LIMITING MODULAR SYMBOLS AND THE LYAPUNOV SPECTRUM

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1. Introduction

This paper consists of variations upon a theme, that of *limiting modular symbols* introduced in [16]. We show that, on level sets of the Lyapunov exponent for the shift map T of the continued fraction expansion, the limiting modular symbol can be computed as a Birkhoff average. We show that the limiting modular symbols vanish almost everywhere on T–invariant subsets for which a corresponding transfer operator has a good spectral theory, thus improving the weak convergence result proved in [16]. We also show that, even when the limiting modular symbol vanishes, it is possible to construct interesting non–trivial homology classes on modular curves that are associated to non–closed geodesics. These classes are related to "automorphic series", defined in terms of successive denominators of continued fraction expansion, and their integral averages are related to certain Mellin transforms of modular forms of weight two considered in [16]. We discuss some variants of the Selberg zeta function that sum over certain classes of closed geodesics, and their relation to Fredholm determinants of transfer operators. Finally, we argue that one can use T–invariant subsets to enrich the picture of non–commutative geometry at the boundary of modular curves presented in [16].

2. Modular symbols and geodesics

Consider the classical compactification of modular curves obtained bye adding cusp points $G\backslash\mathbb{P}^1(\mathbb{Q})$. Given two points $\alpha,\beta\in\mathbb{H}\cup\mathbb{P}^1(\mathbb{Q})$, a real homology class $\{\alpha,\beta\}_G\in H_1(X_G,\mathbb{R})$ is defined by integrating the lifts by $\phi:\mathbb{H}\to X_G$ of differentials of the first kind on X_G along the geodesic arc connecting α and β,

$$\int_{\{\alpha,\beta\}_G}\omega := \int_\alpha^\beta \phi^*\omega.$$

The modular symbols $\{\alpha,\beta\}_G$ satisfy the additivity and invariance properties

$$\{\alpha,\beta\}_G+\{\beta,\gamma\}_G=\{\alpha,\gamma\}_G,$$

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and

$$\{g\alpha, g\beta\}_G = \{\alpha, \beta\}_G,$$

for all $g \in G$.

Because of additivity, it is sufficient to consider modular symbols of the form $\{0, \alpha\}$ with $\alpha \in \mathbb{Q}$,

$$\{0, \alpha\}_G = -\sum_{k=1}^{n} \{g_k(0), g_k(i\infty)\}_G,$$

where α has continued fraction expansion $\alpha = [a_0, \dots, a_n]$, and

$$g_k = \begin{pmatrix} p_{k-1}(\alpha) & p_k(\alpha) \\ q_{k-1}(\alpha) & q_k(\alpha) \end{pmatrix},$$

with p_k/q_k the successive approximations, and $p_n/q_n = \alpha$.

In [16] it was argued that the full $\mathbb{P}^1(\mathbb{R})$ contributes to the compactification of X_G, and a corresponding generalization of modular symbols was introduced, using infinite geodesics in the upper half plane $\mathbb{H}$, which end at an *irrational* point in $\mathbb{P}^1(\mathbb{R})$.

Let γ_β be an infinite geodesic in $\mathbb{H}$ with one end at $i\infty$ and the other end at $\beta \in \mathbb{R} \smallsetminus \mathbb{Q}$. Let $x_0 \in \gamma_\beta$ be a fixed base point, τ be the geodesic arc length, and $y(\tau)$ be the point along γ_β at a distance τ from x_0, towards the end β. Let $\{x, y(\tau)\}_G$ denote the homology class in X_G determined by the image of the geodesic arc $\langle x, y(\tau)\rangle$ in $\mathbb{H}$.

The *limiting modular symbol* is defined as

$$\{\{*, \beta\}\}_G := \lim \frac{1}{\tau} \{x, y(\tau)\}_G \in H_1(X_G, \mathbf{R}), \tag{2.1}$$

whenever such limit exists. The limit is independent of the choice of x_0. As discussed in [16], the limit (2.1) is in fact defined for all infinite geodesics in $\mathbb{H}$ that end at the same β, and the limit is independent of the other end of the geodesic.

2.1. **Shift.** We shall study the limiting modular symbols using properties of the dynamical system given by a generalization of the shift of the continued fraction expansion. For a modular curve

$$X_G = \mathrm{PGL}(2, \mathbb{Z}) \backslash (\mathbb{H} \times \mathbb{P}),$$

where $\mathbb{P} = \mathrm{PGL}(2, \mathbb{Z})/G$, we consider the shift map

$$T : [0,1] \times \mathbb{P} \to [0,1] \times \mathbb{P}$$

$$T(x, t) = \left(\frac{1}{x} - \left[\frac{1}{x}\right], \begin{pmatrix} -[1/x] & 1 \\ 1 & 0 \end{pmatrix} \cdot t \right). \tag{2.2}$$

2.2. **Lyapunov spectrum.** Recall that the *Lyapunov exponent* for a map $T : [0,1] \to [0,1]$ is defined as

$$\lambda(\beta) := \lim_{n\to\infty} \frac{1}{n} \log |(T^n)'(\beta)| = \lim_{n\to\infty} \frac{1}{n} \log \prod_{k=0}^{n-1} |T'(T^k\beta)|. \tag{2.3}$$

The function $\lambda(x)$ is T–invariant. Moreover, in the case of the classical continued fraction shift $Tx = 1/x - [1/x]$ on $[0,1]$, the Lyapunov exponent is given by

$$\lambda(\beta) = 2 \lim_{n\to\infty} \frac{1}{n} \log q_n(\beta), \tag{2.4}$$

with $q_n(\beta)$ the successive denominators of the continued fraction expansion. In particular, the Khintchine–Lévy theorem [12] shows that, for almost all β's, the limit (2.4) is equal to $2C$, with

$$C = \frac{\pi^2}{12 \log 2}. \tag{2.5}$$

The *Lyapunov spectrum* is introduced (cf. [20]) by decomposing the unit interval in level sets of the Lyapunov exponent $\lambda(\beta)$ of (2.3). Let $L_c = \{\beta \in [0,1] \,|\, \lambda(\beta) = c \in \mathbb{R}\}$. These sets provide a T–invariant decomposition of the unit interval,

$$[0,1] = \bigcup_{c\in\mathbb{R}} L_c \cup \{\beta \in [0,1] \,|\, \lambda(\beta) \text{ does not exist}\}.$$

These level sets are uncountable dense T–invariant subsets of $[0,1]$, of varying Hausdorff dimension [20]. The Lyapunov spectrum measures how the Hausdorff dimension varies, as a function $h(c) = \dim_H(L_c)$.

We introduce a function $\varphi : \mathbb{P} \to H_1(\overline{X_G}, cusps, \mathbb{R})$ of the form

$$\varphi(s) = \{g(0), g(i\infty)\}_G, \tag{2.6}$$

where $g \in \mathrm{PSL}(2,\mathbb{Z})$ is a representative of the coset $s \in \mathbb{P}$. We can easily generalize the result §2.3 of [16], where the following is proved for the level set L_C, with C as in (2.5).

Theorem 2.1. *For a fixed $c \in \mathbb{R}$, and for all $\beta \in L_c$, the limiting modular symbol* (2.1) *is computed by the limit*

$$\lim_{n\to\infty} \frac{1}{cn} \sum_{k=1}^{n} \varphi \circ T^k(t_0), \tag{2.7}$$

where $T : [0,1] \times \mathbb{P} \to [0,1] \times \mathbb{P}$ is the shift operator (2.2)*, and $t_0 \in \mathbb{P}$ is the base point.*

Proof. Without loss of generality, we can consider the geodesic γ_β in $\mathbb{H}$ with one end at $i\infty$ and the other at β. Following the argument given in §2.3 of [16], we estimate the geodesic distance $\tau \sim -\log \Im y + O(1)$, as $y(\tau) \to \beta$. (Here $\Im y$ denotes the imaginary part.) Moreover, if y_n is the intersection of γ_β and the geodesic with ends at $p_{n-1}(\beta)/q_{n-1}(\beta)$ and $p_n(\beta)/q_n(\beta)$, then we can estimate

$$\frac{1}{2q_nq_{n+1}} < \Im y_n < \frac{1}{2q_nq_{n-1}}.$$

Moreover, notice that the matrix $g_k^{-1}(\beta)$, with

$$g_k(\beta) = \begin{pmatrix} p_{k-1}(\beta) & p_k(\beta) \\ q_{k-1}(\beta) & q_k(\beta) \end{pmatrix},$$

acts on points $(\beta, t) \in [0,1] \times \mathbb{P}$ as the k–th power of the shift operator T of (2.2). Thus, for $\varphi(t_0) = \{0, i\infty\}_G$, we obtain

$$\varphi(T^k t_0) = \{g_k^{-1}(\beta)\,(0), g_k^{-1}(\beta)\,(i\infty)\}_G = \left\{\frac{p_{k-1}(\beta)}{q_{k-1}(\beta)}, \frac{p_k(\beta)}{q_k(\beta)}\right\}_G.$$

Finally, we can replace the path $\langle x_0, y_n \rangle$ with the union of arcs

$$\langle x_0, y_0\rangle \cup \langle y_0, p_0/q_0\rangle \cup \bigcup_{k=1}^{n} \langle p_{k-1}/q_{k-1}, p_k/q_k\rangle \cup \langle p_n/q_n, y_n\rangle$$

representing the same homology class in $H_1(\overline{X}_G, \mathbb{Z})$.

□

2.3. **Exceptional set.** There is a set of $\beta \in [0,1]$ of Hausdorff dimension equal to one, where the Lyapunov exponent does not exist (Theorem 3 of [20]). On this exceptional set the limiting modular symbols cannot be expressed as the limit (2.7). The typical example of points in the exceptional set shows that, in this case, the corresponding geodesic spends increasingly long times winding around different closed geodesics.

