ON ZARISKI DECOMPOSITION PROBLEM

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Abstract. We discuss different generalizations of Zariski decomposition, relations between them and connections with finite generation of divisorial algebras.

Contents

The purpose of this survey is to clarify some details in §§3-4 of [PLF] related to connections between different generalizations of Zariski decomposition and finite generation of divisorial algebras. The survey almost does not contain new results. All the discussed results were known earlier (see [1], [4], [9], [12], [15], [16], [17], [PLF], [21], [22]).

1. Some preliminary facts

All varieties are assumed to be normal, projective and defined over the field of complex numbers $\mathbb{C}$. However, almost all results can be generalized to the relative situation X/Z. Everywhere, if is not is indicated converse by a divisor we mean an $\mathbb{R}$-divisor, i.e., an $\mathbb{R}$-linear combination of prime Weil divisors. For any divisor D, we put

$$\mathcal{O}_X(D) = \mathcal{O}_X(\lfloor D \rfloor), \qquad |D| = \{F \mid F \sim D, \quad f \geq 0\}.$$

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For definitions and main properties of b-divisors we refer to Iskovskikh paper [Isk].

Recall that a divisor D is said to be *big,* if $\kappa(X, D) = \dim X$, where

$$\kappa(X, D) \stackrel{\text{def}}{=} \begin{cases} -\infty, & \text{if } |nD| = \varnothing \text{ for all } n \in \mathbb{N}, \\ \max\limits_{n \in \mathbb{N}} \dim \Phi_{|nD|}(X), & \text{otherwise,} \end{cases}$$

is the *Iitaka D-dimension* of (X, D). A divisor D is said to be (*semi*) *ample* if $D = \sum \alpha_i H_i$, where H_i are integral (semi) ample divisors and $\alpha_i \in \mathbb{R}_{>0}$. By the Kleiman criterion, the ampleness is equivalent to the positiveness of the divisor on $\overline{NE}(X) \setminus \{0\}$. The semiampleness is equivalent to the existence of a contraction $\alpha\colon X \to Y$ and an ample divisor H on Y such that $D \sim_{\mathbb{R}} \alpha^* H$.

Recall that a divisor F on a variety X is said to be *b-semiample* (*b-nef*) if there exists a model Y dominating X and an $\mathbb{R}$-Cartier divisor L on Y such that the b-divisor $\overline{L}$ is b-semiample (respectively, b-nef) and $F = (\overline{L})_X$. Equivalent: there exists a birational contraction $f\colon Y \to X$ and a semiample (respectively, nef) $\mathbb{R}$-Cartier divisor L on Y such that $f_* L = F$. On a nonsingular variety the b-semiampleness and semiampleness (respectively, b-nef and nef properties) are equivalent [23, Th. 6.1].

1.1. A *rational* 1-*contraction* is a dominant rational map $\alpha\colon X \dashrightarrow Y$ with connected fibers such that

$$\dim \operatorname{Center}_Y G < \dim G \quad (= \dim X - 1)$$

for any prime exceptional b-divisor G on X.

According to Hironaka for a rational 1-contraction $\alpha\colon X \dashrightarrow Y$ there exists a hut

$$\begin{tikzcd} & W \arrow[dl, "g"'] \arrow[dr, "h"] & \\ X \arrow[rr, dashed, "\alpha"] & & Y \end{tikzcd} \tag{1.2}$$

satisfying the following conditions:

(i) morphisms $g\colon W \to X$ and $h\colon W \to Y$ are contractions, and the contraction g is birational,

(ii) if a prime divisor A on W is contracted by g, then it also is contracted by h.

Example 1.3. Let $Y = \mathbb{P}^1$ and let $W = \mathbb{P}_Y(\mathcal{E})$ be the projectivization of the vector bundle $\mathcal{E} = \underbrace{\mathcal{O}_{\mathbb{P}^1} \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}}_{n} \oplus \mathcal{O}_{\mathbb{P}^1}(d)$, $d \geq 1$. Let $g\colon W \to$

$X \subset \mathbb{P}^{d+n}$ be the morphism defined by the linear system $\left|\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)\right|$. Then g is a birational contraction and we have diagram (1.2). Here $X \subset \mathbb{P}^{d+n}$ is a cone over a rational curve $C_d \subset \mathbb{P}^d$ with vertex $\mathbb{P}^{n-1}$ and the map α is defined by the linear system of planes on X. The base locus of this system is precisely the vertex of the cone.

1.4. For a rational 1-contraction $\alpha\colon X \dashrightarrow Y$, we may define the *pull-back* of any $\mathbb{R}$-Cartier divisor D as follows: $\alpha^*D \overset{\text{def}}{=} g_*h^*D$ (it is easy to show that this definition does not depend on the choice of the hut (1.2)). Note however that the map α^* is not functorial: it is possible that $(\alpha \circ \beta)^*$ does not coincide with $\beta^*\alpha^*$. Similarly we can define α_*. For α_* we always have $\alpha_*\beta_* = (\alpha\beta)_*$ whenever β is a rational 1-contraction.

For a closed subset $V \subset X$, denote by $[V]^r$ the subset consisting of all components $V_i \subset V$ of dimension $\geq r$.

A divisor E on X is said to be *exceptional* with respect to a rational 1-contraction α if $\dim \alpha(E_i) < \dim E_i$ and $\alpha(E_i) \neq Y$ for all components $E_i \subset \operatorname{Supp}(E)$. In this situation E is said to be *very exceptional* if for any prime divisor G on Y the divisorial fiber $\alpha^\bullet(G)$ over G is not contained in $\operatorname{Supp}(E)$ (see [PLF, Def. 3.2] or [9, Prop. 1.10]). Here the *divisorial fiber* is defined as $\alpha^\bullet(G) \overset{\text{def}}{=} [\alpha^{-1}(G)]^{\dim X-1}$.

Remark 1.5. Let α be a morphism. To check that an exceptional divisor E is very exceptional it is sufficient to verify the following property:

> for any component $E_i \subset \operatorname{Supp}(E)$ such that $\dim \alpha(E_i) = \dim Y - 1$ we have $\operatorname{Supp}(\alpha^*\alpha(E_i)) \not\subset \operatorname{Supp}(E)$.

Assume that α is a morphism and let $d = \dim X - \dim Y$. Define the following closed subset in Y:

$$\mathfrak{E}(\alpha, E) \overset{\text{def}}{=} \left\{y \in Y \mid \left[\alpha^{-1}(y)\right]^d \subset \operatorname{Supp}(E)\right\},$$

where $\alpha^{-1}(y) = f(h^{-1}(y))$. It is clear that $\mathfrak{E}(\alpha, E) \subset \alpha(\operatorname{Supp}(E))$. Then E is very exceptional if and only if $\operatorname{codim} \mathfrak{E}(\alpha, E) \geq 2$.

The following fact will be frequently used without references:

Lemma 1.6 ([19, §1.1], [20, 2.15])**.** *Let $f\colon X \to Z$ be a birational contraction and A be an $\mathbb{R}$-Cartier divisor on X such that*

(i) *f contracts all components of A with negative coefficients;*
(ii) *for a sufficiently general curve C in a fiber $A_i/f(A_i)$ in each f-exceptional divisor A_i having negative coefficient in A, we have $A \cdot C \leq 0$.*

Then the divisor A is effective.

We need also the following result:

Lemma 1.7. *Let $f\colon X \to Z$ be a contraction (not necessary birational) and let A be a divisor on X. Write $A = A^+ - A^-$, where A^+ and A^- are effective divisors without common component. Assume that $-A$ is f-nef and A^- is very exceptional on Z. Then $A^- = 0$, i.e., A is effective.*

Proof. Assume that $\dim X > \dim Z$. Let $A^- \neq 0$. If $\dim f(\mathrm{Supp}(A^-)) > 0$, then we can replace Z with its general hyperplane section $Z' \subset Z$, X with $f^{-1}(Z)'$, and A with $A|_{f^{-1}(Z)'}$. The very exceptionality of A^- is preserved. Indeed, it is sufficient to choose a hyperplane section $Z' \subset Z$ so that it does not contain components of the set $\mathfrak{E}(\alpha, E)$ of codimension 2. Continuing the process we get the situation when $\dim f(\mathrm{Supp}(A^-)) = 0$. We may also assume that Z is a sufficiently small affine neighborhood of some fixed point $o \in Z$ (and $f(\mathrm{Supp}(A^-)) = o$). Further, all the conditions of lemma are preserved if we replace X with its general hyperplane section X'. If $\dim Z > 1$, then we can reduce our situation to the case $\dim X = \dim Z$. Then the statement of the lemma follows by Lemma 1.6 and from the existence of the Stein factorization. Finally, consider the case $\dim Z = 1$ (here we may assume that $\dim X = 2$). In this instance, $A \cdot A^- = A^+ \cdot A^- - (A^-)^2 \leq -(A^-)^2$. By the Zariski lemma the last number is positive, a contradiction. $\square$

Movable and fixed parts of a divisor. For a divisor D, we put

$$\mathrm{Mov}(D) = \begin{cases} -\inf\limits_{s\in K(X)^*} \{(s) \mid D + (s) \geq 0\} & \text{if } |D| \neq \varnothing, \\ -\infty & \text{otherwise.} \end{cases} \tag{1.8}$$

If $H^0(\mathcal{O}(D)) \neq 0$ (that is equivalent to $|D| \neq \varnothing$), we put

$$\mathrm{Fix}(D) = \inf\{L \mid L \sim D, \quad L \geq 0\} = D - \mathrm{Mov}(D).$$

Divisors $\mathrm{Mov}(D)$ and $\mathrm{Fix}(D)$ are called *mobile* and *fixed* parts of a divisor D respectively. If D is effective, then so is $\mathrm{Mov}(D)$:

$$\mathrm{Mov}(D) \geq -(\mathrm{Const}) = 0.$$

Obviously, $\mathrm{Mov}(D) = \mathrm{Mov}(\lfloor D \rfloor)$.

Lemma 1.9 (cf. [PLF, Prop. 4.15])**.** *Let D be a divisor such that $H^0(\mathcal{O}(D)) \neq 0$. Then $M = \mathrm{Mov}(D)$ satisfies the following properties:*

(i) *$M \leq D$;*
(ii) *the linear system $|M|$ has no fixed components (i.e., the divisor M is b-free);*
(iii) *if a divisor $L \leq D$ is b-free, then $L \leq M$.*

Conversely, if an (integral) divisor M satisfies conditions (i)-(iii)*, then* $M = \mathrm{Mov}(D)$*.*

Proof. It is clear that

$$D - M = \inf_{s \in K(X)^*} \{D + (s) \mid D + (s) \geq 0\} \geq 0.$$

Since

$$0 = \inf_{s \in K(X)^*} \{M + (s) \mid D + (s) \geq 0\}$$

and $D \geq M$, we have that the divisor $D + (s)$ is effective if and only if so is $M + (s)$. Hence,

$$0 = \inf_{s \in K(X)^*} \{M + (s) \mid M + (s) \geq 0\}.$$

This means that the linear system has no fixed components. Finally, let a divisor $L \leq D$ b-free. Then

$$L = - \inf_{s \in K(X)^*} \{(s) \mid L + (s) \geq 0\} \leq - \inf_{s \in K(X)^*} \{(s) \mid D + (s) \geq 0\} = M.$$

The last statement follows from the uniqueness of a divisor M satisfying conditions (i)-(iii). $\square$

Lemma 1.10. *Let D be a divisor such that $H^0(\mathcal{O}(D)) \neq 0$. Then $M = \mathrm{Mov}(D)$ satisfies the following properties:*

(i) *$M \leq D$;*
(ii) *$H^0(\mathcal{O}(M)) = H^0(\mathcal{O}(D))$;*
(iii) *if for a divisor $L \leq D$ the equality $H^0(\mathcal{O}(L)) = H^0(\mathcal{O}(D))$ holds, then $L \geq M$.*

Conversely, if an (integral) divisor M satisfies conditions (i)-(iii)*, then* $M = \mathrm{Mov}(D)$*.*

Proof. (ii) Since $M \leq D$, $H^0(\mathcal{O}(M)) \subset H^0(\mathcal{O}(D))$. Let $s \in H^0(\mathcal{O}(D))$. Then $D + (s) \geq 0$ and according to (1.8) we have $M + (s) \geq 0$, i.e., $s \in H^0(\mathcal{O}(M))$.

(iii) Let $H^0(\mathcal{O}(L)) = H^0(\mathcal{O}(D))$. Then $L + (s) \geq 0$ if and only if $D + (s) \geq 0$. Hence,

$$M = - \inf_{s \in K(X)^*} \{(s) \mid D + (s) \geq 0\} = - \inf_{s \in K(X)^*} \{(s) \mid L + (s) \geq 0\}.$$

Therefore,

$$L - M = \inf_{s \in K(X)^*} \{L + (s) \mid L + (s) \geq 0\} \geq 0.$$

$\square$

2. Classical Zariski decomposition

We say that a divisor D is *effective modulo $\mathbb{Q}$-linear ($\mathbb{R}$-linear) equivalence* if $D \sim_{\mathbb{Q}} D'$ (respectively, $D \sim_{\mathbb{R}} D'$), where D' is effective. The effectiveness modulo $\mathbb{Q}$-linear equivalence is equivalent to that $H^0(\mathcal{O}(\alpha D)) \neq 0$ for some $\alpha \in \mathbb{N}$.

Zariski decomposition and its various generalizations have the form $D = P + N$, where N is the effective part and P is the "maximal positive" part. In general such a decomposition is defined for effective divisors though many statements work in a more general situation:

Definition 2.1 ([7]). A divisor D on projective variety X is said to be *pseudo-effective* if there exists an ample divisor H such that $D + \varepsilon H$ is effective for any $\varepsilon > 0$.

If D is an $\mathbb{R}$-Cartier divisor, then its pseudo-effectiveness is equivalent to any of the following conditions (see [14]):

(i) the class of D is contained in the closure of the cone of effective divisors (in numerical sense), i.e., there exists a sequence of effective divisors $D^{(n)}$ such that $\lim \left(D^{(n)} \cdot C\right) = D \cdot C$ for any curve C;
(ii) the divisor $D + \varepsilon H$ is effective (modulo $\sim_{\mathbb{R}}$) for any ample divisor H and any $\varepsilon > 0$;
(iii) $D \cdot \ell \geq 0$ for any nef 1-cycle ℓ on X.

It is easy to see that the property of an $\mathbb{R}$-Cartier divisor to be pseudo-effective is closed under taking pull-backs f^*.

