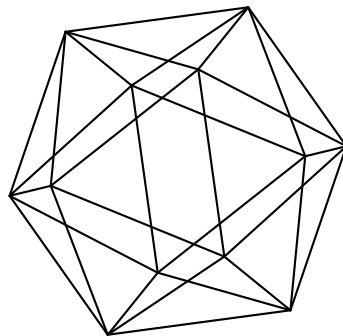# Max-Planck-Institut für Mathematik Bonn

Some conjectures on continuous rational maps into spheres

by

Wojciech Kucharz
Krzysztof Kurdyka

Some conjectures on continuous rational maps into spheres

Wojciech Kucharz

Krzysztof Kurdyka

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Institute of Mathematics
Faculty of Mathematics and Computer Science
Jagiellonian University
ul. Łojasiewicza 6
30-348 Kraków
Poland

Laboratoire de Mathématiques
UMR 5175 du CNRS
Université de Savoie
Campus Scientifique
73376 Le Bourget-du-Lac Cedex
France

Some conjectures on continuous rational maps into spheres

Wojciech Kucharz and Krzysztof Kurdyka

Abstract. Recently continuous rational maps between real algebraic varieties have attracted the attention of several researchers. In this paper we continue the investigation of approximation properties of continuous rational maps with values in spheres. We propose a conjecture concerning such maps and show that it follows from certain classical conjectures involving transformation of compact smooth submanifolds of nonsingular real algebraic varieties onto subvarieties. Furthermore, we prove our conjecture in a special case and obtain several related results.

Key words. Real algebraic variety, regular map, continuous rational map, approximation, homotopy, homology.

Mathematics subject classification (2010). 14P05, 14P25, 26C15.

1   Introduction and main results

Recently several authors devoted their papers to the investigation of continuous rational maps between real algebraic varieties, cf. [3, 7, 8, 13, 15, 16, 17, 18, 19, 20]. Continuing this line of research, we propose Conjecture A, whose proof would completely clarify many problems concerning homotopical and approximation properties of continuous rational maps with values in unit spheres. We prove this conjecture in a special case and also obtain some related results. Furthermore, we show that Conjecture A is a consequence of another conjecture, which has nothing to do with continuous rational maps and originates from the celebrated paper of Nash [23] and the subsequent developments due to Tognoli [25], Akbulut and King [1], and other mathematicians. All results announced in this section are proved in Section 2.

Throughout the present paper we use the term *real algebraic variety* to mean a locally ringed space isomorphic to an algebraic subset of $\mathbb{R}^n$, for some n, endowed with the Zariski topology and the sheaf of real-valued regular functions (such an object is called an affine real algebraic variety in [4]). The class of real algebraic varieties is identical with the class of quasiprojective real varieties, cf. [4, Proposition 3.2.10, Theorem 3.4.4]. Nonsingular varieties are assumed to be of pure dimension. Morphisms of real algebraic varieties are called *regular maps*. Each real algebraic variety carries also the Euclidean topology, which is induced by the usual metric on $\mathbb{R}$. Unless explicitly stated otherwise, all topological notions relating to real algebraic varieties refer to the Euclidean topology.

Let X and Y be real algebraic varieties. A map $f\colon X \to Y$ is said to be *continuous rational* if it is continuous on X and there exists a Zariski open and dense subvariety U of X such that the restriction $f|_U\colon U \to Y$ is a regular map. Let $X(f)$ denote the union of all such U. The complement $P(f) = X \setminus X(f)$ of $X(f)$ is the smallest Zariski closed subvariety of X for which

the restriction $f|_{X\setminus P(f)}\colon X\setminus P(f)\to Y$ is a regular map. If $f(P(f))\neq Y$, we say that f is a *nice* map. There exist continuous rational maps that are not nice, cf. [15, Example 2.2 (ii)]. Continuous rational maps have only recently become the object of serious investigation, cf. [3, 7, 8, 13, 15, 16, 17, 18, 19, 20]. They form a natural intermediate class between regular and continuous maps. Having many desirable features of regular maps they are more flexible.

The space $\mathcal{C}(X,Y)$ of all continuous maps from X into Y will always be endowed with the compact-open topology. There are the following inclusions

$$\mathcal{C}(X,Y)\supseteq\mathcal{R}^0(X,Y)\supseteq\mathcal{R}_0(X,Y)\supseteq\mathcal{R}(X,Y),$$

where $\mathcal{R}^0(X,Y)$ is the set of all continuous rational maps, $\mathcal{R}_0(X,Y)$ consists of the nice maps in $\mathcal{R}^0(X,Y)$, and $\mathcal{R}(X,Y)$ is the set of regular maps. By definition, a continuous map from X into Y can be approximated by continuous rational maps if it belongs to the closure of $\mathcal{R}^0(X,Y)$ in $\mathcal{C}(X,Y)$. Approximation by nice continuous rational maps or regular maps is defined in the analogous way.

Henceforth we assume that the variety X is compact and nonsingular, and concentrate our attention on maps with values in the unit p-sphere

$$\mathbb{S}^p=\{(u_0,\ldots,u_p)\in\mathbb{R}^{p+1}\mid u_0^2+\cdots+u_p^2=1\}$$

for $p\geq 1$. Regular maps from X into $\mathbb{S}^p$ have been extensively studied, cf. [4] and the literature cited there. Here we only recall that the closure of $\mathcal{R}(X,\mathbb{S}^p)$ in $\mathcal{C}(X,\mathbb{S}^p)$ can be a much smaller set than the closure of $\mathcal{R}_0(X,\mathbb{S}^p)$, cf. [17, Example 1.8]. If $\dim X\leq p$, then the set $\mathcal{R}_0(X,\mathbb{S}^p)$ is dense in $\mathcal{C}(X,\mathbb{S}^p)$. This assertion holds for $\dim X<p$ since $\mathbb{R}^p$ is biregularly isomorphic to $\mathbb{S}^p$ with one point removed, whereas for $\dim X=p$ it is proved in [17]. Furthermore, $\mathcal{R}_0(\mathbb{S}^n,\mathbb{S}^p)$ is dense in $\mathcal{C}(\mathbb{S}^n,\mathbb{S}^p)$ for all positive integers n and p, cf. [17]. However, if $\dim X>p$, then it can happen that a continuous map from X into $\mathbb{S}^p$ is not homotopic to any continuous rational map, and hence $\mathcal{R}^0(X,\mathbb{S}^p)$ is not dense in $\mathcal{C}(X,\mathbb{S}^p)$, cf. [17, Theorem 2.8]. There are reasons to believe that homotopical and approximation properties of nice continuous rational maps from X into $\mathbb{S}^p$, investigated in [15] and [17], are actually equivalent and fully determined by certain (co)homological conditions. We give a precise formulation of the last statement in Conjecture A.