2.4. **Periodic continued fractions.** A special case of limiting modular symbols is when the endpoint β is a quadratic irrationality. In this case, β has eventually periodic continued fraction expansion, hence it is a fixed point of some hyperbolic element in G.

Recall that primitive closed geodesics in X_G are parameterized by periodic points of the shift operator T of (2.2), $T^\ell(\beta, t) = (\beta, t)$.

Lemma 2.2. *Consider a geodesic γ_β in $\mathbb{H}$ with an endpoint at β Assume that β has eventually periodic continued fraction expansion. Then the limiting modular symbol is given by*

$$\{\{*, \beta\}\}_G = \frac{\sum_{k=1}^{\ell} \{g_k^{-1}(\beta) \cdot g(0), g_k^{-1}(\beta) \cdot g(i\infty)\}_G}{\lambda(\beta)\ell}. \tag{2.8}$$

Proof. Consider a geodesic γ_β in $\mathbb{H}$ with an endpoint at β. Without loss of generality we may assume the other endpoint is at $i\infty$. There is a lift $\gamma_\beta \times \{t\}$ to $\mathbb{H} \times \mathbb{P}$, such that $T^\ell(\beta, t) = (\beta, t)$, for a minimal non-negative integer ℓ.

For a quadratic irrationality β the limit $\lambda(\beta) = 2\lim_{n\to\infty} \log(q_n(\beta))/n$ exists and belongs to the interval $[2\log((1+\sqrt{5})/2), \infty)$, cf. §4 of [20]. Thus, we can apply Theorem 2.1 and obtain (2.8), where

$$\{g_k^{-1}(\beta) \cdot g(0), g_k^{-1}(\beta) \cdot g(i\infty)\}_G$$

is the homology class in $\overline{X}_G$ of the geodesic

$$\pi\left(\langle \frac{p_{k-1}(\beta)}{q_{k-1}(\beta)}, \frac{p_k(\beta)}{q_k(\beta)} \rangle \times \{g_k^{-1}(\beta) t\}\right).$$

□

In [16] a different but equivalent expression for the limiting modular symbol for quadratic irrationalities is obtained, of the form

$$\{\{*, \alpha_g^+\}\}_G = \frac{\{0, g(0)\}_G}{\lambda(g)}, \tag{2.9}$$

where $\{0, g(0)\}_G$ is the homology class of the image in X_G of the geodesic arc $\langle x_0, gx_0 \rangle$, and $\alpha_g^\pm$ are the pair of attractive and repelling fixed points of an hyperbolic element $g \in G$, with $0 < \Lambda_g^\pm < 1$ the respective eigenvalues, and $\lambda(g) = \log \Lambda_g^-$ is the length of the closed geodesic in X_G obtained from the geodesic in $\mathbb{H}$ with ends at $\alpha_g^\pm$.

3. Transfer and Gauss–Kuzmin operators

We now treat the case of more general T–invariant subsets of $[0,1] \times \mathbb{P}$. It is often possible to recast the dynamical properties of a map like our shift (2.2) in terms of the functional analytic properties of certain transfer operators.

Definition 3.1. *Let $E \subset [0,1] \times \mathbb{P}$ be a T–invariant subset. The Ruelle (or Perron–Frobenius) transfer operator is defined as*

$$(\mathcal{R}_h f)(\beta, t) = \sum_{(\alpha,u)\in T^{-1}(\beta,t)} \exp(h(\alpha, u)) f(\alpha, u).$$

We only consider the case where the function h is of the form $h(\beta, t) = -s \log |T'(\beta)|$, depending on a parameter s.

For a T–invariant subset $E \subset [0,1] \times \mathbb{P}$ of Hausdorff dimension δ_E, we also define a generalized Gauss–Kuzmin operator, which is the adjoint of composition with T under the $L^2(E, \mathcal{H}^{\delta_E})$ inner product, with $\mathcal{H}^{\delta_E}$ the corresponding Hausdorff measure, [22].

Definition 3.2. *Let $E \subset [0,1] \times \mathbb{P}$ be a T–invariant subset of Hausdorff dimension δ_E. The Gauss–Kuzmin operator is defined by*

$$\int_E (\mathcal{L}f) \cdot h \, d\mathcal{H}^{\delta_E} = \int_E f \cdot (h \circ T) \, d\mathcal{H}^{\delta_E}, \tag{3.1}$$

for all f, h in $L^2(E, d\mathcal{H}^{\delta_E})$.

Let $\mathcal{N}_E \subset \mathbb{N}$ be the set

$$\mathcal{N}_E := \left\{ k : \ \left(\left[\frac{1}{k+1}, \frac{1}{k} \right] \times \mathbb{P} \right) \cap E \neq \emptyset \right\}. \tag{3.2}$$

Remark 3.1. The operator $\mathcal{L}$ defined by (3.1) is given by

$$(\mathcal{L}f)(\beta, t) = \sum_{k \in \mathcal{N}_E} \frac{1}{(\beta+k)^{2\delta_E}} f\left(\frac{1}{\beta+k}, \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix} \cdot t \right), \tag{3.3}$$

In this case, we can consider a one–parameter family of deformed Gauss–Kuzmin operators

$$(\mathcal{L}_{\sigma,E} f)(x, t) = \sum_{k \in \mathcal{N}_E} \frac{1}{(x+k)^{\sigma}} f\left(\frac{1}{x+k}, \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix} \cdot t \right) \tag{3.4}$$

With this notation, the operator $\mathcal{L}$ of (3.3) is $\mathcal{L} = \mathcal{L}_{2\delta_E, E}$.

Remark 3.2. If the invariant set is of the form $E = B \times \mathbb{P}$ with B defined by $B = \{\beta \in [0,1] : a_i \in \mathcal{N}\}$, for $\beta = [a_1, \ldots, a_n, \ldots]$, and $\mathcal{N} \subset \mathbb{N}$ a given subset, then we have $\mathcal{R}_h = \mathcal{L}_{2s,E}$, for $h = -s \log |T'|$.

In general, these two operators differ. The Ruelle transfer operator tends to carry more information on the dynamics of T, while the Gauss–Kuzmin operator tends to have better functional analytic properties. The last Remark describes the optimal case.

3.1. **Spectral theory.** We are especially interested in the cases where the Gauss–Kuzmin operator and has a good spectral theory.

Definition 3.3. *We say that a T–invariant set $E \subset [0,1] \times \mathbb{P}$ has good spectral theory (GST) if the following conditions are satisfied:*

1. *The space $\mathbb{V}$ has the property that, for each $f \in \mathbb{V}$, the restriction $f|_E$ lies in $L^2(E, d\mathcal{H}^{\delta_E})$, and the restrictions satisfy*

$$\overline{\{ f|_E \, | f \in \mathbb{V} \}} = L^2(E, d\mathcal{H}^{\delta_E}).$$

2. *The operator $\mathcal{L}_{\sigma,E}$ acting on $\mathbb{V}$ is compact.*

3. *There exists a cone $K \subset \mathbb{V}$, which consists of functions positive at points of E, and an element u in the interior of K, such that $\mathcal{L}_{\sigma,E}$ is u–positive, namely for all non–trivial $f \in K$ there exists $k > 0$ and real $a, b > 0$ such that*
$$au \leq \mathcal{L}^k_{\sigma,E} f \leq bu,$$
where the order is defined by $f \leq g$ iff $g - f \in K$.
4. *For $\sigma = 2\delta_E$ the spectral radius of $\mathcal{L}_{2\delta_E,E}$ is one,*
$$\rho(\mathcal{L}_{2\delta_E,E}) = 1.$$

The techniques developed by Mayer [17] then imply that the following properties are satisfied.

Proposition 3.3. *Let E be a T–invariant set with GST. Then For all $\sigma \in I$, $\lambda_\sigma = \rho(\mathcal{L}_{\sigma,E})$ is a simple eigenvalue of $\mathcal{L}_{\sigma,E}$, and the unique (normalized) eigenfunction satisfying $\mathcal{L}_{\sigma,E} h_\sigma = \lambda_\sigma h_\sigma$ is in the cone K. There exists a unique T–invariant measure on E, which is absolutely continuous with respect to the Hausdorff measure $d\mathcal{H}^{\delta_E}$, with density the normalized eigenfunction $h_{2\delta_E}$.*

Let $\mathcal{L}^*_{\sigma,E}$ be the adjoint operator acting on the dual Banach space. There is a unique eigenfunctional ℓ_σ satisfying $\mathcal{L}^*_{\sigma,E}\ell_\sigma = \lambda_\sigma \ell_\sigma$. For any $f \in \mathbb{V}$, we have
$$\lim_{n\to\infty} \lambda_\sigma^{-n} \mathcal{L}^n_{\sigma,E} f = \ell_\sigma(f) h_\sigma, \tag{3.5}$$
with the eigenfunctional $\ell_{2\delta_E}$ given by $f \mapsto \int_E f \, d\mathcal{H}^{\delta_E}$. The iterates $\mathcal{L}^k_{2\delta_E,E} f$, for $f \in \mathbb{V}$, converge to the invariant density $h_{2\delta_E}$, at a rate of the order of $O(q^n)$, where q is the spectral margin: $|\lambda| < q\lambda_\sigma$ for all points $\lambda \neq \lambda_\sigma$ in the spectrum of $\mathcal{L}_{\sigma,E}$. Thus, for any $f \in \mathbb{V}$ we have
$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n (\mathcal{L}^k f)(x,t) = h_{2\delta_E}(x,t) \cdot \int_E f \, d\mathcal{H}^{\delta_E}. \tag{3.6}$$

In particular, given any function F in $L^2(E, d\mathcal{H}^{\delta_E})$, we have
$$\int_E \frac{1}{n} \sum_{k=1}^n F(T^k(x,t))\, f(x,t)\, d\mathcal{H}^{\delta_E}(x,t) \to \left(\int_E F\, h_{2\delta_E}\, d\mathcal{H}^{\delta_E}\right) \cdot \left(\int_E f \, d\mathcal{H}^{\delta_E}\right), \tag{3.7}$$
for and any test function $f \in L^2(E, d\mathcal{H}^{\delta_E})$.