Theorem 2.2 ([23], [7]). *Let X be a surface and let D be a pseudo-effective divisor on X. Then there exists an effective divisor $N = N(D) = \sum a_i N_i$ on X such that*

(i) *the divisor $P \stackrel{\mathrm{def}}{=} D - N$ is nef;*
(ii) *either $N = 0$ or the matrix $(N_i \cdot N_j)$ is negative definite;*
(iii) *$(P \cdot N_i) = 0$ for all i.*

Furthermore, if D is a $\mathbb{Q}$-divisor, then so is N. The divisor $N(D)$ is uniquely defined by the class of numerical equivalence of D.

A decomposition $D = N + P$ satisfying conditions (i) - (iii) of Theorem 2.2 is called a *Zariski decomposition* of a divisor D. The divisor P is its *positive* and N is its *negative* (or *exceptional* [PLF]) parts. Sometimes, by abuse of language, we say that a Zariski decomposition is the divisor $N = N(D)$.

Note that in the two-dimensional case the $\mathbb{R}$- (or $\mathbb{Q}$)-Cartier condition is not necessary: the intersection theory is defined for any normal surface (see, e.g.., [18]). We give a sketch of proof with running the "D-MMP" (see [18]):

Sketch of the proof. If D is nef, then we put $N(D) = 0$. Otherwise there exists an irreducible curve E such that $D \cdot E < 0$. Since D is pseudo-effective, we have $E^2 < 0$ (otherwise E is a nef curve and by definition 2.1, $D \cdot E \geq 0$). By the Grauert criterion, the curve E is contractible (at least in the category of normal analytic spaces): $f_1 \colon X \to X_1$, where the divisor $f_{1*}D$ is again pseudo-effective. Continuing the process we obtain a model X' on which the image D' of D is nef (in other words, we run "D-minimal model program"). Let $f \colon X \to X'$ be the composition of all contractions. Put $P \overset{\text{def}}{=} f^*D'$ and $N \overset{\text{def}}{=} D - P = D - f^*f_*D$. Since the divisor $-D$ is nef over X', we have $N \geq 0$ (see Lemma 1.6). The uniqueness follows by Proposition 2.5 below. $\square$

Note however that in the category of projective surfaces contraction $X \to X'$ does not necessarily exist. In other words, the divisor P is not always semiample. Our proof shows that we may guarantee the semiampleness of P for divisors of type $D = K_X + B$ whenever the pair (X, B) is log canonical.

In his paper [23] Zariski considered the case of effective and integral divisor D. The generalization to the pseudo-effective case belongs to Fujita [7]. Below, in Example 6.18 we will see that in higher dimensional generalizations one has to consider divisors with irrational coefficients. We discuss the different higher dimensional generalizations of Theorem 2.2 and relations between them.

It is easy to see that Zariski decompositions agree via pull-backs. That is why we have the following.

Proposition 2.3. *For any pseudo-effective divisor D on a surface X, there exists a decomposition $\overline{D} = \mathcal{P} + \mathcal{N}$ of b-divisors in the group* $\mathrm{BCDiv}_{\mathbb{R}}(K(X))$ *of b-Cartier b-divisors (see*[Isk]*) that induce Zariski decompositions $(\overline{D})_Y = \mathcal{P}_Y + \mathcal{N}_Y$ on all normal projective models Y/X.*

Simple examples show also that Zariski decompositions does not agree via f_*. That is why $\mathcal{P}_Y$ and $\mathcal{N}_Y$ are not elements of $\mathrm{BDiv}_{\mathbb{R}}(K(X))$.

Example 2.4. Consider the quadratic birational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, $(x : y : z) \dashrightarrow (yz : xz : xy)$ and let

$$\begin{tikzcd} & X \arrow[dl, "\sigma"'] \arrow[dr, "f"] & \\ \mathbb{P}^2 \arrow[rr, dashed] & & \mathbb{P}^2 \end{tikzcd}$$

be a resolution of the indeterminacy. Here $\sigma\colon X \to \mathbb{P}^2$ and $f\colon X \to \mathbb{P}^2$ are blowups of triples of different points on $\mathbb{P}^2$. Let $D \overset{\text{def}}{=} \sigma^*H + E_1 + E_2 + E_3$, where H is an ample divisor on $\mathbb{P}^2$ and let E_1, E_2, E_3 be exceptional divisors of σ. Then $N(D) = E_1+E_2+E_3$ and $P(D) = \sigma^*H$. But $N(f_*D) = 0 \neq f_*(E_1 + E_2 + E_3)$ and $P(f_*D) = f_*D \neq f_*\sigma^*H$.

Proposition 2.5. *Let X be a surface and let D be a pseudo-effective divisor on X. Let $D = N(D)+P(D)$ be a Zariski decomposition. Then for any nef divisor L such that $L \leq D$ we have $L \leq P(D)$.*

Proof. Write $N = N(D) = \sum a_i N_i$. Let $L \leq D$, where L is nef. For all i we have

$$N_i \cdot (P - L) = -N_i \cdot L \leq 0.$$

Write $P - L = (D - L) - N = F^\sharp - N^\sharp$, where $F^\sharp$ and $N^\sharp$ are effective divisors without common components. Since $N^\sharp \leq N$,

$$0 \geq N^\sharp \cdot (P - L) = N^\sharp \cdot F^\sharp - N^{\sharp 2} \geq 0.$$

This gives us $N^{\sharp 2} = 0$, $N^\sharp = 0$ and $P - L \geq N$. □

Proposition 2.5 follows also by Lemma 1.6 applied to the contraction f from the proof of Theorem 2.2.

2.6. Generalizations. It is clear that Zariski decomposition cannot be generalized in higher dimensions without significant modifications. We formulate general scheme for eventual generalizations. Let D be an $\mathbb{R}$-Cartier divisor on a variety X and let $N(D)$ be the negative part in a (generalized) Zariski decomposition. It is reasonable to claim the following:

(i) the divisor $N(D)$ is effective;
(ii) the divisor $D - N(D)$ is "positive" in some sense;
(iii) $N(D)$ is "minimal".

Also it is reasonable to claim that the decompositions agree via pullbacks: $N(D)$ is $\mathbb{R}$-Cartier and if $f\colon Y \to X$ is a birational contraction, then

$$N(f^*D) = f^*N(D). \tag{2.7}$$

Remark 2.8. The last property allow us to introduce Zariski decompositions of b-Cartier b-divisors: for $\mathcal{D} \in \mathrm{BCDiv}_{\mathbb{R}}(K(X))$ there exists $\mathcal{N}(\mathcal{D}) \in \mathrm{BCDiv}_{\mathbb{R}}(K(X))$ and a projective model Y of field $K(X)$ such that $\mathcal{N}(\mathcal{D})_{Y'}$ is a (generalized) Zariski decomposition for any model of Y' of $K(X)$ dominating Y.

3. On finite generation of divisorial algebras

One of fundamental problems of algebraic geometry is the question about finite generation of algebras

$$\mathcal{R}_X D = \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(nD)).$$

Here $\mathcal{R}_X D$ is considered as a subalgebra of the algebra $K(X)[t]$, where each space $H^0(\mathcal{O}_X(nD))$ is enclosed in the component $K(X)t^n$. Below we present several well known facts.

Proposition 3.1 (see, for example, [2, ch. III, §1, 3^o], [PLF, Th. 4.6])**.** *Let $\mathcal{R} = \bigoplus_{n \geq 0} \mathcal{R}_n$ be a graded algebra over a field k such that $\mathcal{R}_0 = k$. Then*

(i) *the algebra $\mathcal{R}$ is finitely generated if and only so is the* truncated *algebra $\mathcal{R}^{[n_0]} \stackrel{\mathrm{def}}{=} \bigoplus_{n \geq 0} \mathcal{R}_{nn_0}$, $n_0 \in \mathbb{N}$;*

(ii) *if the algebra $\mathcal{R}$ is finitely generated, then there exists $n_0 \in \mathbb{N}$ such that the truncated algebra $\mathcal{R}^{[n_0]}$ is generated by elements degrees* 1*.*

Proposition 3.2. *Let D be an integral divisor such that $\mathrm{Bs}\,|D| = \varnothing$. Then the algebra $\mathcal{R}_X D$ is finitely generated.*

Proof. First, we prove this in the case, when D is ample. Let $X \subset \mathbb{P}^N$ be an embedding corresponding to a suitable multiplicity $n_0 D$ of D and let $\mathcal{J}_X \subset \mathcal{O}_{\mathbb{P}^N}$ be the ideal sheaf. From the exact sequence

$$0 \longrightarrow \mathcal{J}_X(n) \longrightarrow \mathcal{O}_{\mathbb{P}^N}(n) \longrightarrow \mathcal{O}_X(nn_0D) \longrightarrow 0$$

and Serre's vanishing theorem we obtain that the restrictions $H^0(\mathcal{O}_{\mathbb{P}^N}(n)) \longrightarrow H^0(\mathcal{O}_X(n_0D))$ are surjective for $n \geq n_1$. Hence there is a surjective map $\mathcal{R}^{[n_1]}_{\mathbb{P}^N}\mathcal{O}(1) \to \mathcal{R}^{[n_0n_1]}_X D$. According to Proposition 3.1 this is proves finite generation of $\mathcal{R}_X D$.

Now consider the general case. Let $X \to \bar{X} \subset \mathbb{P}^N$ be a morphism defined by the linear system $|n_0 D|$, where $n_0 \gg 0$ and $\bar{X} = \varphi(X)$. Consider its Stein factorization $X \xrightarrow{\varphi} X' \xrightarrow{\psi} \bar{X}$. Then the divisor $H = \psi^* \mathcal{O}_{\bar{X}}(1)$ is ample and $\mathrm{Mov}(nD) = \varphi^* H$. Since $\varphi_* \mathcal{O}_X = \mathcal{O}_{X'}$, $\mathcal{R}^{[n]} D \simeq \mathcal{R}_{X'} H$. According to the above the last algebra is finitely generated. $\square$

Zariski decomposition and finite generation of divisorial algebras.

3.3. For a Zariski decomposition $D = P + N$, there is an isomorphism of graded algebras

$$\mathcal{R}_X D \simeq \mathcal{R}_X P. \tag{3.4}$$

Indeed, for each $n \in \mathbb{N}$ we have

$$H^0(\mathcal{O}_X(nP)) \subset H^0(\mathcal{O}_X(nD)) = H^0(\mathcal{O}_X(\mathrm{Mov}(nD))).$$

On the other hand, the divisor $\mathrm{Mov}(nD)$ is nef and by Proposition 2.5 we have $\mathrm{Mov}(nD) \leq nP$. This gives us the inverse inclusion

$$H^0(\mathcal{O}_X(\mathrm{Mov}(nD))) \subset H^0(\mathcal{O}_X(nP)).$$

Thus the question about finite generation of a divisorial algebra $\mathcal{R}_X D$ can be reduced to the question about finite generation of the divisorial algebra $\mathcal{R}_X P$, where the divisor P is nef. It is well known that the algebra $\mathcal{R}_X D$ is not always finitely generated:

Proposition 3.5 ([23], cf. [PLF, Th. 4.28])**.** *Suppose that an integral effective Cartier divisor D satisfies the following two conditions:*

(i) $\mathrm{Fix}\,|nD| \neq \varnothing$ *for all* $n \in \mathbb{N}$*;*

(ii) *multiplicities of components* $\mathrm{Fix}\,|nD|$ *are bounded as* $n \to \infty$*.*

Then the algebra $\mathcal{R}_X D$ is not finitely generated.

The proposition above enable us to construct a great number of divisors with non-finitely generated algebras $\mathcal{R}_X D$ (see Example 3.9 below).

Proof. Suppose that the algebra $\mathcal{R}_X D$ is generated by the finite number of elements $u_1, \dots, u_r$. Let $d_i = \deg u_i$ and $d \overset{\mathrm{def}}{=} \max\{d_1, \dots, d_r\}$. Then the vector space $H^0(\mathcal{O}_X(nD))$ is generated by the monomials of the form

$$u_1^{\nu_1} u_2^{\nu_2} \cdots u_r^{\nu_r},$$

where

$$\nu_i \in \mathbb{Z}_{\geq 0}, \qquad \sum_{i=1}^{r} d_i \nu_i = n.$$

Therefore,

$$\mathrm{Fix}\,|nD| \geq \mathrm{Min}\left\{ \sum_{i=1}^{r} \nu_i \,\mathrm{Fix}\,|d_i D| \;\middle|\; \sum_{i=1}^{r} d_i \nu_i = n \right\}.$$

On the other hand,

$$\frac{d!}{k}\,\mathrm{Fix}\,|kD| \geq \mathrm{Fix}\,|d!D| \neq 0$$

for all $k = 1, \dots, d$. Thus divisors $\operatorname{Fix}|d_1D|, \dots, \operatorname{Fix}|d_rD|$ have at least one common component. This gives us a contradiction. $\square$

The condition (ii) in Proposition 3.5 is automatically satisfied if the divisor D is nef and big:

Corollary 3.6. *Let D be a nef and big integral Cartier divisor on a surface X such that $\operatorname{Fix}|nD| \neq \varnothing$ for all $n \in \mathbb{N}$. Then the algebra $\mathcal{R}_X D$ is not finitely generated.*

Proof. Follows by Proposition 3.7 below. $\square$

We say that the base locus $\operatorname{Bs}|nD|$ is *bounded as $n \to \infty$* if for any birational contraction $f \colon Y \to X$ multiplicities of components of $\operatorname{Fix}|nf^*D|$ are bounded as $n \to \infty$. In other words multiplicities of components of the b-divisor $\operatorname{Fix}(\overline{nD})$ are bounded (see 8.1).

Proposition 3.7 ([24, Th. 2.2])**.** *Let D be an (integral) Cartier divisor on a variety X. If the base locus $\operatorname{Bs}|nD|$ is bounded as $n \to \infty$, then the divisor D is nef. Conversely, if D is nef and big, then $\operatorname{Bs}|nD|$ is bounded.*

Sketch of the proof. We give an outline of the proof only for $\dim X = 2$. If $D \cdot C < 0$ for some curve C, then $C \subset \operatorname{Fix}|nD|$ for any $n \in \mathbb{N}$. Furthermore, $(nD - mC) \cdot C < 0$ whenever $m < n\frac{D \cdot C}{C^2}$, i.e., $mC \leq \operatorname{Fix}|nD|$ for such m. This is means that the multiplicity C in $\operatorname{Fix}|nD|$ goes to infinity, a contradiction.

