ON THE ELEMENTS OF THE SECOND TYPE IN SURGERY GROUPS

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Abstract. In 1987 Kharshiladze introduced the concept of type for an element in a Wall group, and proved that the elements of the first and second types cannot be realized by normal maps of closed manifolds. This approach is sufficiently easy for computation and sometimes very effective. In this paper we give a geometrical interpretation of this approach by using the Browder-Quinn surgery obstruction groups for filtered manifolds. Then we study some algebraic and geometrical properties of the elements of the second type, and apply the obtained results for computing the assembly map for some classes of groups. Further applications about the realization problem of the surgery and splitting obstructions complete the paper.

1. Introduction.

Let X^n be a closed connected topological n–dimensional manifold with fundamental group $\pi = \pi_1(X)$ and orientation homomorphism $w : \pi_1(X) \to \{\pm 1\}$. The *structure set* $\mathcal{S}^{TOP}(X)$ is the set of s–cobordism classes of closed connected topological n–dimensional manifolds which are simple homotopy equivalent to X (see [25], [26], and [31]). A representative of an element from $\mathcal{S}^{TOP}(X)$ is given by a simple homotopy equivalence $f : M^n \to X$ of n–dimensional manifolds, and it is called an *s–triangulation* of the manifold X. For $n \geq 5$, the set $\mathcal{S}^{TOP}(X)$ fits into the surgery exact sequence (see [25], [26], and [31])

$$\cdots \to L_{n+1}(\pi) \to \mathcal{S}^{TOP}(X) \to [X, G/TOP] \xrightarrow{\sigma_n} L_n(\pi) \to \cdots \tag{1.1}$$

The groups $L_*(\pi)$ denote the surgery obstruction groups up to simple homotopy equivalence, the set $[X, G/TOP]$ is the set of normal invariants, where TOP is the stable group of homeomorphisms of $\mathbb{R}^n$ with a base point and G is the monoid of stable homotopy self–equivalences of spheres. To classify the manifolds which are

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simple homotopy equivalent to X, one must compute all groups and maps in the surgery exact sequence.

To know the map σ_n, it is sufficient to consider it only for closed manifolds (see [31, §13B]). Let $C_n(\pi)$ be the subgroup of $L_n(\pi)$ generated by the images of type $\sigma_n(f, b)$, where

$$(f, b) : M^n \to X^n \tag{1.2}$$

is a closed manifold surgery problem. There exists a geometrical approach for computing the map σ_n. It is based on the investigation of the problem of splitting a simple homotopy equivalence $f : M \to X$ along a one-sided submanifold $Y \subset X$ (see [4], [6], [8], [10], [15], [16], [17], [20], [22], and [23]).

Let (X^n, Y^{n-q}) be a pair of closed manifolds, and U a tubular neighborhood of Y in X with boundary ∂U. Let

$$F = \begin{pmatrix} \pi_1(\partial U) & \to & \pi_1(X \setminus Y) \\ \downarrow & & \downarrow \\ \pi_1(Y) & \to & \pi_1(X) \end{pmatrix} \tag{1.3}$$

be the square of the fundamental groups together with the natural maps.

Recall that a simple homotopy equivalence $f : M \to X$ *splits along the submanifold* Y if it is homotopic to a map $g : M \to X$ which is transversal to the submanifold Y with $N = g^{-1}(Y)$, and the restrictions

$$g|_N : N \to Y \quad \text{and} \quad g|_{(M \setminus N)} : M \setminus N \to X \setminus Y \tag{1.4}$$

are simple homotopy equivalences (see [26, §7] and [31, §11]). The splitting obstruction groups $LS_{n-q}(F)$ are well defined (see [26, §7] and [31, §11]). These groups depend functorially on the square F and on the dimension $n - q$ modulo 4.

A pair of closed topological manifolds (X, Y) is called a *Browder-Livesay pair* if Y^{n-1} is a one–sided submanifold of codimension one, $n \geq 6$, and the horizontal maps in the square F are isomorphisms. In this case, the square F has the following form

$$F = \begin{pmatrix} \pi_1(\partial U) & \to & \pi_1(X \setminus Y) \\ \downarrow & & \downarrow \\ \pi_1(Y) & \to & \pi_1(X) \end{pmatrix} = \begin{pmatrix} \rho & \overset{\cong}{\to} & \rho \\ \downarrow i_- & & \downarrow i_+ \\ \pi^- & \overset{\cong}{\to} & \pi^+ \end{pmatrix}.$$

The vertical maps in this square are inclusions of index 2, and the orientation homomorphisms on the groups $\pi_1(X) = \pi^+$ and $\pi_1(Y) = \pi^-$ coincide on the images of vertical maps and differ outside these images. This change of orientations is denoted by the symbols "+" and "−" in the square F. We shall omit "+" if this does not lead to any confusion.

The splitting obstruction groups for a Browder-Livesay pair are denoted by

$$LN_{n-1}(\pi_1(X \setminus Y) \to \pi_1(X)) = LN_{n-1}(\rho \to \pi) = LS_{n-1}(F)$$

and are called the *Browder–Livesay groups* (see [6], [8], [11], [20], [26, §7], and [31, §11]).

Let $i : \rho \to \pi$ be an inclusion of index 2 between oriented groups. Then an algebraic definition of the Browder–Livesay groups $LN_n(\rho \to \pi)$, $n = 0, 1, 2, 3$

(mod 4), can be found in [11] and [27]. These groups are related to the surgery obstruction groups by means of the following braid of exact sequences (see [11], [26, §7], [27], and [31, §11])

(1.5)

$$\begin{array}{ccccccccccc}
\to & & L_n(\rho) & & \xrightarrow{i_*} & & L_n(\pi) & & \xrightarrow{\partial} & LN_{n-2}(\rho\to\pi) & \to \\
 & \nearrow & & \searrow & & s\nearrow & & \searrow & & \nearrow \quad \searrow & \\
 & & & & L_n(i^!_-) & & \Gamma\downarrow & & L_n(\rho\to\pi) & & \\
 & \searrow & & \nearrow & & q\searrow & & \nearrow & & \searrow \quad \nearrow & \\
\to & & LN_{n-1}(\rho\to\pi) & & \xrightarrow{c_-} & & L_{n-1}(\pi^-) & & \xrightarrow{i^!_-} & L_{n-1}(\rho) & \to
\end{array}$$

In Diagram (1.5), the groups $L_*(\rho \to \pi)$ are the relative groups for the induced map $i_* : L_*(\rho) \to L_*(\pi)$, and the groups $L_*(i^*_-)$ are the relative groups for the transfer map $i^!_- : L_*(\pi^-) \to L_*(\rho)$. The rows of Diagram (1.5) are chain complexes, and the maps Γ provides isomorphisms of the corresponding homology groups. If an element $x \in L_n(\pi)$ represents a homology class $[x] \in \operatorname{Ker}\partial/\operatorname{Im} i_*$, then the class $\Gamma([x])$ is represented by an element $q(y)$, where $y \in L_n(i^!_-)$ and $s(y) = x$. Thus the class $\Gamma([x])$ consists of the elements $qs^{-1}(x)$. Note that the maps c and ∂ have a good algebraic description on the level of rings with antistructures (see [11] and [27]).

In the case of a Browder-Livesay pair $Y \subset X$, there are isomorphisms

$$L_n(i^!_-) \cong LP_{n-1}(F),$$

where $LP_*(F)$ are the surgery obstruction groups for the manifold pair $Y \subset X$ (see [26, §7]). Note that there exist also isomorphisms

$$LN_n(\rho \to \pi) \cong LN_{n+2}(\rho \to \pi^-) \tag{1.6}$$

of the Browder-Livesay groups.

It was proved in [8] that if $x \in L_n(\pi)$ and $\partial(x) \neq 0$, then the element $x \in L_n(\pi)$ cannot be realized by a normal map of closed manifolds. Thus the map ∂ is the first obstruction for realizing an element of the group $L_n(\pi)$ by a normal map of closed manifolds. This map is called the *Browder–Livesay invariant*.

Let us denote by $C^p_n(\pi)$ the image of the group $C_n(\pi)$ in the projective Novikov group $L^p_n(\pi)$ under the natural map $L_n(\pi) \to L^p_n(\pi)$. In [11] Hambleton constructed the second Browder-Livesay invariant and, using these two invariants, he described the subgroup $C^p_n(\pi)$ for any finite abelian 2-group π. Kharshiladze (see [16] and [17]) introduced the concept of iterated Browder-Livesay invariants, which generalize the first and the second invariants. Then he applied them to the investigation of a closed manifold surgery problem. Note that in the original definitions (see [11], [16] and [17]) the k-invariant is defined if and only if the $(k-1)$–invariant is trivial.

To study the behaviour of the elements in a Wall group relatively to the iterated Browder-Livesay invariants, Kharshiladze defined three possible types of elements in a Wall group. The elements of the first type are the elements for which one of the iterated Browder-Livesay invariants is nontrivial. All the Browder-Livesay invariants of the elements of the second type are trivial but they are nontrivial in the homology groups of a certain chain complex, up to infinity. The remaining elements of the Wall group are of the third type. According to [16] and [17], the elements of the first and the second types cannot be realized by normal maps of closed manifolds. Sometimes this approach becomes very effective (see [17]). For example, let π be an elementary abelian 2-group with nontrivial orientation. Then all the elements of the groups $L_{2n+1}(\pi)$ are of the second type. There exists also a natural problem concerning with the realization of the elements in a Browder-Livesay group by simple homotopy equivalences of closed manifolds (see [1] and [4]). In this paper we define the types of elements in a Browder-Livesay group, and describe connections between these concepts and the realization problem mentioned above (see Section 1). In Section 2 we define the elements of the first and second types in a Wall group following the original definition given by Kharshiladze in [16] and [17]. In Section 3 we illustrate some necessary technical results in algebraic surgery theory (see [2], [12], [14], [22], [25], and [34]) and their relations to surgery on filtered manifolds (see [3], [4], [7], [18], [19], [24], and [34]). Then we give another definition of elements of the second type, and prove that it is equivalent to the original one. Our definition is more algebraic and describes a level of indeterminate for the algebraic passing to surgery on a codimension k submanifold of a given filtration. In Section 4 we apply the previous results to describe some algebraic properties of the elements of various types. In Section 5 we establish deep relations between the properties of the elements of various types and the Browder-Quinn surgery obstruction groups for filtered manifolds. Then we give a necessary condition for an element in a Wall group to lie in the image of the assembly map. This condition is formulated on the level Browder-Quinn surgery obstruction groups. In Section 6 we apply our results to the problem of realizing the surgery and splitting obstructions. For some classes of fundamental groups, we obtain very explicit and full results.

2. Types of elements in Wall's groups.

First we recall the definitions of elements of the first and second types in a Wall group $L_n(\pi)$ according to [16] and [17].

Let $i : \rho \to \pi$ be an inclusion of index 2 between oriented groups. This inclusion induces Diagram (1.5) [27] which we shall denote by $\mathcal{D}$. Let π^- be the group π whose orientation is changed outside the image of i. We obtain another inclusion $i_- : \rho \to \pi$ of oriented groups, which coincides with i as inclusion of groups. Similarly to i, the inclusion i_- yields the diagram

(2.1)

$$\begin{array}{ccccccccc}
\to & & L_n(\rho) & \xrightarrow{i_*^-} & L_n(\pi^-) & \xrightarrow{\partial^-} & LN_{n-2}(\rho\to\pi^-) & & \to \\
& \nearrow & & \searrow \quad \nearrow & & \searrow \quad \nearrow & & \searrow & \\
& & & L_n(i^!) & \Gamma\downarrow & L_n(\rho\to\pi^-) & & & \\
& \searrow & & \nearrow \quad \searrow & & \nearrow \quad \searrow & & \nearrow & \\
\to & & LN_{n-1}(\rho\to\pi^-) & \longrightarrow & L_{n-1}(\pi) & \xrightarrow{i^!} & L_{n-1}(\rho) & & \to
\end{array}$$

which we shall denote by $\mathcal{D}^-$. In general, the diagrams $\mathcal{D}$ and $\mathcal{D}^-$ are different. They have period 4, and their subscripts are defined modulo 4. Using the diagrams $\mathcal{D}$ and $\mathcal{D}^-$ we can write down the following diagram which is infinite in two vertical directions:

$$
\begin{array}{ccccc}
 & & \vdots & & \\
 & & \| & & \\
\xrightarrow{i_*} & & L_n(\pi) & & \xrightarrow{\partial_0} \\
 & s\nearrow & & \searrow & \\
L_n(i^!_-) & & \Gamma\downarrow & & L_n(\rho\to\pi) \\
 & q\searrow & & \nearrow & \\
\longrightarrow & & L_{n-1}(\pi^-) & & \longrightarrow \\
 & & \| & & \\
\longrightarrow & & L_{n-1}(\pi^-) & & \xrightarrow{\partial_1} \\
 & s\nearrow & & \searrow & \\
L_{n-1}(i^!) & & \Gamma\downarrow & & L_{n-1}(\rho\to\pi^-) \\
 & q\searrow & & \nearrow & \\
\longrightarrow & & L_{n-2}(\pi) & & \longrightarrow \\
 & & \| & & \\
\longrightarrow & & L_{n-2}(\pi) & & \xrightarrow{\partial_2} \\
 & s\nearrow & & \searrow & \\
L_{n-2}(i^!_-) & & \Gamma\downarrow & & L_{n-2}(\rho\to\pi) \\
 & q\searrow & & \nearrow & \\
\longrightarrow & & L_{n-3}(\pi^-) & & \longrightarrow \\
 & & \| & & \\
 & & \vdots & &
\end{array}
\tag{2.2}
$$

This diagram has period 4 in the vertical direction. We shall continue to denote by s and q the similar maps in different dimensions. In fact, it is clear from the groups in what dimension we consider the map.

Let us consider the infinite part of Diagram (2.2) starting from the group $L_n(\pi)$ and going down toward the bottom. Let $x \in L_n(\pi)$ be an element. If $\partial(x) = 0$ and $x \notin \operatorname{Im} i_*$, then it represents a nontrivial homology class $[x]$ of the chain complex given by the row.

Definition 1. *For an element $x \in L_n(\pi)$, set $\Gamma^0(x) = \{x\}$. Then $\Gamma^0(x)$ is trivial if $x = 0$, and nontrivial otherwise. If $\Gamma^0(x)$ is nontrivial, then the first Browder-Livesay invariant of the element x with respect to the inclusion $i : \rho \to \pi$ is, by definition, the set*

$$\partial_0(\Gamma^0(x)) = \{\partial_0 x\}. \tag{2.3}$$

It is nontrivial if $\partial_0 x \neq 0$.

Note that the condition $x = 0$ is equivalent to the condition $0 \in \Gamma^0(x)$, and the condition $\partial_0 x \neq 0$ is equivalent to the condition $0 \notin \partial_0(\Gamma^0(x))$. Later we shall use similar properties to define the iterated Browder-Livesay invariants.

Let us consider the following part of Diagram (2.2):

$$
\begin{array}{ccccc}
 & & L_j(\pi^\pm) & & \xrightarrow{\partial} \\
 & s\nearrow & & \searrow & \\
L_j(i^!_\mp) & & \Gamma\downarrow & & L_j(\rho\to\pi^\pm) \\
 & q\searrow & & \nearrow & \\
 & & L_{j-1}(\pi^\mp) & &
\end{array}
\tag{2.4}
$$

where $\partial = \partial_{n-j}$.

Definition 2. *Let $\mathcal{B} \subset L_j(\pi^{\pm})$ be an arbitrary subset. We say that $\Gamma(\mathcal{B})$ is undefined if $0 \notin \partial(\mathcal{B})$; otherwise, a set $\Gamma(\mathcal{B}) \subset L_{j-1}(\pi^{\mp})$ can be defined as*

$$\Gamma(\mathcal{B}) = \{qs^{-1}(x) | x \in \mathcal{B}, \partial(x) = 0\}. \tag{2.5}$$

Following [16] and [17], we now give an inductive definition of the *iterated Browder-Livesay invariants* and the sets

$$\Gamma^j(x) \subset L_{n-j}(\pi^*) \tag{2.6}$$

where $* = -$ for j odd and $* = +$ for j even, for any $j \geq 0$. In this definition, we have fixed an inclusion $i : \rho \to \pi$ of index 2.

Definition 3. *For an element $x \in L_n(\pi)$, set $\Gamma^0(x) = \{x\}$.*

Let the set $\Gamma^{j-1}(x)$, $j \geq 1$, be defined. It is called trivial if $0 \in \Gamma^{j-1}(x)$.

If $\Gamma^{j-1}(x)$ is defined and nontrivial, then the $j-th$ Browder-Livesay invariant with respect to the inclusion i is the set $\partial_{j-1}(\Gamma^{j-1}(x))$. The $j-th$ Browder-Livesay invariant is nontrivial if $0 \notin \partial_{j-1}(\Gamma^{j-1}(x))$.

