Bubbling out of Einstein Manifolds

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—— in memory of late Dr. Osamu Tezuka

In [1], [8], and [4] the following compactness theorem of the space of Einstein metrics is obtained in the spirit of Gromov theory.

Theorem A. *Let (X_i, g_i) be a sequence of n-dimensional ($n \geq 4$) smooth manifolds and Einstein metrics on them with uniformly bounded Einstein constants $\{e_i\}$ satisfying*

$$\mathrm{diam}(X_i, g_i) \leq D, \ \mathrm{vol}(X_i, g_i) \geq V \text{ and } \int_{X_i} |R_{g_i}|^{n/2}\, dV_i \leq R$$

for some positive constants D, V and R, where we denote curvature tensor of a metric g by R_g . Then there exist a subsequence $\{j\} \subset \{i\}$ and a compact Einstein orbifold (X_∞, g_∞) with a finite singular set $S = \{x_1, x_2, \dots, x_s\} \subset X_\infty$ (possibly empty) for which the following statement holds:

1) *(X_j, g_j) converges to (X_∞, g_∞) in the Hausdorf distance.*
2) *There exists an into diffeomorphism $F_j : X_\infty \backslash S \longrightarrow X_j$ for each j such that $F_j^* g_j$ converges to g_∞ in the C^∞-topology on $X_\infty \backslash S$.*
3) *For every $x_a \in S$ ($a = 1, 2, \dots, s$) and j, there exist $x_{a,j} \in X_j$ and a positve number r_j such that*
   - 3.a) *$B(x_{a,j}; \delta)$ converges to $B(x_a; \delta)$ in the Hausdorff distance for all $\delta > 0$.*
   - 3.b) *$\lim_{j \to \infty} r_j = \infty$.*
   - 3.c) *$((X_j, r_j g_j), x_{a,j})$ converges to $((M_a, h_a), x_{a,\infty})$ in the pointed Hausdorff distance, where (M_a, h_a) is a complete, non-compact, Ricci-flat, non-flat n-manifold which is ALE of order $n-1$ in general, of order n if (M_a, h_a) is Kähler or $n = 4$.*
   - 3.d) *There exists an into diffeomorphism $G_j : M_a \longrightarrow X_j$ such that $G_j^*(r_j g_j)$ converges to h_a in the C^∞-topology on M_a.*
4) *It holds*

$$\lim_{j \to \infty} \int_{X_j} |R_{g_j}|^{n/2}\, dV_j \geq \int_{X_\infty} |R_{g_\infty}|^{n/2}\, dV_\infty + \sum_a \int_{M_a} |R_{h_a}|^{n/2}\, dV_{h_a}.$$

Moreover if (X_i, g_i) are Kähler, then (X_∞, g_∞) and (M_a, h_a) are also Kähler.

Here we call a smooth n-dimensional complete Riemannian orbifold (X, g) asymtotically locally Euclidean (ALE) of order $\tau > 0$, if there exists a compact subset $K \subset X$ such that $X \backslash K$ has coordinates at infimity; namely there are $R > 0$, $0 < \alpha < 1$, a finite subgroup $\Gamma \subset O(n)$ acting freely on $\mathbf{R}^n \backslash B(0; R)$, and a C^∞-diffeomorphism $\mathcal{Z} : X \backslash K \longrightarrow (\mathbf{R}^n \backslash B(0; R))/\Gamma$ such that $\varphi = \mathcal{Z}^{-1} \circ \mathrm{proj}$ satisfies (where proj is the natural projection of $\mathbf{R}^n$ to $\mathbf{R}^n/\Gamma$)

$$
\begin{gathered}
(\varphi^* g)_{ij}(z) = \delta_{ij} + O(|z|^{-\tau}), \qquad \partial_k(\varphi^* g)_{ij}(z) = O(|z|^{-\tau-1}), \\
\frac{|\partial_k(\varphi^* g)_{ij}(z) - \partial_k(\varphi^* g)_{ij}(w)|}{|z-w|^\alpha} = O(\min\{|z|, |w|\}^{-\tau-1-\alpha}) \\
\text{for } z, w \in \mathbf{R}^n \backslash B(0; R).
\end{gathered}
$$

(For simplicity we assumed that (X, g) has only one end. So is our case.)