Some preparation is required. Let M be a compact smooth (of class $\mathcal{C}^\infty$) codimension p submanifold of X. If the normal bundle to M in X is oriented, we denote by τ_M^X the Thom class of M in the cohomology group $H^p(X,X\setminus M;\mathbb{Z})$, cf. [22, p. 118]. The image of τ_M^X by the restriction homomorphism $H^p(X,X\setminus M;\mathbb{Z})\to H^p(X;\mathbb{Z})$, induced by the inclusion map $X\hookrightarrow(X,X\setminus M)$, will be denoted by $[\![M]\!]^X$ and called the cohomology class represented by M. If X is oriented as a smooth manifold, then $[\![M]\!]^X$ is up to sign Poincaré dual to the homology class in $H_*(X;\mathbb{Z})$ represented by M, cf. [22, p. 136]. Similarly, without any orientability assumption, we define the cohomology class $[M]^X$ in $H^p(X;\mathbb{Z}/2)$ represented by M. The cohomology class $[M]^X$ is Poincaré dual to the homology class in $H_*(X;\mathbb{Z}/2)$ represented by M. Furthermore, if the normal bundle to M in X is oriented, then

$$\rho([\![M]\!]^X)=[M]^X,$$

where

$$\rho\colon H^*(X;\mathbb{Z})\to H^*(X;\mathbb{Z}/2)$$

is the reduction mod 2 homomorphism.

We say that a cohomology class $u\in H^p(X;\mathbb{Z})$ (resp. $v\in H^p(X;\mathbb{Z}/2)$) is *adapted* if there exists a nonsingular codimension p Zariski locally closed subvariety Z of X such that Z is a compact smooth submanifold with trivial normal bundle and

$$u=[\![Z]\!]^X\quad(\text{resp. }v=[Z]^X),$$

where the first equality holds when the normal bundle to Z is suitably oriented. Here Z need not be Zariski closed in X, but the nonsingular locus of its Zariski closure coincides with Z. Denote

by $A^p(X;\mathbb{Z})$ (resp. $A^p(X;\mathbb{Z}/2)$) the subgroup of $H^p(X;\mathbb{Z})$ (resp. $H^p(X;\mathbb{Z}/2)$) generated by all adapted cohomology classes. By construction,

$$\rho(A^p(X;\mathbb{Z})) = A^p(X;\mathbb{Z}/2).$$

The groups $A^p(-;\mathbb{Z})$ and $A^p(-;\mathbb{Z}/2)$ can be explicitly computed for some real algebraic varieties.

Example 1.1. Let $X = X_1 \times \cdots \times X_r$, where X_i is a nonsingular real algebraic variety diffeomorphic to the n_i-sphere for $1 \le i \le r$. Then, by the Künneth formula,

$$A^p(X;\mathbb{Z}) = H^p(X;\mathbb{Z}) \quad \text{and} \quad A^p(X;\mathbb{Z}/2) = H^p(X;\mathbb{Z}/2)$$

for every $p \ge 0$.

Let s_p (resp. $\bar{s}_p$) be a generator of the cohomology group $H^p(\mathbb{S}^p;\mathbb{Z}) \cong \mathbb{Z}$ (resp. $H^p(\mathbb{S}^p;\mathbb{Z}/2) \cong \mathbb{Z}/2$), $p \ge 1$. Recall that a cohomology class $u \in H^p(X;\mathbb{Z})$ (resp. $v \in H^p(X;\mathbb{Z}/2)$) is said to be *spherical* if $u = f^*(s_p)$ (resp. $v = f^*(\bar{s}_p)$) for some continuous map $f\colon X \to \mathbb{S}^p$. Without loss of generality, the map f can be assumed to be smooth. In that case, if $y \in \mathbb{S}^p$ is a regular value of f, then the inverse image $f^{-1}(y)$ is a compact smooth codimension p submanifold of X, embedded with trivial normal bundle, for which

$$f^*(s_p) = [\![f^{-1}(y)]\!]^X \quad \text{and} \quad f^*(\bar{s}_p) = [f^{-1}(y)]^X,$$

where the first equality holds provided that the normal bundle to $f^{-1}(y)$ in X is suitably oriented (this is a well known fact whose proof is recalled in [18, p. 258]). Conversely, if M is a compact smooth codimension p submanifold of X with normal bundle trivial and oriented, then the cohomology classes $[\![M]\!]^X \in H^p(X;\mathbb{Z})$ and $[M]^X \in H^p(X;\mathbb{Z}/2)$ are spherical (this is a consequence of a classical result in framed cobordism, cf. [21, p. 44]). We denote by $H^p_{\mathrm{sph}}(X;\mathbb{Z})$ (resp. $H^p_{\mathrm{sph}}(X;\mathbb{Z}/2)$) the subgroup of $H^p(X;\mathbb{Z})$ (resp. $H^p(X;\mathbb{Z}/2)$) generated by all spherical cohomology classes. As explained above,

$$A^p(X;\mathbb{Z}) \subseteq H^p_{\mathrm{sph}}(X;\mathbb{Z}) \quad \text{and} \quad A^p(X;\mathbb{Z}/2) \subseteq H^p_{\mathrm{sph}}(X;\mathbb{Z}/2).$$

Conjecture A. *Let X be a compact nonsingular real algebraic variety and let p be a positive integer. For a continuous map $f\colon X \to \mathbb{S}^p$, the following conditions are equivalent:*

(A$_1$) *f can be approximated by nice continuous rational maps.*

(A$_2$) *f is homotopic to a nice continuous rational map.*

(A$_3$) *The cohomology class $f^*(s_p) \in H^p(X;\mathbb{Z})$ is adapted.*

(A$_4$) *The cohomology class $f^*(\bar{s}_p) \in H^p(X;\mathbb{Z}/2)$ is adapted.*

(A$_5$) *$f^*(s_p) \in A^p(X;\mathbb{Z})$.*

(A$_6$) *$f^*(\bar{s}_p) \in A^p(X;\mathbb{Z}/2)$.*

The known implications between conditions (A$_1$) through (A$_6$) are indicated as follows

$$(\text{A}_1) \Longrightarrow (\text{A}_2) \Longrightarrow (\text{A}_3) \Longrightarrow (\text{A}_4)$$
$$\Downarrow \qquad\qquad \Downarrow$$
$$(\text{A}_5) \Longrightarrow (\text{A}_6)$$

The implication (A$_2$) $\Rightarrow$ (A$_3$) is proved in [18, Proposition 1.1], while all the others are obvious. The maps satisfying (A$_1$) (resp. (A$_2$)) are characterized in [17] (resp. [15]). The results of [15, 17] are crucial in the proof of

$$(\text{A}_2) \Rightarrow (\text{A}_1) \text{ for } \dim X + 3 \le 2p,$$

given in [19].