Conversely, assume that D is nef and big. Let E_i be fixed components of $|nD|$. Obviously, we may assume that the surface X and all the E_i are nonsingular. Choose a very ample divisor H such that divisors $H - K_X - E_i$ are ample for all E_i. We prove that $H^0(E_i, \mathcal{O}_{E_i}(mD + H)) \neq 0$. By the Riemann-Roch Theorem this is satisfied if

$$(mD + H) \cdot E_i > p_a(E_i) - 1 = \frac{1}{2}(K_X + E_i) \cdot E_i.$$

The last is equivalent to

$$0 < (2mD + 2H - K_X - E_i) \cdot E_i = 2mD \cdot E_i + H \cdot E_i + (H - K_X - E_i) \cdot E_i,$$

that, obviously, is satisfied. Further, $H^1(\mathcal{O}_X(mD + H - E_i)) = 0$. From exact sequence

$$\begin{aligned} 0 \longrightarrow H^0(\mathcal{O}_X(mD + H - E_i)) \longrightarrow H^0(\mathcal{O}_X(mD + H)) \\ \longrightarrow H^0(\mathcal{O}_{E_i}(mD + H)) \longrightarrow 0 \end{aligned}$$

we obtain that E_i is not a fixed component of $|mD + H|$. Therefore the linear system $|mD + H|$ has no fixed component for $m \in \mathbb{Z}_{\geq 0}$. Finally, since D is big, for some $a \in \mathbb{N}$ we have $aD \sim H + F$, where F is an effective divisor. Therefore, $\operatorname{Fix}|(a+m)D| = \operatorname{Fix}|F+H+mD|$ is bounded by the divisor F, a contradiction. $\square$

From Corollary 3.6 (see also Example 4.11 below) taking into account isomorphism (3.4) we obtain the following criterion.

Theorem 3.8 ([23], cf. [PLF, Th. 4.28] and Theorem 4.10)**.** *Let X be a surface and let D be an effective modulo $\mathbb{Q}$-linear equivalence $\mathbb{Q}$-Cartier divisor on X. Then the algebra $\mathcal{R}_X D$ is not finitely generated if and only if $\kappa(X,D)=2$ and the divisor $P(D)$ is not semiample.*

Example 3.9 (Zariski)**.** Consider a nonsingular cubic curve $C \subset \mathbb{P}^2$. Pick 12 points $P_1,\dots,P_{12} \in C$ so that the divisor $\mathcal{O}_C(4) - \sum P_i$ is not a torsion in $\operatorname{Pic}(C)$. Let $\sigma\colon X \to \mathbb{P}^2$ be the blowup of $P_1,\dots,P_{12}$ and let $E_1,\dots,E_{12}$ be the corresponding exceptional divisors. Put $D \overset{\text{def}}{=} \sigma^*\mathcal{O}_{\mathbb{P}^2}(4) - \sum E_i$. It is easy to show that the divisor D is effective modulo linear equivalence, nef and big. Therefore in a Zariski decomposition one has $N(D)=0$. Furthermore, $D\cdot\widetilde{C}=0$ and the birational transform $\widetilde{C} \overset{\text{def}}{=} \sigma^{-1}(C)$ of the curve C is a fixed component of the linear system $|nD|$ for any $n\in\mathbb{N}$ (otherwise $nD|_{\widetilde{C}}=0$). Therefore D satisfies the conditions of Proposition 3.5 and the algebra $\mathcal{R}_X D$ is not finitely generated.

For a big divisor, there is also the following criterion of finite generation of the algebra $\mathcal{R}_X D$:

Theorem 3.10 ([24, Th. 1.2], cf. [PLF, Th. 3.18])**.** *Let D be an (integral) big Cartier divisor. Then the following conditions are equivalent:*

(i) *the algebra $\mathcal{R}_X D$ is finitely generated;*
(ii) *there exists $n\in\mathbb{N}$ and a birational contraction $f\colon \tilde{X}\to X$ such that the linear system $\operatorname{Mov}|nf^*D|$ defines a birational morphism contracting all the components of $\operatorname{Fix}|nf^*D|$.*

4. s-, σ- and sectional decompositions

s-decomposition. Let D be an effective modulo $\mathbb{Q}$-linear equivalence divisor. Put

$$M_n \overset{\text{def}}{=} \operatorname{Mov}(nD).$$

Thus $nD = M_n + F_n$, where $F_n \geq 0$. Put also

$$P_{\mathrm{s}}(D) \overset{\text{def}}{=} \limsup_{n\to\infty}(M_n/n). \tag{4.1}$$

Since $M_{n_1n_2} \geq n_2 M_{n_1}$ for all $n_1,n_2\in\mathbb{N}$, we have

$$nP_{\mathrm{s}}(D) \geq M_n \quad \text{for all } n\in\mathbb{N}.$$

We obtain a decomposition

$$D = P_{\mathrm{s}}(D) + N_{\mathrm{s}}(D), \qquad N_{\mathrm{s}}(D) \geq 0,$$

which we call an *s-decomposition*. The divisor $P_{\mathrm{s}}(D)$ is called its *positive* and $N_{\mathrm{s}}(D)$ its *negative* part. In case, when D is not effective modulo $\mathbb{Q}$-linear equivalence, we put $P_{\mathrm{s}}(D) = -\infty$. Obviously,

$$N_{\mathrm{s}}(D) = \inf_{\substack{n \in \mathbb{N} \\ s \in K(X)^*}} \{D + (s)/n \mid D + (s)/n \geq 0\}.$$

In other words

$$N_{\mathrm{s}}(D) = \inf\{L \mid L \sim_{\mathbb{Q}} D,\ L \geq 0\}. \tag{4.2}$$

Decompositions of such type were used by many authors: [13, §2], [17, §2], [PLF, §§3-4]. Note that $P_{\mathrm{s}}(D)$ and $N_{\mathrm{s}}(D)$ are not necessarily $\mathbb{Q}$-divisors, even if the divisor D is integral (see Example 6.18).

Remark 4.3. Let X be a surface and let D be an effective modulo $\sim_{\mathbb{Q}}$ divisor on X. Then the divisor P_{s} is nef and by Proposition 2.5 we have $P \geq P_{\mathrm{s}}$, where $P = P(D)$ is the positive part of the classical Zariski decomposition. In general case, this inequality is not an equality (see Example 4.5). If $\kappa(X, D) \geq 1$ and surface $\mathbb{Q}$-factorial, then equality $P = P_{\mathrm{s}}$ holds, i.e., the s-decomposition coincides with the classical Zariski decomposition.

Indeed, if the divisor D is big, then so is the divisor P and by the Kodaira lemma $P = A + F$, where A is ample and F is an effective divisor. Since for any $0 < \varepsilon < 1$, the divisor $(1-\varepsilon)P + \varepsilon A$ is ample and $(1-\varepsilon)P + \varepsilon A \leq P$, we have $(1-\varepsilon)P + \varepsilon A \leq P_{\mathrm{s}}$. Therefore, $P \leq P_{\mathrm{s}}$ (see also Proposition 4.21 below).

If $\kappa(X, D) = 1$, then $\kappa(X, P) = 1$, $P^2 = 0$ and for some $n \in \mathbb{N}$ the linear system $\operatorname{Mov}(nP)$ defines a contraction $f\colon X \to Z$ onto a curve. It is easy to see that

$$\begin{aligned} P \cdot \operatorname{Mov}(nP) = P \cdot \operatorname{Fix}(nP) = \operatorname{Mov}(nP)^2 = \\ \operatorname{Mov}(nP) \cdot \operatorname{Fix}(nP) = \operatorname{Fix}(nP)^2 = 0. \end{aligned}$$

Hence the divisor $\operatorname{Fix}(nP)$ is contained in fibers and is a pull-back of some divisor on Z. Therefore we can write $nP \sim f^*L$. Then $N_{\mathrm{s}}(P) = \frac{1}{n} N_{\mathrm{s}}(f^*L)$ (see 4.6 below). Since $\operatorname{Mov}(mf^*L) \geq f^* \operatorname{Mov}(mL)$, $P_{\mathrm{s}}(f^*L) \geq f^*P_{\mathrm{s}}(L) = f^*L$. Thus $P_{\mathrm{s}}(f^*L) = f^*L$, $N_{\mathrm{s}}(P) = N_{\mathrm{s}}(f^*L) = 0$, $P_{\mathrm{s}}(P) = P$ and $P_{\mathrm{s}}(D) \geq P_{\mathrm{s}}(P) = P$.

σ-decomposition. In the work [17] Nakayama defined a similar type of decompositions for any pseudo-effective divisor D, so-called, *σ-decomposition*:

$$D = P_\sigma(D) + N_\sigma(D).$$

If D is a big divisor, then

$$N_\sigma(D) \overset{\text{def}}{=} N_{\mathrm{s}}(D) = \inf\{L \mid L \sim_{\mathbb{Q}} D, \quad L \geq 0\}.$$

If D is a pseudo-effective, but not big divisor, then we put

$$N_\sigma(D) \overset{\text{def}}{=} \lim_{\varepsilon \to 0} N_\sigma(D + \varepsilon A),$$

where A is an arbitrary ample divisor. (It is easy to show that this definition does not depend on the choice of A).

From now on $\overline{\mathrm{Mv}}(X)$ denotes the closed convex cone in $\mathrm{N}^1(X)$ generated by the classes of mobile Cartier divisors and $\mathrm{Mv}^o(X)$ denotes the interior of $\overline{\mathrm{Mv}}(X)$. By the Kodaira lemma ewe have that if the class of any $\mathbb{Q}$-divisor D is contained in $\mathrm{Mv}^o(X)$, then some multiplicity nD, $n \in \mathbb{N}$ is an integral mobile divisor [13, §2].

Proposition 4.4 ([17, 2.1. 10])**.** *σ-decomposition of a pseudo-effective divisor D on a nonsingular* (*enough: on $\mathbb{Q}$-factorial*) *variety satisfies the following properties:*

(i) $[P_\sigma(D)] \in \overline{\mathrm{Mv}}(X)$*;*
(ii) *for any divisor L such that $L \leq D$ and $[L] \in \overline{\mathrm{Mv}}(X)$ we have $L \leq P_\sigma(D)$.*

In particular, on a nonsingular surface the σ-decomposition coincides with the classical Zariski decomposition.

Thus $P_\sigma(D) \geq P_{\mathrm{s}}(D)$ for any effective modulo $\sim_{\mathbb{Q}}$ divisor D. If the divisor D is big, then (by definition) $P_\sigma(D) = P_{\mathrm{s}}(D)$. However, this is not true for arbitrary effective divisors even in the two-dimensional case (see Example 4.5 below).

Example 4.5. Let $C \subset \mathbb{P}^2$ be a nonsingular cubic curve and let $P_1, \ldots, P_9 \in C$ be distinct points such that $\mathcal{O}_C(3) - \sum P_i$ is not a torsion in $\mathrm{Pic}(C)$. Let $\sigma\colon X \to \mathbb{P}^2$ be the blowup of points $P_1, \ldots, P_9$ and let D be the birational transform C. Then $\dim |nD| = 0$ for all $n \in \mathbb{N}$. Therefore, $P_{\mathrm{s}}(D) = 0$. On the other hand, D is nef. Hence, $P_\sigma(D) = P(D) = D$.

Properties of s-decompositions. The following properties are immediate consequences of the definition.

4.6. (i) The negative part of an s-decomposition depends only on the class of $\mathbb{Q}$-linear equivalence of the divisor D. More precisely, if D and D' are $\mathbb{Q}$-linearly equivalent and effective modulo $\sim_{\mathbb{Q}}$ divisors, then $N_{\rm s}(D) = N_{\rm s}(D)'$.

(ii) If D' and D'' are effective modulo $\sim_{\mathbb{Q}}$ divisors, then

$$P_{\rm s}(D)' + P_{\rm s}(D'') \leq P_{\rm s}(D' + D''), \qquad N_{\rm s}(D)' + N_{\rm s}(D'') \geq N_{\rm s}(D' + D'').$$

If additionally $D \geq D'$, then $P_{\rm s}(D) \geq P_{\rm s}(D)'$.

(iii) Let D be an effective modulo $\sim_{\mathbb{Q}}$ divisor. Then for any $\alpha \in \mathbb{Q}_{>0}$ we have $N_{\rm s}(\alpha D) = \alpha N_{\rm s}(D)$.

(iv) Computing $P_{\rm s}$ we always can replace the limit (4.1) on the "truncated" limit:

$$P_{\rm s}(D) = \limsup_{n\to\infty} \frac{M_{nn_0}}{nn_0}.$$

(v) For an ample divisor D, we have $P_s(D) = D$.

Proposition 4.7. *Let D be an effective modulo $\mathbb{Q}$-linear equivalence divisor. If a $\mathbb{Q}$-divisor L is b-semiample and $L \leq D$, then $L \leq P_{\rm s}(D)$.*

Proof. Write $L = \sum \alpha_i H_i$, where H_i are integral b-free divisors and $\alpha_i \in \mathbb{R}_{\geq 0}$. Since the coefficients of the divisor L are rational, we can choose numbers α_i also to be rational. Therefore nL is an (integral) b-free divisor for some $n \in \mathbb{N}$. By Lemma 1.9 we have $L \leq M_n/n \leq P_{\rm s}$. □

The following easy statement shows how s-decompositions can be used in the study of divisorial algebras.

Proposition 4.8. *An s-decomposition of an effective modulo $\mathbb{Q}$-linear equivalence divisor satisfies the following properties:*

(i) $P_{\rm s}(D) \leq D$;

(ii) $\mathcal{R}_X D = \mathcal{R}_X P_{\rm s}(D)$;

(iii) *for any divisor L such that $L \leq D$ and $\mathcal{R}_X D = \mathcal{R}_X L$ we have $L \geq P_{\rm s}(D)$ (i.e., $P_{\rm s}$ is the* smallest *divisor satisfying properties* (i) *and* (ii)).

Thus this proposition and [PLF, Remark 3.30] explain and justify introduction of the concept "s-decomposition".

Proof. Since $P_{\rm s} \geq M_n/n$ we have by Lemma 1.10 that for any $n \in \mathbb{N}$ we have

$$H^0(\mathcal{O}_X(M_n)) = H^0(\mathcal{O}_X(nD)) \supset H^0(\mathcal{O}_X(nP_{\rm s})) \supset H^0(\mathcal{O}_X(M_n)).$$

This proves (ii).

We prove (iii). Since $H^0(\mathcal{O}(nD)) = H^0(\mathcal{O}(nL))$, $nL \geq M_n$ (again by Lemma 1.10). Hence, $L \geq P_{\rm s}(D)$.

The converse follows from the fact that there exists at most one divisor satisfying properties (i) - (iii). □

Remark 4.9. Proposition 4.8 remains to be true if we replace the condition (iii) with the following:

(iii′) for any divisor L such that $L \le P_{\mathrm{s}}(D)$ and $\mathcal{R}_X D = \mathcal{R}_X L$ we have $L = P_{\mathrm{s}}(D)$.

(i.e., P_{s} is the *minimal* divisor satisfying properties (i) and (ii)). Indeed, by Proposition 4.8 the s-decomposition satisfies properties (i), (ii), (iii)′. Conversely, let P'_{s} satisfies properties (i), (ii), (iii)′. Then from (iii) applied to P_{s} we have $P'_{\mathrm{s}} \ge P_{\mathrm{s}}$ and from (iii)′ we obtain $P'_{\mathrm{s}} = P_{\mathrm{s}}$.

