I\}$ be a Hilbert basis for $H$. Let $\overline{f}$ be the composition

$$H \otimes l^2(G) \xrightarrow{p} \mathrm{im}(p) \xrightarrow{u} V \xrightarrow{f} V \xrightarrow{u^{-1}} \mathrm{im}(p) \hookrightarrow H \otimes l^2(G).$$

Define the *von Neumann trace* of $f : V \to V$ by

$$\mathrm{tr}_{\mathcal{N}(G)}(f) \quad := \quad \sum_{i \in I} \langle \overline{f}(b_i \otimes e), b_i \otimes e \rangle_{H \otimes l^2(G)} \qquad \in [0, \infty]. \tag{1.4}$$

This is indeed independent of the choice of $p$, $u$ and the Hilbert basis $\{b_i \mid i \in I\}$. If $V$ is finitely generated, then $\mathrm{tr}_{\mathcal{N}(G)}(f) < \infty$ is always true. Define the von Neumann dimension of a Hilbert $\mathcal{N}(G)$-module $V$ by

$$\dim_{\mathcal{N}(G)}(V) \quad := \quad \mathrm{tr}_{\mathcal{N}(G)}(\mathrm{id} : V \to V) \qquad \in [0, \infty]. \tag{1.5}$$

If $V$ is a finitely generated Hilbert $\mathcal{N}(G)$-module, we define the *universal center-valued von Neumann dimension*

$$\dim^u_{\mathcal{N}(G)}(V) \quad := \quad \operatorname{tr}^u_{\mathcal{N}(G)}(\mathrm{id} : V \to V) \ \in \mathcal{Z}(\mathcal{N}(G)) \tag{1.6}$$

analogously to $\dim_{\mathcal{N}(G)}(V)$ replacing $\operatorname{tr}_{\mathcal{N}(G)}$ by $\operatorname{tr}^u_{\mathcal{N}(G)}$. Given a finitely generated Hilbert $\mathcal{N}(G)$-module $V$, we have $\operatorname{tr}_{\mathcal{N}(G)}(\dim^u_{\mathcal{N}(G)}(V)) = \dim_{\mathcal{N}(G)}(V)$.

Define the $L^2$-cohomology $H^p_{(2)}(\overline{D}^*)$ to be $\ker(\overline{d}^p)/\operatorname{clos}(\operatorname{im}(\overline{d}^{p-1}))$, where $\operatorname{clos}(\operatorname{im}(\overline{d}^{p-1}))$ is the closure of the image of $\overline{d}^{p-1}$. Define the $p$-th Laplacian by $\overline{\Delta}_p = (\overline{d}^p)^*\overline{d}^p + \overline{d}^{p-1}(\overline{d}^{p-1})^*$. By the $L^2$-Hodge-deRham Theorem we get a $G$-equivariant isometric isomorphism $\ker(\overline{\Delta}_p) \xrightarrow{\cong} H^p_{(2)}(\overline{D}^*)$. Thus $H^p_{(2)}(\overline{D}^*)$ inherits the structure of a Hilbert $\mathcal{N}(G)$-module. Moreover, it turns out to be a finitely generated Hilbert $\mathcal{N}(G)$-module. This can be deduced from the results of [12], where an index already over $C^*_r(G)$ is defined and the problem of getting finitely generated modules over $C^*_r(G)$ is treated. Namely, one can deduce from [12] after passing to the group von Neumann algebra, that there are finitely generated Hilbert $\mathcal{N}(G)$-modules $U_1$, $U_2$, $V_1$ and $V_2$ and Hilbert $\mathcal{N}(G)$-modules $W_1$ and $W_2$ together with a morphism $v : V_1 \to V_2$ and isomorphisms of Hilbert $\mathcal{N}(G)$-modules $w : W_1 \xrightarrow{\cong} W_2$, $u_1 : \overline{D}^p \oplus U_1 \xrightarrow{\cong} V_1 \oplus W_1$ and $u_2 : \overline{D}^p \oplus U_2 \xrightarrow{\cong} V_2 \oplus W_2$ such that $u_2 \circ (\Delta_p \oplus 0) = (v \oplus w) \circ u_1$. Obviously the kernel of $v$ and hence the kernel of $\overline{\Delta}_p$ are finitely generated Hilbert $\mathcal{N}(G)$-modules.

Define the *center-valued $L^2$-index* and the *$L^2$-index*

$$\operatorname{index}^u_{\mathcal{N}(G)}(\overline{D}^*) \quad := \quad \sum_{p \geq 0} \dim^u_{\mathcal{N}(G)}(H^p_{(2)}(\overline{D}^*)) \ \in \mathcal{Z}(\mathcal{N}(G)); \tag{1.7}$$

$$\operatorname{index}_{\mathcal{N}(G)}(\overline{D}^*) \quad := \quad \sum_{p \geq 0} \dim_{\mathcal{N}(G)}(H^p_{(2)}(\overline{D}^*)) \ \in \mathbb{R}. \tag{1.8}$$

The rest of this section is devoted to the proof of Theorem 0.4

**Notation 1.9** *Denote by $\operatorname{con}(G)_{cf}$ the set of conjugacy classes $(g)$ of elements $g \in G$ such that the set $(g)$ is finite, or, equivalently, the centralizer $C_g(g) = \{g' \in G \mid g'g = gg'\}$ has finite index in $G$. For $c \in \operatorname{con}(G)_{cf}$ let $N_c$ be the element $\sum_{g \in c} g \in \mathbb{C}G$. In the sequel $L_c$ resp. $L_g$ denotes left multiplication with $N_c$ resp. $g$ for $c \in \operatorname{con}(G)_{cf}$ resp. $g \in G$.*

Notice for the sequel that $N_c \in \mathcal{Z}(\mathcal{N}(G))$ and $L_c$ is $G$-equivariant and commutes with all $G$-operators.

**Lemma 1.10** *Consider $a \in \mathcal{Z}(\mathcal{N}(G))$. Then we have $a = 0$ if and only if $\operatorname{tr}_{\mathcal{N}(G)}(N_c a) = 0$ holds for any $c \in \operatorname{con}_{cf}(G)$.*

**Proof :** Consider $a \in \mathcal{N}(G) = \mathcal{B}(l^2(G))^G$ which belongs to $\mathcal{Z}(\mathcal{N}(G))$. Write $a(e) = \sum_{g \in G} \lambda_g \cdot g \in l^2(G)$. Since $aR_g = R_g a$ holds for $g \in G$ and $R_g : l^2(G) \to l^2(G)$ given by right multiplication with $g \in G$, we get $\lambda_g = \lambda_{hgh^{-1}}$ for $g, h \in G$. This implies that $\lambda_g = 0$ if the conjugacy class $(g)$ is infinite. On easily checks for an element $g$ with finite $(g)$

$$\lambda_g \quad = \quad \operatorname{tr}_{\mathcal{N}(G)}(N_{(g^{-1})} a). \qquad \blacksquare$$

**Lemma 1.11** *We get under the conditions above.*

$$\operatorname{tr}_{\mathcal{N}(G)}\left(\operatorname{index}^u_{\mathcal{N}(G)}(\overline{D}^*)\right) = \operatorname{index}(D^*).$$

**Proof :** The $L^2$-index theorem of Atiyah [1, (1.1)] says

$$\operatorname{index}_{\mathcal{N}(G)}(\overline{D}^*) \;=\; \operatorname{index}(D^*).$$

We have

$$\begin{aligned}
\operatorname{tr}_{\mathcal{N}(G)}\left(\operatorname{index}^u_{\mathcal{N}(G)}(\overline{D}^*)\right) &= \operatorname{tr}_{\mathcal{N}(G)}\left(\sum_{p\geq 0}(-1)^p \dim^u_{\mathcal{N}(G)}(H^p_{(2)}(\overline{D}^*))\right)\\
&= \sum_{p\geq 0}(-1)^p \operatorname{tr}_{\mathcal{N}(G)}\left(\dim^u_{\mathcal{N}(G)}(H^p_{(2)}(\overline{D}^*))\right)\\
&= \sum_{p\geq 0}(-1)^p \dim_{\mathcal{N}(G)}\left(H^p_{(2)}(\overline{D}^*)\right)\\
&= \operatorname{index}_{\mathcal{N}(G)}(\overline{D}^*). \qquad \blacksquare
\end{aligned}$$

Next we want to prove

**Lemma 1.12** *Consider an element $c \in \operatorname{con}(G)_{cf}$ with $c \neq (1)$. Then*

$$\operatorname{tr}_{\mathcal{N}(G)}\left(N_c \cdot \operatorname{index}^u_{\mathcal{N}(G)}(\overline{D}^*)\right) \;=\; 0.$$