3.2. **Generalized Gauss problem.** Let $E = B \times \mathbb{P}$ be a T–invariant subset of $[0,1]$ with GST.

Let $t_0 \in \mathbb{P}$ be a base point, and let $m_n(x,t)$ be defined as
$$m_n(x,t) := \mathcal{H}^{\delta_B}\left(\left\{\alpha \in B \,\middle|\, x_n(\alpha) \leq x \text{ and } g_n^{-1}(\alpha) \cdot t_0 = t\right\}\right). \tag{3.8}$$

Lemma 3.4. *The measures* (3.8) *satisfy the recursive relation*

$$m_{n+1}(x,t) = \sum_{k=1}^{\infty} \left(m_n \left(\frac{1}{k}, \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix} \cdot t \right) - m_n \left(\frac{1}{x+k}, \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix} \cdot t \right) \right),$$

and the densities satisfy

$$m'_{n+1}(x,t) = (\mathcal{L}_{2\delta_E,E} m'_n)(x,t).$$

The measures $m_n(x,t)$ converge to the unique T–invariant measure

$$m(x,t) := \int_E h_{2\delta_E}(x,t)\, d\mathcal{H}^{\delta_E}(x,t),$$

at a rate $O(q^n)$ for some $0 < q < 1$. In this case the density satisfies

$$h_{2\delta_E}(x,t) = \frac{1}{|\mathbb{P}|} h_{2\delta_B}(x),$$

where $h_{2\delta_B}$ is the top eigenfunction for the Gauss–Kuzmin operator $\mathcal{L}_{2\delta_B}$ on B.

The proof of this Lemma is a straightforward generalization of the proof of Theorem 0.1.2 of [16].

4. Variation operator

We introduce another operator, which gives the variation of the Gauss–Kuzmin operator along the 1–parameter family, $\mathcal{A}_{\sigma,E} := \frac{d}{d\sigma}\mathcal{L}_{\sigma,E}$. This can be written in the form

$$(\mathcal{A}_{\sigma,E} f)(x,t) = \sum_{k\in\mathcal{N}_E} \frac{\log(x+k)}{(x+k)^\sigma} f\left(\frac{1}{x+k}, \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix} \cdot t \right), \tag{4.1}$$

with $\mathcal{N}_E \subset \mathbb{N}$ defined as in (3.2).

We have the following result.

Lemma 4.1. *Let E be a T–invariant set with GST, and let λ_σ be the top eigenvalue of the Gauss–Kuzmin operator $\mathcal{L}_{\sigma,E}$ acting on $\mathbb{V}$, and let h_σ be the corresponding unique (normalized) eigenfunction*

$$\mathcal{L}_{\sigma,E} h_\sigma = \lambda_\sigma h_\sigma. \tag{4.2}$$

Then we have

$$\int_E (\mathcal{A}_{2\delta_E,E} h_{2\delta_E})\, d\mathcal{H}^{\delta_E} = \lambda'_\sigma|_{\sigma=2\delta_E}, \tag{4.3}$$

with the operator $\mathcal{A}_{\sigma,E}$ as in (4.1), *and with $\lambda'_\sigma = \frac{d}{d\sigma}\lambda_\sigma$.*

Proof. We differentiate the eigenvalue equation (4.2) with respect to σ, and evaluate at $\sigma = 2\delta_E$. This gives

$$\mathcal{A}_{2\delta_E,E} h_{2\delta_E} + \mathcal{L}_{2\delta_E,E} h'_{2\delta_E} = \lambda'_{2\delta_E} h_{2\delta_E} + h'_{2\delta_E}, \tag{4.4}$$

where we used $\lambda_{2\delta_E} = 1$, and we set

$$h'_{2\delta_E} = \frac{d}{d\sigma} h_\sigma|_{\sigma=2\delta_E}.$$

Notice that

$$\int_E (h'_{2\delta_E} - \mathcal{L}_{2\delta_E,E} h'_{2\delta_E}) d\mathcal{H}^{\delta_E} = 0,$$

hence the result follows by integrating both sides of (4.4).

□

Now we return to the computation of limiting modular symbols (2.1). The following result complements the result of Theorem 2.1.

Theorem 4.2. *Let $E = B \times \mathbb{P}$ be a T–invariant subset of $[0,1] \times \mathbb{P}$, such that the assumptions of section 3.3 are satisfied. Then, for $\mathcal{H}^{\delta_B}$–almost every $\beta \in B$, the Lyapunov exponent satisfies*

$$\lambda(\beta) = 2\lambda'_{2\delta_B}, \tag{4.5}$$

hence the limiting modular symbol (2.1) *is given by*

$$\lim_{n\to\infty} \frac{1}{2\lambda'_{2\delta_B}\, n} \sum_{k=1}^{n} \varphi(T^k t_0), \tag{4.6}$$

where φ is defined as in (2.6).

Proof. The identification (4.5) can be proved using a strong law of large numbers, cf. [11], and §V.9 of [21]. We compute expectations. For $\beta = [a_1, a_2, \dots, a_{n-1}, a_n + x_n]$, we have

$$\frac{1}{x_1 \cdots x_n} = q_{n+1}\left(1 + \frac{x_{n+2} q_n}{q_{n+1}}\right),$$

with q_n the successive denominators satisfying $q_{n+1} = a_{n+1} q_n + q_{n-1}$. We can write

$$\frac{-1}{n+1} \sum_{k=1}^{n+1} \log(x_n) = \frac{1}{n+1} \log q_{n+1} + \frac{1}{n+1} \log\left(1 + \frac{x_{n+2} q_n}{q_{n+1}}\right).$$

Thus, we can estimate the asymptotic behavior of $\log q_n/n$ by the behavior of the average on the left hand side, as $n \to \infty$. By the convergence of the measures m_n, we have $\int_B \log(x_n)\, d\mathcal{H}^\delta = \int_B \log(x)\, h_{2\delta}(x)\,(1 + O(q^n)) d\mathcal{H}^\delta(x)$. Here $h_{2\delta}(x)$ is the normalized eigenfunction of the Gauss–Kuzmin operator on B, and $\delta_E = \delta_B = \delta$ Moreover, we have $\int_B \log(x)\, h_{2\delta} d\mathcal{H}^\delta = \int_B (\mathcal{L}_{2\delta,B} F)\, d\mathcal{H}^\delta$, for $F(x) := \log(x) h_{2\delta}(x)$. Notice that by (4.1) we also have $\mathcal{L}_{2\delta,B} F = -\mathcal{A}_{2\delta,B} h_{2\delta}$, hence we obtain

$$\int_B \log(x_n)\, d\mathcal{H}^\delta = -\int_B (\mathcal{A}_{2\delta,B} h_{2\delta})(1 + O(q^n)) d\mathcal{H}^\delta = -\lambda'_{2\delta}(1 + O(q^n)).$$

Thus, we have expectation

$$-\int_B \sum_{k=1}^{n+1} \log(x_n) d\mathcal{H}^\delta = (n+1)\lambda'_{2\delta} + O(1).$$

The variances can be computed following the argument given in §V.9 of [21] on the classical Khintchine–Levy theorem, so that

$$D^2(\sum_{k=1}^{n+1} \log(x_n)) := \int_B \left(\sum_{k=1}^{n+1} \log(x_n) \right)^2 d\mathcal{H}^\delta - (\lambda'_{2\delta})^2$$

is estimated by evaluating $\int_B \log(x_k)\log(x_{k+j}) d\mathcal{H}^\delta = \int_B \log(x)\log(y) dm(x,y)$, with $dm(x,y)$ the distribution for the measure of $\{x_k(\beta) \leq x$ and $x_{k+j}(\beta) \leq y\}$. By replacing x_k with a truncation at some large N, $x_k^* = [a_{k+1}, a_{k+2}, \ldots, a_{k+N}]$, it is possible to ensure that $dm(x,y) = dm_k(x) \cdot dm_{k+j}(y)(1 + O(q^j))$. The argument then follows as in §V.9 of [21].

∎

5. Vanishing results

In this section we generalize the weak vanishing result of [16], and improve the convergence in average to convergence almost everywhere.

5.1. **Weak convergence.**

The result of Proposition 3.3 imply the following vanishing in the average for limiting modular symbols.