Remark 1.2. In order to prove Conjecture A it would suffice to show that (A_6) implies (A_1).

We record the following observation.

Remark 1.3. Let X be a compact nonsingular real algebraic variety and let p be a positive integer. If Conjecture A holds, then the following conditions are equivalent:

(a) The set $\mathcal{R}_0(X, \mathbb{S}^p)$ of nice continuous rational maps is dense in $\mathcal{C}(X, \mathbb{S}^p)$.

(b) Each continuous map from X into $\mathbb{S}^p$ is homotopic to a nice continuous rational map.

(c) $A^p(X; \mathbb{Z}) = H^p_{\mathrm{sph}}(X; \mathbb{Z})$.

(d) $A^p(X; \mathbb{Z}/2) = H^p_{\mathrm{sph}}(X; \mathbb{Z}/2)$.

As already mentioned at the beginning of this section, Conjecture A is related to seemingly quite different problems investigated in [1, 23, 25]. We now explain this in detail.

Let V be a real algebraic variety. A bordism class in the nth unoriented bordism group $\mathfrak{N}_n(V)$ of V is said to be *algebraic* if it can be represented by a regular map from an n-dimensional compact nonsingular real algebraic variety into V, cf. [1, 2].

Approximation Conjecture. *For any nonsingular real algebraic variety V, the following condition is satisfied: If M is a compact smooth submanifold of V and the unoriented bordism class of the inclusion map $M \hookrightarrow X$ is algebraic, then M is ε-isotopic to a nonsingular Zariski locally closed subvariety of V.*

Here "ε-isotopic" means isotopic via a smooth isotopy that can be chosen arbitrarily close, in the $\mathcal{C}^\infty$ topology, to the inclusion map $M \hookrightarrow V$. A slightly weaker assertion than the one in the Approximation Conjecture is known to be true: If the unoriented bordism class of the inclusion map $M \hookrightarrow V$ is algebraic, then the smooth submanifold $M \times \{0\}$ of $V \times \mathbb{R}$ is ε-isotopic to a nonsingular Zariski locally closed subvariety of $V \times \mathbb{R}$, cf. [1, Theorem F].

The following is a special case of the Approximation Conjecture.

Conjecture B(p). *For any compact nonsingular real algebraic variety V, the following condition is satisfied: If M is a compact smooth codimension p submanifold of V, embedded with trivial normal bundle, and the unoriented bordism class of the inclusion map is algebraic, then M is ε-isotopic to a nonsingular Zariski locally closed subvariety of V.*

Presumably, Conjecture B(p) should be easier to prove than the Approximation Conjecture. In the context of this paper, Conjecture B(p) is of particular interest.

Proposition 1.4. *If Conjecture B(p) holds, then so does Conjecture A.*

Using a method independent of Conjecture B(p), we prove the following special case of Conjecture A.

Theorem 1.5. *Let X be a compact nonsingular real algebraic variety of dimension n. Let d and p be positive integers satisfying $n + 1 \leq p$ and $n + d + 2 \leq 2p$. Then for a continuous map $f \colon X \times \mathbb{S}^d \to \mathbb{S}^p$, the following conditions are equivalent:*

(a_1) *f can be approximated by nice continuous rational maps.*

(a_2) *f is homotopic to a nice continuous rational map.*

(a_3) *The cohomology class $f^*(s_p) \in H^p(X \times \mathbb{S}^d; \mathbb{Z})$ is adapted.*

(a_4) *The cohomology class $f^*(\bar{s}_p) \in H^p(X \times \mathbb{S}^d; \mathbb{Z}/2)$ is adapted.*

(a_5) *$f^*(s_p) \in A^p(X \times \mathbb{S}^d; \mathbb{Z})$.*

(a_6) *$f^*(\bar{s}_p) \in A^p(X \times \mathbb{S}^d; \mathbb{Z}/2)$.*

For any positive integers n, d and p, we have

$$A^p(\mathbb{S}^n \times \mathbb{S}^d; \mathbb{Z}/2) = H^p(\mathbb{S}^n \times \mathbb{S}^d; \mathbb{Z}/2)$$

and hence, if Conjecture A holds, then the set $\mathcal{R}_0(\mathbb{S}^n \times \mathbb{S}^d, \mathbb{S}^p)$ of nice continuous rational maps is dense in $\mathcal{C}(\mathbb{S}^n \times \mathbb{S}^d, \mathbb{S}^p)$. Making no use of Conjecture A, we obtain the following weaker result.

Example 1.6. If n, d and p are positive integers satisfying $n + d + 2 \leq 2p$, then $\mathcal{R}_0(\mathbb{S}^n \times \mathbb{S}^d, \mathbb{S}^p)$ is dense in $\mathcal{C}(\mathbb{S}^n \times \mathbb{S}^d, \mathbb{S}^p)$. Indeed, we may assume that $n \leq d$. Then $n + 1 \leq p$ and hence the assertion follows from Theorem 1.5.

Some special techniques are available when one studies regular or continuous rational maps with values in $\mathbb{S}^p$ for p equal to 1, 2 or 4, cf. [4, 5, 15, 16, 17, 18, 20]. According to [18, Theorem 1.2], each continuous map $f \colon X \to \mathbb{S}^2$ with $f^*(s_2) \in A^2(X; \mathbb{Z})$ can be approximated by continuous rational maps. For maps with values in $\mathbb{S}^4$ we have the following result.

Theorem 1.7. *Let X be a compact nonsingular real algebraic variety. Assume that for each integer $k \geq 3$, the only torsion in the cohomology group $H^{2k}(X; \mathbb{Z})$ is relatively prime to $(k-1)!$. A continuous map $f \colon X \to \mathbb{S}^4$ can be approximated by continuous rational maps, provided that the cohomology class $f^*(s_4) \in H^4(X; \mathbb{Z})$ is adapted.*

The assumption in Theorem 1.7 that the cohomology class $f^*(s_4) \in H^4(X; \mathbb{Z})$ is adapted is not very convenient and it would be desirable to replace it by the condition $f^*(s_4) \in A^4(X; \mathbb{Z})$. This can be done at least for $\dim X \leq 7$.