Kronheimer classified all ALE hyper-Kähler surfaces of order 4 in his thesis [6], he calls such manifolds ALE gravitational instantons. In particular he proved the following;

Theorem B. *An ALE gravitational instanton is diffeomorpfic to a minimal resolution of $\mathbf{C}^2/\Gamma$, where Γ is a finite subgroup of $SU(2)$.*

We remark that a simply connected Ricci-flat Kähler surface is hyper-Kähler. Thus in Einstein-Kähler surfaces case we have rather good understanding on the nature of degeneration. Only missing point is the knowledge of the neck $B(x_{a,j}; \delta) \backslash B(x_{a,j}; r_j)$, i.e. how an instanton is glued to a singular point on X_∞. The purpos of this paper is to clarify it, namely we get the following theorem stated in terms of the above notations.

Theorem. *Assume that the sequence (X_i, g_i) consists of Einstein-Kähler surfaces. If we fix a sufficiently small constant $\delta > 0$, then for sufficiently large j, the geodesic ball $B(x_{a,j}; \delta)$ in X_j is diffeomorphic to a cyclic quotient of ALE gravitational instanton.*

Remark. In 4-dimensional case, for a compact Einstein manifold X the curvature integral $\int_X |R|^2 = \mathrm{const}\ \chi(X)$ is a topological invariant.

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1. Preparation from Analysis

Let M be a complete n-dimensinal ($n \geq 3$) Riemannian manifold with a fixed point $o \in M$. For $0 < r_1 < r_2$ we denote $B(o; r_2) \backslash B(o; r_1)$ by $D(r_1, r_2)$. We assume that there is a domain $D = D(r_0, r_\infty)$ in M with $0 \leq r_0 < r_\infty$ which satisfies the following conformally invariant conditions:

$$\left\{ \int_D v^{2\gamma} \right\}^{1/\gamma} \leq S \int_D |\nabla v|^2 \qquad \text{for all } v \in C_c^1(D),$$
$$\mathrm{vol}(D(r_1, r_2)) \leq V r_2^n \qquad \text{for all } \ r_0 \leq r_1 \leq r_2 \leq r_\infty$$

with some positive constants S, V and $\gamma = n/(n-2)$. Let u be a non-negative function defined on D which satisfies

$$\Delta u \geq -fu \qquad \text{on } \ D$$

with a non-negative function f. Then we have following lemmas. Proofs are essentially same as those of corresponding lemmas in [4; §4], so we omit them.

Lemma 1. *Suppose $f \in L^{n/2}$, and $u \in L^p$ for some $p \in [p_0, p_1]$ where $p_0 > 1$. Then $u \in L^q$ for all $q \geq p$, and there exists $\epsilon_1 = \epsilon_1(S, V, p_0, p_1) > 0$ such that if*

$$\int_{D(r,8r)} f^{n/2} \leq \epsilon_1 \qquad \text{with } \ r_0 \leq r < 8r \leq r_\infty,$$

then we have

$$\left\{ \int_{D(2r,4r)} u^{p\gamma} \right\}^{1/\gamma} \leq C_1 r^{-2} \int_{D(r,8r)} u^p,$$

where $C_1 = C_1(S, V, p_0)$. Moreover if $r_0 = 0$ and

$$\int_{B(o;2r)} f^{n/2} \leq \epsilon_1 \qquad \text{with } \ 2r \leq r_\infty,$$

then it holds that

$$\left\{ \int_{B(o;r)} u^{p\gamma} \right\}^{1/\gamma} \leq C_1 r^{-2} \int_{B(o;2r)} u^p.$$

Lemma 2. *Suppose $f \in L^{n/2}$, and $u \in L^p$ for some $p \in [p_0, p_1]$ where $p_0 > \gamma$. Then there exists $\epsilon_2 = \epsilon_2(S, V, p_0, p_1) > 0$ such that if*

$$\left\{\int_D f^{n/2}\right\} \leq \epsilon_2,$$

then it holds that for $r_0 \leq r_1 < 2r_1 < r_2 < 2r_2 \leq r_\infty$

$$\int_{D(2r_1,r_2)} u^p \leq C_2 \int_{D(r_1,2r_1)\cup D(r_2,2r_2)} u^p,$$

$$\int_{D(r_1,r_2)} u^p \leq C_2 \max\left\{\left(\frac{r_0}{r_1}\right)^{\epsilon_3}, \left(\frac{r_2}{r_\infty}\right)^{\epsilon_3}\right\} \int_D u^p,$$

where $C_2 = C_2(S, V, p_0)$, $\epsilon_3 = \epsilon_3(S, V, p_0) > 0$.