**Proof :** In the sequel we denote by $\overline{\operatorname{pr}}_p : \overline{D}^p \to \overline{D}^p$ the projection onto the kernel of the $p$-th Laplacian $\overline{\Delta}_p = (\overline{d}^p)^*\overline{d}^p + \overline{d}^{p-1}(\overline{d}^{p-1})^*$. By the $L^2$-Hodge-deRham Theorem we get a $G$-equivariant isometric isomorphism $\operatorname{im}(\overline{\operatorname{pr}}_p) \xrightarrow{\cong} H^p_{(2)}(\overline{D}^*)$. This implies

$$\begin{aligned}
&\operatorname{tr}_{\mathcal{N}(G)}\left(N_c \cdot \operatorname{index}^u_{\mathcal{N}(G)}(\overline{D}^*)\right)\\
&= \sum_{p\geq 0}(-1)^p \cdot \operatorname{tr}_{\mathcal{N}(G)}\left(N_c \cdot \operatorname{tr}^u_{\mathcal{N}(G)}\left(\operatorname{id} : H^p_{(2)}(\overline{D}^*) \to H^p_{(2)}(\overline{D}^*)\right)\right)\\
&= \sum_{p\geq 0}(-1)^p \cdot \operatorname{tr}_{\mathcal{N}(G)}\left(L_c : H^p_{(2)}(\overline{D}^*) \to H^p_{(2)}(\overline{D}^*)\right)\\
&= \sum_{p\geq 0}(-1)^p \cdot \operatorname{tr}_{\mathcal{N}(G)}\left(L_c \circ \overline{\operatorname{pr}}_p : \overline{D}^p \to \overline{D}^p\right).
\end{aligned} \tag{1.13}$$

The operator $e^{-t\overline{\Delta}_p} : \overline{D}^p \to \overline{D}^p$ is a bounded $G$-equivariant operator and has a smooth kernel $e^{-t\overline{\Delta}_p}(\overline{x},\overline{y}) : \overline{E}^p_{\overline{x}} \to \overline{E}^p_{\overline{y}}$ for $\overline{x},\overline{y} \in \overline{M}$. Thus $e^{-t\overline{\Delta}_p}(\omega)$ applied to a section $\omega$ is given at $\overline{y} \in \overline{M}$ by $\int_{\overline{M}} e^{-t\overline{\Delta}_p}(\overline{x},\overline{y})(\omega(\overline{x}))d\mathrm{vol}_{\overline{x}}$. The operator $L_c \circ e^{-t\overline{\Delta}_p}$ is also a bounded $G$-equivariant operator and has a smooth kernel $\left(L_c \circ e^{-t\overline{\Delta}_p}\right)(\overline{x},\overline{y})$ satisfying

$$\left(L_c \circ e^{-t\overline{\Delta}_p}\right)(\overline{x},\overline{y}) \;=\; \sum_{g\in c} L_g \circ e^{-t\overline{\Delta}_p}(\overline{x}, g^{-1}\overline{y}).$$

If $\mathcal{F}$ is a fundamental domain for the $G$-action, then [1, Proposition 4.6].

$$\begin{aligned}\operatorname{tr}_{\mathcal{N}(G)}(L_c \circ e^{-t\overline{\Delta}_p}) &= \int_{\mathcal{F}} \operatorname{tr}_{\mathbb{C}}\left(\left(L_c \circ e^{-t\overline{\Delta}_p}\right)(\overline{x},\overline{x})\right) d\mathrm{vol}_{\overline{x}};\\ &= \sum_{g\in c}\int_{\mathcal{F}} \operatorname{tr}_{\mathbb{C}}\left(L_g \circ e^{-t\overline{\Delta}_p}(\overline{x},g^{-1}\overline{x})\right) d\mathrm{vol}_{\overline{x}}. \end{aligned} \tag{1.14}$$

where $\operatorname{tr}_{\mathbb{C}}$ is the trace of an endomorphism of a finite-dimensional complex vector space. Since $M$ is compact, we can find $\epsilon > 0$ such that the distance of $\overline{x}$ and $g\overline{x}$ is bounded from below by $\epsilon$ for all $\overline{x} \in \overline{M}$ and $g \in c$. We have

$$\lim_{t\to 0} \ \sup\left\{||e^{-t\overline{\Delta}_p}(\overline{x},g^{-1}\overline{x})|| \mid \overline{x}\in\mathcal{F}\right\} \quad = \quad 0, \tag{1.15}$$

where $||e^{-t\overline{\Delta}_p}(\overline{x},g^{-1}\overline{x})||$ is the operator norm of the linear map $e^{-t\overline{\Delta}_p}(\overline{x},g^{-1}\overline{x})$ of finite-dimensional Hilbert spaces. This follows from the finite propagation speed method of [6]. There only the standard Laplacian on 0-forms is treated, but the proof presented there carries over to the Laplacian $\overline{\Delta}_p$ associated to the lift $\overline{D}^*$ to the $G$-covering $\overline{M}$ of an elliptic complex $D^*$ of differential operators of order 1 on a closed Riemannian manifold $M$ in any dimension $p$. The point is that $\overline{M}$ has bounded geometry, $\overline{\Delta}_p$ is essentially selfadjoint and positive so that $\sqrt{\overline{\Delta}_p}$ makes sense, and $\frac{\partial^2}{\partial t^2} + \overline{\Delta}_p$ is strictly hyperbolic. Now one applies the results of [6, Section 1] and uses the estimate in [9, page 475], where the special case of $D^*$ being the deRham complex is treated.

Since

$$\left|\operatorname{tr}_{\mathbb{C}}\left(L_g \circ e^{-t\overline{\Delta}_p}(\overline{x},g^{-1}\overline{x})\right)\right| \ \leq \ \dim_{\mathbb{C}}(E^p)\cdot ||e^{-t\overline{\Delta}_p}(\overline{x},g^{-1}\overline{x})||$$

and $\mathcal{F}$ is compact, we conclude from (1.14) and (1.15)

$$\lim_{t\to 0}\operatorname{tr}_{\mathcal{N}(G)}(L_c\circ e^{-t\overline{\Delta}_p}) \quad = \quad 0. \tag{1.16}$$

Since the trace $\operatorname{tr}_{\mathcal{N}(G)}$ is ultraweakly continuous and $\lim_{t\to\infty} e^{-t\overline{\Delta}_p} = \overline{\mathrm{pr}}_p$ in the weak topology, we get

$$\lim_{t\to\infty}\operatorname{tr}_{\mathcal{N}(G)}(L_c\circ e^{-t\overline{\Delta}_p}) \quad = \quad \operatorname{tr}_{\mathcal{N}(G)}(L_c\circ\overline{\mathrm{pr}}_p). \tag{1.17}$$

We conclude from (1.13) and (1.17)

$$\operatorname{tr}_{\mathcal{N}(G)}\left(N_c\cdot \operatorname{index}^u_{\mathcal{N}(G)}(\overline{D}^*)\right) \quad = \quad \lim_{t\to\infty}\sum_{p\geq 0}(-1)^p\cdot\operatorname{tr}_{\mathcal{N}(G)}\left(L_c\circ e^{-t\overline{\Delta}_p}\right). \tag{1.18}$$

We have

$$
\begin{aligned}
&\frac{d}{dt}\sum_{p\geq 0}(-1)^p\cdot \mathrm{tr}_{\mathcal{N}(G)}\left(L_c\circ e^{-t\overline{\Delta}_p}:\overline{D}^p\to\overline{D}^p\right)\\
= &\sum_{p\geq 0}(-1)^p\cdot \mathrm{tr}_{\mathcal{N}(G)}\left(L_c\circ \frac{d}{dt}e^{-t\overline{\Delta}_p}\right)\\
= &\sum_{p\geq 0}(-1)^p\cdot \mathrm{tr}_{\mathcal{N}(G)}\left(L_c\circ (-\overline{\Delta}_p)\circ e^{-t\overline{\Delta}_p}\right)\\
= &-\sum_{p\geq 0}(-1)^p\cdot \mathrm{tr}_{\mathcal{N}(G)}\left(L_c\circ \overline{d}^{p-1}\circ(\overline{d}^{p-1})^*\circ e^{-t\overline{\Delta}_p}\right)\\
&\qquad\qquad -\sum_{p\geq 0}(-1)^p\cdot \mathrm{tr}_{\mathcal{N}(G)}\left(L_c\circ (\overline{d}^{p})^*\circ\overline{d}^{p}\circ e^{-t\overline{\Delta}_p}\right)\\
= &-\sum_{p\geq 0}(-1)^p\cdot \mathrm{tr}_{\mathcal{N}(G)}\left(L_c\circ \overline{d}^{p-1}\circ(\overline{d}^{p-1})^*\circ e^{-\frac{t}{2}\overline{\Delta}_p}\circ e^{-\frac{t}{2}\overline{\Delta}_p}\right)\\
&\qquad\qquad -\sum_{p\geq 0}(-1)^p\cdot \mathrm{tr}_{\mathcal{N}(G)}\left(L_c\circ (\overline{d}^{p})^*\circ e^{-t\overline{\Delta}_{p+1}}\circ\overline{d}^{p}\right)\\
= &-\sum_{p\geq 0}(-1)^p\cdot \mathrm{tr}_{\mathcal{N}(G)}\left(L_c\circ e^{-\frac{t}{2}\overline{\Delta}_p}\circ\overline{d}^{p-1}\circ(\overline{d}^{p-1})^*\circ e^{-\frac{t}{2}\overline{\Delta}_p}\right)\\
&\qquad\qquad +\sum_{p\geq 0}(-1)^p\cdot \mathrm{tr}_{\mathcal{N}(G)}\left(L_c\circ (\overline{d}^{p-1})^*\circ e^{-t\overline{\Delta}_{p}}\circ\overline{d}^{p-1}\right)\\
= &-\sum_{p\geq 0}(-1)^p\cdot \mathrm{tr}_{\mathcal{N}(G)}\left(L_c\circ (\overline{d}^{p-1})^*\circ e^{-\frac{t}{2}\overline{\Delta}_p}\circ e^{-\frac{t}{2}\overline{\Delta}_p}\circ\overline{d}^{p-1}\right)\\
&\qquad\qquad +\sum_{p\geq 0}(-1)^p\cdot \mathrm{tr}_{\mathcal{N}(G)}\left(L_c\circ (\overline{d}^{p-1})^*\circ e^{-t\overline{\Delta}_{p}}\circ\overline{d}^{p-1}\right)\\
= &-\sum_{p\geq 0}(-1)^p\cdot \mathrm{tr}_{\mathcal{N}(G)}\left(L_c\circ (\overline{d}^{p-1})^*\circ e^{-t\overline{\Delta}_p}\circ\overline{d}^{p-1}\right)\\
&\qquad\qquad +\sum_{p\geq 0}(-1)^p\cdot \mathrm{tr}_{\mathcal{N}(G)}\left(L_c\circ (\overline{d}^{p-1})^*\circ e^{-t\overline{\Delta}_{p}}\circ\overline{d}^{p-1}\right)\\
= &\ 0.
\end{aligned}
\tag{1.19}
$$