Theorem 4.10 (cf. [PLF, Th. 4.28])**.** *Let D be an effective modulo $\mathbb{Q}$-linear equivalence divisor. If the divisorial algebra $\mathcal{R}_X D$ is finitely generated, then $P_{\mathrm{s}}(D) = \mathrm{Mov}(n_0 D)/n_0$ for some $n_0 \in \mathbb{N}$ (in other words the limit* (4.1) stabilizes).

Thus in the case when the algebra $\mathcal{R}_X D$ is finitely generated, the positive part of s-decomposition is a b-semiample $\mathbb{Q}$-divisor. Under additional conditions for the positive part we obtain decompositions discussed in §§5-6. Theorem 4.10 is not a criterion: in this form the converse is not true. In fact, the condition $P_{\mathrm{s}}(D) = \mathrm{Mov}(n_0 D)/n_0$ is divisorial and is preserved under small birational contractions, while finite generation is essentially more subtle condition. In order to obtain a criterion of finite generation, we must consider the condition for the stabilization of limits on all blowups of the initial variety, i.e., to pass to a b-divisor (see [PLF, Th. 4.28] and Theorem 8.9). We notice that the condition of stabilization does not mean that $P_{\mathrm{s}}(D) = \mathrm{Mov}(nD)/n$ for all $n \gg 0$. However, this condition implies that $P_{\mathrm{s}}(D) = \mathrm{Mov}(nn_0 D)/(nn_0)$ for all $n \in \mathbb{N}$.

Proof. According to Proposition 3.1 there exists $n_0 \in \mathbb{N}$ such that the algebra $\mathcal{R}_X(n_0 D)$ is generated by the elements $u_1, \dots, u_r \in H^0(X, \mathcal{O}_X(n_0 D))^* = H^0(X, \mathcal{O}_X(n_0 D)) \setminus \{0\}$. For any $n \in \mathbb{N}$ and for any $s \in H^0(X, \mathcal{O}_X(nn_0 D))$, we have

$$s = \sum_{\nu_1 + \dots + \nu_r = n} a_{\nu_1, \dots, \nu_r} u_1^{\nu_1} \cdots u_r^{\nu_r}, \quad \nu_1, \dots, \nu_r \in \mathbb{Z}_{>0}, \quad a_{\nu_1, \dots, \nu_r} \in \mathbb{C}.$$

Hence,

$$\begin{aligned}\mathrm{ord}_F(s) \ge \min_{\nu_1 + \dots + \nu_r = n} (\nu_1 \,\mathrm{ord}_F(u_1) + \dots + \nu_r \,\mathrm{ord}_F(u_r)) \\ \ge n \inf_{u \in H^0(\mathcal{O}_X(n_0 D))^*} \mathrm{ord}_F(u).\end{aligned}$$

By definition we obtain $\mathrm{Mov}(nn_0 D) \le n \,\mathrm{Mov}(n_0 D)$. Since the inverse inequality always holds, we have $\mathrm{Mov}(nn_0 D) = n \,\mathrm{Mov}(n_0 D)$. This proves the theorem. □

Examples 4.11. (i) If $\kappa(X, D) = 0$, then $M_n = 0$ or $-\infty$ for all n. Moreover there exists $n_0 \in \mathbb{N}$ such that $H^0(\mathcal{O}(n_0 m D)) = 1$ for all $m \in \mathbb{N}$. Therefore, $M_{n_0 m} = 0$. This gives us $P_{\mathrm{s}}(D) = 0$.

(ii) If the class of the divisor D is contained in the open cone $\mathrm{Mv}^o(X)$, then D we may be approximated from below by b-semiample $\mathbb{Q}$-divisors. Therefore, $P_{\mathrm{s}}(D) = D$.

(iii) Let D be an effective divisor with $\kappa(X, D) = 1$. Then there exists $n_0 \in \mathbb{N}$ such that $\dim |nD| > 0$ for all $n \geq n_0$. Thus for every $n \geq n_0$ the linear system $|M_n|$ defines a map $X \dashrightarrow Y_n \subset \mathbb{P}^{N_n}$. It is clear that $K(Y_n)$ is a subfield in $K(X)$ generated by $H^0(\mathcal{O}_X(M_n))$. Since $M_n \leq M_{n+1}$, $K(Y_n) \subset K(Y_{n+1})$. All the fields $K(Y_n)$ have transcendence degree 1 over $\mathbb{C}$. Therefore the algebraic closure $\overline{K(Y_{n_0})}$ of $K(Y_{n_0})$ in $K(Y)$ also has transcendence degree 1. Then there exists $n_1 \geq n_0$ such that $K(Y_n) = \overline{K(Y_{n_0})}$ for $n \geq n_1$. Let Y be a nonsingular curve such that $K(Y) = \overline{K(Y_{n_0})}$. All rational maps $X \dashrightarrow Y_n$, $n \geq n_0$ factorise through Y:

$$X \overset{g}{\dashrightarrow} Y \xrightarrow{h_n} Y_n,$$

where h_n is a finite morphism. Therefore,

$$M_n = g^*\bigl(\sup\{L \in \mathrm{Div}(Y) \mid nD - f^*L \geq 0\}\bigr)$$

for $n \gg 0$. Hence the divisor

$$P_{\mathrm{s}}(D) = \lim_{n\to\infty} M_n/n = g^*\bigl(\sup\{F \in \mathrm{Div}_{\mathbb{Q}}(Y) \mid D - f^*F \geq 0\}\bigr)$$

is b-semiample. Obviously, the last divisor is a $\mathbb{Q}$-divisor whenever so is D. In particular, we obtain that if D is a $\mathbb{Q}$-divisor, then the algebra $\mathcal{R}_X D \simeq \mathcal{R}_X P_{\mathrm{s}}$ is finitely generated. The constructed map is a particular case of the *Iitaka fibration* of (X, D).

(iv) (see also Proposition 3.7) Suppose that an $\mathbb{R}$-Cartier divisor D is nef and big. Then by the Kodaira lemma there exists an effective divisor F such that the divisor $D - F$ is ample. Therefore so is $D - \varepsilon F$ for any $0 < \varepsilon < 1$. Thus,

$$P_{\mathrm{s}}(D) \geq P_{\mathrm{s}}(D - \varepsilon F) = D - \varepsilon F.$$

Passing to the limit we obtain $P_{\mathrm{s}}(D) = D$ (cf. with Example 4.5).

Remark 4.12. If $N_{\mathrm{s}}(D) = 0$, then the divisor D is the limit of b-semiample divisors M_n/n, i.e., its class is contained in the closure of the cone $\mathrm{Mv}(X)$. In general case, this does not imply the b-semiampleness of D (see Example 3.9).

Proposition 4.13 (cf. [12])**.** *Let D be a big divisor on a $\mathbb{Q}$-factorial variety. Then the negative part of s-decomposition depends only on the class of numerical equivalence of D.*

Proof. Let D, D' be big numerically equivalent divisors and let $T \stackrel{\text{def}}{=} D' - D \equiv 0$. Since the divisor $P_{\mathrm{s}}(D)$ is big, the Kodaira lemma gives us $P_{\mathrm{s}}(D) = H + F$, where $F \geq 0$ and H is an ample divisor. Write

$$D' = D + T = P_{\mathrm{s}}(D) + T + N_{\mathrm{s}}(D) = P_{\mathrm{s}}(D) - \varepsilon F + T + \varepsilon F + N_{\mathrm{s}}(D).$$

For any $0 < \varepsilon < 1$, the class of $P_{\mathrm{s}}(D) - \varepsilon F + T = (1-\varepsilon)P_{\mathrm{s}}(D) + \varepsilon H + T$ is contained in the cone $\mathrm{Mv}^o(X)$. According to Example 4.11, (ii) we have

$$P_{\mathrm{s}}(D)' \geq P_{\mathrm{s}}(P_{\mathrm{s}}(D) - \varepsilon F + T) = P_{\mathrm{s}}(D) - \varepsilon F + T.$$

Passing to the limit as $\varepsilon \to 0$, we obtain $P_{\mathrm{s}}(D) + T \leq P_{\mathrm{s}}(D)'$, i.e., $N_{\mathrm{s}}(D)' \leq N_{\mathrm{s}}(D)$. By symmetry we have also the inverse inequality. □

Proposition 4.14. *Let $f\colon X \to Y$ be a birational contraction and let D be an effective modulo $\sim_{\mathbb{Q}}$ divisor on X. Then*

$$f_*P_{\mathrm{s}}(D) \leq P_{\mathrm{s}}(f_*D). \tag{4.15}$$

Proof. For any $n \in \mathbb{N}$, we have $f_* \operatorname{Mov}(nD) \leq f_*nD$ and the divisor $f_* \operatorname{Mov}(nD)$ is b-free. Hence, $f_* \operatorname{Mov}(nD) \leq \operatorname{Mov}(nf_*D)$. This proves the statement. □

Example 2.4 shows that in general case inequality (4.15) is not an equality.

From Proposition 4.8 we obtain:

Corollary 4.16 (cf. [17, Th. 3.5. 3])**.** *Let $f\colon Y \to X$ be a birational contraction of $\mathbb{Q}$-factorial varieties and let D be an effective modulo $\sim_{\mathbb{Q}}$ divisor on X. Then*

$$f^*P_{\mathrm{s}}(D) \geq P_{\mathrm{s}}(f^*D).$$

Note that the s-decomposition does not satisfy condition (2.7):

Example 4.17. Let X be a blowup of $\mathbb{P}^3$ in two distinct points P_1 and P_2, and let D be the birational transform of a plane passing through P_1 and P_2. Since the divisor D is mobile, $N_{\mathrm{s}}(D) = 0$. Now let $f\colon Y \to X$ be the blowup of the birational transform of the line passing through P_1 and P_2 and let E be the exceptional divisor. Then $N_{\mathrm{s}}(f^*D) = E \neq f^*N_{\mathrm{s}}(D)$.

Sectional decomposition.

Definition 4.18 ([13, §2])**.** A decomposition $D = P_{\mathrm{sec}}(D) + N_{\mathrm{sec}}(D)$ is called a *sectional decomposition* if the following conditions are satisfied:

(i) $N_{\mathrm{sec}}(D) \geq 0$;
(ii) $P_{\mathrm{sec}}(D) \in \overline{\mathrm{Mv}}(X)$ (in particular, P_{sec} is $\mathbb{R}$-Cartier);

(iii) there is an isomorphism of graded algebras

$$\mathcal{R}_X D \simeq \mathcal{R}_X P_{\text{sec}}.$$

From definition and Proposition 4.8 we immediately obtain:

Proposition 4.19. *Let D be an effective modulo $\sim_{\mathbb{Q}}$ divisor on a $\mathbb{Q}$-factorial variety. Then an s-decomposition is also a sectional decomposition. For any sectional decomposition we have $P_{\text{sec}} \geq P_{\text{s}}$*

Remarks 4.20. (i) Note that unlike s- and the σ-decompositions a sectional decomposition not necessary unique (in Example 4.5 there exists infinitely many sectional decompositions $P_{\text{sec}}(D) = tD$, $0 \leq t \leq 1$, while $P_{\text{s}}(D) = 0$).

(ii) The set of all sectional of decompositions is closed: if there exists a sequence $D = P_{\text{sec}}^{(r)} + N_{\text{sec}}^{(r)}$ of sectional decompositions and the limit $P = \lim_{r\to\infty} P_{\text{sec}}^{(r)}$ exists, then and $D = P + (D - P)$ is also a sectional decomposition.

Proposition 4.21 ([13, §2])**.** *Let D be a big divisor on a $\mathbb{Q}$-factorial variety. Then a sectional decomposition is unique* (*and $P_{\text{sec}} = P_{\text{s}} = P_{\sigma}$*)*. Furthermore, by Proposition* 4.13 *the negative part $N_{\text{sec}}(D)$ is uniquely defined by the numerical class of D.*

Sketch of the proof. According to Proposition 4.19 it is sufficient to prove the inequality $P_{\text{sec}} \leq P_{\text{s}}$. Note that the divisor P_{sec} is big. As in the proof of Proposition 4.13 by the Kodaira lemma we write $P_{\text{sec}} = H + F$, where H is ample and F is effective. Then for any $0 < \varepsilon < 1$ the class of $P_{\text{sec}} - \varepsilon F = (1-\varepsilon)P_{\text{sec}} + \varepsilon H$ is contained in the cone $\mathrm{Mv}^o(X)$. According to Example 4.11, (ii) we have

$$P_{\text{s}}(D) \geq P_{\text{s}}(P_{\text{sec}} - \varepsilon F) = P_{\text{sec}} - \varepsilon F.$$

Passing to the limit, we obtain $P_{\text{sec}} \leq P_{\text{s}}$. □

5. Zariski decomposition in Shokurov's sense and bss-ampleness

Bss-ampleness. We need the following, almost obvious lemma.

Lemma 5.1. *Let $\mathcal{D}$ be a b-divisor of the fields $K(X)$ and let L be a divisor on X such that $\mathcal{D}_X \sim_{\mathbb{R}} L$. Then there exists a unique b-divisor $\Lambda(L, \mathcal{D})$ such that $\Lambda(L, \mathcal{D})_X = L$ and $\Lambda(L, \mathcal{D}) \sim_{\mathbb{R}} \mathcal{D}$.*

Definition 5.2 ([PLF, Def. 3.2], see also [9, Prop. 1.10])**.** A divisor D on X is said to be *bss-ample* if there exists a rational 1-contraction $\alpha\colon X \dashrightarrow Y$, a (numerically) ample divisor H on Y, and a decomposition

$$D = D^{\mathrm{m}} + E, \tag{5.3}$$

satisfying the following conditions:

(i) $D^{\mathrm{m}} \sim_{\mathbb{R}} \alpha^* H$ and there exists a b-semiample b-divisor $\mathcal{H}$ such that $\mathcal{H}_X = \alpha^* H$;
(ii) the divisor E is effective and very exceptional with respect to the map α;
(iii) for any b-semiample b-divisor $\mathcal{L}$ such that $\mathcal{L}_X \leq D$ we have $\mathcal{L} \leq \Lambda(D^{\mathrm{m}}, \mathcal{H})$.

Remark 5.4. (i) According to (iii) the b-divisor $\Lambda(D^{\mathrm{m}}, \mathcal{H})$ is uniquely defined (i.e., it depends only on D). Therefore divisors D^{m}, E, and the contraction α also are uniquely defined. The b-divisor $\mathcal{H}$ is defined up to $\mathbb{R}$-linear equivalence.
(ii) In the notation of Definition 5.2 the divisor D is big if and only if the rational 1-contraction α is birational (see [PLF, Prop. 3.20]).
(iii) The condition of (iii) in the definition is automatically satisfied in cases $\dim X - \dim Y \leq 1$ and $\dim Y = 0$ (see [PLF, 3.4.3]).