If the $j-th$ Browder-Livesay invariant is defined and trivial, then the set $\Gamma(\Gamma^{j-1}(x))$ is well defined by Definition 2. So we can define the set $\Gamma^j(x)$ as

$$\Gamma^j(x) = \Gamma(\Gamma^{j-1}(x)). \tag{2.7}$$

By Definitions 2 and 3 the set $\partial_j(\Gamma^j(x))$ is defined and nonempty if the $j-th$ Browder-Livesay invariant is trivial. Note that the $j-th$ Browder-Livesay invariant is defined only if the $(j-1)-th$ Browder-Livesay invariant is defined and trivial and the set $\Gamma^j(x)$ is nontrivial.

Definition 4. *An element $x \in L_n(\pi)$ is of the first type with respect to an inclusion $i : \rho \to \pi$ if there exists a nontrivial Browder-Livesay invariant of x with respect to the same inclusion.*

Definition 5. *An element $x \in L_n(\pi)$ is of the second type with respect to an inclusion $i : \rho \to \pi$ if all the $j-th$ Browder-Livesay invariants of x with respect to the same inclusion are trivial for every $j \geq 1$ and all the sets $\Gamma^j(x)$ are defined and nontrivial for every $j \geq 0$.*

It follows from Definition 3 that the existence of the sets $\Gamma^j(x)$ for all $j \geq 0$ implies that they are nontrivial and all Browder-Livesay invariants of x with respect to i are trivial.

Definition 6. *An element $x \in L_n(\pi)$ is of the first type if it has the first type with respect to an inclusion $i : \rho \to \pi$. An element $x \in L_n(\pi)$ is of the second type if it is not an element of the first type and it has the second type with respect to an inclusion $i : \rho \to \pi$.*

Theorem 1. [17] *Any element $x \in L_n(\pi)$, $n \geq 5$, of the first (resp. second) type cannot be realized by a normal map of closed manifolds.*

3. Algebraic surgery theory and iterated Browder-Livesay invariants.

The algebraic surgery theory of Ranicki provides a realization of various groups in surgery theory and natural maps between them on the spectra level (see [2], [4], [10], [12], [14], [19], [24], [25], [26], [31], and [34]). In this section we illustrate some necessary results in algebraic surgery theory and use this approach to give another definition of elements of the first type (see [10] and [11]). Then we prove the equivalence of these definitions. Theorems 2 and 3 describe the main algebraic properties of the iterated Browder-Livesay invariants.

Let π be a group with an orientation homomorphism $w : \pi \to \{\pm 1\}$. Let us denote by

$$\mathbb{L}(\pi, w) = \mathbb{L}(\pi) = \{\mathbb{L}_{-k}(\pi) : k \in \mathbb{Z}\} \tag{3.1}$$

the surgery obstruction group Ω-spectrum whose homotopy groups $\pi_n(\mathbb{L}(\pi)) = L_n(\pi)$ are the surgery obstruction groups of π (see [25]). An orientation preserving homomorphism $\pi \to \pi'$ induces a cofibration of spectra

$$\mathbb{L}(\pi) \to \mathbb{L}(\pi') \to \mathbb{L}(\pi \to \pi') \tag{3.2}$$

whose homotopy long exact sequence is the relative exact sequence of L–groups for the map $\pi \to \pi'$ (see [2], [12], and [26]). In particular, we have

$$\pi_n(\mathbb{L}(\pi \to \pi')) = L_n(\pi \to \pi').$$

For any block bundle

$$(D^q, S^{q-1}) \to (E, \partial E) \xrightarrow{p} X \tag{3.3}$$

over a closed topological manifold X, there is a homotopy commutative diagram of spectra (see [2], [12], [26], and [31, p.253]):

$$\begin{array}{ccc} \mathbb{L}(\pi_1(X)) & \xrightarrow{p^\sharp} & \Omega^q\mathbb{L}(\pi_1(\partial E) \to \pi_1(E)) \\ & \searrow & \downarrow \delta^\sharp \\ & & \Omega^{q-1}\mathbb{L}(\pi_1(\partial E)) \end{array} \tag{3.4}$$

where the maps $p^\sharp$ and $\delta^\sharp p^\sharp$ are the transfer maps on the spectra level and $\delta^\sharp$ is the connecting map in the cofibration sequence for the natural map $\pi_1(\partial E) \to \pi_1(E)$.

¿From now on, we assume that all considered closed manifold pairs (X^n, Y^{n-q}) are manifold pairs in the sense of Ranicki [26, p.570]. In particular, Y is a locally flat submanifold of X, and it is endowed with the associated normal block bundle ξ. Let (X, Y, ξ) be a closed manifold pair of codimension q. Then a tubular neighborhood $U = E(\xi)$ of Y in X with boundary $\partial U = S(\xi)$ is a disk bundle as in (3.3) [26]. Moreover, we have a homotopy commutative diagram of spectra (see [2] and [3]):

$$\begin{array}{ccccc} \mathbb{L}(\pi_1(Y)) & \xrightarrow{p^\sharp} & \Omega^q\mathbb{L}(\pi_1(\partial U) \to \pi_1(U)) & \xrightarrow{\alpha} & \Omega^q\mathbb{L}(\pi_1(X\setminus Y) \to \pi_1(X)) \\ & \searrow & \downarrow \delta^\sharp & & \downarrow \delta_1^\sharp \\ & & \Omega^{q-1}\mathbb{L}(\pi_1(\partial U)) & \xrightarrow{\beta} & \Omega^{q-1}\mathbb{L}(\pi_1(X\setminus Y)) \end{array} \tag{3.5}$$

where the right horizontal maps are induced by the natural inclusions, and the vertical maps are the connecting maps in the cofibration sequences for the inclusions $\partial U \to U$ and $X \setminus Y \to X$. For the manifold pair (X^n, Y^{n-q}) the splitting obstruction groups $LS_{n-q}(F)$ and the surgery obstruction groups $LP_{n-q}(F)$ have been defined (see [26] and [31]). The spectra $\mathbb{L}S(F)$ and $\mathbb{L}P(F)$ for these groups fit in the following homotopy commutative diagram (see [2-4]):

$$\begin{array}{ccccc} \mathbb{L}(\pi_1(Y)) & \stackrel{\alpha p^\sharp}{\to} & \Omega^q\mathbb{L}(\pi_1(X\setminus Y)\to\pi_1(X)) & \to & \Omega^{-1}\mathbb{L}S(F) \\ \| & & \downarrow \delta_1^\sharp & & \downarrow \\ \mathbb{L}(\pi_1(Y)) & \stackrel{\beta\delta^\sharp p^\sharp}{\to} & \Omega^{q-1}\mathbb{L}(\pi_1(X\setminus Y)) & \to & \Omega^{-1}\mathbb{L}P(F) \end{array} \tag{3.6}$$

in which the rows are cofibrations and the right square is a pullback.

Now let (X, Y) be a Browder-Livesay pair of manifolds. Setting $\rho = \pi_1(X\setminus Y) = \pi_1(\partial U)$ and $\pi = \pi_1(X) = \pi_1(Y)$, the map $i : \pi_1(X \setminus Y) = \rho \to \pi = \pi_1(X)$ is an inclusion of index 2. We denote $\mathbb{L}N(\rho \to \pi) = \mathbb{L}S(F)$. Then we have

$$\pi_n(\mathbb{L}N(\rho\to\pi)) = LN_n(\rho\to\pi) = LS_n(F),\ \pi_n(\mathbb{L}P(F)) = LP_n(F) = L_{n+1}(i^!_-).$$

Diagram (3.6) provides a homotopy pullback square of spectra (see [2-4] and [14]):

$$\begin{array}{ccccc} & & \mathbb{L}(\pi) & & \\ & \nearrow & & \searrow & \\ \Sigma\mathbb{L}P(F) & & & & \mathbb{L}(\rho\to\pi) \\ & \searrow & & \nearrow & \\ & & \Sigma\mathbb{L}(\pi^-) & & \end{array} \tag{3.7}$$

where Σ is a suspension functor (see [30]). Diagram (3.7) realizes on the spectra level the diagram $\mathcal{D}$ in (1.5). Using the same arguments, we can realize on the spectra level the diagram $\mathcal{D}^-$ in (2.1). Putting such diagrams together and using the suspension functors, we obtain the following infinite homotopy commutative diagram of spectra (see [2], [10], [14], and [19]):

$$
\begin{array}{ccccc}
 & & \mathbb{L}(\pi) & & \\
 & \nearrow & & \searrow & \\
\Sigma\mathbb{L}P(F) & & & & \mathbb{L}(\rho\to\pi) \\
 & \searrow & & \nearrow & \\
 & & \Sigma\mathbb{L}(\pi^-) & & \\
 & \nearrow & & \searrow & \\
\Sigma^2\mathbb{L}P(F^-) & & & & \Sigma\mathbb{L}(\rho\to\pi^-) \\
 & \searrow & & \nearrow & \\
 & & \Sigma^2\mathbb{L}(\pi) & & \\
 & \nearrow & & \searrow & \\
\Sigma^3\mathbb{L}P(F) & & & & \Sigma^2\mathbb{L}(\rho\to\pi) \\
 & \searrow & & \nearrow & \\
 & & \Sigma^3\mathbb{L}(\pi^-) & & \\
 & & \vdots & & \\
 & & \vdots & & \\
 & & \Sigma^{2k}\mathbb{L}(\pi) & & \\
 & \nearrow & & \searrow & \\
\Sigma^{2k+1}\mathbb{L}P(F) & & & & \Sigma^{2k}\mathbb{L}(\rho\to\pi) \\
 & \searrow & & \nearrow & \\
 & & \Sigma^{2k+1}\mathbb{L}(\pi^-) & & \\
 & & \vdots & &
\end{array}\tag{3.8}
$$

which realizes Diagram (2.2). For our purposes we do not mention in Diagram (3.8) the equivalences of spectra that correspond to the vertical equalities in (2.2). Now we briefly recall the main steps of [14] (see also [2], [4], [10], and [19]) for constructing a surgery spectral sequence.

Let

$$\mathbb{X}_{j,j} = \Sigma^j\mathbb{L}(\pi^*),\ \mathbb{X}_{j+1,j} = \Sigma^{j+1}\mathbb{L}P(F^*),\ \mathbb{X}_{j,j+1} = \Sigma^j\mathbb{L}(\rho \to \pi^*),\quad j \geq 0, \tag{3.9}$$

where $* = +$ for j even and $* = -$ otherwise. For $j \geq 1$, Diagram (3.8) provides the maps of spectra

$$
\begin{array}{ccc}
 & & \mathbb{X}_{j,j-1} \\
 & & \downarrow \\
\mathbb{X}_{j+1,j} & \to & \mathbb{X}_{j,j}.
\end{array}
$$

For $j \geq 1$, we define the spectra $\mathbb{X}_{j+1,j-1}$ which fit in the pull-back squares

$$
\begin{array}{ccc}
\mathbb{X}_{j+1,j-1} & \to & \mathbb{X}_{j,j-1} \\
\downarrow & & \downarrow \\
\mathbb{X}_{j+1,j} & \to & \mathbb{X}_{j,j}.
\end{array}\tag{3.10}
$$

Iterating this construction for $j \geq 3$ and $j - k \geq 3$, we can define the spectra $\mathbb{X}_{j,k}$ which fit in the pull-back squares

$$
\begin{array}{ccc}
\mathbb{X}_{j,k} & \to & \mathbb{X}_{j-1,k} \\
\downarrow & & \downarrow \\
\mathbb{X}_{j,k+1} & \to & \mathbb{X}_{j-1,k+1}.
\end{array}\tag{3.11}
$$

Thus we have extended Diagram (3.8) toward the left direction. Using similar arguments and pushout squares, we can extend Diagram (3.8) toward the right direction, too. As a consequence, we obtain the following homotopy commutative diagram of spectra

$$\begin{tikzcd}[column sep=small]
 & & & & \mathbb{X}_{0,0} \arrow[dr] & & & & \\
 & & & \mathbb{X}_{1,0} \arrow[ur] \arrow[dr] & & \mathbb{X}_{0,1} \arrow[dr] & & & \\
 & & \mathbb{X}_{2,0} \arrow[ur] \arrow[dr] & & \mathbb{X}_{1,1} \arrow[ur] \arrow[dr] & & \mathbb{X}_{0,2} \arrow[dr] & & \\
 & \cdots \arrow[ur] & & \mathbb{X}_{2,1} \arrow[ur] \arrow[dr] & & \mathbb{X}_{1,2} \arrow[ur] \arrow[dr] & & \cdots & \\
 & & \cdots \arrow[ur] & & \mathbb{X}_{2,2} \arrow[ur] & & \cdots & & \\
 & & & & \vdots & & & &
\end{tikzcd} \tag{3.12}$$

whose squares are all pullback.

Following [10], we now give another definition of elements of the first type, and prove that it is equivalent to the definition of Section 2. Let

$$\varphi^j : \mathbb{X}_{j,0} \to \mathbb{X}_{0,0} \quad \text{and} \quad \psi^j : \mathbb{X}_{j,0} \to \mathbb{X}_{j,j}, \quad j \geq 0 \tag{3.13}$$

be the maps of spectra that are compositions of maps from Diagram (3.12). For $j \geq 0$, we denote by $\mathbb{F}_j$ a homotopy fiber of the map φ^j and by $\mathbb{H}_j$ a homotopy fiber of the map ψ^j. Thus, for $j \geq 0$, we obtain the cofibrations

$$\mathbb{F}_j \to \mathbb{X}_{j,0} \to \mathbb{X}_{0,0} \quad \text{and} \quad \mathbb{H}_j \to \mathbb{X}_{j,0} \to \mathbb{X}_{j,j}. \tag{3.14}$$

Proposition 1. *There exists a homotopy commutative diagram of spectra*

$$\begin{tikzcd}
 & \mathbb{H}_j \arrow[r, equal] \arrow[d] & \mathbb{H}_j \arrow[d] \\
\mathbb{F}_j \arrow[r] \arrow[d, equal] & \mathbb{X}_{j,0} \arrow[r, "\varphi^j"] \arrow[d, "\psi^j"] & \mathbb{X}_{0,0} \arrow[d, "\mu^j"] \\
\mathbb{F}_j \arrow[r] & \mathbb{X}_{j,j} \arrow[r, "\nu^j"] & \mathbb{X}_{0,j}
\end{tikzcd} \tag{3.15}$$

whose rows and columns are all cofibrations.

Proof. All the small squares in Diagram (3.12) are pullback. Hence the right bottom square in Diagram (3.15) is a pullback, too. □

The cofibrations in (3.14) provides the homotopy long exact sequences

$$\cdots \to \pi_n(\mathbb{F}_j) \xrightarrow{f_j} \pi_n(\mathbb{X}_{j,0}) \xrightarrow{\varphi^j_*} \pi_n(\mathbb{X}_{0,0}) \to \cdots \tag{3.16}$$

and

$$\cdots \to \pi_n(\mathbb{H}_j) \xrightarrow{h_j} \pi_n(\mathbb{X}_{j,0}) \xrightarrow{\psi_*^j} \pi_n(\mathbb{X}_{j,j}) \to \cdots \tag{3.17}$$

where the map φ_*^j is induced by φ^j and the map ψ_*^j is induced by ψ^j.

Let

$$\partial_j : L_{n-j}(\pi^*) \to LN_{n-j-2}(\rho \to \pi^*),\ j \geq 0 \tag{3.18}$$

be the map fitting in the diagram $\mathcal{D}^+$ for j even and in $\mathcal{D}^-$ otherwise.

The exact sequence in (3.16) and the commutative diagram in (3.15) provide the composite map

$$c_j = \psi_*^j \circ f_j : \pi_n(\mathbb{F}_j) \to \pi_n(\mathbb{X}_{j,j}) = L_{n-j}(\pi^*) \tag{3.19}$$

with

$$\operatorname{Im} c_j = \mathcal{C}_j \subset L_{n-j}(\pi^*). \tag{3.20}$$

Let us denote by d_j the composition

$$\pi_n(\mathbb{F}_j) \xrightarrow{c_j} L_{n-j}(\pi^*) \xrightarrow{\partial_j} LN_{n-j-2}(\rho \to \pi^*) \tag{3.21}$$

and by p_j the natural projection

$$LN_{n-j-2}(\rho \to \pi^*) \to LN_{n-j-2}(\rho \to \pi^*)/\{\operatorname{Im} d_j\}. \tag{3.22}$$

Now we give an inductive definition of the iterated Browder-Livesay invariants with respect to an inclusion $i : \rho \to \pi$ of index 2 (see [10]). In this definition the inclusion i is fixed.