Here are some justifications for the calculation above. Recall that $L_c$ is a bounded $G$-operator and commutes with any $G$-equivariant operator. We can commute $\mathrm{tr}_{\mathcal{N}(G)}$ and $\frac{d}{dt}$ since $\mathrm{tr}_{\mathcal{N}(G)}$ is ultraweakly continuous. We conclude $e^{-t\overline{\Delta}_{p+1}}\circ\overline{d}^p=\overline{d}^p\circ e^{-t\overline{\Delta}_p}$ from the fact that $\overline{\Delta}_{p+1}\circ\overline{d}^p=\overline{d}^p\circ\overline{\Delta}_p$ holds on $C^\infty(\overline{E}^{p-1})$. We have used at several places the typical trace relation $\mathrm{tr}_{\mathcal{N}(G)}(AB)=\mathrm{tr}_{\mathcal{N}(G)}(BA)$ which is in each case justified by [1, section 4]. In order to be able to apply this trace relation we have splitted $e^{-t\overline{\Delta}_p}$ into $e^{-\frac{t}{2}\overline{\Delta}_p}\circ e^{-\frac{t}{2}\overline{\Delta}_p}$ in the calculation above.

Hence $\sum_{p\geq 0}(-1)^p\cdot\mathrm{tr}_{\mathcal{N}(G)}\left(L_c\circ e^{-t\overline{\Delta}_p}:\overline{D}^p\to\overline{D}^p\right)$ is independent of $t$ and we conclude from (1.18)

$$
\mathrm{tr}_{\mathcal{N}(G)}\left(N_c\cdot\mathrm{index}^u_{\mathcal{N}(G)}(\overline{D}^*)\right) \;=\; \lim_{t\to 0}\sum_{p\geq 0}(-1)^p\cdot\mathrm{tr}_{\mathcal{N}(G)}\left(L_c\circ e^{-t\overline{\Delta}_p}:\overline{D}^p\to\overline{D}^p\right). \tag{1.20}
$$

Now Lemma 1.12 follows from (1.16) (1.20). ■

Finally Theorem 0.4 follows from Lemma 1.10, Lemma 1.11 and Lemma 1.12.

# 2. Modules over a category

In this section we recall some facts about modules over the category $\mathsf{Sub} = \mathsf{Sub}(G;\mathcal{F}(X))$ for a proper $G$-$CW$-complex $X$ as far as needed here. For more information about modules over a category we refer to [10].

Let $\mathsf{Sub} := \mathsf{Sub}(G;\mathcal{F}(X))$ be the following category. Objects are the elements of the set $\mathcal{F}(X)$ of subgroups $H \subset G$, for which $X^H \neq \emptyset$. For two finite subgroups $H$ and $K$ in $\mathcal{F}(X)$ denote by $\operatorname{conhom}_G(H,K)$ the set of group homomorphisms $f : H \to K$, for which there exists an element $g \in G$ with $gHg^{-1} \subset K$ such that $f$ is given by conjugation with $g$, i.e. $f = c(g) : H \to K, \quad h \mapsto ghg^{-1}$. Notice that $c(g) = c(g')$ holds for two elements $g, g' \in G$ with $gHg^{-1} \subset K$ and $g'H(g')^{-1} \subset K$ if and only if $g^{-1}g'$ lies in the centralizer $C_GH = \{g \in G \mid gh = hg \text{ for all } h \in H\}$ of $H$ in $G$. The group of inner automorphisms of $K$ acts on $\operatorname{conhom}_G(H,K)$ from the left by composition. Define the set of morphisms $\operatorname{mor}_{\mathsf{Sub}}(H,K)$ by $\operatorname{Inn}(K)\backslash \operatorname{conhom}_G(H,K)$. Let $N_GH$ be the normalizer $\{g \in G \mid gHg^{-1} = H\}$ of $H$. Define $H \cdot C_GH = \{h \cdot g \mid h \in H, g \in C_GH\}$. This is a normal subgroup of $N_GH$ and we define $W_GH := N_GH/(H \cdot C_GH)$. One easily checks that $W_GH$ is a finite group and that there is an isomorphism from $W_GH$ to $\operatorname{aut}_{\mathsf{Sub}}(H)$ which sends $g(H \cdot C_GH) \in W_GH$ to the automorphism of $H$ represented by $c(g) : H \to H$. Notice that there is a morphism from $H$ to $K$ if and only if $H$ is subconjugated to $K$. There is an isomorphism from $H$ to $K$ if and only if $H$ and $K$ are conjugated. The category $\mathsf{Sub}$ is a so called EI-category, i.e. any endomorphism in $\mathsf{Sub}$ is an isomorphism.

Let $R$ be a commutative associative ring with unit. A *covariant resp. contravariant $R\mathsf{Sub}$-module* $M$ is a covariant resp. contravariant functor from $\mathsf{Sub}$ to the category of $R$-modules. Morphisms are natural transformations. The structure of an abelian category on the category of $R$-modules carries over to the category of $R\mathsf{Sub}$-modules. In particular the notion of a projective $R\mathsf{Sub}$-module is defined. Given a contravariant $R\mathsf{Sub}$-module $M$ and a covariant $R\mathsf{Sub}$-module $N$, one can define a $R$-module, their *tensor product over* $\mathsf{Sub}$

$$M \otimes_{R\mathsf{Sub}} N \;=\; \oplus_{H \in \mathcal{F}(X)} M(H) \otimes_R N(H)/\sim,$$

where $\sim$ is the typical tensor relation $mf \otimes n = m \otimes fn$, i.e. for each morphism $f : H \to K$ in $\mathsf{Sub}$, $m \in M(K)$ and $n \in N(H)$ we introduce the relation $M(f)(m) \otimes n - m \otimes N(f)(n) = 0$.

Given a left $R[W_GH]$-module $N$ for $H \in \mathcal{F}(X)$, define a covariant $R\mathsf{Sub}$-module $E_HM$ by

$$(E_HM)(K) \;:=\; R\operatorname{mor}_{\mathsf{Sub}}(H,K) \otimes_{R[W_GH]} N \qquad \text{for } K \subset G, |K| < \infty, \tag{2.1}$$

where $R\operatorname{mor}_{\mathsf{Sub}}(H,K)$ is the free $R$-module generated by the set $\operatorname{mor}_{\mathsf{Sub}}(H,K)$. Given a covariant $R\mathsf{Sub}$-module $M$ and $H \in \mathcal{F}(X)$, define $M(H)_s$ to be the left $R$-submodule of $M(H)$, which is spanned by the images of all $R$-maps $M(f) : M(K) \to M(H)$, where $f$ runs through all morphisms $f : K \to H$ in $\mathsf{Sub}$, which have $H$ as target and are not isomorphisms. Obviously $M(H)_s$ is an $R[W_GH]$-submodule of $M(H)$. Define a left $R[W_GH]$-module $S_HM$ by

$$S_HM \;:=\; M(H)/M(H)_s. \tag{2.2}$$

Both functors $E_H$ and $S_H$ respect direct sums and the property finitely generated and the property projective. Given a left $R[W_GH]$-module $M$, $S_K \circ E_HM$ is $M$, if $H = K$ and is 0, if $H$ and $K$ are not conjugated in $G$.