Examples 5.5. (i) Any semiample divisor D is bss-ample (we can put $E = 0$ and $D^{\mathrm{m}} = D$). A b-semiample divisor D is bss-ample if and only if there exists the largest b-semiample b-divisor $\mathcal{D}^m = \Lambda(D^{\mathrm{m}}, \mathcal{H})$ with $\mathcal{D}^m_X = D$.

(ii) In the case $\kappa(X, D) = 0$ the divisor D is bss-ample if and only if $D \sim_{\mathbb{R}} 0$.

(iii) Consider the map α from Example 1.3 and let D be an n-dimensional plane on X. Then D is a bss-ample divisor because $D = \alpha^* \mathcal{O}_{\mathbb{P}^1}(1)$.

(iv) A divisor D with $\kappa(X, D) > 0$ on a surface is bss-ample if and only if the positive part $P(D)$ of the classical Zariski decomposition is semiample. In this instance decomposition (5.3) coincides with the Zariski decomposition, i.e., $N(D) = E$.

Indeed, by Zariski's Main Theorem the map α is a morphism. In the case $\dim Y = 2$ the contraction α is birational and the decomposition $D = D^{\mathrm{m}} + E$ coincides with the Zariski decomposition by definition and because the intersection matrix of exceptional divisors is negative definite. If $\dim Y = 1$, then the intersection matrix $(E_i \cdot E_j)$ is negative definite by the property of very exceptionality and semi-negativity of the intersection matrix in fibers of $X \to Y$.

Conversely, if $P(D)$ is a semiample divisor, then it defines a contraction $\alpha \colon X \to Y$, for which the divisor $N(D)$ is exceptional. Since the intersection matrix of components of $N(D)$ is negative definite, in case $\dim Y = 1$, a fiber of the contraction α cannot be contained in $\mathrm{Supp}(N(D))$.

Remark 5.6. Let D be a bss-ample divisor on X and let $\alpha\colon X \dashrightarrow Y$ be the corresponding rational 1-contraction. Then there exists a diagram

$$
\begin{tikzcd}
 & W \arrow[dl, "g"'] \arrow[dr, "h"] & \\
X \arrow[rr, dashed, "\alpha"] & & Y
\end{tikzcd}
\tag{5.7}
$$

and a numerically ample divisor H on Y such that

$$g^{-1}D + F_W \sim_{\mathbb{R}} h^*H + E_W,$$

where divisors F_W and E_W are exceptional on X and Y, respectively, E_W is effective, and the divisor $g(E_W)$ is very exceptional on Y. It is clear that $\mathcal{H} = \overline{h^*H}$.

Note however that divisors F_W and E_W are defined ambiguously. We can take as E_W the birational transform of E and in this instance E_W will be very exceptional on Y.

Obviously any bss-ample divisor is effective (more precisely, $D \sim_{\mathbb{R}} D'$, where $D' \geq 0$). However, the converse is not always true: the divisor D from Example 3.9 is effective and nef but it is not bss-ample. Indeed, this divisor is not semiample (because its restriction to $\widetilde{C}$ is a numerically trivial divisor which is not a torsion).

Proposition 5.8. *Let D be a big divisor. Assume that the D-MMP holds (including D-abundance conjecture). Then D is bss-ample.*

The conditions of the proposition are satisfied when $\dim X \leq 3$ (and in any dimension modulo LMMP) in most important cases (see 6.8 below).

Proof. First replace (X, D) with its $\mathbb{Q}$-factorialization, and then run the D-MMP. It is clear that bss-ampleness is invariant under birational isomorphisms in codimension 1. Thus it is sufficient to show that bss-ampleness is preserved under divisorial contractions:

Lemma 5.9. *Let $\varphi\colon X \to X'$ be a divisorial extremal D-negative contraction and let $D' = \varphi_* D$. Assume that D' is big and bss-ample. Then so is D.*

Proof. Let S be an (irreducible) φ-exceptional divisor. Then $D = \varphi^*D' + aS$, where $a \geq 0$ according to Lemma 1.6. Consider Hironaka's hut

$$\begin{tikzcd}
 & W \arrow[dl, "\delta"'] \arrow[d, "\pi"] \arrow[dr, "\tau"] & \\
X \arrow[r, "\varphi"] & X' \arrow[r, dashed, "\beta"] & Y
\end{tikzcd}$$

and let $\alpha \overset{\mathrm{def}}{=} \beta \circ \varphi$. Then α is a rational 1-contraction. Furthermore, since D' is big, the map α is birational. We have $D' = D'^{\mathrm{m}} + E'$, where $D'^{\mathrm{m}} \sim_{\mathbb{R}} \beta^* H$ and the divisor H is ample. Put

$$E = \varphi^* \pi_* \tau^* H - \delta_* \tau^* H + \varphi^* E' + aS, \qquad D^{\mathrm{m}} = D - E.$$

Then

$$D^{\mathrm{m}} = \varphi^* D' + aS - (\varphi^* \pi_* \tau^* H - \delta_* \tau^* H + \varphi^* E' + aS) =$$
$$\varphi^*(D' - \beta * H' - E') + \alpha^* H \sim_{\mathbb{R}} \alpha^* H.$$

Further,

$$\varphi_* E = \pi_* \tau^* H - \varphi_* \delta_* \tau^* H + E' = E' \geq 0.$$

One can also see that the divisor E is exceptional on Y. Finally, by Lemma 1.6 the divisor $\varphi^* \pi_* \tau^* H + \varphi^* E' + aS - \delta_* \tau^* H$ is effective. $\square$

After a finite number of contractions and flips $X \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_n$ we obtain a model (X_n, D_n), on which the divisor D_n is nef (and big). By our abundance hypothesis the divisor D_n is semiample. Hence it is bss-ample. $\square$

Note that it is not possible to omit the bigness condition in Proposition 5.8:

Example 5.10. Let $X = \mathbb{P}^1 \times \mathbb{F}_1$, let F be a fiber of the projection $\mathbb{P}^1 \times \mathbb{F}_1 \to \mathbb{P}^1$, and let $G = \mathbb{P}^1 \times C_0 \subset X$, where C_0 is the negative section of $\mathbb{F}_1$. Consider the divisor $D = F + G$. Then the contraction $\varphi \colon X \to \mathbb{P}^1 \times \mathbb{P}^2$ is a (unique) step of D-MMP and the divisor $\varphi_* D$ is nef. However, D is not bss-ample. Indeed, suppose that there is a decomposition such as in (5.3): $D = D^{\mathrm{m}} + E$. Since the divisor D^{m} is b-semiample, we have that there exist at most a finite number of curves negatively intersecting D^{m}. On the other hand, X is a smooth Fano variety. Hence the divisor D^{m} is nef and the rational 1-contraction α is a morphism. Further, X is a quasihomogeneous variety. We obtain that for E there is only one possibility $E = \lambda G$, $\lambda \geq 0$. Then $D^{\mathrm{m}} = F + \lambda G$, where $0 \leq \lambda \leq 1$. However, G is a fixed component of the linear system $|n(F + \lambda G)|$ for $\lambda > 0$. Therefore, $\lambda = 0$, $D^{\mathrm{m}} = F$, and the rational

1-contraction α coincides with the projection $X \to \mathbb{P}^1$. But the divisor G is not exceptional on $\mathbb{P}^1$, a contradiction.

Note that (X, Δ) is 0-pair for a suitable boundary Δ.

Zariski decomposition in Shokurov's sense.

Definition 5.11. A decomposition $D = D^{\mathrm{m}} + D^{\mathrm{e}}$ is called a *Zariski decomposition in sense of Shokurov* (or simply *Shokurov decomposition*) if the following conditions are satisfied:

(i) $D^{\mathrm{e}} \geq 0$;

(ii) there exists a b-semiample b-divisor $\mathcal{D}^{\mathrm{m}}$ such that $\mathcal{D}^{\mathrm{m}}_X = D^{\mathrm{m}}$;

(iii) for any b-semiample b-divisor $\mathcal{L}$ such that $\mathcal{L}_X \leq D$ we have $\mathcal{L} \leq \mathcal{D}^{\mathrm{m}}$.

Thus any bss-ample divisor has a Shokurov decomposition (and it coincides with (5.3)). In general case, the converse not always true:

Example 5.12. Let X and D be such as in Example 5.10. Then we have the following Shokurov decomposition:

$$D^{\mathrm{m}} = P_{\mathrm{s}}(D) = F, \qquad D^{\mathrm{e}} = G.$$

However, the divisor D is not exceptional for the morphism $X \to \mathbb{P}^1$ defined by the linear system $|F|$. Therefore D is not bss-ample.

Immediately from definition we obtain the following properties:

5.13. Let D be an effective modulo $\mathbb{Q}$-linear equivalence divisor.

(i) If a Shokurov decomposition of D exists, then $D^{\mathrm{m}} \geq P_{\mathrm{s}}(D)$. If furthermore D^{m} is a $\mathbb{Q}$-divisor, then $D^{\mathrm{m}} = P_{\mathrm{s}}(D)$ (see Proposition 4.7).

(ii) If a Shokurov decomposition for of D exists, then $\mathcal{R}_X D^{\mathrm{m}} = \mathcal{R}_X D = \mathcal{R}_X P_{\mathrm{s}}(D)$ (see Proposition 4.8).

Remark 5.14. Let $P_{\mathrm{s}}(D)$ be a b-semiample $\mathbb{Q}$-divisor. Then for the decomposition $D = P_{\mathrm{s}} + N_{\mathrm{s}}$ conditions (i) and (ii) of Definition 5.11 are satisfied while (iii) is not necessary satisfied. Instead the following weaker condition holds:

(iii)$'$ for any b-semiample divisor L such that $L \leq D$ we have $L \leq D^{\mathrm{m}}$.

Indeed, it is sufficient to show that for any b-semiample divisor L such that $L \leq D$ the inequality $P_{\mathrm{s}} \geq L$ holds.

Assume that $L \not\leq P_{\mathrm{s}}$. Write $P_{\mathrm{s}} = \sum \alpha_i L_i$ and $L = \sum \beta_i L_i$, where the L_i are integral b-free divisors, $\alpha_i \in \mathbb{Q}_{>0}$, and $\beta_i \in \mathbb{R}_{>0}$. Consider the finite dimensional real vector space V generated by the components of divisors P_{s} and L. The conditions $\mathrm{Min}(P_{\mathrm{s}}, L) \leq x \leq \mathrm{Max}(P_{\mathrm{s}}, L)$ define a parallelepiped $\mathfrak{K}$ (possible not maximal dimension) in V and the b-semiample divisors L_i generate a convex rational polyhedron $\mathfrak{R} \subset V$

containing the diagonal $[P_s, L]$ of $\mathfrak{K}$. So it is easy to see that there exists a rational divisor L' into the interior of the set $\mathfrak{K} \cap \mathfrak{N}$. From Proposition 4.7 we obtain $L' \leq P_s$. Contradiction with the fact that $L \nleq P_s$.

From Theorem 4.10 and 5.14 we obtain:

Corollary 5.15 (cf. [PLF, Remark 3.30, Example 4.30]). *Suppose that a divisorial algebra $\mathcal{R}_X D$ nontrivial and finitely generated. Then there exists a Shokurov decomposition (furthermore, $D^{\mathrm{m}} = P_s$ is a $\mathbb{Q}$-divisor).*

Remark 5.14 and Corollary 5.15 are particular cases of [PLF, Example 4.30] (see also Proposition 8.12).

Proposition 5.16. *Let D be a big divisor on a $\mathbb{Q}$-factorial variety X. If a Shokurov decomposition for D exists, then $D^{\mathrm{m}} = P_s$. In particular, D^{e} depends only on the class of D modulo numerical equivalence.*

Proof. Suppose that $D^{\mathrm{m}} \gneqq P_s(D)$. Similar to the proof of Proposition 4.21, we can write $D^{\mathrm{m}} = H + F$, where H is ample and F is an effective divisor. Then for any $0 < \varepsilon < 1$ the class of the divisor $D^{\mathrm{m}} - \varepsilon F = (1-\varepsilon)D^{\mathrm{m}} + \varepsilon H$ is contained in $\mathrm{Mv}^o(X)$. According to Example 4.11, (ii) we have

$$P_s(D) \geq P_s(D^{\mathrm{m}} - \varepsilon F) = D^{\mathrm{m}} - \varepsilon F.$$

Passing to the limit, we obtain $D^{\mathrm{m}} \leq P_s$, a contradiction. □

Example 5.17. The integral divisor D from Example 3.9 has no Shokurov decompositions. In fact, we can assert that if a divisor D on variety of any dimension is nef and for any $n \in \mathbb{N}$ the linear system $|nD|$ has fixed components of bounded multiplicity, then D has no Shokurov decompositions.

Example 5.18. Let D be a divisor from Example 4.5. Since $D^2 = 0$ and $\dim |nD| = 0$ for all $n \in \mathbb{N}$, we have that D is not semiample. Therefore D has the following Shokurov decomposition $D^{\mathrm{m}} = 0$ and $D^{\mathrm{e}} = D$. It does not coincide with the classical Zariski decomposition $N(D) = 0$.

In the two-dimensional case there is a criterion of the existence of Shokurov decomposition (see Example 5.5 and Proposition 2.5):

Proposition 5.19. *Let X be a surface with $\mathbb{Q}$-factorial singularities and let D be a divisor on X with the (classical) Zariski decomposition $D = P + N$. Then the following conditions are equivalent:*

(i) *the divisor P is semiample;*
(ii) *the divisor D is bss-ample;*

(iii) *there exists a Shokurov decomposition* $D = D^{\mathrm{m}} + D^{\mathrm{e}}$ *such that* $D^{\mathrm{e}} = N$ *and* $\mathcal{D}^{\mathrm{e}} = \overline{N}$.

Example 5.20. If in the conditions of Proposition 5.19 on X there exists a Shokurov decomposition $D = D^{\mathrm{m}} + D^{\mathrm{e}}$ and $\kappa(X, D) \geq 1$, then $D^{\mathrm{e}} = N(D)$.

Indeed, by Proposition 2.5 we always have $P(D) \geq D^{\mathrm{m}} \geq P_{\mathrm{s}}$ and according to Remark 4.3 the equality holds.

Theorem 5.21 ([PLF, Th. 3.33], cf. Proposition 5.8, Theorem 7.9)**.** *Let* (X, B) *be a* 0*-pair, i.e., a Kawamata log terminal pair with* $K_X + B \equiv 0$ *and let* D *be an effective modulo* $\mathbb{R}$*-linear equivalence divisor. Assume the LMMP* (*including the abundance conjecture*)*. Then there exists a Shokurov decomposition for the divisor* D.