Lemma 5.1. *Let $E = B \times \mathbb{P}$ be a T–invariant set with GST. Then, for all test functions $f \in L^2(E, d\mathcal{H}^{\delta_E})$, we have*

$$\lim_{\tau\to\infty} \frac{1}{\tau} \int_E \{x_0, y(\tau)\} \cdot f\, d\mathcal{H}^{\delta_E} = 0. \tag{5.1}$$

Proof. The proof is analogous to Lemma 2.3.1 of [16]. We write the limiting modular symbol as

$$\lim_{n\to\infty} \frac{1}{2\lambda'_{2\delta}\, n} \sum_{k=1}^{n} \varphi(T^k t).$$

Moreover, we have

$$\int_{B\times\mathbb{P}} \frac{1}{n}\sum_{k=1}^{n} \varphi\circ T^k\, f d\mathcal{H}^{\delta_E} = \int_{B\times\mathbb{P}} \frac{1}{n}\sum_{k=1}^{n}(\mathcal{L}^k f)\varphi\, d\mathcal{H}^{\delta_E}.$$

This converges to

$$\left(\int_{B\times\mathbb{P}} \varphi h_{2\delta_E}\, d\mathcal{H}^{\delta_E}\right)\left(\int_{B\times\mathbb{P}} f\, d\mathcal{H}^{\delta_E}\right)$$

where $h_{2\delta_E} = h_{2\delta_B}/|\mathbb{P}|$. This gives

$$\left(\frac{1}{|\mathbb{P}|}\sum_{s\in\mathbb{P}}\varphi(s)\right)\left(\int_{B\times\mathbb{P}} f\, d\mathcal{H}^{\delta_E}\right).$$

Arguing as in Lemma 2.3.1 of [16] we see that $\sum_{s\in\mathbb{P}}\varphi(s) = 0$, hence the result follows. □

5.2. **Strong convergence.** Now we show that, using a strong law of large numbers it is possible to improve the weak convergence of Theorem 5.1 to convergence $\mathcal{H}^{\delta_E}$–almost everywhere in E.

Theorem 5.2. *Let $E = B\times\mathbb{P}$ be a T–invariant subset of $[0,1]\times\mathbb{P}$, with GST. Then, for $\mathcal{H}^{\delta_E}$–almost every $\beta\in E$ we have $\{\{*,\beta\}\} = 0$.*

Proof. Here we treat $\varphi_k = \varphi(T^k t_0)$ as random variables, and prove that we can apply the strong law of large numbers. The argument is similar to the one used in the proof of Theorem 4.2. The result of Theorem 5.1 implies that the expectation $E = E(\varphi_k)$ is zero. We evaluate deviations,

$$D^2 = \int_E \left|\sum_{k=1}^{n}\varphi(T^k t_0)\right|^2 d\mathcal{H}^{2\delta_E}.$$

Here we write $|\varphi_k|^2$ for the pairing $|\varphi_k|^2 = \langle\varphi_k,\varphi_k\rangle$, where

$$\langle\varphi(s),\varphi(t)\rangle := \{g(0), g(i\infty)\}_G \bullet \{h(0), h(i\infty)\}_G,$$

with $s = gG$, $t = hG$, and $\bullet$ the intersection product, cf. [19]. The purpose is to show $D^2 = O(n)$. This implies $D \le K\sqrt{n}$ for some constant K, by which the strong law of large numbers applies (cf. §V.9 of [21]).

When writing

$$\sum_{k=1}^{n}\int|\varphi_k|^2 + \sum_{k=1}^{n}\sum_{j=1}^{n-k}\int\langle\varphi_k,\varphi_{k+j}\rangle,$$

we want to estimate the difference

$$\int\langle\varphi_k,\varphi_{k+j}\rangle - \langle\int\varphi_k, \int\varphi_{k+j}\rangle. \tag{5.2}$$

Let $P(g_k^{-1}(\beta)\cdot t_0 = t)$ be the probability that $g_k^{-1}(\beta)\cdot t_0 = t$, that is

$$P(g_k^{-1}(\beta)\cdot t_0 = t) = \int_B dm_k(x,t) d\mathcal{H}^{2\delta}(x) = m_k(1,t),$$

for $m_n(x,t)$ defined as in (3.8). We can write (5.2) as

$$\begin{array}{l} \sum_{s\in\mathbb{P}}\sum_{t\in\mathbb{P}}\langle\varphi(s),\varphi(t)\rangle \quad (P(g_k^{-1}(\beta)\cdot t_0 = s \text{ and } g_{k+j}^{-1}(\beta)\cdot t_0 = t) \\ \qquad\qquad\qquad -P(g_k^{-1}(\beta)\cdot t_0 = s)\cdot P(g_{k+j}^{-1}(\beta)\cdot t_0 = t)). \end{array}$$

It is sufficient to show that these are weakly dependent events, namely that we have $P(g_k^{-1}(\beta)\cdot t_0 = s \text{ and } g_{k+j}^{-1}(\beta)\cdot t_0 = t) = P(g_k^{-1}(\beta)\cdot t_0 = s)\cdot P(g_{k+j}^{-1}(\beta)\cdot t_0 = t)(1+O(q^j))$.

By the result of Lemma 3.4, we know that the measures $m_n(x,t)$ converge to the limit T–invariant measure $m(x,t)$ at a rate $O(q^n)$. However, the probability distributions of $x_k(\beta)$ and $x_{k+j}(\beta)$ do not satisfy the weak dependence condition

$$m(x,y) = m_n(x)m_{n+j}(y)(1+O(q^j)).$$

However, by proceeding as in the proof of Theorem 4.2 (cf. §V.9 of [21]), we consider a truncation at some large N, $x_k^* = [a_{k+1}, a_{k+2}, \dots, a_{k+N}]$, so that the probabilities

$$m_n^*(x,t) = \mathcal{H}^{2\delta}\left(\{\alpha\in[0,1]\,|x_k^*(\alpha)\le x \text{ and } g_k^{-1}(\alpha)\cdot t_0 = t\}\right)$$

satisfy the weak dependence condition. This gives the desired result for the probabilities $P(g_k^{-1}(\beta)\cdot t_0 = s \text{ and } g_{k+j}^{-1}(\beta)\cdot t_0 = t)$.

□

It is possible to give a different proof of this result based on the ergodic theorem.

6. Examples

We discuss some examples of invariant sets with GST that are relevant to the boundary geometry of modular curves.

6.1. **Generalized Gauss–Kuzmin operator.** This is the case analyzed in [16]. In this case the function space $\mathbb{V}$ is the real Banach space $V(\mathbb{D}\times\mathbb{P})$ of functions holomorphic on each sheet of $\mathbb{D}\times\mathbb{P}$ and continuous to the boundary, real at the real points of each sheet, with $\mathbb{D} = \{z\in\mathbb{C}\,|\,|z-1| < 3/2\}$. The space is endowed with the supremum norm. The generalized Gauss–Kuzmin operator acts on this space $\mathbb{V}$, for $\sigma > 1/2$. An additional hypothesis on the subgroup $G\subset \mathrm{PGL}(2,\mathbb{Z})$ is needed, namely the *transitivity condition* that the coset space $\mathbb{P}$ contains no proper invariant subset under the action of the semigroup $\mathrm{Red} = \cup_{n\ge 1}\mathrm{Red}_n \subset \mathrm{GL}(2,\mathbb{Z})$, with

$$\mathrm{Red}_n = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix} \,|\, a_1,\dots,a_n\ge 1;\ a_i\in\mathbb{Z} \right\}, \tag{6.1}$$

This ensures that $[0,1]\times\mathbb{P}$ has GST. In this case, the T–invariant measure on $[0,1]\times\mathbb{P}$ is given by

$$\mu(x,t)=\frac{1}{|\mathbb{P}|\log 2}\log(1+x), \tag{6.2}$$

where $1/(1+x)\log 2$ is the invariant density of the classical Gauss problem for the shift $Tx=1/x-[1/x]$. Thus, in this case we obtain an improvement on the weak convergence result of [16].

Corollary 6.1. *Assume that* $\mathrm{Red}(t)=\mathbb{P}$ *for each* $t\in\mathbb{P}$*. Then the limiting modular symbol satisfies* $\{\{*,\beta\}\}_G=\lim_{\tau\to\infty}\frac{1}{\tau}\{x_0,y(\tau)\}_G=0$*, for almost all* $\beta\in[0,1]$*.*

6.2. **Continued fractions Cantor sets.** There is a family of Cantor sets associated to the continued fraction expansion of numbers in $[0,1]$, namely the sets E_N given by the numbers $\alpha\in[0,1]$, $\alpha=[a_1,a_2,\ldots,a_\ell,\ldots]$, with all $a_i\leq N$. These sets have Hausdorff dimensions which tend to one as $N\to\infty$ according to the asymptotic formula [10]

$$\delta_N:=\dim_H(E_N)=1-\frac{6}{\pi^2 N}-\frac{72\log N}{\pi^4 N^2}+O(1/N^2).$$

As before, let G be a finite index subgroup of $\mathrm{PGL}(2,\mathbb{Z})$, and $\mathbb{P}$ be the coset space. There is an action of the shift operator T restricted to the space $E_N\times\mathbb{P}$, where E_N is one of the Cantor sets as above.

The Gauss–Kuzmin operator is of the form

$$(\mathcal{L}_{\sigma,N}f)(x)=\sum_{k=1}^{N}\frac{1}{(x+k)^\sigma}f\left(\frac{1}{x+k},\begin{pmatrix}0&1\\1&k\end{pmatrix}\cdot t\right), \tag{6.3}$$

acting on the Banach space B defined in §1.1 of [16].

In the case $G=\mathrm{PGL}(2,\mathbb{Z})$ the properties of this operator were studied by Hensley in [9], [10], where, using a functional analytic setup similar to Babenko's [1], it is shown that for $\sigma=2\delta_N$, there is a unique eigenfunction $f_N(x)$ of $\mathcal{L}_{2\delta_N,N}$, with eigenvalue $\lambda=1$, which gives the density of an invariant measure.

Following Hensley, we can consider the truncated Gauss–Kuzmin operators $\mathcal{L}_{\sigma,N}$ as perturbations of the Gauss–Kuzmin operator $\mathcal{L}_s$ on $[0,1]$.