Let $M$ be a covariant $R\mathsf{Sub}$-module. We want to check whether it is projective or not. A necessary (but not sufficient) condition is that $S_HM$ is a projective $R[W_GH]$-module. Assume that $S_HM$ is $R[W_GH]$-projective for all objects $H$ in $\mathsf{Sub}$. We can choose a $R[W_GH]$-splitting $\sigma_H : S_HM \to M(H)$ of the canonical projection $M(H) \to S_HM = M(H)/M(H)_s$. For a finite subgroup $H \subset G$ define the morphism of covariant $R\mathsf{Sub}$-modules

$$i_HM : E_H(M(H)) \to M$$

by $(i_H M)(K)((f : H \to K) \otimes_{R[W_G H]} m) = M(f)(m)$. We obtain after a choice of representatives $H \in (H)$ for any conjugacy class $(H)$ of subgroups $H \in \mathcal{F}(X)$ a morphism of covariant $R\mathsf{Sub}$-modules

$$T : \oplus_{(H),H\in\mathcal{F}(X)} E_H S_H M \xrightarrow{\oplus_{(H),H\in\mathcal{F}(X)} E_H(\sigma_H)} \oplus_{(H),H\in\mathcal{F}(X)} E_H(M(H)) \xrightarrow{\oplus_{(H),H\in\mathcal{F}(X)} i_H M} M. \quad (2.3)$$

We get as a special case of [11, Theorem 2.11]

**Theorem 2.4** *The morphism $T$ is always surjective. It is bijective if and only if $M$ is a projective $R\mathsf{Sub}$-module.*

# 3. Some representation theory for finite groups

Denote for a finite group $H$ by $\mathrm{Rep}_{\mathbb{Q}}(H)$ resp. $\mathrm{Rep}_{\mathbb{C}}(H)$ the ring of finite dimensional $H$-representations over the field $\mathbb{Q}$ resp. $\mathbb{C}$. Recall for the sequel that these are finitely generated free abelian groups. Given an inclusion of finite groups $H \subset G$, we denote by $\mathrm{ind}_H^G : \mathrm{Rep}_{\mathbb{Q}}(H) \to \mathrm{Rep}_{\mathbb{Q}}(G)$ and $\mathrm{res}_G^H : \mathrm{Rep}_{\mathbb{Q}}(G) \to \mathrm{Rep}_{\mathbb{Q}}(H)$ the induction and restriction homomorphism and similar for $R \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{Q}}$, $\mathrm{Rep}_{\mathbb{C}}$ and $R \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}$ for a commutative ring $R$ with $\mathbb{Z} \subset R$. Let $\mathrm{con}_{\mathbb{Q}}(H)$ be the set of $\mathbb{Q}$-conjugacy classes of elements in $H$, where $h$ and $h'$ are called $\mathbb{Q}$-conjugated if the cyclic subgroups $\langle h\rangle$ and $\langle h'\rangle$ are conjugated in $G$. Let $\mathrm{con}(G)$ be the set of conjugacy classes of elements in $G$. Denote by $\mathrm{class}_{\mathbb{Q}}(H)$ resp. $\mathrm{class}_{\mathbb{C}}(H)$ the rational resp. complex vector space of functions $\mathrm{con}_{\mathbb{Q}}(H) \to \mathbb{Q}$ resp. $\mathrm{con}(G) \to \mathbb{C}$. Character theory yields isomorphisms [15, page 68 and Theorem 29 on page 102]

$$\begin{aligned} \chi_{\mathbb{Q}} : \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{Q}}(H) &\xrightarrow{\cong} \mathrm{class}_{\mathbb{Q}}(H); \\ \chi_{\mathbb{C}} : \mathbb{C} \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(H) &\xrightarrow{\cong} \mathrm{class}_{\mathbb{C}}(H). \end{aligned}$$

For a finite cyclic group $C$ denote by $\theta_C \in \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{Q}}(C)$ the element whose character $\chi_{\mathbb{Q}}(\theta_C)$ sends $c \in C$ to 1, if $c$ generates $C$, and to 0 otherwise.

Let $C \subset H$ be a cyclic subgroup of the finite group $H$. Then we get for $h \in H$

$$\frac{1}{[H:C]} \cdot \chi_{\mathbb{Q}}\left(\mathrm{ind}_C^H \theta_C\right)(h) = \frac{1}{[H:C]} \cdot \frac{1}{|C|} \cdot \sum_{l\in H, l^{-1}hl\in C} \chi_{\mathbb{Q}}(\theta_C)(l^{-1}hl) = \frac{1}{|H|} \cdot \sum_{l\in H, \langle l^{-1}hl\rangle = C} 1.$$

Denote by $[\mathbb{Q}] \in \mathrm{Rep}_{\mathbb{Q}}(H)$ the class of the trivial $H$-representation $\mathbb{Q}$. Notice that $\chi_{\mathbb{Q}}([\mathbb{Q}])$ is the constant function with values 1. We get in $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{Q}}(H)$

$$1 \otimes_{\mathbb{Z}} [\mathbb{Q}] = \sum_{C\subset H, C \text{ cyclic}} \frac{1}{[H:C]} \cdot \mathrm{ind}_C^H \theta_C, \quad (3.1)$$

since for any $l \in H$ and $h \in H$ there is precisely one cyclic subgroup $C \subset H$ with $C = \langle l^{-1}hl\rangle$ and $\chi_{\mathbb{Q}}$ is bijective. In particular we get for a finite cyclic group $C$ in $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{Q}}(C)$

$$\Theta_C = 1 \otimes_{\mathbb{Z}} [\mathbb{Q}] - \sum_{D\subset C, D\neq C} \frac{1}{[C:D]} \cdot \mathrm{ind}_D^C \theta_D. \quad (3.2)$$

Now one easily checks by induction over the order of the finite cyclic subgroup $C$ that the element $\Theta_C$ satisfies

$$\Theta_C \in \mathbb{Z}\left[\frac{1}{|C|}\right] \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{Q}}(C). \quad (3.3)$$

Obviously $\Theta_C$ is an idempotent in $\mathbb{Z}\left[\frac{1}{|C|}\right] \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{Q}}(C)$. By the obvious change of rings homomorphism, $\mathbb{Z}\left[\frac{1}{|C|}\right] \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C)$ becomes a $\mathbb{Z}\left[\frac{1}{|C|}\right] \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{Q}}(C)$-module. Hence multiplication with $\theta_C$ defines an idempotent endomorphism

$$\theta_C : \mathbb{Z}\left[\frac{1}{|C|}\right] \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C) \to \mathbb{Z}\left[\frac{1}{|C|}\right] \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C).$$

It is natural with respect to group automorphisms of $C$, since $\Theta_C$ is invariant under group automorphisms of $C$.

**Lemma 3.4** *(a) For a finite group $H$ the map*

$$\oplus_{C\subset H, C \text{ cyclic}} \operatorname{ind}_C^H : \oplus_{C\subset H, C \text{ cyclic}} \mathbb{Z}\left[\frac{1}{|H|}\right] \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C) \to \mathbb{Z}\left[\frac{1}{|H|}\right] \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(H)$$

*is surjective;*

*(b) Let $C$ be a finite cyclic group. Then the image resp. cokernel of*

$$\oplus_{D\subset C, D\neq C} \operatorname{ind}_D^C : \oplus_{D\subset C, D\neq C} \mathbb{Z}\left[\frac{1}{|C|}\right] \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(D) \to \mathbb{Z}\left[\frac{1}{|C|}\right] \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C)$$

*is equal resp. isomorphic to the kernel resp. image of the idempotent endomorphism*

$$\theta_C : \mathbb{Z}\left[\frac{1}{|C|}\right] \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C) \to \mathbb{Z}\left[\frac{1}{|C|}\right] \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C).$$

**Proof :** (a) follows from the following calculation for $x \in \mathbb{Z}\left[\frac{1}{|H|}\right] \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(H)$ based on (3.1)

$$x = (1 \otimes_{\mathbb{Z}} [\mathbb{Q}]) \cdot x = \left(\sum_{C\subset H, C \text{ cyclic}} \frac{1}{[H:C]} \cdot \operatorname{ind}_C^H \theta_C\right) \cdot x = \sum_{C\subset H, C \text{ cyclic}} \frac{1}{[H:C]} \cdot \operatorname{ind}_C^H(\theta_C \cdot \operatorname{res}_H^C x).$$

(b) follows from the following two calculations based on (3.2) for $x \in \mathbb{Z}\left[\frac{1}{|C|}\right] \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(H)$

$$\begin{aligned} x - \theta_C \cdot x &= (1 \otimes [\mathbb{Q}] - \theta_C) \cdot x \\ &= \left(\sum_{D\subset C, D\neq C} \frac{1}{[C:D]} \cdot \operatorname{ind}_D^C \theta_D\right) \cdot x \\ &= \sum_{D\subset C, D\neq C} \frac{1}{[C:D]} \cdot \operatorname{ind}_D^C(\theta_D \cdot \operatorname{res}_C^D x) \end{aligned}$$

and for $D \subset C, D \neq C$ and $y \in \mathbb{Z}\left[\frac{1}{|C|}\right] \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(D)$

$$\theta_C \cdot \operatorname{ind}_D^C y = \operatorname{ind}_D^C(\operatorname{res}_C^D \theta_C \cdot y) = \operatorname{ind}_D^C(0 \cdot y) = 0.$$

This finishes the proof of Lemma 3.4. ■

The next result is analogous to [11, Lemma 7.4] but we have to go through its proof again because here we want to invert only the orders of finite subgroups of $G$, whereas in [11] we have considered everything over $\mathbb{Q}$.