Lemma 3. *If $f \in L^q$ for some $q > n/2$, $u \in L^p$ for some $p > 1$, and it holds that for any r such that $r_0 \leq r < 8r \leq r_\infty$*

$$\int_{D(r,8r)} f^q \leq Ar^{-(2q-n)}$$

with some constant A, then we have

$$\sup_{D(2r,4r)} u^p \leq C_3 r^{-n} \int_{D(r,8r)} u^p,$$

where $C_3 = C_3(A, S, V, p, q)$. Moreover if $r_0 = 0$ and

$$\int_{B(o;2r)} f^q \leq Ar^{-(2q-n)},$$

Then it holds that

$$\sup_{B(o;r)} u^p \leq C_3 r^{-n} \int_{B(o;2r)} u^p.$$

Let (M, g) be an n-dimensinal Einstein manifold, then applying the Weitzenböck formula we get

$$\Delta|R| \geq -C_4|R|^2.$$

Moreover we have the following inequality using Yau's trick. For the proof see [2], [4], [9].

Lemma 4. *There exist positive constants $\delta = \delta(n)$ and $C_5 = C_5(n)$ such that*

$$\Delta |R|^{1-\delta} \geq -C_5 |R|^{2-\delta}.$$

If $n = 4$ or (M, g) is Kähler we can take $\delta = 4/(n+2)$.

One can show the following lemma via L^2-Hodge theory.

Lemma 5. *Let (X, g) be an n-dimensional ($n \geq 4$), complete, non-compact, Ricci-flat, ALE orbifold. Then its first cohomology group $H^1(X; \mathbf{R})$ vanishes.*

Here we recall the existence theorem of Ricci-flat Kähler metrics on open Kähler orbifolds in [3], which is stated in the case of manifolds but its proof equally works for orbifolds.

Definition. A complete n-dimensional Riemannian orbifold (X, g) is called of $C^{k,\alpha}$-asymptotically flat geometry if for each point $p \in X$ with distance from a fixed point o in X, there exists a quasi-coordinate map $\phi : B^n \longrightarrow X$ centered at p from the unit ball B^n in the Euclidian space (i.e. ϕ gives a local uniformization and $\phi(0) = p$), such that with respect to the standard coordinates $x = (x^1, x^2, \ldots, x^n)$ of the Euclidian space it satisfies the following conditions:

(i) If we write $\phi^* g = \sum g_{ij}(x)\, dx^i\, dx^j$, then the matrix $(r^2+1)^{-1}(g_{ij})$ is bounded from below by a constant positive matrix independent of p.

(ii) The $C^{k,\alpha}$-norms of $(r^2+1)^{-1} g_{ij}$, as functions in x, are uniformly bounded.

On such a orbifold we can define the Banach space $C^{k,\alpha}_\delta$ of weighted $C^{k,\alpha}$-bounded functions: The norm of a function $u \in C^{k,\alpha}_\delta$ is given by the supremum of the $C^{k,\alpha}$-norms of $(r^2+1)^{\delta/2} u$ with respect to the coordinates x.

Theorem C. *Let (X, ω) be an n-dimensional ($n \geq 2$) complete open Kähler orbifold of $C^{k,\alpha}$-asymptotically flat geometry with $k \geq 2$, $0 < \alpha < 1$. Assume that the singularities sit in a compact set and there exists a barrier function ρ. If X admits a Ricci-flat volume form V such that $\omega^n = e^f V$ with $f \in C^{k,\alpha}_{\delta+2}$ and $\delta > 0$, then X admits a complete Ricci-flat Kähler metric asymptotically equal to ω.*

Here a barrier function ρ means that outside a compact set ρ satisfies the following conditions:

(i) ρ is compatible to the distance function d from o; there exists a positive constant c_1 such that $c_1 d \leq \rho \leq c_1^{-1} d$.
(ii) The function $\rho^{-\delta}$ belongs to $C_\delta^{k+2,\alpha}$.
(iii) There exists a positive constant c_2 such that

$$\Box \rho^{-\delta} \leq -c_2 \rho^{-2-\delta}.$$

(iv) There exists a positive constant c_3 such that for any positive number K and sufficiently large d

$$(\omega + \sqrt{-1}\partial\bar{\partial}\, K\rho^{-\delta})^n \leq (1 - c_3 K\rho^{-2-\delta})\omega^n,$$
$$(\omega + \sqrt{-1}\partial\bar{\partial}\, - K\rho^{-\delta})^n \geq (1 + c_3 K\rho^{-2-\delta})\omega^n.$$

2. Einstein Manifolds

Let (X_j, g_j) be a sequence of Einstein manifolds which enjoyes the properties stated in Theorem A. Then by [5] we have the Sobolev inequality on (X_j, g_j) with uniform Sobolev constants, and the following proposition holds. For the proof see [1], [8].