Proposition 6.2. *Under the assumption* $\mathrm{Red}(t)=\mathbb{P}$ *for each* $t\in\mathbb{P}$*, and for sufficiently large* $N\geq N_0$*, there exists a unique eigenfunction* $f_N(x,t)$ *of the operator* $\mathcal{L}_{\delta_N,N}$ *of* (6.3)*, with eigenvalue* $\lambda=1$*. This satisfies*

$$f_N(x,t)=\frac{1}{|\mathbb{P}|}f_N(x), \tag{6.4}$$

where $f_N(x)$ *is the unique eigenfunction of the truncated Gauss–Kuzmin operator of* [9], [10]*, for* $\sigma=2\delta_N$*, and* $\mathbb{P}=\{1\}$*.*

Proof. The assumption $\mathrm{Red}(t) = \mathbb{P}$ for each $t \in \mathbb{P}$ ensures that the Cantor set E_N has GST, for $\sigma > 1/2$, as proved in [16].

The function (6.4) satisfies the relation $\mathcal{L}_{\delta_N,N} f_N = f_N$, for the operator (6.3), at $\sigma = 2\delta_N$.

For large N, we consider the operator $\mathcal{L}_{\sigma,N}$ as a perturbation of $\mathcal{L}_\sigma$. We estimate the operator norm of $T_{\sigma,N} = \mathcal{L}_\sigma - \mathcal{L}_{\sigma,N}$. We have

$$T_{\sigma,N} f(x,t) = \sum_{k=N+1}^{\infty} k^{-\sigma} \frac{1}{(x/k+1)^\sigma} f\left(\frac{1}{x+k}, \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix}(t)\right),$$

hence

$$\|T_{\sigma,N} f\| \leq \sum_{k=N+1}^{\infty} k^{-\eta} \left\| \frac{1}{(1+x/k)^\sigma} f\left(\frac{1}{x+k}, \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix}(t)\right)\right\|$$

$$\leq \frac{C}{N^\eta} \|f\|,$$

for σ complex with real part $\Re(\sigma) = \eta > 1/2$.

One can then apply the Crandall–Rabinowitz bifurcation Lemma [4] (cf. Lemma 7 of [10]), which guarantees the existence of a $\delta > 0$, such that, for $\|T_{\sigma,N}\| < \delta$, there is a unique $\rho_{\sigma,N} \in \mathbb{C}$, with $|\rho_{\sigma,N}| < \delta$, for which $L_\sigma - (\lambda_\sigma + \rho_{\sigma,N}) I$ is singular. The map $T_{\sigma,N} \mapsto \rho_{\sigma,N}$ is analytic, in the sense specified in [4], [10]. Moreover, there exists a unique $f_{\sigma,N}$ satisfying

$$(\mathcal{L}_\sigma + T_{\sigma,N}) f_{\sigma,N} = (\lambda_\sigma + \rho_{\sigma,N}) f_{\sigma,N},$$

and the map $T_{\sigma,N} \mapsto f_{\sigma,N}$ is also analytic.

□

The eigenfunction $f_N(x,t)$ of the truncated Gauss–Kuzmin operator defines the density of a T–invariant measure on $E_N \times \mathbb{P}$ absolutely continuous with respect to the δ_N–dimensional Hausdorff measure $d\mathcal{H}^{\delta_N}$.

We then obtain the following vanishing result.

Lemma 6.3. *Consider infinite geodesics in the upper half plane with an end at $i\infty$ and the other end at a point of E_N. For sufficiently large $N \geq N_0$, the limiting modular symbol satisfies*

$$\{\{*, \beta\}\}_G = \lim_n \frac{1}{2\lambda'_{2\delta_N}\, n} \sum_{k=1}^{n} \varphi(g_k(x)^{-1} t_0) = 0.$$

Notice that the constant N_0 provided by the Crandall–Rabinowitz bifurcation Lemma is independent of $G \subset \mathrm{PGL}(2,\mathbb{Z})$.

7. Intersection numbers

In the classical theory of the modular symbols of [14], [19] (cf. also [8], [18]) cohomology classes obtained from cusp forms are evaluated against relative homology classes given by modular symbols, and the corresponding intersection numbers are interpreted in terms of special values of L–functions associated to the automorphic forms which determine the cohomology class. In this and the next section, we seek to extend this viewpoint to the theory of limiting modular symbols. We first recall the setting of [19], which provides a convenient setting for computing intersection numbers with limiting modular symbols. We consider finite index subgroups $G \subset \mathrm{PSL}(2,\mathbb{Z})$. The arguments given for $\mathrm{PGL}(2,\mathbb{Z})$ adapt with minor modifications, cf. [13], [16].

Let I and R, be the elliptic points on the modular curve X_G, namely the image under the quotient map $\phi : \mathbb{H} \to X_G$ of the $\mathrm{PSL}(2,\mathbb{Z})$ orbits of i and $\rho = e^{2\pi i/3}$, respectively. With the notation of [19], we set $H_A^B := H_1(\overline{X_G} \smallsetminus A, B; \mathbb{Z})$. The modular symbols $\{g(0), g(i\infty)\}$, for $gG \in \mathbb{P}$, define classes in $H^{\rm cusps}$. For σ and τ the generators of $\mathrm{PSL}(2,\mathbb{Z})$ with $\sigma^2 = 1$ and $\tau^3 = 1$, we set $\mathbb{P}_I = \langle\sigma\rangle\backslash\mathbb{P}$ and $\mathbb{P}_R = \langle\tau\rangle\backslash\mathbb{P}$. There is an isomorphism $\mathbb{Z}^{|\mathbb{P}|} \cong H^R_{{\rm cusps}\cup I}$. Given the exact sequences

$$0 \to H_{\rm cusps} \stackrel{\iota'}{\to} H^R_{\rm cusps} \stackrel{\pi_R}{\to} \mathbb{Z}^{|\mathbb{P}_R|} \to \mathbb{Z} \to 0$$

and

$$0 \to \mathbb{Z}^{|\mathbb{P}_I|} \to H^R_{{\rm cusps}\cup I} \stackrel{\pi_I}{\to} H^R_{\rm cusps} \to 0,$$

the image $\pi_I(\tilde{x}) \in H^R_{\rm cusps}$ of an element $\tilde{x} = \sum_{s\in\mathbb{P}} \lambda_s s$ in $\mathbb{Z}^{|\mathbb{P}|} \cong H^R_{{\rm cusps}\cup I}$ represents an element $x \in H_{\rm cusps}$ iff the image $\pi_R(\pi_I(\tilde{x})) = 0$ in $\mathbb{Z}^{|\mathbb{P}_R|}$. As proved in [19], for $s = gG \in \mathbb{P}$, the intersection pairing $\bullet : H^{\rm cusps} \times H_{\rm cusps} \to \mathbb{Z}$ gives

$$\{g(0), g(i\infty)\} \bullet x = \lambda_s - \lambda_{\sigma s}.$$

We define the function $\Delta_x : \mathbb{P} \to \mathbb{R}$ by

$$\Delta_x(s) = \lambda_s - \lambda_{\sigma s}, \tag{7.1}$$

where x is given as above.

We list some examples of results involving intersection numbers that easily follow from the general results of the previous section.

Example 7.1. For a fixed $x \in H_{\rm cusps}$, represented by a linear combination $\tilde{x} = \sum_{s\in\mathbb{P}} \lambda_s s$ in $\mathbb{Z}^{|\mathbb{P}|}$ satisfying $\pi_R(\pi_I(\tilde{x})) = 0$ in $\mathbb{Z}^{|\mathbb{P}_R|}$, the asymptotic intersection number

$$\{\{*, \beta\}\} \bullet x := \lim_{\tau\to\infty} \frac{1}{\tau}\{x_0, y(\tau)\} \bullet x \tag{7.2}$$

is computed by the limit

$$\lim_{n\to\infty} \frac{1}{\lambda(\beta)\, n} \sum_{k=1}^{n} \Delta_x(g_k^{-1}(\beta)\cdot t_0), \tag{7.3}$$

where $\lambda(\beta)$ is the Lyapunov exponent. These asymptotic intersection numbers vanish, for $\mathcal{H}^{\delta_E}$–almost all β in any T–invariant set E with GST.

In the following example, we consider a cusp form Φ on $\mathbb{H}$, obtained as the pullback $\Phi = \varphi^*(\omega)/dz$ under the quotient map $\varphi : \mathbb{H} \to X_G$, and intersection numbers $\Delta_\omega(s) = \int_{g_s(0)}^{g_s(i\infty)} \Phi(z)dz$.

Example 7.2. Let Φ be a cusp form on $\mathbb{H}$, and let β be a number in $[0,1]$ with eventually periodic continued fraction expansion of period ℓ. Then we have

$$\lim_{\tau\to\infty} \frac{1}{\tau} \int_{x_0}^{y(\tau)} \Phi(z)\, dz = \frac{1}{\lambda(\beta)\ell} \sum_{k=1}^{\ell} \int_{\frac{p_{k-1}(\beta)}{q_{k-1}(\beta)}}^{\frac{p_k(\beta)}{q_k(\beta)}} \Phi(z)\, dz = \frac{1}{\lambda(g)} \int_0^{g(0)} \Phi(z)\, dz. \tag{7.4}$$

Here Φ is a cusp form, $\Phi(z)dz = \varphi^*(\omega)$, with ω representing the dual of the class x. The integral homology class $\{0, g(0)\}_G$ is determined by the periodic geodesic with $\beta = \alpha_g^+$, $g \in G$, and with $\lambda(g) = \log \Lambda_g^-$.

The following example follows by arguing as in Theorem 3.5 of [14].