Definition 7. [10] *For an element $x \in L_n(\pi)$, let $\mathcal{B}_0 = \{x\}$ be a set. This set is trivial if $x = 0$. If the set $\mathcal{B}_0$ is nontrivial, then the set $\partial_0\mathcal{B}_0$ consisting of the element $\partial_0 x \in LN_{n-2}(\rho \to \pi)$ is called the first Browder-Livesay invariant of the element x with respect to the inclusion i. It is trivial if $\partial_0 x = 0$.*

For any $j \geq 2$, we define inductively the sets $\mathcal{B}_j$ and the $j-th$ Browder-Livesay invariants of an element $x \in L_n(\pi)$ with respect to the inclusion i. Let a set $\mathcal{B}_{j-1}$ be defined. It is called trivial if $\mathcal{B}_{j-1} \subset \mathcal{C}_{j-1}$, and nontrivial otherwise. If the set $\mathcal{B}_{j-1}$ is defined and nontrivial, then the $j-th$ Browder–Livesay invariant of x is the subset

$$p_{j-1}\partial_{j-1}(\mathcal{B}_{j-1}) \subset LN_{n-j-1}(\rho \to \pi^*)/\{\operatorname{Im} d_{j-1}\}.$$

It is trivial if and only if it equals $\{0\}$ in $LN_{n-j-1}(\rho \to \pi^*)/\{\operatorname{Im} d_{j-1}\}$.

If the $j-th$ Browder–Livesay invariant is defined and trivial, then we consider the subset

$$\mathcal{B}'_{j-1} \subset \mathcal{B}_{j-1} \subset L_{n-j+1}(\pi^*)$$

defined by

$$\mathcal{B}'_{j-1} = \{x \in \mathcal{B}_{j-1} | \partial_{j-1}(x) = 0\}. \tag{3.23}$$

If the set $\mathcal{B}'_{j-1}$ is defined and nonempty, then we define the set $\mathcal{B}_j$ by

$$\mathcal{B}_j = qs^{-1}\left(\mathcal{B}'_{j-1}\right) \tag{3.24}$$

where q and s are the maps arising from Diagram (2.4).

Let $x \in \pi_n(\mathbb{X}_{0,0}) = L_n(\pi)$ be an element. Now we define the sets $\mathcal{A}_j$, for any $j \geq 0$ [10]. Let $\mathcal{A}_0 = \{x\}$. For $j \geq 1$, we define $\mathcal{A}_j$ by setting

$$\mathcal{A}_j = (\varphi^j_*)^{-1}(x) \subset \pi_n(\mathbb{X}_{j,0}). \tag{3.25}$$

Note that if the set $\mathcal{A}_j$ is empty for some j, then all the sets $\mathcal{A}_k,\ k \geq j$, are empty, too.

The following theorem describes some algebraic properties of the Browder-Livesay invariants with respect to an inclusion i in the sense of Definition 7.

Theorem 2. [10]
i) Let the $j-th$ Browder-Livesay invariant, $j \geq 1$, of an element $x \in L_n(\pi)$ with respect to the inclusion i be defined. Then it is trivial if and only if

$$x \in \operatorname{Im}\varphi^j_* \subset L_n(\pi). \tag{3.26}$$

ii) Let the $j-th$ Browder-Livesay invariant, $j \geq 1$, of an element $x \in L_n(\pi)$ with respect to the inclusion i be trivial. Then the set $\mathcal{B}'_{j-1}$ is nonempty and

$$\mathcal{B}_j = \psi^j_*(\mathcal{A}_j). \tag{3.27}$$

iii) Let the $j-th$ Browder–Livesay invariant, $j \geq 1$, of an element $x \in L_n(\pi)$ with respect to the inclusion i be trivial. Then the set $\mathcal{B}_j$ is trivial if and only if

$$x \in \operatorname{Im}(\varphi^j_* \circ h_j). \tag{3.28}$$

In the next theorem we prove the equivalence between Definition 3 and Definition 7 of Browder-Livesay invariants and the equality of the sets $\Gamma^j(x)$ and $\mathcal{B}_j$ given in these definitions. In particular, the concepts of triviality for these sets coincide.

Theorem 3. *Let $i : \rho \to \pi$ be an inclusion of index 2 and $x \in L_n(\pi)$ an element.*
i) The sets $\mathcal{B}_j$ and $\Gamma^j(x)$ coincide, that is, one of them is defined if and only if the other is defined, and in this case the sets are equal.
ii) The set $\Gamma^j(x)$ is trivial in the sense of Definition 3 if and only if the set $\mathcal{B}_j$ is trivial in the sense of Definition 7.
iii) The element x has nontrivial $j-th$ Browder-Livesay invariant with respect to i in the sense of Definition 3 if and only if it has nontrivial $j-th$ Browder-Livesay invariant with respect to i in the sense of Definition 7.

Proof. (By induction on $j \geq 1$). For $j = 0$ we have $\mathcal{B}_0 = \{x\} = \Gamma^0(x)$ by definition. The set $\Gamma^j(x)$ is trivial if and only if $x = 0$. The set $\mathcal{B}_0$ is trivial if and only if $x = 0$. The first Browder-Livesay invariant in the sense of Definition 7 is defined and it is trivial if and only if $\partial_0(x) = 0$. This is equivalent to the condition

$0 \in \{\partial_0(x)\} = \partial_0\Gamma^0(x)$, that is, the triviality of the first Browder–Livesay invariant in the sense of Definition 3.

i) Let $\mathcal{B}_{j-1} = \Gamma^{j-1}(x)$ be defined and nontrivial. The $j-th$ Browder-Livesay invariant is defined in the sense of the two definitions, and it is trivial in the sense of Definition 3 and in the sense of Definition 7. Then it follows immediately from Definition 3 and Definition 7 that

$$\mathcal{B}_j = \Gamma^j(x) = \{qs^{-1}(y) | y \in \mathcal{B}_{j-1} = \Gamma^{j-1}(x), \partial_{j-1}(y) = 0\}. \tag{3.29}$$

ii) Let $\Gamma^j(x)$ be trivial. Then, by definition, $0 \in \Gamma^j(x)$ and by (3.29) $0 \in \mathcal{B}_j$. By Theorem 2, we have $\mathcal{B}_j = \psi_*^j(\mathcal{A}_j)$ and

$$0 \in \psi_*^j(\mathcal{A}_j) = \psi_j\left((\varphi_*^j)^{-1}(x)\right). \tag{3.30}$$

Let $a \in \mathcal{A}_j$ be an element such that

$$\psi_*^j(a) = 0 \quad \text{and} \quad \varphi_*^j(a) = x \tag{3.31}$$

by definition of $\mathcal{A}_j$. Let $z \in \mathcal{B}_j$ be an element. By Theorem 2, we have

$$z = \psi_*^j(b) \quad \text{and} \quad \varphi_*^j(b) = x \tag{3.32}$$

by definition of $\mathcal{A}_j$. Now it follows from (3.31) and (3.32) that

$$\varphi_*^j(b-a) = 0 \in \pi_n(\mathbb{X}_{0,0}) = L_n(\pi) \quad \text{and} \quad z = \psi_*^j(b-a). \tag{3.33}$$

By the exact sequence in (3.16) there exists an element $t \in \pi_n(\mathbb{F}_j)$ such that $f_j(t) = b-a$. From (3.19) and (3.33) we obtain $c_j(t) = \psi_*^j \circ f_j(t) = \psi_*^j(b-a) = z$. Hence any element $z \in \mathcal{B}_j$ lies in the set $\mathcal{C}_j = \operatorname{Im} c_j$, so $\mathcal{B}_j$ is trivial in the sense of Definition 7.

Now we prove the reverse implication. Let the set $\mathcal{B}_j$ be trivial in the sense of Definition 7, that is, $\Gamma^j(x) = \mathcal{B}_j \subset \mathcal{C}_j = \operatorname{Im} c_j$. Let $z \in \Gamma^j(x) = \mathcal{B}_j$ be an element satisfying the conditions in (3.32). Since the set $\mathcal{B}_j$ is trivial, we have

$$z = c_j(t), \ t \in \pi_n(\mathbb{F}_j). \tag{3.34}$$

Hence, by (3.32), (3.19), and (3.34),

$$\psi_*^j(b - f_j(t)) = \psi_*^j(b) - \psi_*^j \circ f_j(t) = z - c_j(t) = z - z = 0 \tag{3.35}$$

and, by (3.16) and (3.32),

$$\varphi_*^j(b - f_j(t)) = \varphi_*^j(b) - \varphi_*^j \circ f_j(t) = x - 0 = x. \tag{3.36}$$

It follows from (3.35), (3.36), and (3.25) that $0 \in \psi_*^j(\mathcal{A}_j)$. Hence by Theorem 2, we get $0 \in \mathcal{B}_j = \Gamma^j(x)$, so the set $\Gamma^j(x)$ is trivial in the sense of Definition 3.

iii) Let $\mathcal{B}_{j-1} = \Gamma^{j-1}(x)$ be defined and nontrivial in the sense of the corresponding definitions. Let the $j-th$ Browder-Livesay invariant of the element x be trivial in the sense of Definition 7. Then, by Theorem 2, we have

$$x \in \operatorname{Im} \varphi_*^j, \quad x = \varphi_*^j(a). \tag{3.37}$$

Let us consider the following commutative diagram fitting in Diagram (3.12):

$$\begin{array}{ccccc} \mathbb{X}_{j-1,0} & \xrightarrow{\psi^{j-1}} & \mathbb{X}_{j-1,j-1} & & \\ \gamma\uparrow & & \uparrow & & \\ \mathbb{X}_{j,0} & \longrightarrow & \mathbb{X}_{j,j-1} & \longrightarrow & \mathbb{X}_{j,j} \end{array} \tag{3.38}$$

where the right upper map fits into the cofibration

$$\mathbb{X}_{j,j-1} \to \mathbb{X}_{j-1,j-1} \to \Sigma^{j+1}\mathbb{L}N(\rho \to \pi^*). \tag{3.39}$$

This follows from the construction of (3.12) and from Diagrams (1.5) and (2.1). The right map in (3.39) induces the map ∂_{j-1}. It follows from Theorem 2, (3.37), and (3.39) that

$$\psi^{j-1}\gamma_*(a) \in \mathcal{B}_{j-1} = \Gamma^{j-1}(x), \tag{3.40}$$

where $\gamma_* : \pi_n(\mathbb{X}_{j,0}) \to \pi_n(\mathbb{X}_{j-1,0})$ is the map induced by γ. ¿From commutative Diagram (3.38) and from the cofibration in (3.39) we get

$$\partial_{j-1}(\psi^{j-1}\gamma_*(a)) = 0 \in \pi_n(\Sigma^{j+1}\mathbb{L}N(\rho \to \pi^*)) = LN_{n-j-1}(\rho \to \pi),$$

and hence $0 \in \partial(\Gamma^{j-1}(x))$. Thus the $j-th$ Browder-Livesay invariant is trivial in the sense of Definition 3. Now we prove the reverse implication. Let $\mathcal{B}_{j-1} = \Gamma^{j-1}(x)$ be defined and nontrivial, and assume that the $j-th$ Browder-Livesay invariant of the element x is trivial in the sense of Definition 3. This means, by definition, that

$$0 \in \partial_{j-1}(\Gamma_{j-1}(x)) = \partial_{j-1}(\mathcal{B}_{j-1}). \tag{3.41}$$

By Theorem 2 and the definition of $\mathcal{A}_{j-1}$ there exists an element $a \in \mathcal{A}_{j-1} \subset \pi_n(\mathbb{X}_{j-1,0})$ with the following properties:

$$\partial_{j-1}(\psi_*^{j-1}(a)) = 0 \ \text{ and } \ \varphi_*^{j-1}(a) = x. \tag{3.42}$$

Let us consider the homotopy commutative diagram of spectra

$$\begin{array}{ccccc} \mathbb{X}_{j,0} & \xrightarrow{\gamma} & \mathbb{X}_{j-1,0} & \longrightarrow & \Sigma^{j+1}\mathbb{L}N(\rho \to \pi^*) \\ \downarrow & & \downarrow\psi^{j-1} & & \| \\ \mathbb{X}_{j,j-1} & \longrightarrow & \mathbb{X}_{j-1,j-1} & \longrightarrow & \Sigma^{j+1}\mathbb{L}N(\rho \to \pi^*) \end{array} \tag{3.43}$$

which arises from the left pullback square in Diagram (3.38) and from the cofibration in (3.39). The left square in (3.43) is a pullback. It follows from (3.42) and (3.43) that there exists an element $b \in \pi_n(\mathbb{X}_{j,0})$ such that $\gamma_*(b) = a$. Then we get

$$x = \varphi_*^{j-1}\gamma(b) = \varphi_*^{j}(b), \ b \in \pi_n(\mathbb{X}_{j,0}). \tag{3.44}$$

Thus, by Theorem 2(i), the $j-th$ Browder-Livesay invariant of the element x in the sense of Definition 7 is trivial. $\square$

4. On the elements of the second type.

In this section we give an equivalent definition of elements of the second type in a Wall surgery obstruction group. Then we describe some algebraic and geometrical properties of such elements. We shall use notations from the previous sections. Let $i : \rho \to \pi$ be an inclusion of index 2 between groups with orientations. For an element $x \in L_n(\pi)$, we define the sets $\mathcal{B}_j$ according to Definition 7. By Definition 7 and Theorem 2, the set $\mathcal{B}_j$ $(j \geq 1)$ is defined and nonempty if and only if the set $\mathcal{B}_{j-1}$ is defined and nontrivial and the $j - th$ Browder-Livesay invariant is trivial (see [10]).

Definition 8. *An element $x \in L_n(\pi)$ is an element of the second type with respect to the inclusion i if the sets $\mathcal{B}_j$ are defined for all $j \geq 0$.*

For the element x of the second type in Definition 8, all the sets $\mathcal{B}_j$ are nontrivial and all the Browder-Livesay invariants are trivial with respect to the inclusion i. The following result is an immediate consequence of Theorem 3.

Theorem 4. *An element $x \in L_n(\pi)$ is of the second type with respect to the inclusion i in the sense of Definition 8 if and only if x has the second type with respect to the inclusion i in the sense of Definition 5.*

Let us consider the commutative diagram (induced by Diagram (3.15))

$$\begin{tikzcd}
 & \pi_n(\mathbb{H}_j) \arrow[r, equal] \arrow[d, "h_j"] & \pi_n(\mathbb{H}_j) \arrow[d, "\varphi^j_* \circ h_j"] \\
\pi_n(\mathbb{F}_j) \arrow[r] \arrow[d, equal] & \pi_n(\mathbb{X}_{j,0}) \arrow[r, "\varphi^j_*"] \arrow[d, "\psi^j_*"] & \pi_n(\mathbb{X}_{0,0}) \arrow[d, "\mu^j_*"] \\
\pi_n(\mathbb{F}_j) \arrow[r] & \pi_n(\mathbb{X}_{j,j}) \arrow[r, "\nu^j_*"] & \pi_n(\mathbb{X}_{0,j})
\end{tikzcd} \tag{4.1}$$

in which the map φ^j_* splits into the composition

$$\pi_n(\mathbb{X}_{j,0}) \xrightarrow{\gamma_*} \pi_n(\mathbb{X}_{j-1,0}) \xrightarrow{\varphi^{j-1}_*} \pi_n(\mathbb{X}_{0,0}). \tag{4.2}$$

Theorem 5. *Let $x \in L_n(\pi)$ be an element of the second type with respect to the inclusion i. Then the sets $\mathcal{A}_j = (\varphi^j_*)^{-1}(x) \subset \pi_n(\mathbb{X}_{j,0})$ are nonempty for all $j \geq 0$ and $\gamma_*(\mathcal{A}_j) \subset \mathcal{A}_{j-1}$ for all $j \geq 1$. If $a \in \mathcal{A}_j$, then the element $\mu^j_* \circ \varphi^j_*(a) = \nu^j_* \circ \psi^j_*(a) \in \pi_n(\mathbb{X}_{0,j})$ is nontrivial.*

Proof. The first statement follows from the definition of the sets $\mathcal{A}_j$ and Theorem 2 (ii). Let $\mu^j_* \circ \varphi^j_*(a) = 0 \in \pi_n(\mathbb{X}_{0,j})$. Since $\varphi^j_*(a) = x$ by definition of $\mathcal{A}_j$, we get

$$\mu^j_*(x) = 0 \in \pi_n(\mathbb{X}_{0,j}). \tag{4.3}$$

The right vertical row in Diagram (4.1) is exact. Hence, it follows from (4.3) that $x = \varphi^j_* \circ h_j(b)$, where $b \in \pi_n(\mathbb{H}_j)$. By Theorem 2 (iii), this implies that the set $\mathcal{B}_j$ is trivial. We have a contradiction which proves the statement. $\square$

Now we illustrate further algebraic properties of the elements of the first and second type with respect to the inclusion i. They are closely related with Diagrams (3.12) and (3.15), and allow us to study some functorial properties of the concept of types of elements in a Wall group. For an inclusion $i : \rho \to \pi$ of index 2, we can write down the following homotopy commutative diagram of spectra (compare with Diagrams (3.12) and (3.15)):

$$
\begin{tikzcd}
\dots \arrow[r] & \mathbb{X}_{j,0} \arrow[r] \arrow[d, "\varphi^j"] & \mathbb{X}_{j-1,0} \arrow[r] \arrow[d, "\varphi^{j-1}"] & \dots \arrow[r] & \mathbb{X}_{2,0} \arrow[r] \arrow[d, "\varphi^2"] & \mathbb{X}_{1,0} \arrow[d, "\varphi^1"] \\
\dots \arrow[r] & \mathbb{L}(\pi) \arrow[r, equal] \arrow[d, "\tau^j"] & \mathbb{L}(\pi) \arrow[r, equal] \arrow[d, "\tau^{j-1}"] & \dots \arrow[r, equal] & \mathbb{L}(\pi) \arrow[r, equal] \arrow[d, "\tau^2"] & \mathbb{L}(\pi) \arrow[d, "\tau^1"] \\
\dots \arrow[r] & \Sigma\mathbb{F}_j \arrow[r] & \Sigma\mathbb{F}_{j-1} \arrow[r] & \dots \arrow[r] & \Sigma\mathbb{F}_2 \arrow[r] & \Sigma\mathbb{F}_1
\end{tikzcd}
\tag{4.4}
$$

where $\mathbb{L}(\pi) = \mathbb{X}_{0,0}$, the columns are cofibrations, and the bottom horizontal maps are induced by upper horizontal maps and central equalities (see [30]). Let us denote by

$$\tau_*^j : L_n(\pi) \to \pi_{n-1}(\mathbb{F}_j) \tag{4.5}$$

the maps induced by τ^j. In particular, we have

$$\tau_*^1 = \partial_0 : L_n(\pi) \to LN_{n-2}(\rho \to \pi).$$

These maps are equivalent to the iterated Browder-Livesay invariants in the sense precised by the next proposition (which follows from Diagram (4.4) and Theorem 2(i)).