Proposition 1. *There exist constants ρ, C_6 and ϵ_4 such that if*

$$\int_{B(x;2r)} |R_{g_j}|^{n/2} \leq \epsilon_4$$

with $2r \leq \rho$, then we have that

$$\sup_{B(x;r)} |R_{g_j}| \leq C_6 r^{-2} \int_{B(x;2r)} |R_{g_j}|^{n/2}.$$

Now we take a positive constant $r_\infty < \rho$ sufficiently small, so that we can assume that for all a

$$\sup_{B(x_{a,j};r_\infty)} |R_{g_j}|^2 = |R_{g_j}|^2(x_{a,j}) \longrightarrow \infty \qquad \text{as } j \longrightarrow \infty$$

and

$$\int_{B(x_a,r_\infty)} |R_{g_\infty}|^{n/2} \leq \frac{\epsilon}{2}$$

with a positve number $\epsilon \leq \epsilon_4/2$ to be determined later. From now on we fix an arbitrary singular point x_a and look at the blowing up process. Since (X_j, g_j) converges to (X_∞, g_∞) in C^∞-topology exept at the singular points, for sufficiently large j we can find a positive number $r_0 = r_{0,j}$ such that

$$\int_{D(r_0,r_\infty)} |R_{g_j}|^{n/2} = \epsilon,$$

where we denote a subset $B(x_{a,j};r_2)\backslash B(x_{a,j};r_1)$ in X_j by $D(r_1, r_2)$. Then we get that

$$r_0 \longrightarrow 0 \qquad \text{as} \qquad j \longrightarrow \infty.$$

Proposition 2. *There is a subsequence $\{k\} \subset \{j\}$ such that the sequence of pointed Einstein manifolds $((X_k, r_0^{-2}g_k), x_{a,k})$ converges to $((Y, h), y_\infty)$*

in the pointed Hausdorrf distance, where (Y, h) is a complete, non-compact, Ricci-flat, non-flat n-orbifold only with finitely many isolated singular points. (Y, h) is ALE of order $n-1$ in general, of order n if $n = 4$ or (Y, h) is Kähler. The convergence is actually in C^∞-topology except at the singular points.

The proof is same as that of Theorem A. We refer to [1], [8] and [4].

Thus we know that for large $1 < K_1 < K_2$ two subsets $D(K_1 r_0, K_2 r_0)$ and $D(K_2^{-1} r_\infty, K_1^{-1} r_\infty)$ in X_k are very close to portions of flat cones $\mathbf{R}^n/\Gamma_0$ and $\mathbf{R}^n/\Gamma_\infty$, respectively. To show that $\Gamma_0 = \Gamma_\infty$ and $D(K_1 r_0, K_2^{-1} r_\infty)$ is also close to a portin of the flat cone, we need the following curvature estimate.

Proposition 3. *Tere exist positive constants C_7 and ϵ_5 such that for $4r_0 \le r < 4r \le r_\infty$ it holds that*

$$r^2 |R_{g_j}| \le C_7 \max\left\{ \left(\frac{r_0}{r}\right)^{\epsilon_5}, \left(\frac{r}{r_\infty}\right)^{\epsilon_5} \right\}.$$

Proof. First apply Lemma 1 to the equation $\Delta |R| \ge -C_4 |R|^2$ on $R = R_{g_j}$, assuming $C_4^{n/2}\epsilon \le \epsilon_3$. Then we get that for $2r_0 \le r < 2r \le r_\infty$

$$\int_{D(r,2r)} |R|^{n/2} \le A r^{-(2q-n)}$$

with a constant A and $q = \gamma n/2$. Next we apply Lemma 2 and Lemma 3 to the equation $\Delta |R|^{1-\delta} \ge -C_5 |R|^{2-\delta}$ with $p = (1-\delta)^{-1} n/2 > \gamma$. If $C_5^{n/2}\epsilon \le \epsilon_2$, we get that for $4r_0 \le r < 4r \le r_\infty$