Example 7.3. Let $G = \Gamma_0(N)$, and let $\Phi = \varphi^*(\omega)/dz$ be an eigenfunction for the Hecke operators T_m, for $(m,N) = 1$, with rational eigenvalues c_m, and let β be a quadratic irrationality. Then we obtain

$$\lim_{\tau\to\infty} \int_{x_0}^{y(T)} \Phi(z)dz = \frac{1}{(\sigma(m) - c_m)\lambda(\beta)\ell} \sum_{d|m} \sum_{\{b \mod d\}} \int_{\gamma_{m,b,d}(\beta)} \Phi(z)dz,$$

with $\sigma(m) = \sum_{d|m} d$, and for infinitely many m with $(m,N) = 1$,

$$\gamma_{m,b,d}(\beta) := \sum_{k=1}^{\ell} \left\{ \frac{p_k(\beta)}{q_k(\beta)}, \frac{m}{d^2} \cdot \frac{p_k(\beta)}{q_k(\beta)} + \frac{b}{d} \right\}_{\Gamma_0(N)} \in H_1(X_{\Gamma_0(N)}, \mathbb{Z}).$$

Example 7.4. Let E be a T–invariant set with GST. For $g \in \mathrm{PSL}(2,\mathbb{Z})$, consider the factor $U(g,z)$ of the form

$$U(g,z) = \frac{1}{(cz+d)^2} \quad \text{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

and set $\tilde{\Phi}(z) := \sum_{t\in\mathbb{P}} \Delta_\omega(t) U(g_t^{-1}, z) \Phi(g_t^{-1} z)$. This satisfies

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (\Delta_\omega(g_k^{-1}(\beta) \cdot t_0))^2 \to \frac{1}{|\mathbb{P}|} \int_0^{i\infty} \tilde{\Phi}(z)dz,$$

for $\mathcal{H}^{\delta_E}$–almost all $\beta \in E$.

8. Automorphic series on the boundary

In [16] it was proved that cusps forms of weight two for congruence subgroups (or rather their Mellin transforms) can be obtained by integrating along the real axis certain "automorphic series" defined in terms of continued fractions and modular symbols.

In particular, the following identity was proved: for $\Re(t) > 0$,

(8.1)

$$\int_0^1 d\beta \sum_{n=0}^{\infty} \frac{q_{n+1}(\beta)+q_n(\beta)}{q_{n+1}(\beta)^{1+t}} \int_{\{0,\frac{q_n(\beta)}{q_{n+1}(\beta)}\}} \omega = \left[\frac{\zeta(1+t)}{\zeta(2+t)} - \frac{L_\omega^{(N)}(2+t)}{\zeta^{(N)}(2+t)^2}\right] \int_0^{i\infty} \Phi(z)dz,$$

where the cusp form $\Phi = \varphi^*(\omega)/dz$ is an eigenform for all Hecke operators, $L_\omega^{(N)}$ is its Mellin transform with omitted Euler N–factor, $\zeta(s)$ is Riemann's zeta, with corresponding $\zeta^{(N)}$. Various generalizations of this average are also discussed in [16]. The main point in considering such averages is in order to recast the theory of modular forms in the upper half plane in terms of some function theory on the boundary $\mathbb{P}^1(\mathbb{R})$, hence in terms of the non–commutative compactification of X_G.

The general type of functions considered in §2 of [16] is of the form

$$\ell(f,\beta) = \sum_{k=1}^{\infty} f(q_k(\beta), q_{k-1}(\beta)), \tag{8.2}$$

for f a complex valued function defined on pairs of coprime integers (q, q') with $q \geq q' \geq 1$ and with $f(q,q') = O(q^{-\epsilon})$ for some $\epsilon > 0$. The identity

$$\int_0^1 d\beta\ \ell(f,\beta) = \sum_{\substack{q \geq q' \geq 1 \\ (q,q')=1}} \frac{f(q,q')}{q(q+q')},$$

which is used in [16] to prove (8.1) and various generalizations, is a result of Lévy, [12]. In our viewpoint, the summing over pairs of successive denominators is what replaces modularity, when "pushed to the boundary". This correspondence between Dirichlet series related to modular forms of weight two and integral averages of such "automorphic series" on the "invisible part" of the boundary of X_G is reminiscent of the physical principle of holography. We shall return to discuss this relation in a separate work.

In [16] only averages $\int_{[0,1]} \ell(f,\beta)d\beta$ were considered, and their relations to modular forms. Here we discuss the pointwise behavior, and prove that it is possible to use such expressions to construct non–trivial homology classes associated to a geodesic with "generic" endpoint in $[0,1]$, and a function f in this Lévy class. We shall only discuss the example of identity (8.1), but similar argument work for other similar functions considered in [16].

Theorem 8.1. *Consider the function*

$$f(q,q') = \frac{q+q'}{q^{1+t}} \int_{\left\{0,\frac{q'}{q}\right\}_{\Gamma_0(N)}} \omega. \tag{8.3}$$

Let $\ell(f,\beta)$ be the corresponding function as in (8.2). We can estimate

$$\ell(f,\beta) \sim \sum_{n=1}^{\infty} e^{-(5+2t)n\lambda(\beta)} \sum_{k=1}^{n} \Delta_\omega(g_k^{-1}(\beta)\cdot t_0).$$

Thus, if E be a T–invariant subset with GST, and with $0 < \lambda'_{2\delta_E} < \infty$, the series defining $\ell(f,\beta)$ converges absolutely, for $\mathcal{H}^{\delta_E}$–almost all $\beta \in E$.

Proof. We can estimate

$$\frac{q_n(\beta)+q_{n+1}(\beta)}{q_{n+1}(\beta)^{1+t}} \sim e^{-(5+2t)n\lambda(\beta)},$$

with $\lambda(\beta) = 2\lim_{n\to\infty}\log(q_n)/n$. Moreover, by equation (2) of §1 of [14], we can write

$$\{0,\frac{b}{a}\} = \sum_{k=-1}^{n}\{\frac{b_{k-1}}{a_{k-1}},\frac{b_k}{a_k}\} = \sum_{k=-1}^{n}\{g_k^{-1}(b/a)(0), g_k^{-1}(b/a)(i\infty)\},$$

where b_k/a_k are the successive convergents of the continued fraction expansion of the rational number $b/a = b_n/a_n$.

Recall that, in terms of continued fractions, we can write the rational numbers

$$\frac{q_n(\beta)}{q_{n+1}(\beta)} = [a_n, a_{n-1}, \dots, a_1],$$

where

$$\beta = [a_1, a_2, \dots, a_{n-1}, a_n, a_{n+1}, \dots].$$

Since the successive denominators q_n, viewed as polynomials

$$q_n(\beta) = Q_n(a_1,\dots,a_n),$$

satisfy

$$Q_n(a_1,\dots,a_n) = Q_n(a_n,\dots,a_1),$$

(cf. §7.5 of [14]), we have, for $\xi_n(\beta) := q_n(\beta)/q_{n+1}(\beta)$,

$$g_k^{-1}(\xi_n(\beta)) = g_k^{-1}(\beta),$$

for $k \leq n$. Hence the estimate on $\ell(f,\beta)$ follows. Then arguing as in Theorem 5.2 we obtain

$$\left|\sum_{k=1}^{n}\Delta_\omega(g_k^{-1}(\beta)\cdot t_0)\right| \sim o(\lambda(\beta)n),$$

hence the series converges absolutely.

∎

In fact, this proves that we have obtained a non–trivial homology class depending on the point β and on the function f.

Theorem 8.2. *Let E be a T–invariant subset of $[0,1] \times \mathbb{P}_{\Gamma_0(N)}$ with GST, and with $0 < \lambda'_{2\delta_E} < \infty$. Then, for $\mathcal{H}^{\delta_E}$–almost all $\beta \in E$, and for $\Re(t) > 0$, the limit*

$$C(f,\beta) := \sum_{n=1}^{\infty} \frac{q_{n+1}(\beta) + q_n(\beta)}{q_{n+1}(\beta)^{1+t}} \left\{0, \frac{q_n(\beta)}{q_{n+1}(\beta)}\right\}_{\Gamma_0(N)} \tag{8.4}$$

defines a class $C(f,\beta)$ in $H_1(X_{\Gamma_0(N)}, \mathbb{R})$. The function $\ell(f,\beta)$, for f as in (8.3), can be written as

$$\ell(f,\beta) = \int_{C(f,\beta)} \omega.$$

Proof. For $G = \Gamma_0(N)$, we have

$$\sum_{n=1}^{\infty} \frac{q_{n+1}(\beta) + q_n(\beta)}{q_{n+1}(\beta)^{1+t}} \left\{0, \frac{q_n(\beta)}{q_{n+1}(\beta)}\right\}_G \sim \sum_{n=1}^{\infty} e^{-(5+2t)n\lambda(\beta)} \sum_{k=1}^{n} \left\{g_k^{-1}(\beta)(0), g_k^{-1}(\beta)(i\infty)\right\}_G.$$

Then the same argument used in Theorem 8.1 proves the result. ∎

The average $\int_{[0,1]} d\beta \int_{C(f,\beta)} \omega$ satisfies (8.1). It would be interesting to know if averages over other T–invariant subsets, $\int_E d\mathcal{H}^{\delta_E}(\beta) \int_{C(f,\beta)} \omega$ also have an interesting number theoretic interpretation.

9. Trace formulae

The Selberg zeta function of a modular curve X_G, for G a finite index subgroup of $\mathrm{PGL}(2,\mathbb{Z})$, can be recovered from the Fredholm determinant of the generalized Gauss–Kuzmin operator, namely

$$\det(I - \mathcal{L}_{2s}) = Z_G(s), \tag{9.1}$$

for $s \in \mathbb{C}$ with $\Re(s) > 1/2$. The result holds for $G \subset \mathrm{PSL}(2,\mathbb{Z})$ with $\mathcal{L}_{2s}$ replaced by $\mathcal{L}_{2s}^2$, see [3], [16] for the general case, and [13] for the case $G = \mathrm{PSL}(2,\mathbb{Z})$ or $\mathrm{PGL}(2,\mathbb{Z})$.