**Theorem 3.5** *Let $G$ be a group and $\Lambda = \Lambda^G(X)$ as defined in (0.5). Consider the covariant $\Lambda\mathsf{Sub}$-module $\Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(?)$ which sends a finite subgroup group $H \subset G$ to $\Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(H)$. Then*

*(a) $S_H\Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(?)$ is trivial if the finite subgroup $H \subset G$ is not cyclic.*

*For a finite cyclic subgroup $C \subset G$, the $\Lambda[W_GC]$-module $S_C\Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(?)$ is isomorphic to the image of the idempotent $\Lambda[W_GC]$-homomorphism*

$$\theta_C : \Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C) \to \Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C).$$

*The isomorphism is given by the composition of the obvious inclusion $\operatorname{im}(\theta_C) \to \operatorname{Rep}_{\mathbb{C}}(C)$ with the obvious projection $\operatorname{Rep}_{\mathbb{C}}(C) \to S_C\Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(?)$;*

*(b) $\Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(?)$ is a projective $\Lambda\mathsf{Sub}$-module;*

*(c) Let $M$ be a contravariant $\Lambda\mathsf{Sub}$-module. There is a natural isomorphism of $\Lambda$-modules*

$$\begin{aligned} \oplus_{(C),C\,\text{cyclic},C\in\mathcal{F}(X)}\ M(C) \otimes_{\Lambda[W_GC]} \operatorname{im}\left(\theta_C : \Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C) \to \Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C)\right) \\ \cong \quad M \otimes_{\Lambda\mathsf{Sub}} \Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(?); \end{aligned}$$

*(d) $\Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(H)$ is a flat $\Lambda\mathsf{Sub}$-module, i.e. for an exact sequence $0 \to M_0 \to M_1 \to M_2 \to 0$ of contravariant $\Lambda\mathsf{Sub}$-modules the induced sequence of $R$-modules $0 \to M_0 \otimes_{\Lambda\mathsf{Sub}} \Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(?) \to M_1 \otimes_{\Lambda\mathsf{Sub}} \Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(?) \to M_2 \otimes_{\Lambda\mathsf{Sub}} \Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(?) \to 0$ is exact.*

**Proof :** (a) Consider $H \in \mathcal{F}(X)$. Since $\operatorname{Rep}_{\mathbb{C}}(H)$ is finitely generated free abelian and $|W_GH|$ is invertible in $\Lambda$, the $\Lambda[W_GH]$- module $\Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(H)$ is projective. We conclude from Lemma 3.4 (a) that $S_H\Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(?)$ is trivial if $H$ is not cyclic. If $H = C$ for a finite cyclic subgroup $C \subset G$, the assertion follows from Lemma 3.4 (b).

(b) Notice that $S_H\Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(?)$ is a projective $\Lambda[W_GH]$-module for all $H \in \mathcal{F}(X)$ by assertion (a). Because of Theorem 2.4 it suffices to show for the morphism $T$ defined in (2.3) that $T(K)$ is injective for any given element $K \in \mathcal{F}(X)$.

Consider an element $u$ in the kernel of $T(K)$. Put $J(H) = \operatorname{mor}_{\mathsf{Sub}}(H,K)/(W_GH)$ for $H \in \mathcal{F}(X)$ and put $I = \{(H) \mid H \in \mathcal{F}(X)\}$. Choose for any $(H) \in I$ a representative $H \in (H)$. Then fix for any element $\overline{f} \in J(H)$ a representative $f : H \to K$ in $\operatorname{mor}_{\mathsf{Sub}}(H,K)$. For the remainder of the proof of assertion (b) we abbreviate $L(?) := \Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(?)$. We can find elements $x_{H,f} \in S_HL$ for $(H) \in I$ and $\overline{f} \in J(H)$ such that only finitely many of the $x_{H,f}$-s are different from zero and $u$ can be written as

$$u = \sum_{(H)\in I} \sum_{\overline{f}\in J(H)} (f : H \to K) \otimes_{\Lambda[W_GH]} x_{H,f}.$$

We want to show that all elements $x_{H,f}$ are zero. Suppose that this is not the case. Let $(H_0)$ be maximal among those elements $(H) \in I$ for which there is $\overline{f} \in J(H)$ with $x_{H,f} \neq 0$, i.e. if for $(H) \in I$ the element $x_{H,f}$ is different from zero for some morphism $f : H \to K$ in $\mathsf{Sub}$ and there is a morphism $H_0 \to H$ in $\mathsf{Sub}$, then $(H_0) = (H)$. In the sequel we choose for any of the morphisms $f : H \to K$ in $\mathsf{Sub}$ a group homomorphism denoted in the same way $f : H \to K$ representing it. Recall that $f : H \to K$ is given by conjugation with an appropriate element $g \in G$. Fix $f_0 : H_0 \to K$ with $x_{H_0 \cdot f_0} \neq 0$. We claim that the composition

$$A : \oplus_{(H)\in I} E_H \circ S_H(L(K)) \xrightarrow{T(K)} L(K) \xrightarrow{\operatorname{res}_K^{\operatorname{im}(f_0)}} L(\operatorname{im}(f_0)) \xrightarrow{\operatorname{ind}_{f_0^{-1}:\operatorname{im}(f_0)\to H_0}} L(H_0) \xrightarrow{\operatorname{pr}_{H_0}} S_{H_0}L$$

maps $u$ to $m \cdot x_{H_0,f_0}$ for some integer $m > 0$ which is invertible in $\Lambda$. This would lead to a contradiction because of $T(K)(u) = 0$ and $x_{H_0,f_0} \neq 0$.

Consider $(H) \in I$ and $\overline{f} \in J(H)$. It suffices to show that $A\left((f : H \to K) \otimes_{\Lambda[W_G H]} x_{H,f}\right)$ is $[K \cap N_G \operatorname{im}(f_0) : \operatorname{im}(f_0)] \cdot x_{H,f}$ if $(H) = (H_0)$ and $\overline{f} = \overline{f_0}$, and is zero otherwise. One easily checks that $A((f : H \to K) \otimes_{\Lambda[W_G H]} x_{H,f})$ is the image of $x_{H,f}$ under the composition

$$a(H,f) : S_H L \xrightarrow{\sigma_H} L(H) \xrightarrow{\operatorname{ind}_{f:H\to\operatorname{im}(f)}} L(\operatorname{im}(f)) \xrightarrow{\operatorname{ind}_{\operatorname{im}(f)}^K} L(K) \xrightarrow{\operatorname{res}_K^{\operatorname{im}(f_0)}} L(\operatorname{im}(f_0))$$
$$\xrightarrow{\operatorname{ind}_{f_0^{-1}:\operatorname{im}(f_0)\to H_0}} L(H_0) \xrightarrow{\operatorname{pr}_{H_0}} S_{H_0} L.$$

The Double Coset formula implies

$$\operatorname{res}_K^{\operatorname{im}(f_0)} \circ \operatorname{ind}_{\operatorname{im}(f)}^K \;=\; \sum_{k \in \operatorname{im}(f_0)\backslash K/\operatorname{im}(f)} \operatorname{ind}_{c(k):\operatorname{im}(f)\cap k^{-1}\operatorname{im}(f_0)k \to \operatorname{im}(f_0)} \circ \operatorname{res}_{\operatorname{im}(f)}^{\operatorname{im}(f)\cap k^{-1}\operatorname{im}(f_0)k}.$$

The composition $\operatorname{pr}_{H_0} \circ \operatorname{ind}_{f_0^{-1}:\operatorname{im}(f_0)\to H_0} \circ \operatorname{ind}_{c(k):\operatorname{im}(f)\cap k^{-1}\operatorname{im}(f_0)k\to\operatorname{im}(f_0)}$ is trivial, if $c(k) : \operatorname{im}(f) \cap k^{-1}\operatorname{im}(f_0)k \to \operatorname{im}(f_0)$ is not an isomorphism. Suppose that $c(k) : \operatorname{im}(f) \cap k^{-1}\operatorname{im}(f_0)k \to \operatorname{im}(f_0)$ is an isomorphism. Then $k^{-1}\operatorname{im}(f_0)k \subset \operatorname{im}(f)$. Since $H_0$ has been choosen maximal among the $H$ for which $x_{H,f} \neq 0$ for some morphism $f : H \to K$, this implies $x_{H,f} = 0$ or that $k^{-1}\operatorname{im}(f_0)k = \operatorname{im}(f)$. Suppose $k^{-1}\operatorname{im}(f_0)k = \operatorname{im}(f)$. Then $(H) = (H_0)$ which implies $H = H_0$. Moreover, the homomorphisms in $\mathsf{Sub}$ represented by $f_0$ and $f$ agree. Hence the group homomorphisms $f_0$ and $f$ agree themselves and we get $k \in N_G \operatorname{im}(f_0) \cap K$. This implies that $a(H,f) = [K \cap N_G \operatorname{im}(f_0) : \operatorname{im}(f_0)] \cdot \operatorname{id}$ if $(H) = (H_0)$ and $\overline{f} = \overline{f_0}$, and that otherwise $a(H,f) = 0$ or $x_{H,f} = 0$ holds. Hence the map $T$ is injective.