$$\begin{aligned} r^2 |R_{g_j}| &\le C_3^{2/n} \left\{ \int_{D(r/2,4r)} |R|^{n/2} \right\}^{2/n} \\ &\le C_3^{2/n} (3C_2 2^{\epsilon_3})^{2/n} \max\left\{ \left(\frac{r_0}{r}\right)^{\epsilon_5}, \left(\frac{r}{r_\infty}\right)^{\epsilon_5} \right\} \end{aligned}$$

with $\epsilon_5 = 2\epsilon_3/n$. We choose ϵ by $\epsilon = \min\{\epsilon_3 C_4^{-n/2}, \epsilon_2 C_5^{-n/2}, \epsilon_4/2\}$, then the proof is complete.

Once we get the curvature estimate, we can construct coordinates as in the proof of the existence theorem of coordinats at infinity [4]. We need only minor changes, so we omit the proof of the following proposition.

Proposition 4. *If one take $1 < K_1 < K_2$ sufficiently large, then the subset $D(K_1 r_0, K_2^{-1} r_\infty)$ is close to a portion of a flat cone $\mathbf{R}^n/\Gamma$ for large j.*

Thus if (Y, h) has no singularity, then the ball $B(x_{a,k}; r_\infty)$ is diffeomorphic to the smooth manifold Y which bubbles out of X_k.

If (Y, h) has a singular point y_s, then we choose a sufficiently small number r'_∞ and the corresponding point $x_{s,k}$ in X_k such that

$$\int_{B(y_s, r'_\infty)} |R_h|^{n/2} \leq \frac{\epsilon}{2}$$

$$\sup_{B(x_{s,k}; r_0 r'_\infty)} |R_{g_k}|^2 = |R_{g_k}|^2(x_{s,k}) \longrightarrow \infty.$$

Choose $r'_0 = r'_{0,k}$ such that

$$\int_{D'(r_0 r'_0, r_0 r'_\infty)} |R_{g_k}|^{n/2} = \epsilon,$$

with $D'(r_1, r_2) = B(x_{s,k}; r_2) \backslash B(x_{s,k}; r_1)$, and consider a sequence of pointed Einstein manifolds $((X_k, (r_0 r'_0)^{-2} g_k), x_{s,k})$. Then we have the same situation as before, and we get a complete, non-compact, Ricci-flat, non-flat, ALE n-orbifold (Y', h') only with finitely many isolated singular points. By the same way we can show the neck is diffeomorphic to a flat cone. If (Y', h') again has a singular point, we repeat the argument. And also we apply the same process at every singular point which appears at each repeated step. Since each singular point contributes at least ϵ to the curvature integral $\int |R|^{n/2}$, the process terminates at finite steps. In this way we get a picture of the small ball $B(x_{a,j}; r_\infty)$.

Theorem 1. *The small ball $B(x_{a,j}; r_\infty)$ in X_j corresponding to a singular point x_a of the limit orbifold X_∞ is deffeormorphic to a connected sum of finite number of complete, non-compact, Ricci-flat, non-flat, ALE n-orbifolds only with finitely many isolated singular points, whose singular points are glued to the infinities.*

Remark. We may also use the following gap theorem to show the process terminates at finite steps.

Theorem 2. *Let (X, g) be an n-dimensional ($n \geq 4$), complete, non-compact, Ricci-flat Riemannian orbifold, which satisfies*

$$\left\{ \int v^{2\gamma} \right\}^{1/\gamma} \leq S \int |\nabla v|^2 \qquad \text{for all } v \in C_c^1(X)$$

with a constant $S > 0$. There exists a constant $\epsilon_6 = \epsilon_6(n, S) > 0$ such that the inequality

$$\int_X |R|^{n/2} \leq \epsilon_6$$

implies that (X, g) is the Euclidian space.

Proof. Apply Lemma 1.