Theorem 1.8. *Let X be a compact nonsingular real algebraic variety of dimension at most 7. Assume that the cohomology group $H^6(X; \mathbb{Z})$ has no 2-torsion. A continuous map $f \colon X \to \mathbb{S}^4$ can be approximated by continuous rational maps, provided that $f^*(s_4) \in A^4(X; \mathbb{Z})$.*

One might wonder whether the condition $f^*(s_4) \in A^4(X; \mathbb{Z})$ can be replaced by

$$f^*(\bar{s}_4) \in A^4(X; \mathbb{Z}/2).$$

We only have a partial result.

Theorem 1.9. *Let X be a compact nonsingular real algebraic variety of dimension at most 7. Assume that the cohomology group $H^6(X; \mathbb{Z})$ has no 2-torsion and $H^4_{\mathrm{sph}}(X; \mathbb{Z}) = H^4(X; \mathbb{Z})$. A continuous map $f \colon X \to \mathbb{S}^4$ can be approximated by continuous rational maps, provided that*

$$f^*(\bar{s}_4) \in A^4(X; \mathbb{Z}/2).$$

The last three theorems can be strengthened if the following holds.

Conjecture C. *Any continuous rational map from a compact nonsingular real algebraic variety X into the p-sphere can be approximated by nice continuous rational maps.*

Conjecture C is known to hold if $p = 1$ or $\dim X \leq p + 1$, cf. [17].

2 Proofs

For the clarity of the exposition we recall the following approximation criterion.

Theorem 2.1 ([17, Theorem 1.2]). *Let X be a compact nonsingular real algebraic variety and let $f \colon X \to \mathbb{S}^p$ be a smooth map. Assume that there exists a regular value $y \in \mathbb{S}^p$ of f such that the smooth submanifold $f^{-1}(y)$ is ε-isotopic to a nonsingular Zariski locally closed subvariety of X. Then f can be approximated by nice continuous rational maps.*

The proof of Proposition 1.4 will be preceded by two lemmas. As usual, given a smooth manifold M, we denote by $w_i(M)$ its ith Stiefel–Whitney class. If the manifold M is compact, let $[M]$ denote its fundamental class in the homology group $H_*(M; \mathbb{Z}/2)$.

Lemma 2.2. *Let N and P be compact smooth manifolds, and let $f\colon N \to P$ be a smooth immersion with trivial normal bundle. If $f_*([N]) = 0$ in $H_*(P; \mathbb{Z}/2)$, then the unoriented bordism class of the map f is zero.*

Proof. According to [6, (17.3)], it suffices to prove that for any nonnegative integer l and any cohomology class u in $H^l(P; \mathbb{Z}/2)$, the equality

$$(\dagger) \qquad \langle w_{i_1}(N) \smile \cdots \smile w_{i_r}(N) \smile f^*(u), [N] \rangle = 0$$

holds for all nonnegative integers $i_1, \ldots, i_r$ satisfying $i_1 + \cdots + i_r = \dim N - l$.

Since the normal bundle of the immersion f is trivial, we have

$$w_i(N) = f^*(w_i(P))$$

for every nonnegative integer i, cf. [22, p. 31]. Setting

$$v = w_{i_1}(P) \smile \cdots \smile w_{i_r}(P) \smile u,$$

we get

$$w_{i_1}(N) \smile \cdots \smile w_{i_r}(N) \smile f^*(u) = f^*(v).$$

Since

$$\langle f^*(v), [N] \rangle = \langle v, f_*([N]) \rangle,$$

equality $(\dagger)$ holds if $f_*([N]) = 0$. The proof is complete. $\square$

Lemma 2.3. *Let X be a compact nonsingular real algebraic variety. Let M be a compact smooth codimension p submanifold of X, embedded with trivial normal bundle. Assume that the cohomology class $[M]^X$ belongs to $A^p(X; \mathbb{Z}/2)$. Then the unoriented bordism class of the inclusion map $e\colon M \hookrightarrow X$ is algebraic.*

Proof. By the definition of $A^p(X; \mathbb{Z}/2)$, we have

$$[M]^X = [Z_1]^X + \cdots + [Z_k]^X,$$

where Z_i is a nonsingular codimension p Zariski locally closed subvariety of X, which is a compact smooth submanifold with trivial normal bundle, $1 \le i \le k$. Let Z be the disjoint union of the Z_i, and let $g\colon Z \to X$ be the map whose restriction to Z_i corresponds to the inclusion map $Z_i \hookrightarrow X$. By construction, Z is a compact nonsingular real algebraic variety and $g\colon Z \to X$ is a regular map. Since the cohomology class $[M]^X$ (resp. $[Z_1]^X + \cdots + [Z_k]^X$) is Poincaré dual to the cohomology class $e_*([M])$ (resp. $g_*([Z])$), we get

$$e_*([M]) = g_*([Z]).$$

Let N be the disjoint union of M and Z, and let $f\colon N \to X$ be the map whose restriction to M (resp. Z) corresponds to e (resp. g). Note that $f\colon N \to X$ is a smooth immersion with trivial normal bundle. Furthermore, $f_*([N]) = 0$. In view of Lemma 2.2, the maps $e\colon M \hookrightarrow X$ and $g\colon Z \to X$ represent the same unoriented bordism class, which completes the proof. $\square$

Proof of Proposition 1.4. Suppose that Conjecture B(p) holds. It suffices to show that (A$_6$) implies (A$_1$). Assume that (A$_6$) is satisfied, that is,

$$f^*(\bar{s}_p) \in A^p(X; \mathbb{Z}/2).$$

We can assume without loss of generality that the map $f\colon X \to \mathbb{S}^p$ is smooth. By Sard's theorem there exists a regular value $y \in \mathbb{S}^p$ of f. The inverse image $f^{-1}(y)$ is a compact smooth submanifold of X, embedded with trivial normal bundle. Since

$$f^*(\bar{s}_p) = [f^{-1}(y)]^X,$$

it follows from Lemma 2.3 that the unoriented bordism class of the inclusion map $f^{-1}(y) \hookrightarrow X$ is algebraic. According to Conjecture B(p), the smooth submanifold $f^{-1}(y)$ is ε-isotopic to a nonsingular Zariski locally closed subvariety of X. Hence, in view of Theorem 2.1, condition (A$_1$) is satisfied. $\qquad\square$

The proof of Theorem 1.5 requires some preparation.