(c) follows from assertion (a) and the bijectivity of the isomorphism $T$ defined in (2.3) because there is a natural isomorphism

$$M \otimes_{\Lambda\mathsf{Sub}} E_H S_H \Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(?) \quad \xrightarrow{\cong} \quad M(H) \otimes_{\Lambda[W_G H]} S_H \Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(?). \tag{3.6}$$

Now (d) follows from (c) and the fact that the $\Lambda[W_G H]$-module $S_H\Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(?) \cong \operatorname{im}(\Theta_C)$ is projective. This finishes the proof of Theorem 3.5. ∎

# 4. The construction of the Chern character

In this section we want to prove Theorem 0.7. There are similarities with the construction in [11]. The main difference is that here we want to give a construction, where we only have to invert the orders of elements in $\mathcal{F}(X)$, whereas in [11] we have worked over the rationals. In [11] we have used the Hurewicz homomorphism from stable homotopy to singular homology, which is only an isomorphism after inverting all primes. We will use the multiplicative structure of $K_*^G$ instead and work with a different source for the equivariant Chern character, which allows us to invert only the orders of finite subgroups of $G$.

In the sequel we denote by $K_p^G(X)$ the equivariant $K$-homology of a proper $G$-$CW$-complex $X$. It is defined by $\operatorname{colim}_{Y\subset X} KK_G^p(C_0(Y),\mathbb{C})$, where $Y$ runs over all cocompact $G$-subcomplexes of $X$ and $KK_G^p(C_0(Y),\mathbb{C})$ denotes equivariant $KK$-theory of the $G$-$C^*$-algebra $C_0(X)$ of continuous functions $X \to \mathbb{C}$, which vanish at infinity, and the $C^*$-algebra $\mathbb{C}$ with the trivial $G$-action. Given a homomorphism $\phi : H \to G$ of groups and a proper $H$-$CW$-complex, then $\operatorname{ind}_\phi X := G \times_\phi X$ is a proper $G$-$CW$-complex and there is an induction homomorphism

$$\operatorname{ind}_\phi : K_0^H(X) \to K_0^G(\operatorname{ind}_\phi X).$$

If the kernel of $\phi$ acts freely on $X$, then $\mathrm{ind}_\phi$ is bijective. In particular we get for a proper $G$-$CW$-complex $X$ a homomorphism

$$K_p^G(X) \xrightarrow{\mathrm{ind}_{G\to\{1\}}} K_p(G\backslash X),$$

which is bijective if $G$ acts freely on $X$. There is an external product

$$\mu : K_p^G(X) \times K_q^{G'}(X') \to K_{p+q}^{G\times G'}(X \times X')$$

for groups $G$ and $G'$, a proper $G$-$CW$-complex $X$ and a proper $G'$-$CW$-complex $X'$. External products and induction are compatible. For more information about equivariant $K$-homology and $KK$-theory we refer to [8] and in particular for the induction homomorphisms to [16].

Let $X$ be a proper $G$-$CW$-complex. We have introduced the ring $\Lambda = \Lambda^G(X)$ in (0.5). We want to construct for $H \in \mathcal{F}(X)$ and $p = 0, 1$ a $\Lambda$-homomorphism

$$\underline{\mathrm{ch}}_p^G(X)(H) : \Lambda \otimes_{\mathbb{Z}} K_p(C_GH\backslash X^H) \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(H) \quad \to \quad \Lambda \otimes_{\mathbb{Z}} K_p^G(X), \tag{4.1}$$

where $K_p(C_GH\backslash X^H)$ is the (non-equivariant) K-homology of the $CW$-complex $C_GH\backslash X^H$. The map will be defined by the following composition

$$\begin{tikzcd}
\Lambda \otimes_{\mathbb{Z}} K_p(C_GH\backslash X^H) \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(H) \\
\Lambda \otimes_{\mathbb{Z}} K_p(EG \times_{C_GH} X^H) \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(H) \arrow[u, "\mathrm{id} \otimes_{\mathbb{Z}} K_p(\mathrm{pr}_1;R) \otimes_{\mathbb{Z}} \mathrm{id}"', "\cong"] \\
\Lambda \otimes_{\mathbb{Z}} K_p^{C_GH}(EG \times X^H) \otimes_{\mathbb{Z}} K_0^H(*) \arrow[u, "\mathrm{id} \otimes_{\mathbb{Z}} \mathrm{ind}_{C_GH\to\{1\}} \otimes j"', "\cong"] \arrow[d, "\mu"'] \\
\Lambda \otimes_{\mathbb{Z}} K_p^{C_GH\times H}(EG \times X^H) \arrow[d, "\mathrm{ind}_{m_H}"', "\cong"] \\
\Lambda \otimes_{\mathbb{Z}} K_p^G(\mathrm{ind}_{m_H} EG \times X^H) \arrow[d, "\mathrm{id} \otimes_{\mathbb{Z}} K_p^G(\mathrm{ind}_{m_H} \mathrm{pr}_2)"'] \\
\Lambda \otimes_{\mathbb{Z}} K_p^G(\mathrm{ind}_{m_H} X^H) \arrow[d, "\mathrm{id} \otimes_{\mathbb{Z}} K_p^G(v_H)"'] \\
K_p^G(X)
\end{tikzcd}$$

Some explanations are in order. We have a left $C_GH$-action on $EG \times X^H$ by $g(e,x) = (ge, gx)$ for $g \in C_GH$, $e \in EG$ and $x \in X^H$. It extends to a $C_GH \times H$-action by letting the factor $H$ acting trivially. The map $\mathrm{pr}_1 : EG \times_{C_GH} X^H \to C_GH\backslash X^H$ is the canonical projection. It induces an isomorphism

$$\Lambda \otimes_{\mathbb{Z}} K_p(\mathrm{pr}_1; R) : \Lambda \otimes_{\mathbb{Z}} K_p(EG \times_{C_GH} X^H) \xrightarrow{\cong} \Lambda \otimes_{\mathbb{Z}} K_p(C_GH\backslash X^H)$$

since each isotropy group of the $C_GH$-space $X^H$ is finite and for any finite group $L$ the projection induces an isomorphism $\Lambda \otimes_{\mathbb{Z}} H_p(BL) \xrightarrow{\cong} \Lambda \otimes_{\mathbb{Z}} H_p(*)$ and hence by the Atiyah-Hirzebruch spectral sequence an isomorphism $\Lambda \otimes_{\mathbb{Z}} K_p(BL) \xrightarrow{\cong} \Lambda \otimes_{\mathbb{Z}} K_p(*)$ for all $p$. The isomorphism $j : K_0^H(*) \xrightarrow{\cong} \mathrm{Rep}_{\mathbb{C}}(H)$ is the canonical isomorphism. The group homomorphism $m_H : C_GH \times H \to G$ sends $(g,h)$

to $gh$. We denote by $\mathrm{pr}_2 : EG \times X^H \to X^H$ the canonical projection. The $G$-map $v_H : \mathrm{ind}_{m_H} X^H = G \times_{m_H} X^H \to X$ sends $(g,x)$ to $gx$.