We investigate possible generalizations of this identity when only geodesics with ends on some smaller T–invariant subset are considered.

Given a T–invariant subset $E \subset [0,1] \times \mathbb{P}$, we can define

$$Z_{G,E}(s) := \prod_{\gamma \in \mathrm{Prim}_E} \prod_{m=0}^{\infty} \left(1 - e^{-(s+m)\mathrm{length}(\gamma)}\right), \tag{9.2}$$

where Prim_E is the set of primitive closed geodesics in X_G that correspond to geodesics in $\mathbb{H} \times \mathbb{P}$ with ends at points of E. Notice that, in general, little is known about the properties of such zeta functions. For instance, whether they still have a meromorphic continuation, or if there is a trace formula relating (9.2) to the spectrum of the Laplacian on some suitable space of functions.

Regarding the relation of (9.2) to the Gauss–Kuzmin operator, the following result is a simple generalization of the results of [16] and [17]. It applies, for instance, to Hensley's Cantor sets of continued fractions $E_N \times \mathbb{P}$.

Proposition 9.1. *Let E be a T–invariant subset of $[0,1]\times\mathbb{P}$ with GST, which is of the form $E = B\times\mathbb{P}$, with*

$$B = \{\beta \in [0,1] : a_i \in \mathcal{N}\} \tag{9.3}$$

for some $\mathcal{N}\subset\mathbb{N}$. We have a trace formula

$$\det(I - \mathcal{L}_{2s,E}) = Z_{G,E}(s), \tag{9.4}$$

where $\mathcal{L}_{2s,E} = \mathcal{R}_h$ is the Ruelle transfer operator, with $\sigma = 2s \in \mathbb{C}$, $\Re(s) > 1/2$, $h = -s\log|T'|$, and the right hand side is defined as in (9.2).

Proof. We denote by $\pi_{\sigma,k}$ the operator

$$(\pi_{\sigma,k} f)(x,t) = \frac{1}{(x+k)^\sigma} f\left(\frac{1}{x+k}, \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix}\cdot t\right),$$

for $\sigma\in\mathbb{C}$, $\Re(\sigma) > 1$. We can write (3.4) as the operator

$$\mathcal{L}_{\sigma,E} f = \sum_{k\in\mathcal{N}} \pi_{\sigma,k} f. \tag{9.5}$$

The same argument of [17] shows that this is a nuclear operator of order zero in the sense of [7] on a complex Banach space of holomorphic functions defined on a domain $\mathbb{D}\times\mathbb{P}$ containing E, for $\Re(\sigma)>1$. Thus, we have

$$-\log\det(I-\mathcal{L}_{\sigma,E}) = \sum_{\ell=1}^\infty \frac{Tr\,\mathcal{L}^\ell_{\sigma,E}}{\ell} = Tr\left(\sum_{\ell=1}^\infty \frac{1}{\ell}\left(\sum_{k\in\mathcal{N}} \pi_{\sigma,k}\right)^\ell\right) =$$

$$Tr\left(\sum_{g\in \mathrm{Red}_E} \frac{1}{\ell(g)}\pi_\sigma(g)\right) = \sum_{g\in\mathrm{Hyp}_E} \frac{1}{\kappa(g)}\chi_\sigma(g)\tau_g,$$

where $\pi_\sigma(g) = \prod_{j=1}^{\ell(g)} \pi_{\sigma,a_j}$, and

$$\mathrm{Red}_E := \left\{ g\in\mathrm{Red} \,\middle|\, g = \prod_{j=1}^{\ell(g)} \begin{pmatrix} 0 & 1 \\ 1 & a_j\end{pmatrix} \; a_i\in\mathcal{N}\right\}.$$

Here we denote by Hyp_E a set or representatives of the conjugacy classes of hyperbolic matrices in $\mathrm{GL}(2,\mathbb{Z})$ which contain reduced representatives in Red_E, and

$$\chi_{2s}(g) = \frac{N(g)^{-s}}{1-\det(g)N(g)^{-1}},$$

for

$$N(g) = \left(\frac{Tr(g) + D(g)^{1/2}}{2}\right)^2,$$

and $D(g) = Tr(g)^2 - 4\det(g)$, and τ_g is defined as $\tau_g := \#\{t \in \mathbb{P} : g \cdot t = t\}$.

The argument now follows the lines of [13]. Under the hypotheses on E, (9.2) is of the form

$$Z_{G,E}(s) := \prod_{g\in \mathrm{Prim}_E} \prod_{m=0}^{\infty} \det\left(1 - \det(g)^m N(g)^{-(s+m)} \rho_{\mathbb{P}}(g)\right), \tag{9.6}$$

where $\rho_{\mathbb{P}}(g)$ is the action of g on $\mathbb{P}$, and Prim_E is the set of primitive hyperbolic elements, $g = h^{k(g)}$, $g \in \mathrm{Hyp}_E$. We write (9.6) as

$$\sum_{g\in\mathrm{Prim}_E} \sum_{m=0}^{\infty} Tr \sum_{k=1}^{\infty} \frac{1}{k} \det(g)^{mk} N(g)^{-(s+m)k} \rho_{\mathbb{P}}(g^k) =$$

$$\sum_{g\in\mathrm{Prim}_E} \sum_{k=1}^{\infty} \frac{1}{k} \frac{N(g)^{-ks}\tau_{g^k}}{1-\det(g^k)N(g)^{-k}} = \sum_{g\in\mathrm{Hyp}_E} \frac{1}{\kappa(g)} \chi_\sigma(g)\tau_g,$$

where $Tr(\rho_{\mathbb{P}}(g)) = \tau_g$. The argument underlying the passage from summing over Prim_E to the summing over Hyp_E is exactly as in [13], by observing that if a conjugacy class in Hyp contains a representative in Red_E, then all the $\ell(g)/k(g)$ representatives in Red are also in Red_E. Thus, we obtain

$$-\log\det(I - \mathcal{L}_{\sigma,E}) = -\log Z_{G,E}.$$

□

It is easy to see where this argument breaks down when less strong assumptions are made on the invariant set E. For instance, for a T–invariant subset $E = B \times \mathbb{P}$, we can consider the sets $\mathrm{Red}_{E,\ell} = \{g \in \mathrm{Red}_\ell : [\overline{a_1,\dots,a_\ell}] \in B\}$. Even assuming that all the corresponding operators $\pi_\sigma(g)$ are of trace class, we no longer have the identification of the term $\sum_{g\in\mathrm{Red}_{E,\ell}} \pi_\sigma(g)$ with $(\sum_n \pi_{\sigma,n})^\ell$, hence we can no longer identify $Tr(\sum_{\ell=1}^\infty \frac{1}{\ell}\sum_{g\in\mathrm{Red}_{E,\ell}} \pi_\sigma(g))$ with the Fredholm determinant of a Gauss–Kuzmin operator. Many interesting examples of invariant subsets, such as level sets of the Lyapunov exponent, do not satisfy the assumptions of Proposition 9.1. However, in certain cases, it is still possible to identify (9.2) with a Fredholm determinant, albeit for a different Gauss-Kuzmin operator.

9.1. **Example: minimal Lyapunov exponent.** The minimal value attained by the Lyapunov exponent at periodic points of the shift T on $[0,1]$ is given by

$$c_0 = 2\log\left(\frac{1+\sqrt{5}}{2}\right).$$

Let L_{c_0} be the level set of the Lyapunov exponent $L_{c_0} = \{\beta \in [0,1] : \lambda(\beta) = c_0\}$.

Proposition 9.2. *Consider the dense T–invariant subset of $L_{c_0} \times \mathbb{P} \subset [0,1] \times \mathbb{P}$. The zeta function (9.2), with Geod_E the set of primitive closed geodesics in X_G which lift to geodesics in $\mathbb{H}$ with end in L_{c_0} satisfies*

$$\det(I - \mathcal{L}_{2s,E_1\times\mathbb{P}}) = Z_{G,L_{c_0}\times\mathbb{P}}(s),$$

for $s \in \mathbb{C}$, $\Re(s) > 1/2$, where $\mathcal{L}_{\sigma,E_1\times\mathbb{P}}$ is the transfer operator associated to the Hensley Cantor set $E_N \times \mathbb{P}$ for $N = 1$.

Proof. The level set L_{c_0} of the Lyapunov exponent consists of all $\beta \in [0,1]$ whose continued fraction expansion has a proportion of 1's which tends asymptotically to 100%, cf. [20]. Thus, L_{c_0} is an uncountable dense subset of $[0,1]$. The only eventually periodic points in L_{c_0} are numbers with partial quotients in the continued fraction expansion eventually equal to 1. Thus, we have

$$-\log Z_{G,L_{c_0}\times\mathbb{P}}(s) = Tr\left(\sum_{\ell=1}^{\infty}\frac{1}{\ell}\sum_{g\in \mathrm{Red}_{L_{c_0}\times\mathbb{P},\ell}}\pi_\sigma(g)\right) = \sum_{\ell=1}^{\infty}\frac{1}{\ell}Tr\,{\mathcal{L}_{2s,E_1}}^{\ell},$$

for

$$(\mathcal{L}_{2s,E_1}f)(x,t) = \frac{1}{(x+1)^{2s}}f\left(\frac{1}{x+1}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\cdot t\right).$$

□

A case which is especially of interest, in relation to Manin's theory of real multiplication [15], and which also does not fit into the assumptions of Proposition 9.1 is the case of geodesics in $\mathbb{H}$ with ends at quadratic irrationalities in a real quadratic field $\mathbb{Q}(\sqrt{d})$. One can separates out the Selberg zeta function into contributions of different real quadratic fields and seek for corresponding Gauss–Kuzmin operators. We shall return to this in future work.

