A weighted version of Zariski's hyperplane section theorem and fundamental groups of complements of plane curves

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§0. Introduction

In this paper, we formulate and prove a weighted homogeneous version of Zariski's hyperplane section theorem on the fundamental groups of the complements of hypersurfaces in a complex projective space, and apply it to the study of $\pi_1(\mathbb{P}^2 \setminus C)$, where $C \subset \mathbb{P}^2$ is a projective plane curve. The main application is to prove a comparison theorem as follows. Let $\varphi : \mathbb{P}^2 \to \mathbb{P}^2$ be the composition of the Veronese embedding $\mathbb{P}^2 \hookrightarrow \mathbb{P}^N$ and the restriction of a general projection $\mathbb{P}^N \cdots\to \mathbb{P}^2$. Our comparison theorem enables us to calculate $\pi_1(\mathbb{P}^2 \setminus \varphi^{-1}(C))$ from $\pi_1(\mathbb{P}^2 \setminus C)$. In [14] and [16], Zariski studied some projective plane curves with interesting properties. An example is sextic curves with 6 cusps. Zariski showed that the fundamental group of the complement depends on the placement of the 6 cusps. Another example is the 3-cuspidal quartic curve, whose complement has a non-abelian and finite fundamental group. This curve is the only known example with this property. Using the comparison theorem, we derive infinite series of curves with these interesting properties from the classical examples of Zariski. As another application, we shall discuss a relation between $\pi_1(\mathbb{P}^2 \setminus C)$ and $\pi_1(\mathbb{P}^2 \setminus (C \cup L_\infty))$, where L_∞ is a line intersecting C transversely.

Let $x_1, \ldots, x_n$ be variables with weights

$$\deg x_1 = d_1, \quad \ldots \quad , \deg x_n = d_n,$$

and let $F(x_1, \ldots, x_n)$ be a non-zero weighted homogeneous polynomial of total degree $d > 0$. Suppose that $n \geq 2$ and $d_i > 0$ for $i = 1, \ldots, n$. In the affine space $\mathbb{A}^n$ with affine coordinates $(x_1, \ldots, x_n)$, the equation $F = 0$ defines a hypersurface $\Sigma \subset \mathbb{A}^n$. We let the multiplicative group $\mathbb{G}_m$ of non-zero complex numbers act on $\mathbb{A}^n$ by

$$\lambda \cdot (x_1, \ldots, x_n) \quad = \quad (\lambda^{d_1} x_1, \ldots, \lambda^{d_n} x_n) \qquad \text{for} \quad \lambda \in \mathbb{G}_m. \tag{0.1}$$

This action leaves the complement $\mathbb{A}^n \setminus \Sigma$ invariant, because F is weighted homogeneous. Thus we have a natural homomorphism

$$\pi_1(\mathbb{G}_m) \quad \longrightarrow \quad \pi_1(\mathbb{A}^n \setminus \Sigma). \tag{0.2}$$

Note that the image of (0.2) is contained in the center of $\pi_1(\mathbb{A}^n \setminus \Sigma)$, so that we have the cokernel of this homomorphism. We fix a projective plane $\mathbb{P}^2$ with homogeneous coordinates $(\xi_0 : \xi_1 : \xi_2)$. Let f_i be homogeneous polynomials of degree d_i in (ξ_0, ξ_1, ξ_2) for $i = 1, \ldots, n$. We denote by f the n-tuple $(f_1, \ldots, f_n)$. The polynomial

$$F_f(\xi_0, \xi_1, \xi_2) := F(f_1(\xi_0, \xi_1, \xi_2), \ldots, f_n(\xi_0, \xi_1, \xi_2))$$

in the variables (ξ_0, ξ_1, ξ_2) is then homogeneous of degree d, and $F_f(\xi_0, \xi_1, \xi_2) = 0$ defines a subscheme of $\mathbb{P}^2$, which we shall denote by C_f. Unless F_f is constantly zero on $\mathbb{P}^2$, C_f is a projective plane curve of degree d.