Proposition 2. *The $j-th$ ($j \geq 1$) Browder-Livesay invariant of an element $x \in L_n(\pi)$ with respect to the inclusion i is nontrivial if and only if $\tau_*^j(x) \neq 0$ and $\tau_*^k(x) = 0$ for every $k < j$.*

Note that, on the contrary to the iterated Browder-Livesay invariants, the maps τ_*^j are defined for all $j \geq 1$, and the condition $\tau_*^j(x) \neq 0$ implies that $\tau_*^k(x) \neq 0$ for $k \geq j$.

Corollary 1. *An element $x \in L_n(\pi)$ is of the first type with respect to the inclusion i if and only if there exists an index $j \geq 1$ such that $\tau_*^j(x) \neq 0$.*

Similarly to Diagram (4.4) we can write down a homotopy commutative diagram of spectra

$$
\begin{tikzcd}
\mathbb{H}_1 \arrow[r] \arrow[d] & \mathbb{H}_2 \arrow[r] \arrow[d] & \dots \arrow[r] & \mathbb{H}_{j-1} \arrow[r] \arrow[d] & \mathbb{H}_j \arrow[r] \arrow[d] & \dots \\
\mathbb{L}(\pi) \arrow[r, equal] \arrow[d, "\mu^1"] & \mathbb{L}(\pi) \arrow[r, equal] \arrow[d, "\mu^2"] & \dots \arrow[r, equal] & \mathbb{L}(\pi) \arrow[r, equal] \arrow[d, "\mu^{j-1}"] & \mathbb{L}(\pi) \arrow[r, equal] \arrow[d, "\mu^j"] & \dots \\
\mathbb{X}_{0,1} \arrow[r] & \mathbb{X}_{0,2} \arrow[r] & \dots \arrow[r] & \mathbb{X}_{0,j-1} \arrow[r] & \mathbb{X}_{0,j} \arrow[r] & \dots
\end{tikzcd}
\tag{4.6}
$$

where the columns are cofibrations and the top horizontal maps are induced by bottom horizontal maps and central equalities (see [30]). For any $j \geq 1$, we have a commutative diagram

$$
\begin{tikzcd}
 & \mathbb{X}_{0,0} \arrow[d, equal] & \\
\mathbb{X}_{j,0} \arrow[r] & \mathbb{L}(\pi) \arrow[r, "\tau^j"] \arrow[d, "\mu^j"] & \Sigma\mathbb{F}_j \\
 & \mathbb{X}_{0,j} \arrow[ur] &
\end{tikzcd}
\tag{4.7}
$$

where the horizontal row is a cofibration. Let us denote by

$$\mu^j_* : L_n(\pi) \to \pi_n(\mathbb{X}_{0,j}) \tag{4.8}$$

the maps induced by μ^j.

Theorem 6. *An element $x \in L_n(\pi)$ is of the second type with respect to the inclusion i if and only if $\mu^j_*(x) \neq 0$ and $\tau^j_*(x) = 0$ for every $j \geq 1$.*

Proof. If the element x is of the second type with respect to i, then the statement of the theorem directly follows from Theorem 5 and Proposition 2. Now we prove the reverse implication. If $\tau^j_*(x) = 0$ for all $j \geq 1$, then all the Browder-Livesay invariants are trivial by Proposition 2. For $j \geq 1$, the sets $\mathcal{A}_j = (\varphi^j_*)^{-1}(x) \subset \pi_n(\mathbb{X}_{j,0})$ are nonempty, as follows from (4.4). Hence, for $j \geq 0$, the sets $\mathcal{B}_j = \psi^j_*(\mathcal{A}_j)$ are defined. The condition $\mu^j_*(x) \neq 0$ and Diagrams (4.1) and (4.6) imply that x does not lie in the image of the map $\varphi^j_* \circ h_j$. Hence, by Theorem 2(iii), the set $\mathcal{B}_j$ is nontrivial. $\square$

Now we describe some functorial properties of the elements of various types. Let $i : \rho \to \pi$ and $i' : \rho' \to \pi'$ be two inclusions of index 2 between oriented groups. These inclusions provide the short exact sequences

$$1 \longrightarrow \rho \xrightarrow{\ i\ } \pi \longrightarrow \{\pm 1\} \longrightarrow 1 \tag{4.9}$$

and

$$1 \longrightarrow \rho' \xrightarrow{\ i'\ } \pi' \longrightarrow \{\pm 1\} \longrightarrow 1. \tag{4.10}$$

Definition 9. *A morphism (f, g) from the inclusion i to the inclusion i' is given by a commutative diagram*

$$
\begin{tikzcd}
1 \arrow[r] \arrow[d, equal] & \rho \arrow[r, "i"] \arrow[d, "f"] & \pi \arrow[r] \arrow[d, "g"] & \{\pm 1\} \arrow[r] \arrow[d, equal] & 1 \arrow[d, equal] \\
1 \arrow[r] & \rho' \arrow[r, "i'"] & \pi' \arrow[r] & \{\pm 1\} \arrow[r] & 1
\end{tikzcd}
\tag{4.11}
$$

in which the homomorphisms f and g agree with respect to the orientations.

As follows from the definition, the morphism (f, g) is also a morphism from the inclusion $i_- : \rho \to \pi^-$ to $i'_- : \rho' \to \pi'^-$. Note that a commutative diagram

$$
\begin{tikzcd}
\rho \arrow[r, "i"] \arrow[d, "f"] & \pi \arrow[d, "g"] \\
\rho' \arrow[r, "i'"] & \pi'
\end{tikzcd}
\tag{4.12}
$$

with orientation preserving homomorphisms is not a morphism of the inclusions in the general case.

Theorem 7. *Let (f,g) be a morphism from i to i', where $i:\rho\to\pi$ and $i':\rho'\to\pi'$ are inclusions of index 2. Let $x\in L_n(\pi)$ be an element and $g_*(x)=x'\in L_n(\pi')$, where g_* is the induced map of L–groups.*

If the element x' is of the first type with respect to the inclusion i', then x has the first type with respect to the inclusion i.

If the element x' is of the second type with respect to the inclusion i', then x may have either the first or the second type with respect to the inclusion i.

Proof. The morphism (f,g) induces a commutative diagram that is given by the maps from the groups of Diagram (1.5) for the inclusion i to the corresponding groups of Diagram (1.5) for the inclusion i'. A similar commutative diagram arises from Diagram (2.1). These two diagrams can be realized on the spectra level. Thus, as follows from the construction of Diagram (3.12), the morphism (f,g) induces a homotopy commutative diagram of spectra. This is given by the maps from the spectra of Diagram (3.12) for the inclusion i to the corresponding spectra of the similar diagram for the inclusion i'. By [30], we obtain a homotopy commutative diagram of spectra

$$\left(\begin{array}{ccccc} & & \mathbb{X}_{0,0} & & \\ & & \| & & \\ \mathbb{X}_{j,0} & \longrightarrow & \mathbb{L}(\pi) & \xrightarrow{\tau^j} & \Sigma\mathbb{F}_j \\ & & \downarrow\mu^j & \nearrow & \\ & & \mathbb{X}_{0,j} & & \end{array}\right) \longrightarrow \left(\begin{array}{ccccc} & & \mathbb{X}'_{0,0} & & \\ & & \| & & \\ \mathbb{X}'_{j,0} & \longrightarrow & \mathbb{L}(\pi') & \xrightarrow{\tau^{j'}} & \Sigma\mathbb{F}'_j \\ & & \downarrow\mu^{j'} & \nearrow & \\ & & \mathbb{X}'_{0,j} & & \end{array}\right) \tag{4.13}$$

that is given by the maps from the spectra of Diagram (4.7) to the corresponding spectra of the similar diagram constructed for the inclusion i'.

Let an element x' be of the first type with respect to the inclusion i'. By definition, this means that a Browder-Livesay invariant of x' with respect to i' is nontrivial. Then, by Proposition 2, ${\tau^{j}}'_*(x')\neq 0$ for some $j'\geq 1$. ¿From the homotopy commutative diagram in (4.13) we obtain a commutative diagram of groups

$$\begin{array}{ccc} L_n(\pi) & \xrightarrow{g_*} & L_n(\pi') \\ \Big\downarrow{\scriptstyle {\tau^j}_*} & & \Big\downarrow{\scriptstyle {\tau^j}'_*} \\ \pi_{n-1}(\mathbb{F}_j) & \xrightarrow{\varsigma} & \pi_{n-1}(\mathbb{F}'_j) \end{array} \tag{4.14}$$

in which the horizontal maps are induced by the morphism (f,g).

¿From the commutativity of (4.14) we get $\varsigma\left(\tau^j_*(x)\right)={\tau^{j}}'_*\left(g_*(x)\right)={\tau^{j}}'_*(x')\neq 0$. Hence ${\tau^j}_*(x)\neq 0$, and by Corollary 1 the element x is of the first type with respect to the inclusion i.

Now let the element x' be of the second type with respect to the inclusion i'. By Theorem 6, we have ${\mu^{j}}'_*(x')\neq 0$ and ${\tau^{j}}'_*(x')=0$ for all $j'\geq 1$. Using the same arguments as above, we can conclude from Diagram (4.13) that $\mu^j(x)\neq 0$ for all $j\geq 1$. If there exists an index j such that $\tau^j_*(x)\neq 0$, then, by Corollary 1, the element x is of the first type with respect to the inclusion i. Otherwise, if $\tau^j_*(x)=0$ for all $j\geq 1$, then, by Theorem 6, the element x is of the second type with respect to the inclusion i. $\square$

Remark 1. *Let $\mathcal{S}$ denote Diagram (3.12) for an inclusion i of index 2. Any morphism (f,g) from i to another inclusion i' of index 2 defines a diagram $\mathcal{S}^r$ of spectra, that is similar to Diagram (3.12), and consists of the spectra of the corresponding relative groups. This follows from the existence of the diagrams $\mathcal{D}$ and $\mathcal{D}^-$ of relative groups. The top spectrum in the diagram $\mathcal{S}^r$ is $\mathbb{L}_n(\pi \xrightarrow{g} \pi')$. Similarly to the above definitions, we can define the elements of the first and second type in the group $\mathbb{L}_n(\pi \xrightarrow{g} \pi')$ with respect to the pair of inclusions (i,i').*

Let us denote by $\mathcal{E} = \pi_n(\mathcal{S})$ the commutative diagram of groups obtained from Diagram (3.12) applying the functor π_n. Note that the central vertical row of squares in $\mathcal{E}$ coincides with the vertical row of squares from Diagram (2.2).

Corollary 2. *Under the assumptions of Theorem 7, suppose that (f,g) has a left inverse morphism (retraction) (r,p) from i' to i. Then the diagram $\mathcal{E}'$ splits into the direct sum of the diagram $\mathcal{E}$, that corresponds to the inclusion i, and the relative diagram $\mathcal{E}^r = \pi_n(\mathcal{S}^r)$ with the top group $L_n(\pi \xrightarrow{g} \pi')$. The element*

$$x' = x \oplus y \in L_n(\pi) \oplus L_n(\pi \to \pi') = L_n(\pi') \tag{4.15}$$

has the first type with respect to the inclusion i if and only if one (or both) of the elements x and y has the first type with respect to i and (i,i'), respectively. The element x' in (4.15) has the second type with respect to the inclusion i if and only if both x and y are not elements of the first type and one of them (or both) has the second type with respect to i and (i,i'), respectively.

Proof. The functoriality of the diagram $\mathcal{E}$ and of the relative groups provides the exact sequence of diagrams

$$0 \to \mathcal{E} \to \mathcal{E}' \to \mathcal{E}^r \to 0$$

where the map $\mathcal{E} \to \mathcal{E}'$ has a left inverse. Thus we obtain a direct sum decomposition $\mathcal{E}' = \mathcal{E} \oplus \mathcal{E}^r$. Now the statement of the corollary follows from Theorem 7. $\square$

5. Elements of the second type and Browder-Quinn surgery obstruction groups.

In this section we describe some relations between the elements of the second type and the Browder–Quinn surgery obstruction groups for manifolds with a filtration (see [4], [6], [10], [19], and [34]). The main results of this section are Theorems 8 and 9. In Theorem 8 we obtain some braids of exact sequences which interpret in geometrical sense the pullback squares

$$\begin{array}{ccc} \mathbb{X}_{i,0} & \to & \mathbb{X}_{0,0} \\ \downarrow & & \downarrow \\ \mathbb{X}_{i,i} & \to & \mathbb{X}_{0,i} \end{array} \tag{5.1}$$

fitting in Diagram (3.12). In Theorem 9 we give a geometrical formulation of the necessary condition for an element in a Wall group to lie in the image of the assembly map. This condition is equivalent to the conditions for an element in a Wall group to have the first or the second type.

Let $i : \rho \to \pi$ be an inclusion of index 2 between groups, and X a closed connected topological n–dimensional manifold with fundamental group $\pi_1(X) = \pi$. Let us consider a map

$$\phi : X^n \to \mathbb{R}P^N \tag{5.2}$$

from X to a real projective space of high dimension which induces an epimorphism of the fundamental groups $\pi \to \mathbb{Z}/2$ with kernel ρ. Let k be a natural number such that $n - k \geq 5$. Changing the map ϕ in its homotopy class, we can make it transversal to all the submanifolds fitting in the standard filtration

$$\mathbb{R}P^{n-k} \subset \mathbb{R}P^{n-k+1} \subset \cdots \subset \mathbb{R}P^{N-1} \subset \mathbb{R}P^N \tag{5.3}$$

of the projective space $\mathbb{R}P^N$. Let $X_j = \phi^{-1}(\mathbb{R}P^{N-j})$ be the transversal preimage for $k \geq j \geq 0$. Then we obtain a filtration $\mathcal{X}$

$$X_k \subset X_{k-1} \subset \cdots \subset X_2 \subset X_1 \subset X_0 = X \tag{5.4}$$

of X by means of locally flat embedded submanifolds. Moreover, every pair of manifolds $X_{j+1} \subset X_j$ is a Browder-Livesay pair with respect to the inclusion i for j even, and it is a Browder-Livesay pair with respect to the inclusion i_- for j odd. Note that every pair of manifolds fitting in the filtration $\mathcal{X}$ (5.4) is a topological manifold pair in the sense of Ranicki [26, §7.2]. The obtained filtration $\mathcal{X}$ in (5.4) will be called the *Browder–Livesay filtration with respect to the inclusion* i (see [4], [10], and [19]).