Lemma 2.4. *Let Y be a nonsingular real algebraic variety and let M be a compact smooth submanifold of Y, embedded with trivial normal bundle. If M is isotopic to a nonsingular Zariski closed subvariety of Y, then it is ε-isotopic to a nonsingular Zariski closed subvariety of Y.*

Proof. This is proved in [14, Theorem 2.1] if the ambient variety Y is compact. In the general case we can argue as follows. Suppose that M is isotopic to a nonsingular Zariski closed subvariety V of Y. According to Hironaka's theorem [9] (cf. also [12] for a very readable exposition), we can assume that Y is a Zariski open subvariety of a compact nonsingular real algebraic variety X. Let A be the Zariski closure of V in X. Then $S := A \setminus V$ is a Zariski closed subvariety of X, contained in $X \setminus Y$. By Hironaka's theorem, there exists a regular map $\pi\colon X' \to X$ such that X' is a compact nonsingular real algebraic variety, the restriction $\pi_0\colon X' \setminus \pi^{-1}(S) \to X \setminus S$ of π is a biregular isomorphism, and the subvariety $\pi^{-1}(V)$ is Zariski closed in X'. Identifying Y, M, V with $\pi^{-1}(Y)$, $\pi^{-1}(M)$, $\pi^{-1}(V)$, respectively, we can assume that V is Zariski closed in X. The proof is complete since X is compact and M is isotopic to V in X. $\qquad\square$

Lemma 2.5. *Let X be a nonsingular real algebraic variety and let d be a positive integer. Let M be a compact smooth submanifold of $X \times \mathbb{S}^d$, embedded with trivial normal bundle. Assume that*

$$2 \dim M + 2 \leq \dim X + d,$$

the unoriented bordism class of the inclusion map $M \hookrightarrow X \times \mathbb{S}^d$ is algebraic, and $\sigma(M) \neq \mathbb{S}^d$, where $\sigma\colon X \times \mathbb{S}^d \to \mathbb{S}^d$ is the canonical projection. Then the smooth submanifold M is ε-isotopic to a nonsingular Zariski locally closed subvariety of $X \times \mathbb{S}^d$.

Proof. The assumption $\sigma(M) \neq \mathbb{S}^p$ implies the existence of a point $u \in \mathbb{S}^p$ for which

$$M \subseteq X \times (\mathbb{S}^d \setminus \{u\}).$$

Since $\mathbb{S}^d \setminus \{u\}$ is biregularly isomorphic to $\mathbb{R}^d$, we identify $\mathbb{S}^d \setminus \{u\}$ with $\mathbb{R}^d$ and regard M as a submanifold of $X \times \mathbb{R}^d$. It remains to prove that M is ε-isotopic to a Zariski locally closed subvariety of $X \times \mathbb{R}^d$.

Let $\varphi\colon M \to X \times \mathbb{R}^{d-1}$ and $\psi\colon M \to \mathbb{R}$ be maps such that

$$f := (\varphi, \psi)\colon M \to (X \times \mathbb{R}^{d-1}) \times \mathbb{R} = X \times \mathbb{R}^d$$

is the inclusion map. Since

$$2 \dim M + 1 \leq \dim X + d - 1,$$

the map φ is homotopic to a smooth embedding $\bar{\varphi}\colon M \to X \times \mathbb{R}^{d-1}$, cf. [10, p. 55]. If $\bar{\psi}\colon M \to \mathbb{R}$ is the constant map sending M to $0 \in \mathbb{R}$, then the map

$$\bar{f} := (\bar{\varphi}, \bar{\psi})\colon M \to (X \times \mathbb{R}^{d-1}) \times \mathbb{R}) = X \times \mathbb{R}^d$$

is a smooth embedding homotopic to f. Since

$$2 \dim M + 2 \leq \dim X + d,$$

the smooth embeddings f and $\bar{f}$ are isotopic, cf. [26, Theorem 6] or [10, p. 183, Exercise 10]. Thus M is isotopic in $(X \times \mathbb{R}^{d-1}) \times \mathbb{R}$ to the smooth submanifold $\bar{f}(M) = \bar{M} \times \{0\}$, where $\bar{M} := \bar{\varphi}(M)$.

We assert that the smooth submanifold $\bar{f}(M)$ is ε-isotopic to a nonsingular Zariski locally closed subvariety of $X \times \mathbb{R}^d$. This can be proved as follows. Since the unoriented bordism class of the inclusion map $e \colon M \hookrightarrow X \times \mathbb{S}^d$ is algebraic, so is the unoriented bordism class of the map $e_X \colon M \to X$, where e_X is the composite of e and the canonical projection $X \times \mathbb{S}^d \to X$. It follows that the unoriented bordism class of the map $\varphi \colon M \to X \times \mathbb{R}^{d-1}$ is algebraic. The maps $\varphi \colon M \to X \times \mathbb{R}^{d-1}$ and $\bar{\varphi} \colon M \to X \times \mathbb{R}^{d-1}$ represent the same unoriented bordism class. Consequently, the unoriented bordism class of the inclusion map $\bar{M} \hookrightarrow X \times \mathbb{R}^{d-1}$ is algebraic. This implies the assertion in view of [1, Theorem F].

By the assertion, M is isotopic to a nonsingular Zariski locally closed subvariety Z of $X \times \mathbb{R}^d$. Let $F \colon M \times [0,1] \to X \times \mathbb{R}^d$ be a smooth isotopy such that F_0 is the inclusion map and $F_1(M) = Z$, where $F_t(x) = F(x,t)$ for all $x \in M$ and $t \in [0,1]$. Let A be the Zariski closure of Z in $X \times \mathbb{R}^d$. Then $S := A \setminus Z$ is a Zariski closed subvariety of $X \times \mathbb{R}^d$ of dimension at most $\dim Z - 1 = \dim M - 1$. In particular, S has a finite stratification into smooth submanifolds of $X \times \mathbb{R}^d$ of dimension at most $\dim S$. Since

$$\dim(M \times [0,1]) + \dim S \leq 2 \dim M < \dim X + d,$$

according to the transversality theorem, there exists a smooth map $G \colon M \times [0,1] \to X \times \mathbb{R}^d$, arbitrarily close to F in the $\mathcal{C}^\infty$ topology, such that

$$G(M \times [0,1]) \subseteq (X \times \mathbb{R}^d) \setminus S \quad \text{and} \quad G_1 = F_1.$$

Note that $G_1(M) = Z$ is a Zariski closed subvariety of $(X \times \mathbb{R}^d) \setminus S$. If G is close enough to F, then G is an isotopy. Consequently, the smooth submanifold $N := G_0(M)$ is isotopic to $X = G_1(M)$ in $(X \times \mathbb{R}^d) \setminus S$. The normal bundle to N in $(X \times \mathbb{R}^d) \setminus S$ is trivial and hence, in view of Lemma 2.4, N is ε-isotopic to a nonsingular Zariski closed subvariety of $(X \times \mathbb{R}^d) \setminus S$. Since the smooth embedding $G_0 \colon M \to X \times \mathbb{R}^d$ is close to the inclusion map $F_0 \colon M \hookrightarrow X \times \mathbb{R}^d$, it follows that M is ε-isotopic to a nonsingular Zariski locally closed subvariety of $X \times \mathbb{R}^d$, as required. $\qquad\square$

Proof of Theorem 1.5. We can assume without loss of generality that the map f is smooth. It suffices to prove that (a_6) implies (a_1). Suppose that (a_6) holds.