In the conditions of this theorem there exist (and coincide between each other) all other Zariski decompositions (see below) for the divisor D.

6. Zariski decomposition in Fujita's sense

Fujita noticed that property 2.5 characterizes Zariski decomposition:

Definition 6.1. A decomposition $D = P_{\mathrm{f}} + N_{\mathrm{f}}$ is called a *Zariski decomposition in Fujita sense* (or simply *Fujita decomposition*) if

(i) $N_{\mathrm{f}} \geq 0$;
(ii) P_{f} is nef;
(iii) for any b-nef divisor L such that $L \leq D$ we have $L \leq P_{\mathrm{f}}$.

It is clear that we may set a question about the existence of such a decomposition only for pseudo-effective divisors.

Remarks 6.2. (i) It follows immediate from the definition that the negative part of a Fujita decomposition depends only on the class of numerical equivalence of D.

(ii) It is clear that $P_{\mathrm{f}} \geq M_n/n$. Therefore, $P_{\mathrm{f}} \geq P_{\mathrm{s}}$. So,

$$\mathcal{R}_X D = \mathcal{R}_X P_{\mathrm{f}} = \mathcal{R}_X P_{\mathrm{s}}. \tag{6.3}$$

Example 4.5 shows that the equality $P_{\mathrm{f}} = P_{\mathrm{s}}$ is not always true (even if the divisor P_{s} is nef). Nevertheless $D = P_{\mathrm{f}} + N_{\mathrm{f}}$ is a σ-decomposition for any pseudo-effective divisor D (see [17]).

(iii) In dimension 2 the Fujita decomposition coincides with the classical Zariski decomposition (see Proposition 2.5).

Example 6.4. Let D be a big bss-ample $\mathbb{R}$-Cartier divisor. We make use the notation of Remark 5.6. Then in diagram (5.7) there exists a unique decomposition

$$g^*D = P_{\mathrm{f}} + N_{\mathrm{f}}, \tag{6.5}$$

where $P_{\mathrm{f}} \sim_{\mathbb{R}} h^*H$ and the divisor N_{f} is effective and (very) exceptional on Y. This decomposition is a Fujita decomposition (with semiample positive part).

Indeed, in this instance h is a birational contraction. Put $N_{\mathrm{f}} = E_W - F_W$. Then the divisor $-N_{\mathrm{f}} \sim_{\mathbb{R}} h^*H - g^*D$ is nef over X. By Lemma 1.6 we have $N_{\mathrm{f}} \geq 0$. It is clear that N_{f} is exceptional on Y. Assume that a divisor $L \leq D$ is nef. Then the divisor $L - P_{\mathrm{f}} \sim_{\mathbb{R}} L - h^*H$ is nef over Y. From Lemma 1.6 we obtain that $L \leq P_{\mathrm{f}}$. Thus $g^*D \sim_{\mathbb{R}} P_{\mathrm{f}} + N_{\mathrm{f}}$ is a Zariski decomposition in Fujita sense. It is unique.

The statement is no longer true if we omit the condition for D to be big (see Example 1.3)

Proposition 6.6. *Let D be an effective $\mathbb{R}$-Cartier divisor. Assume that the decomposition $D = P_{\mathrm{f}} + N_{\mathrm{f}}$ satisfies conditions* (i)-(ii) *of Definition* 6.1. *Then* (iii) *is equivalent to the following*

(iii)′ *for any birational contraction $f\colon \hat{X} \to X$ and for any nef divisor F on $\hat{X}$ such that $F \leq f^*D$ we have $F \leq f^*P_{\mathrm{f}}$.*

Proof. **(iii) ⇒ (iii)′.** Since $-(f^*P_{\mathrm{f}} - F)$ is nef over X and $f_*(f^*P_{\mathrm{f}} - F) \geq 0$, we have $f^*P_{\mathrm{f}} - F \geq 0$. (We used that f_*F is b-nef and $f_*F \leq D$, so $f_*F \leq P_{\mathrm{f}}$).

(iii) ′ ⇒ (iii). Since L is b-nef, there exists a birational contraction $f\colon \hat{X} \to X$ and a nef divisor $\hat{L}$ on $\hat{X}$ such that $f_*\hat{L} = L$. Further, $\hat{L} - f^*D$ is nef over X and $f_*(f^*D - \hat{L}) \geq 0$. Hence, $f^*D - \hat{L} \geq 0$ and $L \leq P_{\mathrm{f}}$. □

However, a Fujita decomposition does not always exists that shows the following simple example:

Example 6.7. Consider the linear system $|D|$ on a nonsingular projective variety X such that

(i) $|D|$ has no fixed components;
(ii) D is not nef.

For example, similar to Example 4.17 we may take as X a blowup of $\mathbb{P}^3$ in two distinct points P_1 and P_2, and as D the birational transform of a plane passing through P_1 and P_2. Assume that there exists a decomposition $N_{\mathrm{f}}(D)$ in Fujita's sense. Then $P_{\mathrm{f}} \geq M_1 = D$, i.e., $P_{\mathrm{f}} = D$, a contradiction.

6.8. The above shows that it is more naturally to construct a Fujita decomposition for a pull-back f^*D of D under some birational contraction $f\colon Y \to X$. However even in such a stating, the problem of the existence of a Fujita decomposition fails (see [17]). Nevertheless, its positive decision is expected (for a pseudo-effective divisor D) in the most important cases:

(i) if (X, B) is 0-pair (see Theorem 5.21);
(ii) if (X, B) is a Fano log variety;
(iii) if $D = K_X + B$, where the pair (X, B) has log canonical singularities and $\kappa(X, K_X + B) \geq 0$ (cf. Theorem 7.9).

Argument justifying this hope is, for example, fact proved by Kawamata [12, Prop. 5]: for any effective Cartier divisor D on a toric variety X there exists a toric birational contraction $f\colon Y \to X$ such that the divisor f^*D has a Fujita decomposition and this decomposition coincides with a Shokurov decomposition and with an CKM-decomposition (see below).

Remark 6.9. It is clear that the Fujita decomposition satisfies condition (2.7). Therefore (see Remark 2.8) if it exists on Y for the divisor $\mathcal{D}_Y$, where $\mathcal{D} \in \mathrm{BCDiv}_{\mathbb{R}}(K(Y))$, then there exists a b-divisor $\mathcal{N}(\mathcal{D}) \in \mathrm{BCDiv}_{\mathbb{R}}(K(Y))$ such that $\mathcal{N}(\mathcal{D})_{Y'}$ is a decomposition in Fujita sense for any model Y' of the field $K(Y)$ dominating Y. In other words a Fujita decomposition is unique *in the birational sense.*

Definition 6.10. A decomposition $D = P_{\mathrm{gf}} + N_{\mathrm{gf}}$ is called *generalized Fujita decomposition* if

(i) $N_{\mathrm{gf}} \geq 0$;
(ii) P_{gf} is b-nef;
(iii) for any b-nef divisor L such that $L \leq D$ we have $L \leq P_{\mathrm{gf}}$.

Another way to generalize Fujita decomposition is to consider the straightforward analog of Definition 5.11: necessary to claim the existence of a b-nef b-divisor $\mathcal{P}$ such that $\mathcal{P}_X = P$ and satisfying condition 5.11 (iii).

Proposition 6.11. *Let D be an effective $\mathbb{R}$-Cartier divisor. Assume that there exists a birational contraction $f\colon \hat{X} \to X$ such that the divisor $D^* = f^*D$ has a Fujita decomposition $D^* = P_{\mathrm{f}}(D^*) + N_{\mathrm{f}}(D^*)$. Then $D = f_*P_{\mathrm{f}}(D^*) + f_*N_{\mathrm{f}}(D^*)$ is a generalized Fujita decomposition.*

Proof. By construction $f_*N_{\mathrm{f}}(D^*)$ is effective and $f_*P_{\mathrm{f}}(D^*)$ is b-nef. Let L be a b-nef divisor on X such that $L \leq D$. We may assume that there exists a nef divisor $\hat{L}$ on $\hat{X}$ such that $f_*\hat{L} = L$. Then $-(f^*D - \hat{L})$ is nef over X and $f_*(f^*D - \hat{L}) \geq 0$. By Lemma 1.6 we have $f^*D \geq \hat{L}$. But then $P_{\mathrm{f}}(D^*) \geq \hat{L}$ and $f_*P_{\mathrm{f}}(D^*) \geq L$. □

Proposition 6.12. *Let D be an effective divisor. Assume that there exists a generalized Fujita decomposition $D = P_{\mathrm{gf}} + F_{\mathrm{gf}}$. Then*

(i) $\mathcal{R}_X D = \mathcal{R}_X P_{\mathrm{gf}}$;

(ii) $P_{\mathrm{gf}} \geq P_{\mathrm{s}}(D)$. *If furthermore the divisor D is big and the variety X is $\mathbb{Q}$-factorial, then $P_{\mathrm{gf}} = P_{\mathrm{s}}(D)$.*

Conversely, if the divisor D is big, X is $\mathbb{Q}$-factorial, and $P_{\mathrm{s}}(D)$ is b-nef, then there exists the generalized Fujita decomposition for D and $P_{\mathrm{gf}} = P_{\mathrm{s}}(D)$.

Proof. The proof of (i) is similar to the proof of Proposition 4.8. We prove (ii). Since $M_n \leq nP_{\mathrm{gf}}$ for all $n \in \mathbb{N}$, we have $P_{\mathrm{s}}(D) \leq P_{\mathrm{gf}}$. Assume now that the divisor D is big. By (i) so is P_{gf}. Fix $\varepsilon > 0$. By the Kodaira lemma $P_{\mathrm{gf}} = A+F$, where A is ample and F is an effective divisor. It is clear that $P_{\mathrm{gf}} - \varepsilon F = (1-\varepsilon)P_{\mathrm{gf}} + \varepsilon A$ is a b-semiample divisor and its class is contained in the open cone $\mathrm{Mv}^o(X)$. Similar to the proof of Proposition 5.16 we have

$$P_{\mathrm{s}}(D) \geq P_{\mathrm{s}}(P_{\mathrm{gf}} - \varepsilon F) = P_{\mathrm{gf}} - \varepsilon F.$$

Passing to the limit, we obtain $P_{\mathrm{gf}} \leq P_{\mathrm{s}}(D)$.

Finally, let $L \leq D$ be a b-nef divisor. We prove that $L \leq P_{\mathrm{s}}(D)$. By the Kodaira lemma $D = A + F$, where the divisor A is ample and F is effective. Take a sufficiently small rational $\varepsilon > 0$. Then $(1+\varepsilon)D \geq L + \varepsilon A$. As above

$$(1+\varepsilon)P_{\mathrm{s}}(D) = P_{\mathrm{s}}((1+\varepsilon)D) \geq P_{\mathrm{s}}(L+\varepsilon A) = L + \varepsilon A.$$

Hence, $L \leq P_{\mathrm{s}}(D)$. This proves the proposition. □

6.13. Zariski decomposition for varieties of intermediate Kodaira dimension.

Proposition 6.14 ([9, Prop. 1.24, 1.10])**.** *Let $f\colon X \to Z$ be a contraction of nonsingular varieties, let D be a divisor on Z, and let R be a very exceptional divisor on X. Then the pull-back of $f^*D + R$ under some birational contraction $g\colon X' \to X$ has a Zariski decomposition in Fujita sense if and only if this decomposition exists for the pull-back of D under some birational contraction $h\colon Z' \to Z$. Furthermore, the divisor $P_{\mathrm{f}}(g^*(f^*D+R))$ is the pull-back of $P_{\mathrm{f}}(h^*D)$ for a suitable choice of g and h: if the diagram*

$$\begin{tikzcd} X & X' \arrow[l, "g"'] \arrow[d, "f'"] \\ Z \arrow[from=u, "f"'] & Z' \arrow[l, "h"] \end{tikzcd}$$

is commutative, then $P_{\mathrm{f}}(g^*(f^*D+R)) = f'^*P_{\mathrm{f}}(h^*D)$.

Using this statement and a formula for canonical divisor of elliptic fibrations Fujita proved the following theorem.

Theorem 6.15 ([9]). *Let $f\colon X \to Z$ be a contraction of a three-dimensional variety with general fiber being an elliptic curve. Assume that $\kappa(X, K_X) \geq 0$. Then there exists a birational contraction $g\colon X' \to X$ such that g^*K_X has a Zariski decomposition in Fujita sense $P_{\mathrm{f}} + N_{\mathrm{f}}$ with semiample positive part P_{f} (and it coincides with the Shokurov decomposition). In particular, the canonical algebra $\mathcal{R}K_X$ is finitely generated.*

Later the last fact was generalized in [6]:

Theorem 6.16 ([6]). *Let (X, B) be a Kawamata log terminal pair with $\kappa(X, K_X + B) = l \geq 0$, where B is a $\mathbb{Q}$-boundary. Then there exists an l-dimensional Kawamata log terminal pair (Z, Δ) with $\kappa(Z, K_Z + \Delta) = l$ such that log canonical algebras $\mathcal{R}(K_X + B)$ and $\mathcal{R}(K_Z + \Delta)$ are quasi-isomorphic* [PLF, Def. 4.3]. *In particular, questions about finite generation of these algebras are equivalent* [PLF, Th. 4.6].

Corollary 6.17. *Let (X, B) be a Kawamata log terminal pair with $0 \leq \kappa(X, K_X + B) \leq 3$. Then the log canonical algebra $\mathcal{R}(K_X + B)$ is finitely generated.*

6.18. An example of a divisor with non-rational Zariski decomposition. Following [4], we present an example of a divisor with a Zariski decomposition in Fujita sense having irrational coefficients.

Let E be an elliptic curve with $\mathrm{End}(E) \simeq \mathbb{Z}$ and let $S = E \times E$. Pick a point $P \in E$ and put $E_1 = \{P\} \times E$ and $E_2 = E \times \{P\}$. Then the diagonal Δ and the curves E_1 and E_2 form the basis of the space $\mathrm{NS}_{\mathbb{R}}(S)$. Put $\Delta' = \Delta - E_1 - E_2$. Then Δ', E_1, E_2 is also a basis. In this basis the quadratic form x^2 has the following form

$$(\alpha_1 E_1 + \alpha_2 E_2 + \beta \Delta')^2 = 2\alpha_1\alpha_2 - 2\beta^2.$$

Therefore the cone of ample divisors in $\mathrm{NS}_{\mathbb{R}}(S)$ is defined by conditions

$$\alpha_1\alpha_2 - \beta^2 > 0, \qquad \alpha_1, \alpha_2 > 0. \tag{6.19}$$

Consider the $\mathbb{P}^1$-bundle $\pi\colon X = \mathbb{P}(\mathcal{O}_S(\Delta') \oplus \mathcal{O}_S) \to S$. Identify S with the zero section. Then $\mathcal{O}_S(S) = \mathcal{O}_S(\Delta')$. Put $S_1 = \pi^*E_1$ and $S_2 = \pi^*E_2$. Let H be an integral ample divisor on X and let $L \overset{\mathrm{def}}{=} H|_S$.