For $0 \leq j \leq k$, let $\mathcal{X}_j$ denote the restricted filtration

$$X_j \subset X_{j-1} \subset \cdots \subset X_2 \subset X_1 \subset X_0 = X. \tag{5.5}$$

The filtration $\mathcal{X}$ yields a filtration of manifolds with boundary

$$(X_{k-1} \setminus X_k, \partial(X_{k-1} \setminus X_k)) \subset (X_{k-2} \setminus X_k, \partial(X_{k-2} \setminus X_k)) \subset \cdots \subset (X \setminus X_k, \partial(X \setminus X_k)) \tag{5.6}$$

which is a $\mathcal{C}$-stratified manifold with boundary (see [4], [7], [10], [19], and [34]). We denote the filtration in (5.6) by $\overline{\mathcal{X}_k} = \overline{\mathcal{X}}$. Let $\partial\overline{\mathcal{X}}$ denote the filtration

$$\partial(X_{k-1} \setminus X_k) \subset \partial(X_{k-2} \setminus X_k) \subset \cdots \subset \partial(X_1 \setminus X_k) \subset \partial(X_0 \setminus X_k) \tag{5.7}$$

of the boundary $\partial(X_0 \setminus X_k)$ which arises from (5.6). There is a natural inclusion $\partial\overline{\mathcal{X}} \to \overline{\mathcal{X}}$ of stratified spaces.

Similarly, any subfiltration $\mathcal{X}_j$ $(1 \leq j \leq k)$ defines the filtrations $\overline{\mathcal{X}_j}$ and $\partial\overline{\mathcal{X}_j}$.

The filtration $\mathcal{X}$ is a stratified manifold in the sense of Browder-Quinn (see [7] and [34]) and hence the groups $L_*^{BQ}(\mathcal{X})$ are defined. These groups are realized on the spectra level by a spectrum $\mathbb{L}_*^{BQ}(\mathcal{X})$, and there are isomorphisms

$$\pi_i\left(\mathbb{L}_*^{BQ}(\mathcal{X})\right) = L_i^{BQ}(\mathcal{X}). \tag{5.8}$$

In our notations the subscript $*$ in the obstruction group $L_*^{BQ}(\mathcal{X})$ of the filtration in (5.4) equals the dimension $n - k$ of the smallest manifold of the filtration.

Now we recall the inductive definition of the spectra $\mathbb{L}^{BQ}(\mathcal{X})$ that realizes the Browder–Quinn surgery obstruction groups $L^{BQ}(\mathcal{X})$ [34, p.129] (see also [4], [10], and [19]).

By definition, we have $\mathbb{L}^{BQ}(\mathcal{X}_0) = \mathbb{L}(\pi)$. The spectrum $\mathbb{L}^{BQ}(\mathcal{X}_1)$ is defined as a homotopical fiber of the composition

$$\mathbb{L}(\pi_1(X_1)) \to \mathbb{L}\left(\pi_1(\partial U)\right) \to \mathbb{L}(\pi_1(X_0 \setminus X_1)) \tag{5.9}$$

where $\partial U = \partial(X_0 \setminus X_1)$, the first map is the transfer map as in (3.5), and the second map is induced by the inclusion $\partial(X_0 \setminus X_1) \subset X_0 \setminus X_1$. It follows from (3.6) and (5.9) that there exists a homotopy equivalence

$$\mathbb{L}^{BQ}(\mathcal{X}_1) \simeq \mathbb{L}P(F) \tag{5.10}$$

where F is the square in the splitting problem for the pair (X_0, X_1).

Let us consider the filtration $\mathcal{X}$ in (5.4) consisting of $k+1$ manifolds. By inductive assumption, we suppose that the spectra $\mathbb{L}^{BQ}(\mathcal{Y})$ are already defined for any filtration $\mathcal{Y}$ consisting of i manifolds $(1 \leq i \leq k)$ with

$$\pi_*(\mathbb{L}^{BQ}(\mathcal{Y})) = L^{BQ}_*(\mathcal{Y}) \tag{5.11}$$

where $L^{BQ}_*(\mathcal{Y})$ is the Browder–Quinn surgery obstruction group for the stratified space $\mathcal{Y}$.

The stratified spaces $\partial\overline{\mathcal{X}_k}$ and $\overline{\mathcal{X}_k}$ consists of k manifolds. Hence, by inductive assumption, the spectra $\mathbb{L}^{BQ}(\partial\overline{\mathcal{X}_k})$ and $\mathbb{L}^{BQ}(\overline{\mathcal{X}_k})$ have been defined. The transfer map

$$L_{n-k}(\pi_1(X_k)) \to L^{BQ}_{n-k}(\partial\overline{\mathcal{X}_k})$$

is realized on the spectra level (see [34]) by a map of spectra

$$\mathbb{L}(\pi_1(X_k)) \to \mathbb{L}^{BQ}(\partial\overline{\mathcal{X}_k}). \tag{5.12}$$

By definition, the spectrum $\mathbb{L}^{BQ}(\mathcal{X}_k)$ fits into a cofibration

$$\mathbb{L}^{BQ}(\mathcal{X}_k) \to \mathbb{L}(\pi_1(X_k)) \to \mathbb{L}^{BQ}(\overline{\mathcal{X}_k}). \tag{5.13}$$

The second map in (5.13) is the composition

$$\mathbb{L}(\pi_1(X_k)) \to \mathbb{L}^{BQ}(\partial\overline{\mathcal{X}_k}) \to \mathbb{L}^{BQ}(\overline{\mathcal{X}_k}) \tag{5.14}$$

where the first map is the map in (5.12) and the second map is induced by the inclusion of filtrations $\partial\overline{\mathcal{X}_k} \subset \overline{\mathcal{X}_k}$. Note that for $1 \leq j \leq k$ we have the following cofibration sequence

$$\cdots \to \mathbb{L}^{BQ}(\mathcal{X}_j) \to \mathbb{L}(\pi_1(\mathcal{X}_j)) \to \mathbb{L}^{BQ}(\overline{\mathcal{X}_j}) \to \Sigma\mathbb{L}^{BQ}(\mathcal{X}_j) \to \cdots \tag{5.15}$$

which follows from the inductive definition.

For $1 \leq j \leq k$ we denote by $\mathcal{Y}_j$ the subfiltration

$$X_j \subset X_{j-1} \subset \cdots \subset X_2 \subset X_1 \tag{5.16}$$

of the filtration $\mathcal{X}$. Then the filtrations $\overline{\mathcal{Y}_j}$ and $\partial\overline{\mathcal{Y}_j}$ are defined as before.

By [10] for $1 \le j \le k$ we have the following braid of exact sequences

$$(5.17)\qquad \begin{array}{ccccccccc} \to & L_{n+j}(\pi_1(X_0\backslash X_1)) & & \to & & L^{BQ}_{n+1}(\mathcal{X}_{j-1}) & \to & & LN_{n-1}(\rho\to\pi_1(X_{j-1})) \to \\ & \nearrow & \searrow & & \nearrow & \searrow & & \nearrow & \searrow \\ & & & L^{BQ}_n(\mathcal{X}_j) & & & L^{BQ}_{n+1}(\mathcal{Y}_{j-1}) & & \\ & \searrow & \nearrow & & \searrow & \nearrow & & \searrow & \nearrow \\ \to & LN_n(\rho\to\pi_1(X_{j-1})) & & \to & & L^{BQ}_n(\mathcal{Y}_j) & \to & & L_{n+j-1}(\pi_1(X_0\backslash X_1)) \to \end{array}$$

which is realized on the spectra level.

Now we recall the definition of spectra $\mathbb{L}SF(\mathcal{X}_j)$ with homotopy groups

$$(5.18)\qquad \pi_n(\mathbb{L}SF(\mathcal{X}_j)) = LSF_n(\mathcal{X}_j)$$

that are the obstruction groups for splitting a simple homotopy equivalence $f : M \to X$ along the subfiltration $\mathcal{Y}_j \subset \mathcal{X}_j$ (see [4]). More precisely, the group $LSF_{n-j}(\mathcal{X}_j)$ is the obstruction group for splitting a simple homotopy equivalence $f : M \to X$ along the subfiltration $\mathcal{Y}_j$.

Forgetting the smallest manifold yields the following natural maps of spectra

$$(5.19)\qquad \mathbb{L}^{BQ}(\mathcal{Y}_k) \to \Omega\mathbb{L}^{BQ}(\mathcal{Y}_{k-1}) \to \cdots \to \Omega^{k-2}\mathbb{L}^{BQ}(\mathcal{Y}_2) \to \Omega^{k-1}\mathbb{L}(\pi_1(X_1)).$$

The composition in (5.19) and the transfer map fitting in (3.6) give the composition ($1 \le j \le k$)

$$(5.20)\qquad \mathbb{L}^{BQ}(\mathcal{Y}_j) \to \Omega^{j-1}\mathbb{L}(\pi_1(X_1)) \to \Omega^j\mathbb{L}(\pi_1(X \setminus X_1) \to \pi_1(X)).$$

We define the spectrum $\mathbb{L}SF(\mathcal{X}_j)$ as the homotopy fiber of the composition in (5.20), and let $LSF_*(\mathcal{X}_j) = \pi_*(\mathbb{L}SF(\mathcal{X}_j))$.

Lemma 1. [4] *The groups $LSF_* = LSF_*(\mathcal{X})$ fit in the commutative braid of exact sequences*

$$(5.21)\qquad \begin{array}{ccccccccc} \to & L_n(\rho) & & \longrightarrow & & L_n(\pi) & \to & & LSF_{m-1} \to \\ & \nearrow & \searrow & & \nearrow & \searrow & & \nearrow & \searrow \\ & & & L^{BQ}_m(\mathcal{X}) & & & L_n(\rho\to\pi) & & \\ & \searrow & \nearrow & & \searrow & \nearrow & & \searrow & \nearrow \\ \to & LSF_m & & \longrightarrow & & L^{BQ}_m(\mathcal{Y}) & \longrightarrow & & L_{n-1}(\rho) \to \end{array}$$

where $\rho = \pi_1(X_0 \setminus X_1)$, $\pi = \pi_1(X_0)$, $\mathcal{X} = \mathcal{X}_k$, $\mathcal{Y} = \mathcal{Y}_k$, and $m = n - k$. Diagram (5.21) is realized on the spectra level.

Note that $\mathbb{L}SF(\mathcal{X}_1) = \mathbb{L}S(F)$. Similarly to the sequence in (5.19), we obtain the following sequence formed by natural maps of spectra

$$(5.22)\qquad \Sigma^k\mathbb{L}^{BQ}(\mathcal{X}_k) \to \Sigma^{k-1}\mathbb{L}^{BQ}(\mathcal{X}_{k-1}) \to \cdots \to \Sigma^1\mathbb{L}^{BQ}(\mathcal{X}_1) \to \mathbb{L}(\pi_1(X)).$$

Let $\mathcal{X}^i_j$ ($0 \le i \le j \le k$) denote the subfiltration

$$(5.23)\qquad X_j \subset X_{j-1} \subset \cdots \subset X_{i+1} \subset X_i$$

of the filtration $\mathcal{X}$. Forgetting the largest manifold of the filtration yields the following natural maps of spectra

$$(5.24)\qquad \Sigma^k\mathbb{L}^{BQ}(\mathcal{X}^0_k) \to \Sigma^k\mathbb{L}^{BQ}(\mathcal{X}^1_k) \to \cdots \to \Sigma^k\mathbb{L}^{BQ}(\mathcal{X}^{k-1}_k) \to \Sigma^k\mathbb{L}(\pi_1(X_k)).$$

Proposition 3. *The sequence of maps in (5.22) coincides with the sequence of maps of spectra*

$$\mathbb{X}_{k,0} \to \mathbb{X}_{k-1,0} \to \cdots \to \mathbb{X}_{k,2} \to \mathbb{X}_{k,k}$$

arising from Diagram (3.12).

The sequence of maps in (5.24) coincides with the sequence of maps of spectra

$$\mathbb{X}_{k,0} \to \mathbb{X}_{k,1} \to \cdots \to \mathbb{X}_{1,0} \to \mathbb{X}_{0,0}$$

arising from Diagram (3.12).

Proof. The first statement was proved in [19] (see also [10]). The second statement follows from the realization of the central squares in Diagram (5.17) on the spectra level. This square coincides with the pullback square

$$\begin{array}{ccc} \mathbb{X}_{j,0} & \to & \mathbb{X}_{j-1,0} \\ \downarrow & & \downarrow \\ \mathbb{X}_{j,1} & \to & X_{j-1,1} \end{array}$$

arising from Diagram (3.12). □

Let $\mathcal{V}_j$ $(0 \leq j \leq k-1)$ be the subfiltration

$$X_k \subset X_j \subset X_{j-1} \cdots \subset X_1 \subset X_0 = X$$

of the filtration $\mathcal{X}$. Let $\mathcal{Z}_j = \overline{\mathcal{V}_j}$ $(0 \leq j \leq k-1)$ be the filtration of manifolds with boundary obtained by cutting the submanifold X_k from the filtration $\mathcal{V}_j$. Note that $\overline{\mathcal{X}_k} = \mathcal{Z}_{k-1}$ and $\mathbb{L}^{BQ}(\overline{\mathcal{X}_k}) = \mathbb{L}^{BQ}(\mathcal{Z}_{k-1})$. Similarly to the sequences in (5.19) and (5.22), we obtain the following sequence of natural maps of spectra

(5.25)
$$\Sigma^{k-1}\mathbb{L}^{BQ}(\overline{\mathcal{X}_k}) = \Sigma^{k-1}\mathbb{L}^{BQ}(\mathcal{Z}_{k-1}) \to \Sigma^{k-2}\mathbb{L}^{BQ}(\mathcal{Z}_{k-2}) \to \cdots \to \Sigma^{1}\mathbb{L}^{BQ}(\mathcal{Z}_{1}) \to \mathbb{L}(\pi_1(X\backslash X_k)).$$

Thus we can write down the composition

$$\Sigma^{j}\mathbb{L}^{BQ}(\mathcal{Z}_j) \to \mathbb{L}(\pi_1(X \setminus X_k)) \to \mathbb{L}(\pi_1(X)) \tag{5.26}$$

where the second map is induced by the natural inclusion $(X \setminus X_k) \subset X$. Let us denote by $\mathbb{L}^{BQ}(\mathcal{Z}_j \to \mathcal{X})$ a homotopical cofiber of the composition in (5.26), and by

$$L_*^{BQ}(\mathcal{Z}_j \to \mathcal{X}) = \pi_*(\mathbb{L}^{BQ}(\mathcal{Z}_j \to \mathcal{X})) \tag{5.27}$$

its homotopy groups.

Let us consider the maps

$$\begin{array}{ccc} \Sigma^k\mathbb{L}^{BQ}(\mathcal{X}_k) & \longrightarrow & \mathbb{L}(\pi_1(X)) \\ \Big\downarrow & & \\ \Sigma^k\mathbb{L}(\pi_1(X_k)) & & \end{array} \tag{5.28}$$

which are given by the compositions in (5.22) and (5.24).

Theorem 8.
i) The spectrum

$$\mathbb{L}^{BQ}(\mathcal{Z}_{k-1} \to \mathcal{X}) = \mathbb{L}^{BQ}(\overline{\mathcal{X}} \to \mathcal{X})$$

fits in the pushout square

$$\begin{array}{ccc} \Sigma^k\mathbb{L}^{BQ}(\mathcal{X}_k) & \longrightarrow & \mathbb{L}(\pi_1(X)) \\ \big\downarrow & & \big\downarrow \\ \Sigma^k\mathbb{L}(\pi_1(X_k)) & \longrightarrow & \mathbb{L}^{BQ}(\overline{\mathcal{X}} \to \mathcal{X}) \end{array} \tag{5.29}$$

defined by Diagram (5.28).
ii) The square in (5.29) coincides with the pullback square

$$\begin{array}{ccc} \mathbb{X}_{k,0} & \xrightarrow{\varphi^k} & \mathbb{X}_{0,0} \\ \big\downarrow{\scriptstyle \psi^k} & & \big\downarrow{\scriptstyle \mu^k} \\ \mathbb{X}_{k,k} & \xrightarrow{\nu^k} & \mathbb{X}_{0,k} \end{array} \tag{5.30}$$

fitting into Diagram (3.12).
iii) The homotopical fiber of the vertical maps in Diagram (5.29) is the spectrum $\Sigma^{k-1}\mathbb{L}^{BQ}(\overline{\mathcal{X}})$, the homotopical fiber of the horizontal maps in Diagram (5.29) is the spectrum $\Sigma^k\mathbb{L}SF(\mathcal{X})$, and there is a homotopy commutative diagram of spectra

$$\begin{array}{ccccc} & & \Sigma^{k-1}\mathbb{L}^{BQ}(\overline{\mathcal{X}}) & = & \Sigma^{k-1}\mathbb{L}^{BQ}(\overline{\mathcal{X}}) \\ & & \big\downarrow & & \big\downarrow \\ \Sigma^k\mathbb{L}SF(\mathcal{X}) & \longrightarrow & \Sigma^k\mathbb{L}^{BQ}(\mathcal{X}) & \longrightarrow & \mathbb{L}(\pi_1(X_0)) \\ \big\| & & \big\downarrow & & \big\downarrow \\ \Sigma^k\mathbb{L}SF(\mathcal{X}) & \longrightarrow & \Sigma^k\mathbb{L}(\pi_1(X_k)) & \longrightarrow & \mathbb{L}^{BQ}(\overline{\mathcal{X}} \to \mathcal{X}) \end{array} \tag{5.31}$$

whose rows and columns are cofibrations.
iv) The homotopy commutative diagram in (5.31) coincides with Diagram (3.15) for $j = k$.