Notice that we obtain a contravariant $\mathsf{Sub}$-module $K_0(C_G?\backslash X^?)$ by assigning to a finite subgroup $H \subset G$ the $\Lambda$-module $\Lambda \otimes_{\mathbb{Z}} K_p(C_G H\backslash X^H)$. We have already introduced the covariant $\Lambda$-module $\Lambda \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(?)$. Analogously to [11] one checks that the various maps $\underline{\mathrm{ch}}_p^G(X)(H)$ defined above induce a map of $\Lambda$-modules

$$\mathrm{ch}_p^G(X) : \Lambda \otimes_{\mathbb{Z}} K_p(C_G?\backslash X^?) \otimes_{\Lambda\mathsf{Sub}} \Lambda \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(?) \quad \to \quad \Lambda \otimes_{\mathbb{Z}} K_p^G(X). \tag{4.2}$$

Notice that for $L \in \mathcal{F}(X)$ and $X = G/L$ the $\Lambda\mathsf{Sub}$-module $K_0(C_G?\backslash (G/L)^?)$ is isomorphic to the $\Lambda\mathsf{Sub}$-module $\Lambda\,\mathrm{mor}_{\mathsf{Sub}}(?,L)$, which sends a finite subgroup $H \subset G$ to the free $\Lambda$-module with base $\mathrm{mor}_{\mathsf{Sub}}(H,K)$. By the Yoneda Lemma one obtaines a canonical isomorphism

$$\Lambda \otimes_{\mathbb{Z}} K_p(C_G?\backslash (G/L)^?) \otimes_{\Lambda\mathsf{Sub}} \Lambda \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(?) \quad \xrightarrow{\cong} \quad \mathrm{Rep}_{\mathbb{C}}(L).$$

One easily checks that under this identification $\mathrm{ch}_0^G(G/L)$ becomes the canonical identification of $\mathrm{Rep}_{\mathbb{C}}(L)$ with $K_0^G(G/L)$. Notice that $K_1(C_G?\backslash (G/L)^?)$ and $K_1^G(G/L)$ are both trivial. Hence $\mathrm{ch}_p^G(G/L)$ is bijective for all $L \in \mathcal{F}(X)$ and $p = 0,1$. Because of Theorem 3.5 (d) the source of $\mathrm{ch}_*^G$ is an equivariant homology theory on proper $G$-$CW$-complexes $Y$ with $\mathcal{F}(Y) \subset \mathcal{F}(X)$. One easily checks that $\mathrm{ch}_*^G$ is compatible with the Mayer-Vietoris sequences. By induction over the number of equivariant cells and the Five-Lemma $\mathrm{ch}_p^G(Y)$ is bijective for any finite proper $G$-$CW$-complex $Y$ with $\mathcal{F}(Y) \subset \mathcal{F}(X)$. Notice that $K_p^G(Y)$ is the colimit $\mathrm{colim}_{Z \subset Y} K_p^G(Z)$, where $Z$ runs through all finite $G$-$CW$-subcomplexes $Z$ of $Y$. The analogous statement holds for the source of $\mathrm{ch}_*^G$. Hence $\mathrm{ch}_p^G(Y)$ is bijective for all proper $G$-$CW$-complexes $Y$ with $\mathcal{F}(Y) \subset \mathcal{F}(X)$ and $p = 0,1$. Now Theorem 0.7 follows from Theorem 3.5 (c). ■

# 5. The Baum-Connes Conjecture and the Trace Conjecture

In the sequel we denote for a proper $G$-$CW$-complex $X$ by

$$\mathrm{asmb}^G : K_0^G(X) \quad \to \quad K_0(C_r^*(G)) \tag{5.1}$$

the assembly map which essentially assigns to an element in $K_0^G(X)$ represented by an equivariant Kasparov cycle its index. Given a homomorphism $\phi : H \to G$ of groups with finite kernel, there is an induction homomorphism $\mathrm{ind}_\phi : K_p(C_r^*(H)) \to K_p(C_r^*(G))$ such that the following diagram commutes [16, Theorem 1]

$$\begin{tikzcd} K_0^H(X) \arrow[r, "\mathrm{asmb}^H"] \arrow[d, "\mathrm{ind}_\phi"'] & K_0(C_r^*(H)) \arrow[d, "\mathrm{ind}_\phi"'] \\ K_0^G(\mathrm{ind}_\phi X) \arrow[r, "\mathrm{asmb}^G"] & K_0(C_r^*(G)) \end{tikzcd}$$

These induction homomorphisms, the assembly maps and the change of rings homomorphisms associated to the passage from $C_r^*(G)$ to $\mathcal{N}(G)$ are compatible with the external products

$$\begin{aligned} \mu : K_p^G(X) \times K_q^{G'}(X') &\to K_{p+q}^{G\times G'}(X \times X'); \\ \mu : K_p(C_r^*(G)) \times K_q(C_r^*(G')) &\to K_{p+q}(C_r^*(G \times G')); \\ \mu : K_p(\mathcal{N}(G)) \times K_q(\mathcal{N}(G')) &\to K_{p+q}(\mathcal{N}(G \times G')) \end{aligned}$$

for groups $G$ and $G'$, a proper $G$-$CW$-complex $X$ and a proper $G'$-$CW$-complex $X'$. We will use in the sequel the elementary fact that for any $G$-map $f : X \to Y$ of proper $G$-$CW$-complexes the composition $K_0^G(X) \xrightarrow{K_0^G(f)} K_0^G(Y) \xrightarrow{\mathrm{asmb}^G} K_0(C_r^*(G))$ is $\mathrm{asmb}^G : K_0^G(X) \to K_0(C_r^*(G))$. In the sequel the letter $i$ denotes change of rings homomorphism for the canonical map $C_r^*(G) \to \mathcal{N}(G)$.

Let $X$ be a proper $G$-$CW$-complex. We have introduced $J = J^G(X)$ in (0.6). Define the homomorphism

$$\xi_1 : \oplus_{(C)\in J}\ \Lambda \otimes_{\mathbb{Z}} K_0(C_GC\backslash X^C) \otimes_{\Lambda[W_GC]} \mathrm{im}\,(\theta_C : \Lambda \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(C) \to \Lambda \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(C)) \to K_0(\mathcal{N}(G)) \quad (5.2)$$

by the composition of the equivariant Chern character of Theorem 0.7

$$\mathrm{ch}_0^G(X) : \oplus_{(C)\in J}\ \Lambda \otimes_{\mathbb{Z}} K_0(C_GC\backslash X^C) \otimes_{\Lambda[W_GC]} \mathrm{im}\,(\theta_C : \Lambda \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(C) \to \Lambda \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(C)) \xrightarrow{\cong} \Lambda \otimes_{\mathbb{Z}} K_0^G(X),$$

the assembly map

$$\mathrm{id} \otimes \mathrm{asmb}^G : \Lambda \otimes_{\mathbb{Z}} K_0^G(X) \to \Lambda \otimes_{\mathbb{Z}} K_0(C_r^*(G))$$

and the change of rings homomorphism

$$\mathrm{id} \otimes i : \Lambda \otimes_{\mathbb{Z}} K_0(C_r^*(G)) \to \Lambda \otimes_{\mathbb{Z}} K_0(\mathcal{N}(G)).$$

This is the homomorphism which we want to understand. In particular we are interested in its image. We will identify it with a second easier to compute homomorphism

$$\xi_2 : \oplus_{(C)\in J}\ \Lambda \otimes_{\mathbb{Z}} K_0(C_GC\backslash X^C) \otimes_{\Lambda[W_GC]} \mathrm{im}\,(\theta_C : \Lambda \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(C) \to \Lambda \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(C)) \to K_0(\mathcal{N}(G)), \quad (5.3)$$

which is defined as follows. Let $l : \mathrm{im}(\Theta_C) \to \mathrm{Rep}_{\mathbb{C}}(C)$ be the inclusion. Let $K_0(\mathrm{pr}) : K_0(C_GC\backslash X^C) \to K_0(*)$ be induced by the projection from $C_GC\backslash X^C$ to the one-point space $*$. We obtain a map

$$(i \circ \mathrm{asmb}^{\{1\}} \circ K_0(\mathrm{pr})) \otimes l : K_0(C_GC\backslash X^C) \otimes \mathrm{im}(\theta_C) \to K_0(\mathcal{N}(\{1\})) \otimes \mathrm{Rep}_{\mathbb{C}}(C).$$

Define

$$\alpha : K_0(\mathcal{N}(\{1\})) \otimes \mathrm{Rep}_{\mathbb{C}}(C) \to \mathrm{Rep}_{\mathbb{C}}(C) \qquad [U] \otimes [W] \mapsto \dim_{\mathbb{C}}(U) \cdot [W].$$

Notice that $\alpha$ is essentially given by the external product and $K_0(\mathcal{N}(H)) = \mathrm{Rep}_{\mathbb{C}}(H)$ holds by definition for any finite group $H$. Induction yields a map

$$\mathrm{ind}_C^G : K_0(\mathcal{N}(C)) \to K_0(\mathcal{N}(G)).$$

The composition of these three maps above induces for any finite cyclic subgroup $C \subset G$ a homomorphism

$$\xi_2(C) : \Lambda \otimes_{\mathbb{Z}} K_0(C_GC\backslash X^C) \otimes_{\Lambda[W_GC]} \mathrm{im}\,(\theta_C : \Lambda \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(C) \to \Lambda \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(C)) \to K_0(\mathcal{N}(G)).$$

Define $\xi_2$ to be the direct sum $\oplus_{(C)\in J}\ \xi_2(C)$ after the choice of a representative $C \in (C)$ for each $(C) \in J$.