Let u be a point in $\mathbb{S}^d$. Since $n + 1 \leq p$, we have

$$f(X \times \{u\}) \neq \mathbb{S}^p.$$

By Sard's theorem, there exists a regular value $y \in \mathbb{S}^p \setminus f(X \times \{u\})$ of f. Then $M := f^{-1}(y)$ is a compact smooth submanifold of $X \times \mathbb{S}^d$, embedded with trivial normal bundle. Since $[M]^X = f^*(\bar{s}_p)$, it follows from Lemma 2.3 that the unoriented bordism class of the inclusion map $M \hookrightarrow X \times \mathbb{S}^d$ is algebraic. Note that $\sigma(M) \subseteq \mathbb{S}^d \setminus \{u\}$, where $\sigma \colon X \times \mathbb{S}^d \to \mathbb{S}^d$ is the canonical projection. Furthermore, $\dim M = n + d - p$, and hence

$$2 \dim M + 2 \leq n + d.$$

Consequently, according to Lemma 2.5, the submanifold M is ε-isotopic to a nonsingular Zariski locally closed subvariety of $X \times \mathbb{S}^d$. Thus, (a_1) holds in view of Theorem 2.1. $\qquad\square$

In the remainder of this paper we will need several results on stratified-algebraic vector bundles, all of which are proved in [20].

Let X be a real algebraic variety. By a *stratification* of X we mean a finite collection $\mathcal{S}$ of pairwise disjoint Zariski locally closed subvarieties whose union is X. Each subvariety in $\mathcal{S}$ is called a stratum of $\mathcal{S}$. A map $f \colon X \to Y$, where Y is a real algebraic variety, is said to be *stratified-regular* if it is continuous and for some stratification $\mathcal{S}$ of X, the restriction $f|_S \colon S \to Y$ of f to each stratum S in $\mathcal{S}$ is a regular map, cf. [20]. The notion of stratified-regular map is closely related to those of hereditarily rational function [13] and fonction régulue [7]. One readily sees that each stratified-regular map is continuous rational.

Let $\mathbb{F}$ stand for $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ (the quaternions). All $\mathbb{F}$-vector spaces will be left $\mathbb{F}$-vector spaces. When convenient, $\mathbb{F}$ will be identified with $\mathbb{R}^{d(\mathbb{F})}$, where $d(\mathbb{F}) = \dim_{\mathbb{R}} \mathbb{F}$.

For any nonnegative integer n, let $\varepsilon_X^n(\mathbb{F})$ denote the standard trivial $\mathbb{F}$-vector bundle on X with total space $X \times \mathbb{F}^n$, where $X \times \mathbb{F}^n$ is regarded as a real algebraic variety.

An algebraic $\mathbb{F}$-vector bundle on X is an algebraic $\mathbb{F}$-vector subbundle of $\varepsilon_X^n(\mathbb{F})$ for some n (cf. [4, Chapters 12 and 13] for various characterizations of algebraic $\mathbb{F}$-vector bundles).

We now recall the fundamental notion introduced in [20]. A *stratified-algebraic* $\mathbb{F}$-vector bundle on X is a topological $\mathbb{F}$-vector subbundle ξ of $\varepsilon_X^n(\mathbb{F})$, for some n, such that for some stratification $\mathcal{S}$ of X, the restriction $\xi|_S$ of ξ to each stratum S in $\mathcal{S}$ is an algebraic $\mathbb{F}$-vector subbundle of $\varepsilon_S^n(\mathbb{F})$.

A topological $\mathbb{F}$-vector bundle on X is said to *admit a stratified-algebraic structure* if it is isomorphic to a stratified-algebraic $\mathbb{F}$-vector bundle on X.

Let $\mathbb{K}$ be a subfield of $\mathbb{F}$, where $\mathbb{K}$ (as $\mathbb{F}$) stands for $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Any $\mathbb{F}$-vector bundle ξ on X can be regarded as a $\mathbb{K}$-vector bundle, which is indicated by $\xi_{\mathbb{K}}$. In particular, $\xi_{\mathbb{K}} = \xi$ if $\mathbb{K} = \mathbb{F}$. If the $\mathbb{F}$-vector bundle ξ admits a stratified-algebraic structure, then so does the $\mathbb{K}$-vector bundle $\xi_{\mathbb{K}}$. The following result will play a crucial role.

Theorem 2.6 ([20, Theorem 1.7]). *Let X be a compact real algebraic variety. A topological $\mathbb{F}$-vector bundle ξ on X admits a stratified-algebraic structure if and only if the $\mathbb{K}$-vector bundle $\xi_{\mathbb{K}}$ admits a stratified-algebraic structure.*

The proof for $\mathbb{K} = \mathbb{R}$, rather involved, is given in [20]. The general case follows since $\xi_{\mathbb{R}} = (\xi_{\mathbb{K}})_{\mathbb{R}}$. In the present paper we make use of this result with $\mathbb{F} = \mathbb{H}$ and $\mathbb{K} = \mathbb{C}$.

Another useful fact is the following.

Theorem 2.7 ([20, Corollary 3.14]). *Let X be a compact real algebraic variety. A topological $\mathbb{F}$-vector bundle on X admits a stratified-algebraic structure if and only if it is stably equivalent to a stratified-algebraic $\mathbb{F}$-vector bundle on X.*

There is a close connection between stratified-algebraic vector bundles and approximation by stratified-regular maps with values in Grassmannians. Let $\mathbb{G}_k(\mathbb{F}^n)$ denote the Grassmannian of k-dimensional $\mathbb{F}$-vector subspaces of $\mathbb{F}^n$. We regard $\mathbb{G}_k(\mathbb{F}^n)$ as a real algebraic variety, cf. [4]. The tautological $\mathbb{F}$-vector bundle $\gamma_k(\mathbb{F}^n)$ on $\mathbb{G}_k(\mathbb{F}^n)$ is algebraic.