Proof. Let us consider the diagram

$$\begin{array}{ccccc} & & \Sigma^{k-1}\mathbb{L}^{BQ}(\overline{\mathcal{X}}) & & \\ & & \big\downarrow & & \\ \Sigma^k\mathbb{L}SF(\mathcal{X}) & \longrightarrow & \Sigma^k\mathbb{L}^{BQ}(\mathcal{X}) & \longrightarrow & \mathbb{L}(\pi_1(X)) \\ & & \big\downarrow & & \\ & & \Sigma^k\mathbb{L}(\pi_1(X_k)) & & \end{array} \tag{5.32}$$

which arises from (5.15) and Diagram (5.21) on the spectra level. The composite map

$$\Sigma^{k-1}\mathbb{L}^{BQ}(\overline{\mathcal{X}}) \to \Sigma^k\mathbb{L}^{BQ}(\mathcal{X}) \to \mathbb{L}(\pi_1(X)) \tag{5.33}$$

from (5.32) fits in the following homotopy commutative diagram of spectra

$$
\begin{array}{ccccc}
\Sigma^{k-1}\mathbb{L}^{BQ}(\overline{\mathcal{X}}) & \longrightarrow & \Sigma^{k}\mathbb{L}^{BQ}(\mathcal{X}) & \longrightarrow & \mathbb{L}(\pi_1(X)) \\
\| & & \| & & \| \\
\Sigma^{k-1}\mathbb{L}^{BQ}(\mathcal{Z}_{k-1}) & \longrightarrow & \Sigma^{k}\mathbb{L}^{BQ}(\mathcal{V}_{k-1}) & \longrightarrow & \mathbb{L}(\pi_1(X)) \\
\downarrow & & \downarrow & & \downarrow \\
\Sigma^{k-2}\mathbb{L}^{BQ}(\mathcal{Z}_{k-2}) & \longrightarrow & \Sigma^{k}\mathbb{L}^{BQ}(\mathcal{V}_{k-2}) & \longrightarrow & \mathbb{L}(\pi_1(X)) \\
\downarrow & & \downarrow & & \downarrow \\
\dots & & \dots & & \dots \\
\downarrow & & \downarrow & & \downarrow \\
\Sigma\mathbb{L}^{BQ}(\mathcal{Z}_{1}) & \longrightarrow & \Sigma^{k}\mathbb{L}^{BQ}(\mathcal{V}_{1}) & \longrightarrow & \mathbb{L}(\pi_1(X)) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{L}^{BQ}(\mathcal{Z}_{0}) & \longrightarrow & \Sigma^{k}\mathbb{L}^{BQ}(\mathcal{V}_{0}) & \longrightarrow & \mathbb{L}(\pi_1(X)) \\
\| & & \| & & \| \\
\mathbb{L}(X\setminus X_k) & \longrightarrow & \Sigma^{k}\mathbb{L}P(F_k) & \longrightarrow & \mathbb{L}(\pi_1(X))
\end{array}
\tag{5.34}
$$

in which the rows are defined for the filtrations $\mathcal{V}_j$ similarly to (5.33) and F_k is the square in the splitting problem for the manifold pair $X_k \subset X$. The vertical maps are the natural maps induced by forgetting the submanifold in the passing from $\mathcal{V}_j$ to $\mathcal{V}_{j-1}$. The commutativity of each square follows from the functoriality of the transfer maps in (5.12). Hence the composition in (5.25) coincides with the composition in (5.33). The row and the column of Diagram (5.32) are cofibrations, and hence the cofibres of the sloping maps

$$\Sigma^{k-1}\mathbb{L}^{BQ}(\overline{\mathcal{X}}) \to \Sigma^{k}\mathbb{L}^{BQ}(\mathcal{X})$$

and

$$\Sigma^{k}\mathbb{L}SF(\mathcal{X}) \to \Sigma^{k}\mathbb{L}(\pi_1(X_k))$$

are naturally homotopy equivalent (see [2], [3], and [22]). Thus we obtain Diagram (5.31) in which the right bottom square is a pushout (and pullback). Now the theorem follows from the unity property for pushout squares and from Proposition 3. □

The left vertical column of squares in (5.34) comes from pushout squares since the fibers of the horizontal maps are naturally homotopy equivalent. We now describe the cofibres of the vertical maps in these squares.

Proposition 4. *Let $\overline{F}_k^{j-1}$ be a square in the splitting problem for the manifold pair $(X_j \setminus X_k) \subset (X_{j-1} \setminus X_k)$. Then the cofiber of the map*

$$\Sigma^{j}\mathbb{L}^{BQ}(\mathcal{Z}_j) \to \Sigma^{j-1}\mathbb{L}^{BQ}(\mathcal{Z}_{j-1})$$

from Diagram (5.34) is $\Sigma^{j+1}\mathbb{L}S(\overline{F}_k^{j-1})$.

Proof. For the filtration $\overline{\mathcal{X}}$ we can construct a diagram which is similar to Diagram (3.43) obtained for the filtration $\mathcal{X}$. Now the result follows. □

Now we describe some relations between various spectra $\mathbb{L}^{BQ}(\overline{\mathcal{X}}_j \to \mathcal{X}_j)$ and $\mathbb{L}^{BQ}(\overline{\mathcal{X}}_j)$ for a given Browder-Livesay filtration $\mathcal{X} = \mathcal{X}_k$ as in (5.4).

Proposition 5. *Let $\mathcal{X}$ be the Browder-Livesay filtration defined above. Then we have the following homotopy commutative diagram of spectra*

(5.35)
$$\begin{array}{ccccc}
\mathbb{L}(\pi_1(X\setminus X_1)) & \to & \Sigma\mathbb{L}^{BQ}(\overline{\mathcal{X}}_2) & \to\cdots\to & \Sigma^{k-1}\mathbb{L}^{BQ}(\overline{\mathcal{X}}_k) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{L}(\pi_1(X)) & = & \mathbb{L}(\pi_1(X)) & =\cdots= & \mathbb{L}(\pi_1(X)) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{L}(\pi_1(X\setminus X_1)\to\pi_1(X)) & \to & \mathbb{L}^{BQ}(\overline{\mathcal{X}}_2\to\mathcal{X}_2) & \to\cdots\to & \mathbb{L}^{BQ}(\overline{\mathcal{X}}_k\to\mathcal{X}_k)
\end{array}$$

whose vertical columns are cofibrations.

Proof. By Theorem 8, Diagram (5.35) coincides with the left part of Diagram (4.6), so the result follows. □

Proposition 6. *Under the assumptions of Proposition 5, we have the following homotopy commutative diagram of spectra*

$$\begin{array}{ccccc}
\mathbb{L}(\pi_1(X\setminus X_1)) & \to & \Sigma\mathbb{L}^{BQ}(\overline{\mathcal{X}}_2) & \to\cdots\to & \Sigma^{k-1}\mathbb{L}^{BQ}(\overline{\mathcal{X}}_k) \\
\downarrow & & \downarrow & & \downarrow \\
\Sigma\mathbb{L}^{BQ}(\mathcal{X}_1) & \leftarrow & \Sigma^2\mathbb{L}^{BQ}(\mathcal{X}_2) & \leftarrow\cdots\leftarrow & \Sigma^k\mathbb{L}^{BQ}(\mathcal{X}_k) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{L}(\pi_1(X)) & = & \mathbb{L}(\pi_1(X)) & =\cdots= & \mathbb{L}(\pi_1(X))
\end{array} \qquad (5.36)$$

whose vertical composite maps coincide with the vertical maps in (5.35) and the maps in the middle row are induced by the natural forgetful maps.

Proof. The result follows from Theorem 8 and Diagrams (3.12), (4.6), and (5.35). □

To complete the section, we give a necessary geometrical condition for an element in a Wall group to lie in the image of the assembly map.

Theorem 9. *Let an element $x \in L_n(\pi)$ be realized by a normal map of closed manifolds. Then for any inclusion $\rho \to \pi$ of index 2 there exist a closed manifold X^n with $\pi_1(X) = \pi$ and a Browder-Livesay filtration $\mathcal{X} = \mathcal{X}_k$ with respect to the inclusion i such that*

$$x \in \mathrm{Im}\left(L^{BQ}_{n-k+1}(\overline{\mathcal{X}}_k) \to L_n(\pi)\right). \qquad (5.36)$$

Proof. Let an element x be realized by a normal map of closed manifolds. Then for any inclusion $i : \rho \to \pi$ of index 2 and for any integer $k \geq 1$ there exists a Browder-Livesay filtration $\mathcal{X}_k$ with respect to i such that (see [10] and [19])

$$x \in \mathrm{Im}\left(L^{BQ}_{n-k}(\mathcal{X}_k) \to L_n(\pi_1(X))\right). \qquad (5.37)$$

But x cannot have the second type with respect to i. Hence, by Theorem 6, there exists an integer $j \geq 1$ such that $\mu^j_*(x) = 0$ and

$$x \in \mathrm{Im}(\pi_n(\mathbb{H}_j) \to L_n(\pi)).$$

Now the result follows from Theorem 8, part iv. □

6. Realizing surgery and splitting obstructions.

In this section we apply the obtained results for computing the assembly map for various cases of groups. Then we obtain further applications on the problem of realizing various surgery and splitting obstructions by simple homotopy equivalences of closed manifolds. The main result of the section are Theorems 10-15.

The representation of an element x in a Wall group $L_n(\pi)$ as a surgery obstruction of a degree-one normal map $(f,b): M^n \to X^n$ between closed manifolds with $\pi_1(X) = \pi$ is one of the basic problems in Surgery Theory. This is equivalent to the fact that the element x belongs to the image of the assembly map $A: H_n(B\pi, \mathbf{L}_\bullet) \to L_n(\pi)$, where $B\pi = K(\pi, 1)$ is the classifying space of the group π (see [31]).

Let $Y^{n-q} \subset X^n$ be a submanifold of codimension q in a closed connected topological n-manifold X, $n-q \geq 5$. Then the splitting obstruction groups $LS_{n-q}(F)$ are defined (see [26] and [31]), where F is the square in (1.3).

Let
$$(f,b): (M^n, \nu_M) \to (X^n, \nu_X)$$
be a degree-one normal map. It is defined an obstruction to the existence of a simple homotopy equivalence which splits along Y
$$f': M'^n \to X^n$$
in the class of the normal cobordism of (f,b). This obstruction lies in the surgery obstruction group $LP_{n-q}(F)$, where F is the square in (1.3). This group depends as well only on F and on the dimension $n-q \pmod 4$. It follows from geometrical definitions that there are obvious maps [31]

$$\begin{array}{cc} LS_{n-q}(F) \to LP_{n-q}(F), & LP_{n-q}(F) \xrightarrow{p_0} L_n(\pi_1(X)) \\ LP_{n-q}(F) \xrightarrow{p_1} L_{n-q}(\pi_1(Y)), & LS_{n-q}(F) \xrightarrow{s} L_{n-q}(\pi_1(Y)) \end{array}. \tag{6.1}$$

All the elements of the groups $LS_*(F)$ and $LP_*(F)$ are realized by maps of manifolds with boundary (see [26] and [31]). For a manifold pair $Y^{n-q} \subset X^n$ we have a commutative diagram

$$\begin{array}{cccccccc} \cdots\to & L_{n+1}(\pi) & \to & \mathcal{S}^{TOP}(X) & \to & [X, G/TOP] & \to & L_n(\pi) \\ & & \searrow & \downarrow\theta & & \downarrow & \nearrow & \\ & & & LS_{n-q}(F) & \to & LP_{n-q}(F) & & \end{array} \tag{6.2}$$

in which the upper row is the surgery exact sequence in (1.3), and the other row lies in the following braid of exact sequences (see [26] and [31])

$$\begin{array}{ccccccccccccc} \to & L_n(\pi_1(X\backslash Y)) & & \longrightarrow & & L_n(\pi_1(X)) & & \to & & LS_{n-q-1}(F) & & \longrightarrow & \\ \nearrow & & \searrow & & p_0\nearrow & & \searrow & & \nearrow & & \searrow & & \\ & & & LP_{n-q}(F) & & & & L_n^{rel} & & & & & \\ \searrow & & \nearrow & & \searrow p_1 & & \nearrow & & \searrow & & \nearrow & & \\ \to & LS_{n-q}(F) & & \xrightarrow{s} & & L_{n-q}(\pi_1(Y)) & & \longrightarrow & & L_{n-1}(\pi_1(X\backslash Y)) & & \to & \end{array} \tag{6.3}$$

where $L_n^{rel} = L_n(\pi_1(X \setminus Y) \to \pi_1(X))$.

All the maps from (6.1) fit into Diagram (6.3). Note that Diagrams (6.2) and (6.3) are realized on the spectra level (see [2], [3], [25], and [31]).

It follows from Diagram (6.2) that the problem of realizing the splitting obstructions by homotopy equivalences of closed manifolds closely relates to the computation of maps in the surgery exact sequence. In particular, for a given manifold X it relates to the computation of the map $\theta : \mathcal{S}^{TOP}(X) \to LS_{n-q}(F)$ from Diagram (6.2). The simplest cases for realizing the splitting obstructions in Browder-Livesay groups are considered in [1] and [20].

Lemma 2. *i) Let $x \in LS_{n-q}(F)$ be an element such that $y = s(x) \in L_{n-q}(\pi_1(Y))$ is not realized by a normal map of closed manifolds. Then x cannot be the splitting obstruction of a simple homotopy equivalence of closed manifolds.*

ii) Let $x \in LP_{n-q}(F)$ be an element such that $y = p_1(x) \in L_{n-q}(\pi_1(Y))$ is not realized by a normal map of closed manifolds. Then x cannot be the obstruction to surgery on a closed manifold pair.

iii) Let $x \in LP_{n-q}(F)$ be an element such that $y = p_0(x) \in L_n(\pi_1(X))$ is not realized by a normal map of closed manifolds. Then x cannot be the obstruction to surgery on a closed manifold pair.

Proof. The results follow from the geometrical definitions of the maps s, p_1, and p_0 in Diagram (6.3) [31]. $\square$

According to notations in [20], we have isomorphisms $LN_*(1 \to \mathbb{Z}/2^{\pm}) \cong BL_{n+1}(\pm)$ and the map $s : LN_*(1 \to \mathbb{Z}/2^{\pm}) \to L_n(\mathbb{Z}/2^{\mp})$ in Lemma 2 coincides with the map $l_{n+1} : BL_{n+1}(\pm) \to L_n(\mathbb{Z}/2^{\mp})$ from [20]. The results of [20, Chapter III.3.2] provides some realization theorems for the elements of the groups $LN_*(1 \to \mathbb{Z}/2^{\pm})$.

Below we shall give several examples in which the splitting obstruction groups are very large but the image of the map θ is trivial or very small.