Consider the divisor $G(\alpha_1, \alpha_2) \overset{\mathrm{def}}{=} L + \alpha_1 E_1 + \alpha_2 E_2$. For positive α_1, α_2, define the function

$$\gamma(\alpha_1, \alpha_2) \overset{\mathrm{def}}{=} \sup\{\beta \mid G(\alpha_1, \alpha_2) + \beta\Delta' \quad \text{is nef}\}$$

Using (6.19), choose $\alpha_1, \alpha_2 \in \mathbb{Q}$, $\alpha_1, \alpha_2 > 0$ so that $\gamma \overset{\mathrm{def}}{=} \gamma(\alpha_1, \alpha_2)$ is irrational. If $\delta > \gamma$, then $G(\alpha_1, \alpha_2) + \delta\Delta'$ is not nef. Therefore there exists an irreducible curve Γ having negative intersection with

$G(\alpha_1,\alpha_2)+\delta\Delta'$. Since S is an abelian surface, there exists a family of such curves $\{\Gamma_\lambda\}$ on S:

$$(G(\alpha_1,\alpha_2)+\gamma\Delta')\cdot\Gamma_\lambda<0. \tag{6.20}$$

On the other hand, if $0<\delta<\gamma$, then $G(\alpha_1,\alpha_2)+\delta\Delta'$ is ample.

Put $D \stackrel{\text{def}}{=} B+rS$, where $B \stackrel{\text{def}}{=} H+\alpha_1S_1+\alpha_2S_2$ and take $r\in\mathbb{Q}$ so that $r>\gamma$.

Claim 6.21. *Let D be the (effective and big) divisor constructed above. Then $N_{\mathrm{f}}(D)=(r-\gamma)S$ is a Fujita decomposition. In particular, for any birational contraction $f\colon Y\to X$ the pull-back f^*D has no Fujita decompositions with rational coefficients.*

Proof. First we show that the divisor $P_{\mathrm{f}}=D-N_{\mathrm{f}}(D)$ is nef. Indeed, suppose that $(B+\gamma S)\cdot C<0$ for some irreducible curve C. Since B is ample, $S\cdot C<0$. Therefore, $(G(\alpha_1,\alpha_2)+\gamma\Delta')\cdot C<0$, a contradiction with our choice of γ.

Further, let $f\colon Y\to X$ be a birational contraction and let F be an effective divisor on Y such that $f^*D-F=f^*(P_{\mathrm{f}}+N_{\mathrm{f}})-F$ is nef. From (6.20) we have $(G(\alpha_1,\alpha_2)+(\gamma+\varepsilon)\Delta')\cdot\Gamma_\lambda<0$ for any $\varepsilon>0$. Hence, $(P_{\mathrm{f}}+\varepsilon S)\cdot\Gamma_\lambda<0$. Let $\{\Gamma'_\lambda\}$ be a family of irreducible curves on Y dominating $\{\Gamma_\lambda\}$. Then $f^*(P_{\mathrm{f}}+\varepsilon S)\cdot\Gamma'_\lambda<0$. Since B is ample, $f^*S\cdot\Gamma'_\lambda<0$. Therefore the family $\{\Gamma'_\lambda\}$ cover the birational transform $\widetilde{S}$ of S. On the other hand,

$$0<(f^*D-F)\cdot\Gamma'_\lambda=(f^*(P_{\mathrm{f}}+\varepsilon S+(r-\gamma-\varepsilon)S)-F)\cdot\Gamma'_\lambda.$$

Hence,

$$-(r-\gamma-\varepsilon)S\cdot\Gamma'_\lambda<f^*(P_{\mathrm{f}}+\varepsilon S-F)\cdot\Gamma'_\lambda<-F\cdot\Gamma'_\lambda.$$

This means that $f_*F>(r-\gamma-\varepsilon)S$. Since ε is an arbitrary positive, $f_*F\geq(r-\gamma)S$. Finally, the inequality $F\geq f^*(r-\gamma)S$ follows from $f_*F\geq(r-\gamma)S$ and the fact that the divisor $f^*(r-\gamma)S-F$ is nef over X (see Lemma 1.6). □

Nakayama [17] using similar construction, constructed an example of a big (integral) divisor on a nonsingular variety such that its pull-back under any blowup has no Fujita decompositions (as well as CKM decompositions, see below).

7. Zariski decomposition in CKM's sense

Definition 7.1 ([4], [12], [15])**.** A decomposition $D=P_{\mathrm{CKM}}(D)+N_{\mathrm{CKM}}(D)$ is called a *Zariski decomposition in sense Cutkosky-Kawamata-Moriwaki* (or simply *CKM-decomposition*) if the following conditions are satisfied:

(i) $N_{\text{CKM}}(D) \geq 0$;
(ii) the divisor $P_{\text{CKM}}(D)$ is nef;
(iii) there is an isomorphism of graded algebras

$$\mathcal{R}_X D \simeq \mathcal{R}_X P_{\text{CKM}}(D).$$

Remark 7.2. Since there is the following embedding of cones

$$\operatorname{Nef}(X) = \overline{\operatorname{Amp}}(X) \subset \overline{\operatorname{Mv}}(X),$$

we have that an CKM-decomposition is also a sectional decomposition. If $D = P_{\text{sec}}(D) + N_{\text{sec}}(D)$ is a sectional decomposition (such as in 4.18), then it is an CKM-decomposition if and only if the divisor $P_{\text{sec}}(D)$ is nef.

Remark 7.3. If D is $\mathbb{R}$-Cartier, then a decomposition $N_{\text{CKM}}(D)$ satisfies condition (2.7).

Indeed, let $f\colon Y \to X$ be a birational contraction. It is sufficient to show the existence of isomorphisms

$$H^0(\mathcal{O}_X(nP_{\text{CKM}})) \simeq H^0(\mathcal{O}_Y(f^*nP_{\text{CKM}})),$$
$$H^0(\mathcal{O}_X(nD)) \simeq H^0(\mathcal{O}_Y(f^*nD)),$$

that follows from the fact that f is a contraction.

Proposition 7.4. (i) *Let D be a big divisor on a $\mathbb{Q}$-factorial variety and let $D = P_{\mathrm{f}}(D) + N_{\mathrm{f}}(D)$ be its Fujita decomposition. Then it is also an CKM-decomposition.*
(ii) *Let D be a big $\mathbb{R}$-Cartier divisor. If a decomposition $D = P_{\text{CKM}}(D) + N_{\text{CKM}}(D)$ exists, then it is a Fujita decomposition.*

Proof. The statement of (i) follows by (6.3) and (ii) follows by Propositions 4.21 and 6.12. □

Remark 7.5. In general case (i.e., if D is not big) the CKM-decomposition is not unique and does not coincide with the Fujita decomposition (and even with the classical Zariski decomposition). For example, the divisor D from Example 4.5 has infinitely many CKM-decompositions: $N_{\text{CKM}}(D) = tD$, $0 \leq t \leq 1$.

Example 7.6. An CKM-decomposition on a surface with $\mathbb{Q}$-factorial singularities satisfies condition (iii) of Theorem 2.2 and instead of (ii) we have only

(ii)$'$ the matrix $(N_i \cdot N_j)$ is seminegative definite.

Indeed, we may assume that $P_{\text{CKM}}^2 = 0$. If $P_{\text{CKM}} \cdot N_i > 0$, then $(P_{\text{CKM}} + \varepsilon N_i)^2 = \varepsilon(2P_{\text{CKM}} \cdot N_i + \varepsilon N_i^2) > 0$ for some $0 < \varepsilon \ll 1$, i.e., the divisor $P_{\text{CKM}} + \varepsilon N_i$ is big, a contradiction. Thus, $P_{\text{CKM}} \cdot N_i = 0$ for all N_i. Now suppose that $N'^2 > 0$, where $N' \overset{\text{def}}{=} \sum \varepsilon_i N_i$, $|\varepsilon_i| \ll 1$. Then

$(P_{\text{CKM}} + N')^2 = N'^2 > 0$ and the divisor $P_{\text{CKM}} + N'$ is big. Again we have a contradiction.

The following theorem is a consequence of the existence of Iitaka fibration and two-dimensional Zariski decomposition.

Theorem 7.7 ([4]). *Let D be an effective Cartier divisor. Assume that $1 \le \kappa(X, D) \le 2$. Then there exists a birational contraction $f\colon Y \to X$ such that f^*D has an CKM-Zariski decomposition $N_{\text{CKM}}(f^*D)$ satisfying the following condition*

$$\kappa(X, D) = \nu(Y, P_{\text{CKM}}(f^*D)). \tag{7.8}$$

Furthermore, if D is a $\mathbb{Q}$-divisor, then so is $N_{\text{CKM}}(D)$. An CKM-decomposition satisfying condition (7.8) *is unique.*

It is expected that Zariski decomposition exists for the log canonical divisor. In this instance Zariski decomposition must have good properties:

Theorem 7.9 ([1], [15], [16], [12]). *Let (X, Δ) be a projective Kawamata log terminal pair such that Δ is a $\mathbb{Q}$-boundary and the divisor $K_X + \Delta$ is big. Assume that there exists a Zariski decomposition in CKM sense for $D = K_X + \Delta$. Then the positive part $P_{\text{CKM}}(D)$ is a semiample $\mathbb{Q}$-divisor. In particular, the algebra $\mathcal{R}_X(K_X+\Delta)$ is finitely generated.*

An CKM-decomposition in this case coincides with Fujita and Shokurov decompositions.

Theorem 7.9 was proved by Moriwaki [15]-[16] and Kawamata [12] in more general, relative situation. Thus the existence of a (relative) Zariski decomposition in CKM sense for the log canonical divisor of small contractions $X \to Z$ is a sufficient condition for the existence of log flips.

8. Decompositions of b-divisors

8.1. Let $\mathcal{D}$ be a b-divisor. Similar to (1.8) we put

$$\operatorname{Mov}(\mathcal{D}) \stackrel{\text{def}}{=} \begin{cases} -\inf\limits_{s \in K(X)^*} \{(s) \mid \mathcal{D} + (s) \ge 0\} & \text{if } H^0(\mathcal{O}(\mathcal{D})) \ne 0, \\ -\infty & \text{otherwise.} \end{cases}$$

In the case $\operatorname{Mov}(\mathcal{D}) \ne -\infty$, we may take a section $s_0 \in K(X)^*$ so that $\mathcal{D} + (s_0) \ge 0$. Then $-(s_0) \le \operatorname{Mov}(\mathcal{D}) \le D$. Therefore $\operatorname{Mov}(\mathcal{D})$ is a b-divisor. If $H^0(\mathcal{O}(\mathcal{D})) \ne 0$, then we also put

$$\operatorname{Fix}(\mathcal{D}) \stackrel{\text{def}}{=} \inf\{\mathcal{L} \mid \mathcal{L} \sim \mathcal{D}, \quad \mathcal{L} \ge 0\} = \mathcal{D} - \operatorname{Mov}(\mathcal{D}).$$

B-divisors $\mathrm{Mov}(\mathcal{D})$ and $\mathrm{Fix}(\mathcal{D})$ defined above are called *mobile* and *fixed* parts of $\mathcal{D}$ respectively. If a b-divisor $\mathcal{D}$ is effective, then so is $\mathrm{Mov}(\mathcal{D})$: $\mathrm{Mov}(\mathcal{D}) \geq -(\mathrm{Const}) = 0$. It is easy to see from the definition that

$$(\mathrm{Mov}(\mathcal{D}))_X \leq \mathrm{Mov}(\mathcal{D}_X), \qquad (\mathrm{Fix}(\mathcal{D}))_X \geq \mathrm{Fix}(\mathcal{D}_X). \tag{8.2}$$

Lemma 8.3. *Let $\mathcal{D}$ be a b-divisor such that $H^0(\mathcal{O}(\mathcal{D})) \neq 0$. Then $\mathcal{M} = \mathrm{Mov}(\mathcal{D})$ satisfies the following properties:*

(i) *$\mathcal{M} \leq \mathcal{D}$;*
(ii) *the b-divisor $\mathcal{M}$ is b-free (in particular it is b-Cartier);*
(iii) *if a b-divisor $\mathcal{L} \leq \mathcal{D}$ is b-free, then $\mathcal{L} \leq \mathcal{M}$.*

Conversely, if an (integral) b-divisor $\mathcal{M}$ satisfies conditions (i) - (iii), *then $\mathcal{M} = \mathrm{Mov}(\mathcal{D})$.*

Proof. Prove (ii). Similar to Lemma 1.9 we have

$$\mathrm{Fix}(\mathcal{M}) = \inf\{\mathcal{L} \mid \mathcal{L} \sim \mathcal{M}, \quad \mathcal{L} \geq 0\} = 0.$$

Hence, $\mathrm{Fix}(\mathcal{M}_X) = 0$, i.e., the linear system $|\mathcal{M}_X|$ has no fixed components. Let $f\colon Y \to X$ be a resolution of base points of the linear system $|\mathcal{M}_X|$ and let $|M|$ be the birational transform of $|\mathcal{M}_X|$ on Y. Using [PLF, Prop. 3.20] we obtain $|\mathcal{M}_X| \simeq |\mathcal{M}_Y| \simeq |M|$. Since $|\mathcal{M}_Y|$ has no fixed components, $|\mathcal{M}_Y| = |M|$ and this linear system has no base points. The rest of the proof is completely similar to Lemma 1.9 □

Similar to Lemma 1.10 one can prove the following

Lemma 8.4. *Let $\mathcal{D}$ be a b-divisor such that $H^0(\mathcal{O}(\mathcal{D})) \neq 0$. Then $\mathcal{M} = \mathrm{Mov}(\mathcal{D})$ satisfies the following properties:*

(i) *$\mathcal{M} \leq \mathcal{D}$;*
(ii) *$H^0(\mathcal{O}(\mathcal{M})) = H^0(\mathcal{O}(\mathcal{D}))$;*
(iii) *if for a b-divisor $\mathcal{L} \leq \mathcal{D}$ the equality $H^0(\mathcal{O}(\mathcal{L})) = H^0(\mathcal{O}(\mathcal{D}))$ holds, then $\mathcal{L} \geq \mathcal{M}$.*

Conversely, if an (integral) b-divisor $\mathcal{M}$ satisfies conditions (i) - (iii), *then $\mathcal{M} = \mathrm{Mov}(\mathcal{D})$.*

Here (as well as everywhere) we assume that sections of the sheaf $\mathcal{O}(\mathcal{D})$ are elements of the field $K(X)$:

$$H^0(\mathcal{O}(\mathcal{D})) = \{s \in K(X) \mid \mathcal{D} + (s) \geq 0\}.$$

Therefore the statement remains to be true for infinitely dimensional spaces $H^0(\mathcal{O}(\cdot))$.