Theorem 2.8 ([20, Theorem 4.10]). *Let X be a compact real algebraic variety. For a continuous map $f \colon X \to \mathbb{G}_k(\mathbb{F}^n)$, the following conditions are equivalent:*

(a) *f can be approximated by stratified-regular maps.*

(b) *f is homotopic to a stratified-regular map.*

(c) *The pullback $\mathbb{F}$-vector bundle $f^*\gamma_k(\mathbb{F}^n)$ on X admits a stratified-algebraic structure.*

As usual, the kth Chern class of a $\mathbb{C}$-vector bundle ξ will be denoted by $c_k(\xi)$. Note that if η is an $\mathbb{H}$-vector bundle, then

$$c_{2l+1}(\eta_{\mathbb{C}}) = 0$$

for every $l \geq 0$.

We now proceed to the investigation of maps with values in $\mathbb{S}^4$. Recall that the Grassmannian $\mathbb{G}_1(\mathbb{H}^2)$ is biregularly isomorphic to $\mathbb{S}^4$. Henceforth we identify $\mathbb{G}_1(\mathbb{H}^2)$ with $\mathbb{S}^4$ and set $\gamma = \gamma_1(\mathbb{H}^2)$. In particular, γ is an algebraic $\mathbb{H}$-line bundle on $\mathbb{S}^4$ and the Chern class $c_2(\gamma_{\mathbb{C}})$ is a generator of the cohomology group $H^4(\mathbb{S}^4; \mathbb{Z})$. We can assume without loss of generality that

$$s_4 = c_2(\gamma_{\mathbb{C}}).$$

Lemma 2.9. *Let X be a compact nonsingular real algebraic variety and let $u \in H^4(X; \mathbb{Z})$ be an adapted cohomology class. Then there exists a stratified-algebraic $\mathbb{H}$-line bundle ξ on X with $c_2(\xi_{\mathbb{C}}) = u$.*

Proof. The cohomology class u is of the form

$$u = [\![Z]\!]^X,$$

where Z is a nonsingular codimension 4 Zariski locally closed subvariety of X, which is a compact smooth submanifold with trivial normal bundle that is suitably oriented. Choose a smooth framing F of the normal bundle to Z in X so that it determines the existing orientation. According to a classical result in framed cobordism, we can find a smooth map $f \colon X \to \mathbb{S}^4$ and a regular value $y \in \mathbb{S}^4$ of f such that the framed submanifolds (Z, F) and $(f^{-1}(y), F_f)$ are equal, where F_f is a framing of $f^{-1}(y)$ induced by f, cf. [21, p. 44]. Then

$$f^*(s_4) = [\![Z]\!]^X = u.$$

Furthermore, according to [15, Theorem 2.4], f is homotopic to a continuous rational map $g \colon X \to \mathbb{S}^4$. In particular,

$$g^*(s_4) = f^*(s_4) = u.$$

Since the variety X is nonsingular, the map g is stratified-regular (this follows from [13, Proposition 8] as commented in [20, Remark 2.3]). The pullback $\mathbb{H}$-line bundle

$$\xi := g^*\gamma$$

on X is stratified-regular, the map g being stratified-regular and the $\mathbb{H}$-line bundle γ on $\mathbb{S}^4$ being algebraic. Furthermore,

$$c_2(\xi_{\mathbb{C}}) = c_2(g^*\gamma_{\mathbb{C}}) = g^*(c_2(\gamma_{\mathbb{C}})) = g^*(s_4) = u,$$

as required. $\qquad\square$

Proof of Theorem 1.7. According to Theorem 2.8, it suffices to prove that the pullback $\mathbb{H}$-line bundle $\eta := f^*\gamma$ on X admits a stratified-algebraic structure. The Chern class

$$c_2(\eta_{\mathbb{C}}) = c_2(f^*\gamma_{\mathbb{C}}) = f^*(c_2(\gamma_{\mathbb{C}})) = f^*(s_4)$$

is an adapted cohomology class in $H^4(X; \mathbb{Z})$, and hence, by Lemma 2.9, there exists a stratified-algebraic $\mathbb{H}$-line bundle ξ on X with

$$c_2(\xi_{\mathbb{C}}) = c_2(\eta_{\mathbb{C}}).$$

Since $c_j(\xi_{\mathbb{C}}) = c_j(\eta_{\mathbb{C}}) = 0$ for $j = 1$ and $j \geq 3$, we get

$$c_k(\xi_{\mathbb{C}}) = c_k(\eta_{\mathbb{C}}) \quad \text{for all } k \geq 0.$$

Hence, according to [24, Theorem 3.2], the $\mathbb{C}$-vector bundles $\xi_{\mathbb{C}}$ and $\eta_{\mathbb{C}}$ are stably equivalent (here the assumption on the torsion of the cohomology groups $H^{2k}(X; \mathbb{Z})$ is needed). Consequently, in view of Theorem 2.7, the $\mathbb{C}$-vector bundle $\eta_{\mathbb{C}}$ admits a stratified-algebraic structure. Finally, by Theorem 2.6, the $\mathbb{H}$-line bundle η admits a stratified-algebraic structure, as required. $\qquad\square$

Proof of Theorem 1.8. According to Theorem 2.8, it suffices to prove that the pullback $\mathbb{H}$-line bundle $\eta := f^*\gamma$ on X admits a stratified-algebraic structure. We modify the proof of Theorem 1.7 as follows.

The Chern class

$$c_2(\eta_{\mathbb{C}}) = f^*(s_4)$$

belongs to $A^4(X;\mathbb{Z})$. Note that if a cohomology class u in $H^4(X;\mathbb{Z})$ is adapted, then so is $-u$. Consequently, each element in $A^4(X;\mathbb{Z})$ can be written as a finite sum of (not necessarily distinct) adapted cohomology classes. In particular,

$$c_2(\eta_{\mathbb{C}}) = u_1 + \cdots + u_k,$$

where the u_i are adapted cohomology classes in $H^4(X;\mathbb{Z})$. By Lemma 2.9, there exists a stratified-algebraic $\mathbb{H}$-line bundle ξ_i on X with

$$c_2((\xi_i)_{\mathbb{C}}) = u_i.$$

Since $\dim X \leq 7$, the stratified-algebraic $\mathbb{H}$-vector bundle $\theta := \xi_1 \oplus \cdots \oplus \xi_k$ can be written as

$$\theta = \xi \oplus \varepsilon,$$

where ξ is a topological $\mathbb{H}$-line bundle and ε is a trivial $\mathbb{H}$-vector bundle, cf. [11, p. 99]. According to Theorem 2.7, the $\mathbb{H}$-line bundle ξ admits a stratified-algebraic structure. Furthermore,

$$c_2(\xi_{\mathbb{C}}) = c_2(\theta_{\mathbb{C}}) = c_2((\xi_1)_{\mathbb{C}}) + \cdots + c_2((\xi_k)_{\mathbb{C}}) = c_2(\eta_{\mathbb{C}}).$$

Since $c_j(\xi_{\mathbb{C}}) = c_j(\eta_{\mathbb{C}}) = 0$ for $j = 1$ and $j \geq 3$, we get

$$c_k(\xi_{\mathbb{C}}) = c_k(\eta_{\mathbb{C}}) \quad \text{for all } k \geq 0.$$

The rest of the proof is the same as that of Theorem 1.7. $\square$

We record the following general fact.