Now we discuss the realization of the splitting obstructions for manifolds X such that $\pi_1(X) = \pi$ is an elementary abelian 2-group of positive rank $r + 1$. Let $i : \rho \to \pi$ be an inclusion of index 2. Then the group ρ is an elementary 2-group of rank r. Denote by π^- the group π equipped with a nontrivial orientation. Up to isomorphism of inclusions, we have two oriented subgroups of index 2 of the group π. The first one is given by an inclusion

$$i : \rho \to \pi^- \tag{6.4}$$

where the subgroup ρ has a trivial orientation. In this case, $\pi^- \cong \rho \oplus \mathbb{Z}/2^-$. The second one is given by an inclusion

$$j : \rho^- \to \pi^- \tag{6.5}$$

where the subgroup ρ^- has a nontrivial orientation, and

$$\pi^- \cong \rho^- \oplus \mathbb{Z}/2^- \cong \rho^- \oplus \mathbb{Z}/2. \tag{6.6}$$

Note that in (6.5) the subgroup ρ^- has a direct summand $\mathbb{Z}/2$ with a nontrivial orientation. Recall that there are the following isomorphisms [31]

$$\begin{aligned} LN_n(\rho \to \pi^-) &\cong LN_{n+2}(\rho \to \pi) \cong L_n(\rho), \\ LN_n(\rho^- \to \pi^-) &\cong LN_{n+2}(\rho^- \to \pi^-) \cong L_n(\rho^-). \end{aligned} \tag{6.7}$$

The surgery and splitting groups for elementary 2-groups are well known (see, for example, [17], [23], and [33]). In the oriented case we have the following isomorphisms

$$L_n(\rho) = \mathbb{Z}^{2^r} \oplus (\mathbb{Z}/2)^{2^r-r-1-\binom{r}{2}}, (\mathbb{Z}/2)^{2^r-r-1-\binom{r}{2}}, \mathbb{Z}/2, (\mathbb{Z}/2)^{2^r-1} \tag{6.8}$$

for $n = 0, 1, 2, 3 \pmod 4$, respectively. In the nonoriented case we have

$$\begin{array}{rll} L_n(\pi^-) = & (\mathbb{Z}/2)^{2^r-r}, & (\mathbb{Z}/2)^{2^r-r-1} \\ L_n(\rho^-) = & (\mathbb{Z}/2)^{2^{r-1}-r+1}, & (\mathbb{Z}/2)^{2^{r-1}-r} \end{array} \tag{6.9}$$

for $n = 0, 1 \pmod 2$, respectively. In particular, we get

$$L_n(\mathbb{Z}/2^-) = \quad \mathbb{Z}/2, \quad 0, \quad \mathbb{Z}/2, \quad 0, \tag{6.10}$$

for $n = 0, 1, 2, 3 \pmod 4$, respectively.

For every $r \geq 1$, the natural inclusion $\mathbb{Z}/2^- \to \rho^-$ induces an inclusion

$$\delta : \mathbb{Z}/2 = LN_{2k}(\mathbb{Z}/2^- \to \mathbb{Z}/2^- \oplus \mathbb{Z}/2^\pm) \to LN_{2k}(\rho^- \to \rho^- \oplus \mathbb{Z}/2^\pm) \tag{6.11}$$

of a direct summand.

Let us denote by S a subgroup of the group $L_3(\rho)$ which is generated by the images of $\mathbb{Z}/2$ under all the inclusions $\mathbb{Z}/2 = L_3(\mathbb{Z}/2) \to L_3(\rho)$ induced by the inclusions $\mathbb{Z}/2 \to \rho$ on the direct summands $\mathbb{Z}/2$ of the group ρ. It follows from [17], [23], and [33] that $S = (\mathbb{Z}/2)^r$. Using the isomorphisms in (6.7), we obtain the inclusions

$$S \to L_3(\rho) \to LN_3(\rho \to \pi^-) \ \text{ and } \ S \to L_3(\rho) \to LN_1(\rho \to \pi^+). \tag{6.12}$$

Theorem 10. *i) Any nontrivial element of the groups*

$$\begin{array}{rclcl} LN_1(\rho^- \to \pi^-) & \cong & LN_1(\rho^- \to \pi) & \cong & LN_3(\rho^- \to \pi^-) \cong \\ \cong LN_3(\rho^- \to \pi) & \cong & (\mathbb{Z}/2)^{2^{r-1}-r} & & \end{array}$$

and

$$LN_3(\rho \to \pi) \cong LN_1(\rho \to \pi^-) \cong (\mathbb{Z}/2)^{2^r-r-1-\binom{r}{2}}$$

cannot be realized as a splitting obstruction of a simple homotopy equivalence of closed manifolds along a Browder-Livesay submanifold of codimension 1.

ii) For every $r \geq 1$, the elements of the groups

$$\begin{array}{rclcl} LN_0(\rho^- \to \pi^-) & \cong & LN_0(\rho^- \to \pi) & \cong & LN_2(\rho^- \to \pi^-) \cong \\ \cong LN_2(\rho^- \to \pi) & \cong & (\mathbb{Z}/2)^{2^{r-1}-r+1} & & \end{array}$$

that do not lie in the direct summand $\mathbb{Z}/2$ given by the image of the map δ in (6.11) cannot be realized as splitting obstructions of simple homotopy equivalences of closed manifolds along Browder-Livesay submanifolds of codimension 1.

iii) For every $r \geq 2$, the elements of the groups

$$LN_3(\rho \to \pi^-) \cong LN_1(\rho \to \pi) \cong (\mathbb{Z}/2)^{2^r-1}$$

that do not lie in the subgroup $S = (\mathbb{Z}/2)^r$ defined above cannot be realized as splitting obstructions of simple homotopy equivalences of closed manifolds along Browder-Livesay submanifolds of codimension 1.

Let (X^{4k+2}, Y^{4k+1}), $4k+1 \geq 5$, be a Browder–Livesay pair of closed topological manifolds with the square

$$\begin{array}{ccc} \rho & \to & \rho \\ \downarrow & & \downarrow \\ \pi^- & \to & \pi \end{array}$$

for a splitting problem. Then every element of the subgroup $S \subset LN_1(\rho \to \pi)$ is realized as a splitting obstruction of a simple homotopy equivalence $f : M^{4k+2} \to X^{4k+2}$ of closed manifolds along the Browder-Livesay submanifold Y.

iv) Any nontrivial element of the group

$$LN_0(\rho \to \pi^-) = (\mathbb{Z})^{2^r} \oplus (\mathbb{Z}/2)^{2^r-r-1-\binom{r}{2}}$$

cannot be realized as a splitting obstruction of a simple homotopy equivalence of closed manifolds along a Browder-Livesay submanifold of codimension 1.

v) Let (X^{4k+3}, Y^{4k+2}), $4k+2 \geq 6$, be a Browder–Livesay pair of closed topological manifolds with the square

$$\begin{array}{ccc} \rho & \to & \rho \\ \downarrow & & \downarrow \\ \pi^- & \to & \pi \end{array}$$

for a splitting problem. Then every element of the group

$$LN_2(\rho \to \pi) = \mathbb{Z}^{2^r} \oplus (\mathbb{Z}/2)^{2^r-r-1-\binom{r}{2}}$$

is realized as a splitting obstruction of a simple homotopy equivalence $f : M^{4k+2} \to X^{4k+2}$ of closed manifolds along the Browder-Livesay submanifold Y.

Proof. i) The diagram chasing in Diagram (1.5) and the isomorphisms (6.6)-(6.9) imply that, in all cases under our consideration, the maps

$$c_\pm : LN_{2k+1}(\rho^* \to \pi^\mp) \to L_{2k+1}(\pi^\pm)$$

in (1.5) are monomorphisms (see also [23]). By [17] all nontrivial elements of the groups $L_1(\pi^-)$, $L_3(\pi^-)$, and $L_1(\pi)$ have the second type with respect to some subgroup of index 2. Hence they cannot be realized by normal maps of closed manifolds. Now the statement i) follows from the isomorphism in (6.6) and Lemma 2, part i).

ii) The same arguments as in part i) provide that all the maps

$$c_- : LN_{2k}(\rho^- \to \pi^\pm) \to L_{2k}(\pi^\mp)$$

in Diagram (1.5) are monomorphisms. Now let us consider the inclusion $\rho^- \to \rho^- \oplus \mathbb{Z}/2 \cong \pi^-$. For $r \geq 1$, we have a commutative diagram

$$\begin{array}{ccc} \mathbb{Z}/2^- & \longrightarrow & \mathbb{Z}/2^- \oplus \mathbb{Z}/2 \\ \downarrow & & \downarrow \\ \rho^- & \longrightarrow & \rho^- \oplus \mathbb{Z}/2 \end{array} \tag{6.13}$$

where the vertical maps give a morphism of inclusions of index 2. As follows from Section 4, Diagram (6.13) induces a map of Diagrams (1.5) for the given inclusions. Thus we obtain a commutative diagram

$$\begin{array}{ccc} \mathbb{Z}/2 & = & L_0(\mathbb{Z}/2^-) \\ \| & & \cong\downarrow \\ LN_0(\mathbb{Z}/2^- \to \mathbb{Z}/2^- \oplus \mathbb{Z}/2) & \xrightarrow{\cong} & L_0(\mathbb{Z}/2^- \oplus \mathbb{Z}/2^-) \\ \downarrow & & \downarrow \\ LN_0(\rho^- \to \rho^- \oplus \mathbb{Z}/2) & \xrightarrow{mono} & L_0(\rho^- \oplus \mathbb{Z}/2^-) \end{array} \tag{6.14}$$

where the right upper vertical map is induced by a natural inclusion. By [17] only the direct summand $\mathbb{Z}/2 \subset L_0(\rho^- \oplus \mathbb{Z}/2^-)$ given by the right vertical composition in (6.14) can be realized by normal maps of closed manifolds. Now the statement ii) for the group $LN_0(\rho^- \to \rho^- \oplus \mathbb{Z}/2)$ follows from Lemma 2 and Diagram (6.14). The discussion of the other cases is similar.

iii) Let us consider the following part

$$L_3(\pi) \to LN_1(\rho \to \pi) \to L_1(\pi^-) \tag{6.15}$$

of Diagram (1.5) for the inclusion $\rho \to \pi$. By [23, Theorem 4] (see also [21] and [33]) the image of the left map in (6.15) is $(\mathbb{Z}/2)^r$, and the sequence in (6.15) is exact. All nontrivial elements of the group $L_1(\pi^-)$ cannot be realized by normal maps of closed manifolds by [17]. Hence it remains only to prove that the image of the left map in (6.15) coincides with the subgroup S. Let us fix a decomposition

$$\rho = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \cdots \oplus Z/2 \tag{6.16}$$

into a direct sum, and consider the commutative diagram

$$\begin{array}{ccc} \mathbb{Z}/2 & \longrightarrow & \mathbb{Z}/2 \oplus \{\mathbb{Z}/2\} \\ \downarrow & & \downarrow \\ \rho & \longrightarrow & \rho \oplus \{\mathbb{Z}/2\} \end{array} \tag{6.17}$$

whose left vertical map is an inclusion of a direct summand into the decomposition in (6.16). ¿From the functoriality of Diagram (1.5) we obtain a commutative diagram

$$\begin{array}{ccccc} & & \mathbb{Z}/2 & & \mathbb{Z}/2 \\ & & \| & & \| \\ L_3(\mathbb{Z}/2 \oplus \mathbb{Z}/2) & \xrightarrow{\mathrm{Im}=\mathbb{Z}/2} & LN_1(\mathbb{Z}/2 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2) & \xleftarrow{\cong} & L_3(\mathbb{Z}/2) \\ mono\downarrow & & mono\downarrow & & mono\downarrow \\ L_3(\pi) & \longrightarrow & LN_1(\rho \to \pi) & \xleftarrow{\cong} & L_3(\rho). \end{array} \tag{6.18}$$

Now the statement about the nonrealization of the elements for the case $LN_1(\rho \to \pi)$ follows from (6.12) and (6.18). The map $L_3(\pi) \to LN_1(\rho \to \pi)$ in (6.18) is the composition

$$L_3(\pi) \xrightarrow{\lambda} \mathcal{S}^{TOP}(X^{4k+2}) \xrightarrow{\theta} LN_1(\rho \to \pi)$$

fitting in Diagram (6.2) for the manifold pair (X, Y). The geometrical definitions of the action λ and the map θ (see [31]) imply that the elements lying in the image of $\theta\lambda$ are realized as splitting obstruction of a simple homotopy equivalences $M^{4k+2} \to X^{4k+2}$ along Browder-Livesay submanifolds $Y^{4k+1} \subset X^{4k+2}$ of codimension 1.

The proof for the group $LN_3(\rho \to \pi^-)$ is similar. It is necessary to start from the chain complex

$$LN_3(\rho \to \pi^-) \xrightarrow{mono} L_3(\pi) \xrightarrow{\text{Image}=(\mathbb{Z}/2)^r} L_3(\rho)$$

fitting in Diagram (1.5) for the inclusion $\rho \to \pi^-$. In the group $L_3(\pi)$ only the elements of the subgroup $\mathbb{Z}/2^{r+1}$ generated by $\mathbb{Z}/2 = L_3(\mathbb{Z}/2) \to L_3(\pi)$ (which are induced by the inclusions $\mathbb{Z}/2 \to \pi$ on the direct summands) are realized by normal maps of closed manifolds [18]. Now arguments similar to those used for the group $LN_1(\rho \to \pi)$ provide the proof.

iv) Let us consider the following composition

$$LN_0(\rho \to \pi^-) \xrightarrow{c} L_0(\pi) \xrightarrow{\partial} LN_2(\rho \to \pi) \tag{6.19}$$

where the first map lies in Diagram (1.5) for the inclusion $\rho \to \pi^-$ and the second map lies in Diagram (1.5) for the inclusion $\rho \to \pi$. The first map is a monomorphism [23], the second map is an epimorphism[23], and the composition is the multiplication by 2 if we use the isomorphism from (6.7) (see [4], [11], and [14]). By [17] only the elements of a direct summand $\mathbb{Z}$ of the group $L_0(\pi)$ can be realized by normal maps of closed manifolds. Since the map c in (6.19) is a monomorphism, we obtain by Lemma 2 that only the elements of infinite order in the group $LN_0(\rho \to \pi^-)$ can be realized as splitting obstructions for simple homotopy equivalences of closed manifolds. But any element $x \in LN_0(\rho \to \pi^-)$ of infinite order maps to a nontrivial element of the group $LN_2(\rho \to \pi)$ since the composition in (6.19) is the multiplication by 2. Hence the element $c(x) \in L_0(\pi)$ has a nontrivial Browder-Livesay invariant $\partial\,(c(x)) \neq 0 \in LN_2(\rho \to \pi^+)$, and the element $c(x)$ cannot be realized by a normal map of closed manifolds. Now the statement follows from Lemma 2 (see also [4]).

v) The proof is similar to that in iii) for realizing the elements of $S \subset LN_1(\rho \to \pi)$. We remark only that the epimorphism $\partial : L_0(\pi) \to LN_2(\rho \to \pi)$ in (6.19) is the composition

$$L_0(\pi) \xrightarrow{\lambda} \mathcal{S}^{TOP}(X^{4k+3}) \xrightarrow{\theta} LN_2(\rho \to \pi).$$

Now the result follows. □

Remark 2. *Note that we have natural isomorphisms*

$$LN_0(\rho \to \pi) \cong LN_0(1 \to \mathbb{Z}/2) \underset{\cong}{\xrightarrow{s}} L_0(\mathbb{Z}/2^-) \cong \mathbb{Z}/2$$

and

$$LN_2(\rho \to \pi^-) \cong LN_2(1 \to \mathbb{Z}/2^-) \underset{\cong}{\xrightarrow{s}} L_2(\mathbb{Z}/2) \cong \mathbb{Z}/2.$$

Now the results of [20, Chapter III.3.2] *provide the realization of the elements in the group*

$$LN_0(\rho \to \pi) \cong LN_2(\rho \to \pi^-) \cong \mathbb{Z}/2.$$

In case ii) of Theorem 10, Example 2.4 from [1] *provides the case of a closed Browder-Livesay manifold pair when a nontrivial element of the group $LN_2(\rho^- \to \pi)$ is realized as an obstruction to splitting along the submanifold.*

In the situation of a manifold pair, we have naturally the problem of realizing various elements of the group $LP_{n-q}(F)$ by normal maps of closed manifolds. This problem is connected with the computation of the map $[X, G/TOP] \to LP_{n-q}(F)$ from Diagram (6.2).

Let $\rho^- \to \pi^\pm = \rho^- \oplus \mathbb{Z}/2^\pm$ be an inclusion of index 2 between elementary 2-groups and $r = \operatorname{rank} \rho \geq 1$. Let

$$F^\pm = \begin{pmatrix} \rho^- & \to & \rho^- \\ \downarrow & & \downarrow \\ \pi^\mp & \to & \pi^\pm \end{pmatrix} \tag{6.20}$$

denote the various squares in the correspondent splitting problems. We have the natural isomorphisms $\pi^- \cong \pi^+$ and $F^- \cong F^+$ of oriented objects.

Proposition 7. *Under these assumptions, there are isomorphisms*

$$LP_n(F^\pm) \cong (\mathbb{Z}/2)^{2^r + 2^{r-1} - 2r} \tag{6.21}$$

for every $n = 0, 1, 2, 3 \pmod 4$.