10. C^*–algebras

We show how considering T–invariant subsets of $[0,1] \times \mathbb{P}$ can be used to enrich the picture of non–commutative geometry at the boundary of modular curves, already illustrated in [16].

Consider a T–invariant set $E = B \times \mathbb{P}$, where B is defined by a condition on the digits (9.3), for $\mathcal{N} \subset \mathbb{N}$. Such sets are totally disconnected compact T–invariant subsets of

$[0,1] \times \mathbb{P}$. Let $C(E,\mathbb{Z})$ be the space of continuous functions from E to the integers. We denote by $C(E,\mathbb{Z})^T$ the invariants $C(E,\mathbb{Z})^T = \{f \in C(E,\mathbb{Z}) \,|\, f - f\circ T = 0\}$, and by $C(E,\mathbb{Z})_T$ the co–invariants $C(E,\mathbb{Z})_T = C(E,\mathbb{Z})/B(E,\mathbb{Z})$, with the set of co–boundaries $B(E,\mathbb{Z}) := \{f - f\circ T \,|\, f \in C(E,\mathbb{Z})\}$.

We recall the following result (see e.g. [2]).

Lemma 10.1. *Let E be a totally disconnected compact Hausdorff space. There is an identification $K_0(C(E)) \cong C(E,\mathbb{Z})$, while $K_1(C(E)) = 0$. Moreover, there is an exact sequence*

$$0 \to C(E,\mathbb{Z})^T \to K_0(C(E)) \stackrel{I-T_*}{\to} K_0(C(E)) \to C(E,\mathbb{Z})_T \to 0.$$

The set of co–invariants $C(E,\mathbf{Z})_T$ is a unital pre–ordered group, $(C(E,\mathbb{Z})_T, C(E,\mathbb{Z})_T^+, [1])$.

The pre–order structure is given by specifying the positive cone $C(E,\mathbb{Z})_T^+$ and an order unit, namely an element $u \in C(E,\mathbb{Z})_T^+$ such that, for all $[f] \in C(E,\mathbb{Z})_T$ there exists a $n \in \mathbb{Z}$, $n \geq 1$, such that $nu - [f] \in C(E,\mathbb{Z})_T^+$. Here we set:

$$C(E,\mathbb{Z})_T^+ := \{[f] \in C(E,\mathbb{Z})_T \,|\, \exists f_0 \in C(E,\mathbb{Z}), [f_0] = [f], f_0 \geq 0\},$$

and the order unit $u = [1]$, the class of the constant function on E.

The pre–ordered groups for T–invariant E defined by (9.3) give a collection of invariants describing some features of the non–commutative geometry at the boundary of the modular curve X_G. The arithmetic relevance of these invariants remains to be understood. The structure of these pre–ordered groups can be studied through traces.

10.1. **Invariant measures and traces.** Recall that a *trace* on a unital pre–ordered group $(C(E,\mathbb{Z})_T, C(E,\mathbb{Z})_T^+, [u])$, where E is a totally disconnected compact Hausdorff space invariant under T, is a positive homomorphism $\tau : C(E,\mathbb{Z})_T \to \mathbb{R}$, where positive means that it sends the cone $C(E,\mathbb{Z})_T^+$ to $\mathbb{R}^{\geq 0}$, which moreover satisfies $\tau(u) = 1$. Normalized Borel T–invariant measures on E define traces by integration,

$$\tau([f]) = \int_E f\, d\mu.$$

Thus, if the invariant set E has GST, so that we can prove the existence of an invariant measure, we have an associated trace on $C(E,\mathbb{Z})_T$

$$f \mapsto \int_E f\, h_{2\delta_E,E}\, d\mathcal{H}^{\delta_E}.$$

10.2. **Cuntz–Krieger algebras.** Consider a T–invariant set $E = B \times \mathbb{P}$, where B is defined by the condition (9.3), with $\mathcal{N} \subset \mathbb{N}$ a finite subset. Then we can further enrich the picture of the non–commutative geometry at the boundary of the modular curve X_G, by considering a C^*–algebra associated to the dynamics of the shift T on E, in the form of a Cuntz–Krieger algebra $\mathcal{O}_{A_E}$ for a matrix A_E, see [5], [6].

Example 10.2. In the case of the Hensley Cantor sets, the shift T on E_N is a full one–sided shift on N symbols. For $\mathbb{P} = \{1\}$, the associated C^*–algebra is the classical Cuntz algebra $\mathcal{O}_N$. In general, we consider the Markov partition given by

$$\mathcal{A}_N := \{((i,t),(j,s)) | U_{i,t} \subset T(U_{j,s})\},$$

where $i, j \in \{1, \dots, N\}$, and $s, t \in \mathbb{P}$, with sets $U_{i,t} = U_i \times \{t\}$, where $U_i \subset E_N$ are the clopen subsets where the local inverses of T are defined,

$$U_j = \left[\frac{1}{j+1}, \frac{1}{j}\right] \cap E_N.$$

The condition $U_{i,t} \subset T(U_{j,s})$ corresponds to

$$\begin{pmatrix} 0 & 1 \\ 1 & j \end{pmatrix} \cdot t = s.$$

The Markov partition determines a corresponding $Np \times Np$ matrix A_N, for $p = |\mathbb{P}|$, with entries $(A_N)_{is,jt} = 1$ if $U_{i,t} \subset T(U_{j,s})$ and zero otherwise. We obtain a Cuntz–Krieger algebra $\mathcal{O}_{A_N}$.

There are numerical invariant that can be extracted from the non–commutative geometry of such Cuntz–Krieger algebras, and our hope is to show that some of these recover the rich arithmetic structure of modular curves. An instance where something of this sort is known to happen is the result of [16], where we showed that from the Pimsner–Voiculescu exact sequence for the C^*–algebra $C(\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}) \rtimes \mathrm{PSL}(2,\mathbb{Z})$ one can recover Manin's presentation of the modular complex.

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References

[1] K. I. Babenko, *On a problem of Gauss.* Dokl. Akad. Nauk SSSR, Tom 238 (1978) No. 5, 1021–1024.

[2] M. Boyle, D. Handelman, *Orbit equivalence, flow equivalence, and ordered cohomology*, Israel J. Math. 95 (1996) 169–210.

[3] C.H. Chang, D. Mayer, *Thermodynamic formalism and Selberg's zeta function for modular groups.* Regul. Chaotic Dyn. 5 (2000), no. 3, 281–312.

[4] M.G. Crandall, P.H. Rabinowitz, *Bifurcation, perturbation of simple eigenvalues and linearized stability*, Arch. Rational Mech. Anal. 52 (1973) 161–180.

[5] J. Cuntz, *A class of C^*–algebras and topological Markov chains II: reducible chains and the Ext–functor for C^*–algebras*, Invent. Math. 63 (1981) 25–40.

[6] J. Cuntz, W. Krieger, *A class of C^*–algebras and topological Markov chains*, Invent. Math. 56 (1980) 251–268.

[7] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires.* Mem. Am. Math. Soc. Vol. 16, 1955.

[8] G. Harder, *General aspects in the theory of modular symbols*, Seminar on number theory, Paris 1981–82 (Paris, 1981/1982), 73–88, Progr. Math., 38, Birkhäuser Boston, Boston, MA, 1983.

[9] D. Hensley, *The Hausdorff dimension of some continued fraction Cantor sets*, J. Number Theory 33 (1989) 182–198.
[10] D. Hensley, *Continued fraction Cantor sets, Hausdorff dimension, and functional analysis*, J. Number Theory 40 (1992) 336–358.
[11] A.Ya. Khintchine, *Continued fractions*, P.Noordhoff, 1963.
[12] P. Lévy. *Sur les lois de probabilité dont dépendent les quotients complets et incomplets d'une fraction continue.* Bull. Soc. Math. France, 557 (1929), 178–194.
[13] J. Lewis, D. Zagier. *Period functions and the Selberg zeta function for the modular group.* In: The Mathematical Beauty of Physics, Adv. Series in Math. Physics 24, World Scientific, Singapore, 1997, pp. 83–97.
[14] Yu.I. Manin, *Parabolic points and zeta-functions of modular curves.* Math. USSR Izvestija, vol. 6, No. 1 (1972), 19–64, and Selected Papers, World Scientific, 1996, pp. 202–247.
[15] Yu.I. Manin, *Real multiplication and non–commutative geometry*, preprint 2001.
[16] Yu.I. Manin, M. Marcolli, *Continued fractions, modular symbols, and non–commutative geometry*, preprint math.NT/0102006.
[17] D. Mayer. *Continued fractions and related transformations.* In: Ergodic Theory, Symbolic Dynamics and Hyperbolic Spaces, Eds. T. Bedford et al., Oxford University Press, Oxford 1991, pp. 175–222.
[18] B. Mazur, *Courbes elliptiques et symboles modulaires*, Séminaire Bourbaki vol.1971/72, Exposés 400–417, LNM 317, Springer Verlag 1973, pp. 277–294.
[19] L. Merel. *Intersections sur les courbes modulaires.* Manuscripta Math., 80 (1993), 283–289.
[20] M. Pollicott, H. Weiss, *Multifractal analysis of Lyapunov exponent for continued fraction and Manneville-Pomeau transformations and applications to Diophantine approximation*, Comm. Math. Phys. 207 (1999), no. 1, 145–171.
[21] A. Rockett, P. Szüsz, *Continued fractions*, World Scientific, 1992.
[22] F. Schweiger. *Ergodic theory of fibred systems and metric number theory.* Clarendon Press, 1995.

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