3. Einstein Kähler Surfaces

In this section we assume that all manifolds (X_j, g_j) are Einstein-Kähler surfaces. Since the limit space X_∞ is an orbifold, there is a neiborhood U of the singular point x_a which is biholomorphic to a quotient B/Γ of the unit ball $B \subset \mathbf{C}^2$ with a finite subgroup $\Gamma \subset U(2)$ acting freely on $\mathbf{C}^2 \backslash \{0\}$. Let $\det : U(2) \longrightarrow S^1$ be a group homomorphism defined by the determinant. Then the image $\det(\Gamma)$ is a finite cyclic group, say, $\mathbf{Z}_m$. Then U has a branched $\mathbf{Z}_m$-covering: $\tilde{U} \longrightarrow U$ with a branched point x_a such that $\tilde{U}$ has trivial canonical line bundle $K_{\tilde{U}}$. Namely, set $\tilde{\Gamma} = \ker \det \cap \Gamma \subset SU(2)$. Then we have a natural projection $\tilde{U} = B/\tilde{\Gamma} \longrightarrow U$ and a non-vanishing holomorphic 2-form $\omega = dz^1 \wedge dz^2$ descends to $\tilde{U}$, where (z^1, z^2) is the standard coordinates in $\mathbf{C}^2$. We have the corresponding result on $x_{a,j} \in X_j$ for large j.

Proposition 5. *There exists a positive constant δ such that for large j there is a smooth $\mathbf{Z}_m$-covering: $\tilde{U}_j \longrightarrow U_j \supset B(x_{a,j}; \delta)$, where $\tilde{U}_j$ has topologically trivial canonical line bundle $K_{\tilde{U}_j}$.*

Proof. We may assume the domain $U \subset X_\infty$ has smooth boundary ∂U. Then there exists a sequence of neiborhoods $U_j \subset X_j$ of $x_{a,j}$ which have smooth boundaries $\partial U_j = F_j(\partial U)$. We take δ so small that $B(x_{a,j}; \delta) \subset U_j$. Then it is sufficient to show that for large j there are sections θ_j of $K_{U_j}^{\otimes m}$ on U_j such that

$$C_8^{-1} \leq |\theta_j| \leq C_8, \qquad \text{and} \qquad |\nabla \theta_j| \leq C_9$$

with positive constants C_8, C_9.

Define operator $\Box = \Box_j$ acting on the space of sections of $K_{X_j}^{\otimes m}$ by

$$\Box = -\bar{\partial}^* \bar{\partial} = \operatorname{tr} \nabla' \nabla'',$$

where we decompose the covariant defferentiation $\nabla = \nabla' + \nabla''$ into $(1,0)$- and $(0,1)$-parts. Let ψ be the local holomorphic uniformization $\psi : B \longrightarrow B/\Gamma \xrightarrow{\simeq} U$ and η be a radial cut-off function on B such that $\eta = 0$ on $B(0; 1/3)$ and $\eta = 1$ on $B \backslash B(0; 2/3)$. Using ψ the section $\eta\omega^{\otimes m}$ of $K_B^{\otimes m}$ defines a section of $K_U^{\otimes m}$, which we still denote by $\eta\omega^{\otimes m}$. For large j we define sections $\theta_0 = \theta_{0,j}$ of $K_{X_j}^{\otimes m}$ on U_j by $\theta_{0,j} = \operatorname{proj}(F_j^{-1})^* \eta\omega^{\otimes m}$, where $\operatorname{proj} = \operatorname{proj}_j$ is the projection map of tensors to $K_{X_j}^{\otimes m}$. (Note that the maps

$F_j : X_\infty \backslash S \longrightarrow X_j$ need not to be holomorphic, but become closer and closer to be holomorphic as j tends to ∞.) We solve the following equation on a section $\theta = \theta_j$ of $K_{X_j}^{\otimes m}$ on U_j

$$\Box \theta = 0 \qquad \text{and} \qquad \theta\big|_{\partial U_j} = \theta_0\big|_{\partial U_j}.$$

Then θ satisfy

$$\Delta\theta = \text{tr } \nabla\nabla\theta = -2me_j\theta.$$

Set $\theta' = \theta - \theta_0$. Then θ' has vanishing boundary value and satisfies

$$\Delta\theta' = -2me_j\theta' + \zeta$$

with $\zeta = \zeta_j$ on which we have good control. We have that

$$\begin{aligned}\lambda \int |\theta'|^2 &\le \int |\nabla|\theta'||^2 \le \int |\nabla\theta'|^2 \\ &= -\int (\theta', \Delta\theta') = 2me_j \int |\theta'|^2 - \int (\theta', \zeta) \\ &\le 2me_j \int |\theta'|^2 + \left(\int |\theta'|^2\right)^{1/2} \left(\int |\zeta|^2\right)^{1/2}\end{aligned}$$

with the first eigenvalue $\lambda = \lambda_j$ of the Laplacian acting on functions on U_j with the Dirichlet condition. If we choose U, hence U_j, sufficiently small such that $\lambda \ge 2m|e_j| + 1$, we get L^2-estimates of θ', $\nabla\theta'$ and those of θ. We apply Lemma 3 to the inequality $\Delta|\theta| \ge -2m|e_j||\theta|$, and get C^0-estimate on θ.