Proof. Let us consider Diagram (1.5) for the inclusion $\rho^- \to \pi^-$. In this diagram all the maps

$$L_n(\rho^-) \to L_n(\pi^-) \quad \text{and} \quad LN_n(\rho^- \to \pi^-) \to L_n(\pi^+)$$

are monomorphisms, and all the maps

$$L_n(\pi^-) \to LN_{n-2}(\rho^- \to \pi^-) \quad \text{and} \quad L_n(\pi^+) \to L_n(\rho^-)$$

are trivial (see [23]). Now the result follows by a diagram chasing. □

Theorem 11. *For every $n = 0, 1, 2, 3 \pmod 4$, there exists a direct summand*

$$S = \mathbb{Z}/2 \subset LP_n(F^\pm) = (\mathbb{Z}/2)^{2^r + 2^{r-1} - 2r}$$

such that the elements that do not lie in S cannot be realized as surgery obstructions of normal maps of closed manifold pairs.

The nontrivial element of the direct summand $S \subset LP_{2k+1}(F^\pm)$ is realized as an obstruction to surgery a normal map $f : (M^n, N^{n-1}) \to (X, Y)$ on pairs of closed topological manifolds.

Proof. We consider only the case $LP_1(F^-)$ since the others are similar. Using Proposition 7, we can write down the following part of Diagram (1.5)

$$\begin{array}{ccccc} L_2(\rho^-) & & \longrightarrow & & L_2(\pi^-) \\ & mono \searrow & & \nearrow epi & \\ & & LP_1(F^-) & & \\ & mono \nearrow & & \searrow epi & \\ LN_1(\rho^- \to \pi^-) & & \longrightarrow & & L_1(\pi). \end{array} \tag{6.22}$$

By [17] we have not nontrivial elements in the group $L_1(\pi)$ that are realized by normal maps of closed manifolds. Furthermore, there exists only one such an element y in the group $L_2(\pi^-)$. We can realize this element by a normal map $f : M^{4k+2} \to X^{4k+2}$. The map ϕ as in (5.2) with respect to the inclusion $\rho^- \to \pi^-$ provides a codimension 1 submanifold $Y \subset X$ with the square F^- in the splitting problem. Thus the normal map f to the pair (X, Y) has an obstruction x to surgery on manifold pairs in the group $LP_1(F^-)$ such that $y = p_0(x)$. Hence $x \neq 0$. Now the result follows from Diagram (6.22) and Lemma 2. □

Remark 3. *Example 2.4 of* [1] *gives the case in which a nontrivial element z of the group $LN_2(\rho^- \to \pi)$ is realized as the splitting obstruction of a simple homotopy equivalence of closed manifolds. From a diagram similar to (6.22), we obtain a monomorphism*

$$s_1 : LN_2(\rho^- \to \pi^+) \to LP_2(F^+). \tag{6.23}$$

¿From the geometrical description of the map s_1 in Diagram (6.3), we obtain that the nontrivial element $s_1(z) \in S \subset LP_{2k}(F^+)$ is realized by a normal map of closed manifold pairs.

Now we consider the case of the trivial orientation on an elementary 2-group ρ of rank $r \geq 0$. Let $\rho \to \pi^{\pm} = \rho \oplus \mathbb{Z}/2^{\pm}$ be an inclusion of index 2 between elementary 2-groups, and let

$$F^{\pm} = \begin{pmatrix} \rho & \to & \rho \\ \downarrow & & \downarrow \\ \pi^{\mp} & \to & \pi^{\pm} \end{pmatrix} \tag{6.24}$$

denote the various squares in the correspondent splitting problems.

Proposition 8. *Let $F^{\pm}$ be the square in (6.24). Then there are isomorphisms*

(6.25)

$$LP_n(F^+) = (\mathbb{Z}/2)^{2^{r+1}-2r-\binom{r}{2}-1}, \ (\mathbb{Z}/2)^{2^r-r}, \ (\mathbb{Z}/2)^{2^{r+1}-r-1}, \ (\mathbb{Z})^{2^r} \oplus (\mathbb{Z}/2)^{2^{r+1}-2r-\binom{r}{2}-2}$$

for $n = 0, 1, 2, 3 \pmod 4$, respectively, and isomorphisms

$$LP_n(F^-) \cong LP_{n+1}(F)$$

for $n = 0, 1, 2, 3 \pmod 4$.

Proof. The proof is similar to that given in Proposition 7. Let us consider Diagram (1.5) for the inclusion $\rho \to \pi$. By diagram chasing we obtain

$$LP_n(F) = L_{n+1}(\rho) \oplus L_n(\pi^-)$$

for any n. Now the result follows from (6.8) and (6.9). The consideration of the case F^- is similar. □

Theorem 12. *Let F^+ be the square in (6.24).*

i) All the elements of the group $LP_0(F)$, except those in the direct summand

$$\mathbb{Z}/2 = \mathrm{Im}\{\mathbb{Z}/2 = LN_0(\rho \to \pi) \xrightarrow{mono} LP_0(F)\},$$

cannot be realized as surgery obstructions of normal maps of closed manifold pairs.

ii) Only the elements of the direct summand

$$\mathbb{Z}/2 = \mathrm{Im}\{\mathbb{Z}/2 = L_2(\rho) \xrightarrow{mono} LP_1(F)\} \subset LP_1(F)$$

are realized as surgery obstructions of normal maps of closed manifold pairs.

iii) Only the elements of a direct summand

$$(\mathbb{Z}/2)^r = S' \subset LP_2(F) = (\mathbb{Z}/2)^{2^{r+1}-r-1}$$

are realized as surgery obstructions of normal maps of closed manifold pairs. The subgroup $S' = p_0^-(S)$ is the preimage of the subgroup $S \subset L_3(\pi)$ which is generated by the images of $\mathbb{Z}/2$ under all the inclusions $\mathbb{Z}/2 = L_3(\mathbb{Z}/2) \to L_3(\pi)$ induced by the inclusions $\mathbb{Z}/2 \to \pi$ on the direct summands $\mathbb{Z}/2$ of the group π.

iv) Only the elements of a direct summand

$$\mathbb{Z} = \mathrm{Im}\{\mathbb{Z} = L_0(1) \to LP_3(F)\} \subset LP_3(F) = (\mathbb{Z})^{2^r} \oplus (\mathbb{Z}/2)^{2^{r+1}-2r-\binom{r}{2}-2}$$

are realized as surgery obstructions of normal maps of closed manifold pairs.

Proof. We treat only the case $LP_3(F)$ since the others are similar. Let us consider the commutative diagram

$$\begin{array}{ccccc}
 & L_0(1) & \xrightarrow{=} & L_0(1) & \\
 & \downarrow & & \downarrow & \\
 & L_0(\rho) & \xrightarrow{mono} & L_0(\pi) & \\
 & {\scriptstyle mono}\searrow & & \nearrow & \\
(6.26) & & LP_3(F) & & \\
 & {\scriptstyle mono}\nearrow & & \searrow{\scriptstyle epi} & \\
 & LN_3(\rho \to \pi) & \xrightarrow{mono} & L_3(\pi^-) &
\end{array}$$

where the two upper vertical maps are induced by natural inclusions and the bottom part arises from Diagram (1.5) similarly to (6.22). In the group $L_0(\pi)$ only the elements of the image $L_0(1) \to L_0(\pi)$, that equals $\mathbb{Z}$, are realized by normal maps of closed manifolds [17]. Denote this image by $T \subset L_0(\pi)$. The same arguments as in Theorem 11 provide that only the elements of $p_0^{-1}(T) \subset LP_3(F)$ can be realized by normal maps of closed manifolds. Furthermore, for every element $b \in T$ there exists an element $b' \in LP_3(F)$ ($p_0(b') = b$) that is realized by a normal map of closed manifold pairs. By Lemma 2 $p_1(b') = 0$. Then $b' \in \mathrm{Im}\{\mathbb{Z} = L_0(1) \to LP_3(F)\}$ since (6.26) is commutative and the upper horizontal map is a monomorphism. Let $x \in LP_3(F)$ be an other element such that $p_0(x) = b$. If the element $p_1(x) \in L_3(\pi^-)$ is not trivial, then it cannot be realized by a normal map of closed manifolds by [17]. If it is trivial, then x lies in the image $\mathrm{Im}\{\mathbb{Z} = L_0(1) \to LP_3(F)\}$ as before, and hence $x = b'$. Now the result follows.

For the groups $LP_*(F^-)$, the results are not such explicit in all dimensions. But in dimension 1 the result is very explicit.

Theorem 13. *Let F^- be the square in (6.24).*

i) All the elements of the group $LP_0(F^-)$, except those in the direct summand

$$\mathbb{Z} = \mathrm{Im}\{\mathbb{Z} = LN_0(1 \to \mathbb{Z}/2^-) \xrightarrow{mono} LP_0(F^-)\},$$

cannot be realized as surgery obstructions of normal maps of closed manifold pairs.

ii) Only the elements of the direct summand

$$\mathbb{Z}/2 = \mathrm{Im}\{\mathbb{Z}/2 = L_2(\rho) \xrightarrow{mono} LP_1(F^-)\} \subset LP_1(F^-)$$

are realized as surgery obstructions of normal maps of closed manifold pairs.

iii) All the elements of the group $LP_2(F^-)$ except the direct summand

$$\mathbb{Z}/2 = \mathrm{Im}\{LN_2(1 \to \mathbb{Z}/2^-) \xrightarrow{mono} LP_2(F^-)\} \subset LP_2(F^-),$$

cannot be realized as surgery obstructions of normal maps of closed manifold pairs.

iv) All the elements of the group $LP_3(F^-)$, except those in a direct summand

$$S = (\mathbb{Z}/2)^r \subset LP_2(F^-),$$

cannot be realized as surgery obstructions of normal maps of closed manifold pairs. The elements of a direct summand $\mathbb{Z}/2 \subset S$ are realized as surgery obstructions of normal maps of closed manifold pairs.

Proof. The proofs are similar to those given in Theorem 11 and Theorem 12. □

We note that the developed methods are applicable not only for elementary 2-groups. Let us consider an index 2 inclusion $\rho \to \pi^-$ between finite abelian 2-groups, where ρ has the trivial orientation while the orientation of π^- is nontrivial.

Theorem 14. [4] *Any element of the group $LN_{2k}(\rho \to \pi^-)$ that does not lie in the torsion subgroup cannot be realized as the splitting obstruction of a simple homotopy equivalence of closed manifolds.*

Proof. We use here a little changed argument from [4]. Let use consider the composition

$$LN_{2k}(\rho \to \pi^-) \to L_{2k}(\pi) \to LN_{2k+2}(\rho \to \pi^+) \xrightarrow{\cong} LN_{2k}(\rho \to \pi^-) \tag{6.27}$$

where the first map fits in Diagram (1.5) for the inclusion $\rho \to \pi^-$ and the second map fits in Diagram (1.5) for the inclusion $\rho \to \pi$. The composition in (6.27) coincides with the first differential in the surgery spectral sequence (see [4], [11], and [14]), and hence it is the multiplication by 2. The second map in (6.27) is the first Browder-Livesay invariant, and any element which does not lie in its kernel cannot be realized by a normal map of closed manifolds. Now the result follows from Lemma 2. □

Corollary 3. [4] *Any nontrivial element of the groups*

$$LN_{2k}(\mathbb{Z}/2^n \to \mathbb{Z}/2^{n+1^-}) = \mathbb{Z}^{2^{n-1}}$$

is not realized as the splitting obstruction of a simple homotopy equivalence of closed manifolds.

The results of Ranicki [29] give a good possibility to apply the given approach to the computation of the assembly map for infinite groups with nontrivial torsion. For the splitting obstruction groups in the case of fundamental group with a nontrivial torsion, the developed methods are applicable, too (as follows from [9] and [26]).

As a final application, we give sufficiently simple and very explicit descriptions of some assembly maps.

Let π be an elementary finite 2-group with an orientation and $G = \pi \oplus \mathbb{Z}$, where the group $\mathbb{Z}$ has the trivial orientation. Let X be a topological space with $\pi_1(X) = \pi$. The algebraic surgery exact sequence

$$\cdots \to L_{n+1}(\pi \oplus \mathbb{Z}) \to \mathcal{S}^s_{n+1}(X \times S^1) \to H_n(X \times S^1, \mathbf{L}_\bullet) \xrightarrow{\Sigma_n} L_n(\pi \oplus \mathbb{Z}) \to \cdots \tag{6.28}$$

splits into the direct sum of the exact sequences

$$\cdots \to L_{n+1}(\pi) \to \mathcal{S}^s_{n+1}(X) \to H_n(X, \mathbf{L}_\bullet) \xrightarrow{\sigma_n} L_n(\pi) \to \cdots \tag{6.29}$$

and

$$\cdots \to L_n(\pi) \to \mathcal{S}^s_n(X) \to H_{n-1}(X, \mathbf{L}_\bullet) \xrightarrow{\sigma_{n-1}} L_{n-1}(\pi) \to \cdots \tag{6.30}$$

by [29] (see also [9] and [26]). Note that for a topological manifold X^n the algebraic surgery exact sequence in (6.29) is isomorphic to the surgery exact sequence in (1.1).

Theorem 15. *i) Let $X = K(\pi^-, 1)$, where π^- is an elementary 2-group of rank $r \geq 1$ with nontrivial orientation. Then the image of the map Σ_n in the exact sequence (6.28) equals $\mathbb{Z}/2$ for every n.*

ii) Let $X = K(\pi, 1)$, where π is an elementary 2-group of rank $r \geq 0$ with trivial orientation. Then the image of the map Σ_n in the exact sequence (6.28) equals

$$\mathbb{Z} \oplus (\mathbb{Z}/2)^r,\ \mathbb{Z},\ \mathbb{Z}/2,\ (\mathbb{Z}/2)^{r+1}$$

for $n = 0, 1, 2, 3 \pmod 4$, respectively.

Proof. We consider only the case π^- since the other is similar. As follows from [17] the image of the map $H_n(X, \mathbf{L}_\bullet) \xrightarrow{\sigma_n} L_n(\pi^-)$ is trivial for n odd and equals $\mathbb{Z}/2$ for n even. Now the result follows from the decomposition of (6.28). Note that the same result implies the existence of an algebraic retraction $L_*(\pi^- \oplus \mathbb{Z}) \to L_*(\pi^-)$, Corollary 2, and the nonrealization of the elements of the second type. □

The classification of the manifolds with a filtration as well as that of the stratified manifolds demand to know what elements of Browder-Quinn obstruction groups are realized by normal maps of closed stratified manifolds. A similar question is very natural for the splitting problem of a manifold with a filtered system of submanifolds (see [4] and [18]). There are a lot of diagrams that generalize Diagrams (6.2) and

(6.3) for filtered manifolds (see [3], [4], [10], [18], and [19]). The relations between L^{BQ} groups obtained in Section 5 and the computational methods developed in Section 6 allow us to solve such questions. We present here only one simple example.

Consider a triple

$$\mathbb{R}P^{2k-1} \subset \mathbb{R}P^{2k} \subset \mathbb{R}P^{2k+1} \tag{6.31}$$

of real projective spaces with $2k-1 \geq 5$. Denote by $\mathcal{P}$ the filtration in (6.31), and by $LT_* = L_*^{BQ}(\mathcal{P})$ the Browder-Quinn surgery obstruction groups for the triple $\mathcal{P}$ (see [18], [19], and [24]). These groups fit in the following braid of exact sequences [21]

$$\begin{tikzcd}[column sep=small]
{} \arrow[r] & L_n(1) \arrow[rr] \arrow[dr] & & LP_{n-1}(F) \arrow[rr] \arrow[dr] & & LN_{n+1}(1\to\mathbb{Z}/2^-) \arrow[r] \arrow[dr] & {} \\
{} \arrow[ur] \arrow[dr] & & LT_{n-2} \arrow[ur] \arrow[dr] & & L_{n-1}(\mathbb{Z}/2^-) \arrow[ur] \arrow[dr] & & {} \\
{} \arrow[r] & LN_{n-2}(1\to\mathbb{Z}/2^-) \arrow[rr] \arrow[ur] & & LP_{n-2}(F^-) \arrow[rr] \arrow[ur] & & L_{n-1}(1) \arrow[r] \arrow[ur] & {}
\end{tikzcd} \tag{6.32}$$

where

$$F^{\pm} = \left(\begin{array}{ccc} 1 & \rightarrow & 1 \\ \downarrow & & \downarrow \\ \mathbb{Z}/2^{\mp} & \rightarrow & \mathbb{Z}/2^{\pm} \end{array} \right). \tag{6.33}$$

In this case, $LT_{2k+1} = \mathbb{Z}/2$ (see [18] and [24]).

Proposition 9. *All the elements of the groups $LT_{2k-1} = L_{2k-1}^{BQ}(\mathcal{P})$ are realized as obstructions to surgery on triples of closed manifolds.*

Proof. The diagram chasing in Diagram (6.22) provides that the maps

$$LT_{2k-1} \to LP_{2k}(F^+)$$

from Diagram (6.32) are isomorphisms. But from a geometrical point of view these maps are natural forgetful maps. Using Theorem 11 and Remark 2, the same arguments as in Corollary 2 provide the result. $\square$

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