Theorem 1. *Suppose that Σ is reduced. If $f_1, \dots, f_n$ are general, then $\pi_1(\mathbb{P}^2 \setminus C_f)$ is isomorphic to the cokernel of the natural homomorphism (0.2).*

Consider the case when $d_1 = \dots = d_n = 1$ and $n > 2$. Then $F = 0$ defines a hypersurface $\overline{\Sigma}$ in the projective space $\mathbb{P}^{n-1}$ with homogeneous coordinates $(x_1 : \dots : x_n)$. In this case, the action (0.1) of $\mathbb{G}_m$ on $\mathbb{A}^n \setminus \Sigma$ is fixed-point free, and the quotient space $(\mathbb{A}^n \setminus \Sigma)/\mathbb{G}_m$ is isomorphic to $\mathbb{P}^{n-1} \setminus \overline{\Sigma}$. Hence the cokernel of the homomorphism (0.2) is isomorphic to $\pi_1(\mathbb{P}^{n-1} \setminus \overline{\Sigma})$. On the other hand, if $f_1, \dots, f_n$ are general linear forms, then C_f is the pull-back of $\overline{\Sigma}$ by the linear embedding $\iota_f : \mathbb{P}^2 \hookrightarrow \mathbb{P}^{n-1}$ defined by $\iota_f^* \, x_i = f_i$. Hence Theorem 1 is nothing but the classical hyperplane section theorem of Zariski [15] in this case. This justifies us in calling Theorem 1 a *weighted Zariski's hyperplane section theorem.*

Let f_0, f_1, f_2 be general homogeneous polynomials of degree k in (ξ_0, ξ_1, ξ_2). We consider the branched covering $\varphi : \mathbb{P}^2 \to \mathbb{P}^2$ of degree k^2 defined by

$$(\xi_0 : \xi_1 : \xi_2) \quad \longmapsto \quad (\ f_0(\xi_0, \xi_1, \xi_2) \ : \ f_1(\xi_0, \xi_1, \xi_2) \ : \ f_2(\xi_0, \xi_1, \xi_2)\ ).$$

Suppose that we are given a reduced projective plane curve $C \subset \mathbb{P}^2$ of degree d. Using Theorem 1, we shall show that, when f_0, f_1, f_2 are general with respect to C, the fundamental group $\pi_1(\mathbb{P}^2 \setminus \varphi^{-1}(C))$ can be computed from $\pi_1(\mathbb{P}^2 \setminus C)$ in a simple way.

We define the *linking number map*

$$\ell \quad : \quad \pi_1(\mathbb{P}^2 \setminus C) \quad \longrightarrow \quad \mathbb{Z}/(d)$$

as follows. It is well known that $H_1(\mathbb{P}^2 \setminus C, \mathbb{Z})$ is naturally isomorphic to the cokernel of the direct sum of the restriction maps

$$H^2(\mathbb{P}^2, \mathbb{Z}) \quad \xrightarrow{r_1 \oplus \dots \oplus r_s} \quad H^2(C_1, \mathbb{Z}) \oplus \dots \oplus H^2(C_s, \mathbb{Z}), \tag{0.3}$$

where $C_1, \dots, C_s$ are the irreducible components of C (cf. [4, §8]). Let $e_i \in H^2(C_i, \mathbb{Z}) \cong \mathbb{Z}$ be the positive generator, which is the Poincaré dual of a point on C_i, and let $[L] \in H^2(\mathbb{P}^2, \mathbb{Z}) \cong \mathbb{Z}$ be the positive generator, which is the Poincaré dual of a line. Since C is reduced, we have $r_i([L]) = (\deg C_i) \cdot e_i$. Because $d = \deg C_1 + \dots + \deg C_s$, the maps $e_i \mapsto 1 \mod d$ induces a well-defined homomorphism from the cokernel of (0.3) to $\mathbb{Z}/(d)$; i.e., $H_1(\mathbb{P}^2 \setminus C, \mathbb{Z}) \to \mathbb{Z}/(d)$. The map ℓ is then defined to be the composition of this homomorphism with the Hurwicz map $\pi_1(\mathbb{P}^2 \setminus C) \to H_1(\mathbb{P}^2 \setminus C, \mathbb{Z})$.

We can define ℓ in the following different way. Let $[\alpha] \in \pi_1(\mathbb{P}^2 \setminus C)$ be an element represented by a loop $\alpha : S^1 \to \mathbb{P}^2 \setminus C$. Since $\pi_1(\mathbb{P}^2) = \{1\}$, there exists a continuous map $\beta : D \to \mathbb{P}^2$ from $D := \{\ z \in \mathbb{C} \ ; \ |z| \le 1\}$ to $\mathbb{P}^2$ such that $\partial\beta = \alpha$. By a small deformation of β homotopically relative to the boundary, we may assume that the image of β intersects the curve C transversely. Let $\operatorname{Im} \beta \cdot C$ be the intersection number. Since the intersection number of any element in $H_2(\mathbb{P}^2, \mathbb{Z})$ with $[C] \in H_2(\mathbb{P}^2, \mathbb{Z})$ is divisible by

d, the number $\mathrm{Im}\,\beta \cdot C$ modulo d is independent of the choice of β. We define $\ell([\alpha])$ to be $\mathrm{Im}\,\beta \cdot C \mod d \in \mathbb{Z}/(d)$. From this definition, we can call $\ell([\alpha])$ the linking number of the loop α around C. It is easy to see that this definition coincides with the previous definition.