For C^1-estimate we differentiate the equation on θ and get the following equations

$$\begin{aligned}\Delta|\nabla'\theta|^2 &= 2|\nabla\nabla'\theta|^2 + 2e_j|\nabla'\theta|^2, \\ \Delta|\nabla''\theta|^2 &= 2|\nabla\nabla''\theta|^2 + 2(-4m+1)e_j|\nabla''\theta|^2, \\ \Delta|\nabla\theta|^2 &\ge 2|\nabla\nabla\theta|^2 - 2(4m-1)|e_j||\nabla\theta|^2, \\ \Delta|\nabla\theta| &\ge -(4m-1)|e_j||\nabla\theta|.\end{aligned}$$

Then again applying Lemma 3, we get C^1-estimate away from boundaries. As to near boundaries, we have good control on the smoothness of the boundaries, the boundary values and the equations and also we have C^0-estimate of θ. So there is no trouble to get C^∞-estimate on $U\backslash B(x_{a,j}; r)$ for any fixed $r > 0$.

Now consider the sequence $\{\text{proj}\, \psi^* F_j^* \theta_j\}$ on $B \backslash \{0\}$ which has uniform C^∞-estimate away from the origin 0. So it has a convergent subsequence with limit, say, $\tilde{\theta}$ defined on $B \backslash \{0\}$. $\tilde{\theta}$ satisfies the equation $\Box \tilde{\theta} = 0$ and has C^1-estimate, so it extends to a smooth solution of the equation across the origin. It must coincide with the unique solution $\omega^{\otimes m}$. So the sequence $\{\text{proj}\, \psi^* F_j^* \theta_j\}$ itself converges to $\omega^{\otimes m}$, and there is a positive constant C_{10} such that for fixed $r > 0$ we have that for large $j \geq j(r)$

$$|\theta_j| \geq C_{10} \qquad \text{on } U_j \backslash B(x_{a,j}; r).$$

By Theorem 1 there exists a constant C_{11} such that every point in $B(x_{a,j}; r)$ can be connected to the boundary $\partial B(x_{a,j}; r)$ with a curve of length at most $C_{11} r$. Thus for $j \geq j(r)$ with $r = C_{10}/(2C_9 C_{11})$ we have that $|\theta_j| \geq C_{10}/2$ on U_j.

Remark. One can also show that $K_{U_j}^{\otimes m}$ is complex analytically trivial for large j.

Hereafter we work on the covering space $\tilde{U}_j$, and denote it simply by U_j. Then we have $m = 1$. We made a trivialization θ of K_{U_j} with uniform C^1-estimate. Thus if we conformally change it, the triviality is preserved in the process of bubbling out of complete, Ricci-flat, ALE, orbifold Kähler surfaces. So the local fundamental groups of the singular points and the fundamental groups at the infinities are contained in $SU(2)$.

Proposition 6. *Let (X, g) be a complete, Ricci-flat, ALE, orbifold Kähler surface. If its canonical line bundle K_X is topologically trivial, then (X, g) is hyper-Kähler.*

Proof. By the assumption K_X is flat and defines an element in $H^1(X; S^1)$. The exact sequence

$$H^1(X; \mathbf{R}) \longrightarrow H^1(X; S^1) \longrightarrow H^2(X; \mathbf{Z})$$

and Lemma 5 imply that the topologically trivial K_X has trivial connection.

Thus our bubbles are all hyper-Kähler. Hence if there is only one bubble coming out, the proof of the main theorem is done.

Theorem 3. *If we take $\delta > 0$ sufficiently small, then for sufficiently large j, the geodesic ball $B(x_{a,j}; \delta)$ in X_j is diffeomorphic to a cyclic quotient of ALE gravitational instanton.*

Remark. We conjecture that $B(x_{a,j}; \delta)$ is biholomorphic to a domain of a cyclic quotient of ALE gravitational instanton.

The proof of Theorem 3 is to apply the following theorem inductively.