Lemma 2.10. *Let X be a compact nonsingular real algebraic variety and let v be a spherical cohomology class in $H^p(X;\mathbb{Z})$, where $p \geq 1$. Then the cohomology class $2v$ is adapted. In particular,*

$$2H^p_{\mathrm{sph}}(X;\mathbb{Z}) \subseteq A^p(X;\mathbb{Z}).$$

Proof. It suffices to prove the first assertion. Recall that $v = [\![M]\!]^X$, where M is a compact smooth codimension p submanifold of X, with trivial and oriented normal bundle. There exists an isotopic copy M' of M such that $M \cap M' = \varnothing$ and the union $M \cup M'$ is the boundary of a compact smooth submanifold with boundary, embedded in X with trivial normal bundle. It follows that the smooth submanifold $M \cup M'$ is isotopic to a nonsingular Zariski closed subvariety V of X (cf. for example [14, Lemma 2.3]). By construction,

$$2v = 2[\![M]\!]^X = [\![V]\!]^X,$$

provided that the normal bundle to V in X is suitably oriented. Hence the cohomology class $2v$ is adapted, as required. $\square$

Proof of Theorem 1.9. Since

$$\rho(A^4(X;\mathbb{Z})) = A^4(X;\mathbb{Z}/2) \quad \text{and} \quad \rho(f^*(s_4)) = f^*(\bar{s}_4) \in A^4(X;\mathbb{Z}/2),$$

by the universal coefficient theorem, the cohomology class $f^*(s_4)$ can be written as

$$f^*(s_4) = u + 2v,$$

where $u \in A^4(X;\mathbb{Z})$ and $v \in H^4(X;\mathbb{Z})$. Making use of the equality $H^4_{\mathrm{sph}}(X;\mathbb{Z}) = H^4(X;\mathbb{Z})$ and Lemma 2.10, we get $f^*(s_4) \in A^4(X;\mathbb{Z})$. The proof is complete in view of Theorem 1.8. $\square$

References

[1] S. Akbulut and H. King, On approximating submanifolds by algebraic sets and a solution to the Nash conjecture, Invent. Math. 107 (1992), 87–98.

[2] S. Akbulut and H. King, Topology of Real Algebraic Sets, Math. Sci. Research Institute Publ., vol. 25, Springer, 1992.

[3] M. Bilski, W. Kucharz, A. Valette, and G. Valette, Vector bundles and regulous maps, Math. Z. 275 (2013), 403–418.

[4] J. Bochnak, M. Coste, and M.-F. Roy, Real Algebraic Geometry, Ergeb. der Math. und ihrer Grenzgeb. Folge 3, vol. 36, Springer, 1998.

[5] J. Bochnak and W. Kucharz, Algebraic approximation of mappings into spheres, Michigan Math. J. 34 (1987), 119–125.

[6] P.E. Conner, Differentiable Periodic Maps, 2nd ed., Lecture Notes in Math. 738, Springer, 1979.

[7] G. Fichou, J. Huisman, F. Mangolte, and J.-Ph. Monnier, Fonctions régulues, arXiv:1112.3800 [math.AG], to appear in J. Reine Angew. Math.

[8] G. Fichou, J.-Ph. Monnier, amd R. Quarez, Continuous functions in the plane regular after one blowing-up, arXiv:1409.8223v2 [math.AG].

[9] H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero, Ann. of Math. 79 (1964), 109–326.

[10] M. Hirsch, Differential Topology, Springer, 1997.

[11] D. Husemoller, Fibre Bundles, Springer, 1975.

[12] J. Kollár, Lectures on Resolution of Singularities, Ann. of Math. Studies, vol. 166, Princeton University Press, Princeton, NJ, 2007.

[13] J. Kollár and K. Nowak, Continuous rational functions on real and p-adic varieties, Math. Z. 279 (2015), 85–97.

[14] W. Kucharz, On homology of real algebraic varieties, Invent. Math. 82 (1985), 19–25.

[15] W. Kucharz, Rational maps in real algebraic geometry, Adv. Geom. 9 (2009), 517–539.

[16] W. Kucharz, Regular versus continuous rational maps, Topology Appl. 160 (2013), 1375–1378.

[17] W. Kucharz, Approximation by continuous rational maps into spheres, J. Eur. Math. Soc. 16 (2014), 1555–1569.

[18] W. Kucharz, Continuous rational maps into the unit 2-sphere, Arch. Math.(Basel) 102 (2014), 257–261.

[19] W. Kucharz, Continuous rational maps into spheres, arxiv:1403.5127 [math.AG].

[20] W. Kucharz and K. Kurdyka, Stratified-algebraic vector bundles, arXiv:1308.4376 [math.AG], to appear in J. Reine Angew. Math.

[21] J.W. Milnor, Topology from the Differentiable Viewpoint, The University Press of Virginia, Charlottesville, Va. 1965.

[22] J.W. Milnor and J.D. Stasheff, Characteristic Classes, Princeton University Press, Princeton, NJ, 1974.

[23] J. Nash, Real algebraic manifolds, Ann. of Math. 56 (1952), 405–421.

[24] F.P. Peterson, Some remarks on Chern classes, Ann. of Math. 69 (1959), 414–420.

[25] A. Tognoli, Su una congettura di Nash, Ann. Sc. Norm. Sup. Pisa Sci. Fis. Mat. 3 (27) (1973), 167–185.

[26] H. Whitney, Differentiable manifolds, Ann. of Math. 37 (1936), 645–680.

Wojciech Kucharz
Institute of Mathematics
Faculty of Mathematics and Computer Science
Jagiellonian University
Łojasiewicza 6
30-348 Kraków
Poland
E-mail address: `Wojciech.Kucharz@im.uj.edu.pl`

Krzysztof Kurdyka
Laboratoire de Mathématiques
UMR 5175 de CNRS
Université de Savoie
Campus Scientifique
73 376 Le Bourget-du-Lac Cedex
France
E-mail address: `kurdyka@univ-savoie.fr`