Definition 8.5. A b-divisor $\mathcal{G}$ is said to be *pbs-ample* if there exists a sequence b-semiample b-divisors $\mathcal{G}_i$ such that $\lim_{i\to\infty} \mathcal{G}_i = \mathcal{G}$.

Definition 8.6. A decomposition of a b-divisor $\mathcal{D} = \mathcal{D}^{\mathrm{m}} + \mathcal{D}^{\mathrm{e}}$ is called a *pbs-decomposition* (pseudo b-semiample decomposition) if it satisfies the following properties:

(i) $\mathcal{D}^{\mathrm{e}} \geq 0$;
(ii) $\mathcal{D}^{\mathrm{m}}$ is pbs-ample;
(iii) for any pbs-ample b-divisor $\mathcal{L}$ such that $\mathcal{D}^{\mathrm{m}} \leq \mathcal{L} \leq \mathcal{D}$ we have $\mathcal{L} = \mathcal{D}^{\mathrm{m}}$.

Here $\mathcal{D}^{\mathrm{m}}$ is the *maximal pbs-ample* part and $\mathcal{D}^{\mathrm{e}}$ is the *fixed part.*

Pbs-decompositions generalize divisorial Shokurov decompositions 5.11. Indeed, let $D = D^{\mathrm{m}} + D^{\mathrm{e}}$ and $\mathcal{D}^{\mathrm{m}}$ be a decomposition and the corresponding b-divisor from 5.11 and let $\mathcal{D}^{\mathrm{e}} = D^{\mathrm{e}}$ (the last is considered as an equality of b-divisors). Put also $\mathcal{D} = \mathcal{D}^{\mathrm{m}} + \mathcal{D}^{\mathrm{e}}$. Then by definition $\mathcal{D} = \mathcal{D}^{\mathrm{m}} + \mathcal{D}^{\mathrm{e}}$ is a pbs-decomposition and $D = \mathcal{D}_X$.

8.7. Similar to the divisorial case we say that a b-divisor $\mathcal{D}$ is *effective modulo $\mathbb{Q}$-linear equivalence* if $\mathcal{D} \sim_{\mathbb{Q}} \mathcal{D}'$, where $\mathcal{D}' \geq 0$. Equivalent: $H^0(\mathcal{O}(n_0\mathcal{D})) \neq 0$ for some $n_0 \in \mathbb{N}$. Let $\mathcal{D}$ be an effective modulo $\mathbb{Q}$-linear equivalence b-divisor. Denote

$$\mathcal{M}_n \stackrel{\text{def}}{=} \operatorname{Mov}(n\mathcal{D}).$$

Since $\mathcal{M}_n/n \leq \mathcal{D}$, there exists the limit

$$\mathcal{P}_{\mathrm{s}}(\mathcal{D}) \stackrel{\text{def}}{=} \limsup_{n\to\infty}(\mathcal{M}_n/n).$$

Since $\mathcal{M}_{n_1n_2} \geq n_2\mathcal{M}_{n_1}$ for all $n_1, n_2 \in \mathbb{N}$, we have

$$n\mathcal{P}_{\mathrm{s}}(\mathcal{D}) \geq \mathcal{M}_n \quad \text{for all } n \in \mathbb{N}.$$

It is clear that $-(s_0) \leq \mathcal{P}_{\mathrm{s}}(\mathcal{D}) \leq \mathcal{D}$, where $0 \neq s_0 \in H^0(\mathcal{O}(n_0\mathcal{D}))$. Hence, $\mathcal{P}_{\mathrm{s}}(\mathcal{D})$ is also a b-divisor. Thus we obtain a decomposition

$$\mathcal{D} = \mathcal{P}_{\mathrm{s}}(\mathcal{D}) + \mathcal{N}_{\mathrm{s}}(\mathcal{D}), \qquad \mathcal{N}_{\mathrm{s}}(\mathcal{D}) \stackrel{\text{def}}{=} \mathcal{D} - \mathcal{P}_{\mathrm{s}}(\mathcal{D}) \geq 0,$$

which we call an *s-decomposition* of a b-divisor. Similar to (4.2) we have

$$\mathcal{N}_{\mathrm{s}}(\mathcal{D}) = \inf\{\mathcal{L} \mid \mathcal{L} \sim_{\mathbb{Q}} \mathcal{D}, sL \geq 0\}. \tag{8.8}$$

By construction $\mathcal{P}_{\mathrm{s}}(\mathcal{D})$ is a pbs-ample b-divisor.

It is easy to see that properties 4.6 hold also for the b-divisorial s-decomposition (if one replaces divisors with b-divisors).

The concept of s-decomposition of b-divisors allows us to formulate a criterion of finite generation of a *b-divisorial algebra* $\mathcal{R}_X\mathcal{D} \stackrel{\text{def}}{=} \oplus_{n\geq 0}H^0(X, \mathcal{O}_X(n\mathcal{D}))$ (cf. Theorem 4.10).

Theorem 8.9 (Limiting Criterion [PLF, Th. 4.28]). *Let $\mathcal{D}$ be an effective modulo $\mathbb{Q}$-linear equivalence b-divisor. Then the b-divisorial algebra $\mathcal{R}_X\mathcal{D}$ is finitely generated if and only if $\mathcal{P}_{\mathrm{s}}(\mathcal{D}) = \mathrm{Mov}(n_0\mathcal{D})/n_0$ for some $n_0 \in \mathbb{N}$.*

8.10. Example 2.4 shows that the divisorial s-decomposition $D = P_{\mathrm{s}}(D) + N_{\mathrm{s}}(D)$ does not agree via pull-backs f_*. Therefore the system $P_{\mathrm{s}}(\mathcal{D}_X)$ does not form a b-divisor. In particular, $\mathcal{P}_{\mathrm{s}}(\mathcal{D})_X \neq P_{\mathrm{s}}(\mathcal{D}_X)$. However, from (8.2) we always have

$$\mathcal{P}_{\mathrm{s}}(\mathcal{D})_X \leq P_{\mathrm{s}}(\mathcal{D}_X), \qquad \mathcal{N}_{\mathrm{s}}(\mathcal{D})_X \geq N_{\mathrm{s}}(\mathcal{D}_X).$$

Where equalities are achieved "birationally asymptotically":

Lemma 8.11.

$$\mathcal{P}_{\mathrm{s}}(\mathcal{D})_X = \inf_{f\colon Y\to X} f_*P_{\mathrm{s}}(\mathcal{D}_Y), \qquad \mathcal{N}_{\mathrm{s}}(\mathcal{D})_X = \sup_{f\colon Y\to X} f_*N_{\mathrm{s}}(\mathcal{D}_Y),$$

where the infimum and supremum are taken over all birational contractions $f\colon Y \to X$.

Proof. Consider the following sets of divisors on X

$$\mathfrak{S} \stackrel{\mathrm{def}}{=} \{\mathcal{L}_X \mid \mathcal{L} \sim_{\mathbb{Q}} \mathcal{D}, \quad \mathcal{L} \geq 0\},$$
$$\mathfrak{S}_Y \stackrel{\mathrm{def}}{=} \{\mathcal{L}_X \mid \mathcal{L} \sim_{\mathbb{Q}} \mathcal{D}, \quad \mathcal{L}_Y \geq 0\},$$

where Y is a birational model dominating X. Then $\mathfrak{S}_{Y'} \subset \mathfrak{S}_Y$ if Y' dominates Y and $\mathfrak{S} = \bigcap\limits_{Y/X} \mathfrak{S}_Y$. According to (4.2) and (8.8) we have $\mathcal{N}_{\mathrm{s}}(\mathcal{D})_X = \inf \mathfrak{S}$ and $f_*N_{\mathrm{s}}(\mathcal{D}_Y) = \inf \mathfrak{S}_Y$. Hence,

$$\mathcal{N}_{\mathrm{s}}(\mathcal{D})_X = \inf \mathfrak{S} = \inf \bigcap_{Y/X} \mathfrak{S}_Y = \sup_{Y/X} \inf \mathfrak{S}_Y = \sup_{f\colon Y\to X} f_*N_{\mathrm{s}}(\mathcal{D}_Y).$$

This proves the statement. □

It follows from definitions that $\mathcal{P}_{\mathrm{s}}(\mathcal{D}) \leq \mathcal{D}^{\mathrm{m}}$ for any effective modulo $\mathbb{Q}$-linear equivalence b-divisor $\mathcal{D}$ (such that a decomposition $\mathcal{D} = \mathcal{D}^{\mathrm{m}} + \mathcal{D}^{\mathrm{e}}$ exists).

Completely similar to 5.14 one can prove the following.

Proposition 8.12. *Suppose that $\mathcal{P}_s(\mathcal{D})$ is a rational b-semiample b-divisor. Then for any b-semiample b-divisor $\mathcal{L} \leq \mathcal{D}$ the inequality $\mathcal{L} \leq \mathcal{P}_s(\mathcal{D})$ holds.*

According to Limiting Criterion 8.9 the conditions of proposition are satisfied, for example, in case when the b-divisorial algebra $\mathcal{R}_X\mathcal{D}$ is finitely generated.

In conclusion we mention an interesting result of Fujita:

Theorem 8.13 ([10]). *Let D be a big Cartier divisor on a d-dimensional variety X and let*

$$v(D) \stackrel{\text{def}}{=} \limsup_{t\to\infty} \left(\frac{d!}{t^n} \dim H^0(X, \mathcal{O}(tD))\right)$$

be its volume. *Then for any $\varepsilon > 0$, there exists a birational contraction $f\colon Y \to X$ and a decomposition*

$$f^*D = P_\varepsilon + N_\varepsilon$$

in a sum of $\mathbb{Q}$-Cartier divisors, where

- $N_\varepsilon \geq 0$*;*
- *the divisor P_ε is semiample;*
- $v(D) - \varepsilon < v(P_\varepsilon) = (P_\varepsilon)^d < v(D)$*.*

9. Analytic Zariski decomposition

In conclusion we mention about complex analytic approach to the constructing of Zariski decompositions [21], [22] (see also [3]). Main definitions and facts on complex currents can be found in [11] or [5].

Let T be a closed positive $(1,1)$-current on the open unit ball in $\mathbb{C}^n$ with center at 0. The *Lelong number* $\Theta(T,0)$ of T at 0 is the number

$$\Theta(T,0) \stackrel{\text{def}}{=} \lim_{r\to 0} \frac{T(\chi(r)\omega^{n-1})}{\pi^{n-1} r^{2(n-1)}}$$

where $\omega = \frac{\sqrt{-1}}{2}\sum dz_i \wedge d\bar{z}_i$ and $\chi(r)$ is the characteristic function of the open ball in $\mathbb{C}^n$ with center at 0 and radius r. This number is invariant under the changes of coordinates. Thus it is possible to define the Lelong number $\Theta(T,x)$ for a closed positive $(1,1)$-current at a point $x \in X$ on any complex variety X. If T is given by integration over an analytic subvariety of codimension 1, then $\Theta(T,x)$ is the usual multiplicity of this subvariety at x.

Further we suppose that X is a nonsingular projective complex variety. For any analytic subset $V \subset X$, we can define

$$\Theta(T,V) \stackrel{\text{def}}{=} \inf\{\Theta(T,x) \mid x \in V\}.$$

If V irreducible, then $\Theta(T,V)$ coincides with the Lelong number $\Theta(T,x)$ at a *very* general point $x \in V$. The last enable us to define *Siu decomposition* (see [5, Ch. III, (8.16)]):

$$T = T' + \sum_V \Theta(T,V)[V], \tag{9.1}$$

where the (infinite in general) sum is taken over all prime divisors V and $[V]$ is current given by integration over V. Here T' is also a closed positive $(1,1)$-current such that the set $\{x \in X \mid \Theta(T,x) \geq \varepsilon\}$ has codimension ≥ 2 for any $\varepsilon > 0$.

Let $f \colon Y \to X$ be a surjective morphism and let T be a closed positive $(1,1)$-current on X. Locally we may write $T = \sqrt{-1}\partial\bar{\partial}\varphi$, where φ is a plurisubharmonic function [5, Ch. III, Prop. 1.19]. Therefore we can define the pull-back of a current T by the formula $f^*T = \sqrt{-1}\partial\bar{\partial}f^*\varphi$ (this definition does not depend on the choice of the function φ).

It is possible also to define a "b-divisorial" version of the Siu decomposition: for a closed positive $(1,1)$-current T we put

$$\mathcal{S}(T) = \sum_V \Theta(T,V)V, \tag{9.2}$$

where the sum is taken over all divisorial discrete valuations V of the field $K(X)$ and $\Theta(T,V)$ is computed on a (nonsingular projective) birational model $f \colon Y \to X$, on which the center of V is a prime divisor: $\Theta(T,V) = \Theta(f^*T,V)$. Strictly speaking $\mathcal{S}(T)$ is not a b-divisor, since $\mathcal{S}(T)_X$ can be an infinite sum. Nevertheless, $\mathcal{S}(T)$ satisfies the condition $\mathcal{S}(T)_{Y_1} = g_*\mathcal{S}(T)_{Y_2}$ whenever there is a birational morphism $g \colon Y_1 \to Y_2$.

Theorem 9.3 ([21]). *Let D be a big divisor on X. Then there exists a (closed, positive) $(1,1)$-current T such that*

(i) *T represents the cohomology class $c_1(D)$ of D in $H^2(X,\mathbb{R})$;*
(ii) *for any birational contraction $f \colon Y \to X$, any nonnegative integer m, and any point $y \in Y$ the following approximation formulas for $\Theta(f^*T,y)$ hold:*

$$\min_{L \in |f^*mD|} (\operatorname{mult}_y L) \geq m\Theta(f^*T,y) \tag{9.4}$$

$$\liminf_{m\to\infty} \left(\frac{1}{m}\min_{L \in |f^*mD|} (\operatorname{mult}_y L)\right) = \Theta(f^*T,y). \tag{9.5}$$

Such a current T is called an *analytic Zariski decomposition* (*AZD*) of D.

Equality (9.5), in particular, asserts that the Siu decomposition of a current T is a divisor and coincides with the s-decomposition $N_{\mathrm{s}}(D)$. Furthermore, the formal sum $\mathcal{S}(T)$ is a b-divisor and $\mathcal{S}(T) = \mathcal{N}_{\mathrm{s}}(\overline{D})$.

Corollary 9.6. *Let D be a divisor on X.*

(i) *If D is nef and big, then $c_1(D)$ can be represented by a closed positive $(1,1)$-current T on X such that $\Theta(T) \equiv 0$.*
(ii) *Conversely, if $c_1(D)$ can be represented by a closed positive $(1,1)$-current T on X such that $\Theta(T) \equiv 0$, then D is nef.*

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