Theorem 4. *Let (X,g) be a complete, hyper-Kähler, ALE, orbifold surface which has a singular point o with local fundamental group $\Gamma \subset SU(2)$, and (Y,h) be an ALE gravitational instanton which is biholomorphic to the minimal resolution of $\mathbf{C}^2/\Gamma$. Then we can glue the infinity of Y to the singular point o of X such that the obtained space $X\sharp Y$ is again a complete, hyper-Kähler, ALE, orbifold surface.*

Proof. First fix a Kähler structure (X,ω_1) on X, where ω_1 is its Kähler form. We can take a holomorphic local uniformization $\psi_1 : B(0;\delta) \subset \mathbf{C}^2 \longrightarrow U \ni o$ such that

$$\psi_1^*\omega_1 = \sqrt{-1}\partial\bar{\partial}\phi_1, \qquad \phi_1 = |z|^2 + O(|z|^3),$$
$$\psi_1^*\omega_1^2 = 2(\sqrt{-1}dz^1 \wedge d\bar{z}^1)(\sqrt{-1}dz^2 \wedge d\bar{z}^2).$$

Let $\psi_2 : \mathbf{C}^2\backslash B(0;K) \longrightarrow Y$ be the holomorphic local uniformization of Y at the infinity. Then by Kronheimer [6] the Kähler form ω_2 of (Y,h) satisfies the following properties. (c.f. [3])

$$\psi_2^*\omega_2 = \sqrt{-1}\partial\bar{\partial}\phi_2, \qquad \phi_2 = |z|^2 + O(|z|^{-2}),$$
$$\psi_2^*\omega_2^2 = 2(\sqrt{-1}dz^1 \wedge d\bar{z}^1)(\sqrt{-1}dz^2 \wedge d\bar{z}^2).$$

For sufficiently small positive numbers δ_1, δ_2, by the map $\psi(z) = z/(\delta_1\delta_2)$ we identify two subsets $\psi_1(D(\delta_1, 4\delta_1)) \subset X$, $\psi_2(D(\delta_2^{-1}, 4\delta_2^{-1})) \subset Y$, and get an orbifold suface $Z = X\sharp Y$. In this construction the parallel holomorphic 2-forms on X and Y are glued to give a holomorphic 2-form on Z. We define a Kähler metric ω on Z as follows.

$$\omega = \begin{cases} \omega_1, & \text{on } X\backslash\psi_1(B(0;4\delta_1)); \\ \sqrt{-1}\partial\bar{\partial}\{\eta_{4\delta_1}\phi_1 + (1-\eta_{4\delta_1})(\delta_1\delta_2)^2\psi^*\phi_2\}, & \text{on } \psi_1(D(\delta_1,4\delta_1)); \\ (\delta_1\delta_2)^2\omega_2, & \text{on } Y\backslash\psi_2(\mathbf{C}^2\backslash B(0;\delta_2^{-1})), \end{cases}$$

where $\eta_\delta(z) = \eta(z/\delta)$ is a cut-off function. Since $\phi_1 - |z|^2$ and $(\delta_1\delta_2)^2\psi^*\phi_2 - |z|^2$ are small on $\psi_1(D(\delta_1, 4\delta_2))$, it is easy to see that ω actually defines a Kähler metric on Z.

By the assumption there is a coordinate $\psi_\infty : \mathbf{R}^4\backslash B(0;K) \longrightarrow X$ at the infinity of X such that

$$\psi_\infty^* g_{ij} = \delta_{ij} + O(|x|^{-4}).$$

Then it is easy to see that $\rho = |x|$ makes a barrier function on X, hence on Z. Thus (Z, ω) satisfys the assumption of Theorem C. That means Z admits a complete, Ricci-flat, ALE, orbifold Kähler metric. It is easy to see that the holomorphic 2-form on Z is parallel, so Z is hyper-Kähler.

Now we prove Theorem 3. Assume the blowing up process of orbifold singular point terminates at l-th steps. Then the bubbles coming out in the l-th steps are all smooth ALE gravitational instantons. So they are diffeomorphic to the minimal resolutions of $\mathbf{C}^2/\Gamma$, $\Gamma \subset SU(2)$. We replace their structrues by those comming from minimal resolutions. Then by Theorem 4 we can glue them to the bubbles of $(l-1)$-th steps, and get smooth ALE gravitational instantons. Repeating this arguement we finally get a smooth ALE gravitational instanton which is given by glueing all bubbles. This implies Theorem 3.

For examples of bubbling out of ALE gravitational instantons we refer [7].

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