**Theorem 5.4** *Let $X$ be a proper $G$-$CW$-complex. Then the maps $\xi_1$ of (5.2) and $\xi_2$ of (5.3) agree.*

**Proof :** In the sequel maps denoted by the letter $\mu$ will be given by external products and pr denotes the projection from a space to the one-point space $*$. Fix a cyclic subgroup $C \in \mathcal{F}(X)$. Notice that the homomorphism $m_C : C_GC \times C \to G \quad (g,c) \mapsto gc$ has a finite kernel so that induction is defined also on the level of the reduced group $C^*$-algebra and the group von Neumann algebra. Denote by $\nu : \Lambda \otimes_{\mathbb{Z}} K_0^{C_GC\times C}(EG \times X^C) \to K_0^G(X)$ the composition of the maps $\mathrm{id} \otimes K_0^G(v_C)$, $\mathrm{id} \otimes K_0^G(\mathrm{ind}_{m_C} \mathrm{pr}_2)$ and $\mathrm{ind}_{m_C}$ appearing in the definition of $\underline{\mathrm{ch}}_0(X)(C)$. Then the following diagram commutes

$$\begin{array}{ccc}
\Lambda \otimes_{\mathbb{Z}} K_0^{C_GC}(EG \times X^C) \otimes_{\mathbb{Z}} K^C(*) & \xrightarrow{\mathrm{id}\otimes i\circ \mathrm{asmb}^{C_GC}\otimes j} & \Lambda \otimes_{\mathbb{Z}} K_0(\mathcal{N}(C_GC)) \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(C) \\
\Big\downarrow{\scriptstyle \mu} & & \Big\downarrow{\scriptstyle \mu} \\
\Lambda \otimes_{\mathbb{Z}} K_0^{C_GC\times C}(EG \times X^C) & \xrightarrow{\mathrm{id}\otimes i\circ \mathrm{asmb}^{C_GC\times C}} & \Lambda \otimes_{\mathbb{Z}} K_0(\mathcal{N}(C_GC \times C)) \\
\Big\downarrow{\scriptstyle \nu} & & \Big\downarrow{\scriptstyle \mathrm{ind}_{m_C}} \\
\Lambda \otimes_{\mathbb{Z}} K_0^G(X) & \xrightarrow{\mathrm{id}\otimes i\circ \mathrm{asmb}^G} & \Lambda \otimes_{\mathbb{Z}} K_0(\mathcal{N}(G))
\end{array}$$

For any group $G$ the map induced by the center-valued von Neumann dimension

$$\dim^u_{\mathcal{N}(G)} : K_0(\mathcal{N}(G)) \quad \to \quad \mathcal{Z}(\mathcal{N}(G))$$

is injective. Given a $CW$-complex $Z$ and an element $\eta \in K_0(Z)$, there is a closed manifold $M$ with a map $f : M \to BG$ and an elliptic complex $D^*$ of differential operators of order 1 over $M$ such that $K_0(f) : K_0(M) \to K_0(Z)$ maps the class $[D^*] \in K_0(M)$ to $\eta$ [2]. In the case $Z = BG$ the composition

$$K_0(M) \xrightarrow{K_0(f)} K_0(BG) \xrightarrow{(\mathrm{ind}_{G\to\{1\}})^{-1}} K_0^G(EG) \xrightarrow{\mathrm{asmb}^G} K_0(C_r^*(G)) \xrightarrow{i} K_0(\mathcal{N}(G)) \xrightarrow{\dim^u_{\mathcal{N}(G)}} \mathcal{Z}(\mathcal{N}(G))$$

resp. the composition

$$K_0(M) \xrightarrow{K_0(\mathrm{pr})} K_0(*) \xrightarrow{\mathrm{asmb}^{\{1\}}} K_0(C^*(\{1\}) \xrightarrow{i} K_0(\mathcal{N}(\{1\})) \xrightarrow{\mathrm{ind}_{\{1\}}^G} K_0(\mathcal{N}(G)) \xrightarrow{\dim^u_{\mathcal{N}(G)}} \mathcal{Z}(\mathcal{N}(G))$$

maps $[D^*]$ to the element $\mathrm{index}^u_{\mathcal{N}(G)}(\overline{D}^*)$ resp. $\mathrm{index}(D^*) \cdot 1_{\mathcal{N}(G)}$, where $\mathrm{index}^u_{\mathcal{N}(G)}(\overline{D}^*)$ resp. $\mathrm{index}(D^*)$ has been defined in (1.7) resp. (1.1). We conclude from Theorem 0.4 and the injectivity of the map $\dim^u_{\mathcal{N}(G)}$ of (5.5) that the following diagram commutes

$$\begin{array}{ccc}
K_0^G(EG) & \xrightarrow{i\circ \mathrm{asmb}^G} & K_0(\mathcal{N}(G)) \\
\Big\downarrow{\scriptstyle K_0(\mathrm{pr})\circ \mathrm{ind}^{-1}_{G\to\{1\}}} & & \Big\uparrow{\scriptstyle \mathrm{ind}_{\{1\}}^G} \\
K_0(*) & \xrightarrow{i\circ \mathrm{asmb}^{\{1\}}} & K_0(\mathcal{N}(\{1\}))
\end{array}$$

Since there is a $C_GC$-map $EG \times X^C \to EC_GC$, we conclude from the diagram above applied to the case $G = C_GC$ that the following diagramm commutes

$$\begin{array}{ccc}
\Lambda \otimes_{\mathbb{Z}} K_0^{C_GC}(EG \times X^C) \otimes_{\mathbb{Z}} K^C(*) & \xrightarrow{\mathrm{id}\otimes i\circ \mathrm{asmb}^{C_GC}\otimes j} & \Lambda \otimes_{\mathbb{Z}} K_0(\mathcal{N}(C_GC)) \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(C) \\
\Big\downarrow{\scriptstyle \mathrm{id}\otimes\left(K_0(\mathrm{pr})\circ \mathrm{ind}_{C_GC\to\{1\}}\right)\otimes j} & & \Big\uparrow{\scriptstyle \mathrm{id}\otimes \mathrm{ind}_{\{1\}}^{C_GC}\otimes \mathrm{id}} \\
\Lambda \otimes_{\mathbb{Z}} K_0(*) \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(C) & \xrightarrow{\mathrm{id}\otimes i\circ \mathrm{asmb}^{\{1\}}\otimes \mathrm{id}} & \Lambda \otimes_{\mathbb{Z}} K_0(\mathcal{N}(\{1\}) \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(C)
\end{array}$$

The composition

$$K_0(\mathcal{N}(\{1\})\otimes_{\mathbb{Z}}\mathrm{Rep}_{\mathbb{C}}(C) \xrightarrow{\mathrm{ind}_{\{1\}}^{C_GC}\otimes \mathrm{id}} K_0(\mathcal{N}(C_GC))\otimes_{\mathbb{Z}}\mathrm{Rep}_{\mathbb{C}}(C) \xrightarrow{\mu} K_0(\mathcal{N}(C_GC\times C)) \xrightarrow{\mathrm{ind}_{m_C}} K_0(\mathcal{N}(G))$$

agrees with the composition

$$K_0(\mathcal{N}(\{1\})) \otimes \mathrm{Rep}_{\mathbb{C}}(C) \xrightarrow{\alpha} \mathrm{Rep}_{\mathbb{C}}(C) = K_0(\mathcal{N}(C)) \xrightarrow{\mathrm{ind}_C^G} K_0(\mathcal{N}(G)).$$

We conclude that the following diagram commutes for any cyclic subgroup $C \in \mathcal{F}(X)$

$$\begin{array}{ccc}
\Lambda \otimes_{\mathbb{Z}} K_0^{C_G C}(EG \times X^C) \otimes_{\mathbb{Z}} K^C(*) & \xrightarrow{(\mathrm{id} \otimes i \circ \mathrm{asmb}^{C_G C}) \circ \nu \circ \mu} & \Lambda \otimes_{\mathbb{Z}} K_0(\mathcal{N}(G)) \\
{\scriptstyle \mathrm{id} \otimes (i \circ \mathrm{asmb}^{\{1\}} \circ K_0(\mathrm{pr}) \circ \mathrm{ind}_{C_G C \to \{1\}}) \otimes j} \downarrow & & \uparrow {\scriptstyle \mathrm{id} \otimes \mathrm{ind}_C^G} \\
\Lambda \otimes_{\mathbb{Z}} K_0(\mathcal{N}(\{1\})) \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(C) & \xrightarrow{\mathrm{id} \otimes \alpha} & \Lambda \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(C)
\end{array}$$

Hence the following diagram commutes for any cyclic subgroup $C \in \mathcal{F}(X)$

$$\begin{array}{ccc}
\Lambda \otimes_{\mathbb{Z}} K_p(C_G C \backslash X^C) \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(C) & \xrightarrow{\mathrm{id} \otimes (\alpha \circ (i \circ \mathrm{asmb}^{\{1\}} \circ K_0(\mathrm{pr})) \otimes \mathrm{id})} & \Lambda \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(C) \\
{\scriptstyle \mathrm{id} \otimes \underline{\mathrm{ch}}_0^G(X)(C)} \downarrow & & \downarrow {\scriptstyle \mathrm{id} \otimes \mathrm{ind}_C^G} \\
\Lambda \otimes_{\mathbb{Z}} K_0^G(X) & \xrightarrow{\mathrm{id} \otimes (i \circ \mathrm{asmb}^G)} & \Lambda \otimes_{\mathbb{Z}} K_0(\mathcal{N}(G))
\end{array}$$

Now Theorem 5.4 (and hence also Theorem 0.8) follow. ■

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