We define the *extended linking number map* by

$$\begin{array}{cccc} \tilde{\ell} & : & \mathbb{Z} \times \pi_1(\mathbb{P}^2 \setminus C) & \longrightarrow & \mathbb{Z}/(d) \\ & & (\nu, [\alpha]) & \mapsto & \ell([\alpha]) - \nu \mod d. \end{array}$$

Theorem 2. *Let $C \subset \mathbb{P}^2$ be a reduced curve of degree d. Suppose that f_0, f_1 and f_2 are general homogeneous polynomials of degree k in (ξ_0, ξ_1, ξ_2). Then $\pi_1(\mathbb{P}^2 \setminus \varphi^{-1}(C))$ is isomorphic to* $\mathrm{Ker}\ \tilde{\ell}/((kd) \times \{1\})$.

Corollary 1. *Suppose the same assumptions as in Theorem 2.*

(1) The fundamental group $\pi_1(\mathbb{P}^2 \setminus \varphi^{-1}(C))$ is a central extension of $\pi_1(\mathbb{P}^2 \setminus C)$ by $\mathbb{Z}/(k)$. In particular, if $\pi_1(\mathbb{P}^2 \setminus C)$ is finite, then $\pi_1(\mathbb{P}^2 \setminus \varphi^{-1}(C))$ is also finite of order k times the order of $\pi_1(\mathbb{P}^2 \setminus C)$.

(2) The fundamental group $\pi_1(\mathbb{P}^2 \setminus \varphi^{-1}(C))$ is abelian if and only if so is $\pi_1(\mathbb{P}^2 \setminus C)$.

Remark that, when f_0, f_1 and f_2 are general, the morphism $\varphi : \mathbb{P}^2 \to \mathbb{P}^2$ is étale over a Zariski open neighborhood of the singular points of C. Hence, for example, if the singular locus $\mathrm{Sing}\,C$ consists of δ nodes and κ cusps, then the singular locus of $\varphi^{-1}(C)$ consists of $k^2\delta$ nodes and $k^2\kappa$ cusps.

Examples.

(1) *Zariski pairs.* A couple of reduced projective plane curves C_1 and C_2 is said to make a *Zariski pair* if they satisfy the following conditions [1]; (i) $\deg C_1 = \deg C_2$, (ii) there exist tubular neighborhoods $T(C_i) \subset \mathbb{P}^2$ of C_i for $i = 1, 2$ such that $(T(C_1), C_1)$ and $(T(C_2), C_2)$ are diffeomorphic, and (iii) the pairs $(\mathbb{P}^2, C_1)$ and $(\mathbb{P}^2, C_2)$ are *not* homeomorphic. That is, the singularities of C_1 and C_2 are topologically equivalent, but the embeddings $C_1 \hookrightarrow \mathbb{P}^2$ and $C_2 \hookrightarrow \mathbb{P}^2$ are not topologically equivalent.

The first example of Zariski pair was discovered and studied by Zariski. In [14] and [16], he showed that there exist projective plane curves C_1 and C_2 of degree 6 such that $\mathrm{Sing}\,C_1$ consists of 6 cusps lying on a conic, while $\mathrm{Sing}\,C_2$ consists of 6 cusps *not* on any conic, and that $\pi_1(\mathbb{P}^2 \setminus C_1)$ is isomorphic to the free product $\mathbb{Z}/(2) * \mathbb{Z}/(3)$ of cyclic groups of order 2 and 3, while $\pi_1(\mathbb{P}^2 \setminus C_2)$ is cyclic of order 6. Thus C_1 and C_2 make a Zariski pair. (See also [8].) After this example, only few Zariski pairs have been constructed (cf. [1], [12]).

Let $C_1(k)$ and $C_2(k)$ be the pull-backs of the sextic curves C_1 and C_2 above by the covering $\varphi : \mathbb{P}^2 \to \mathbb{P}^2$ in Theorem 2. Both of them are of degree $6k$, and each of their singular loci consists of $6k^2$ cusps. Combining Corollary 1 (2) with Zariski's result, we see that $\pi_1(\mathbb{P}^2 \setminus C_1(k))$ is non-abelian, while $\pi_1(\mathbb{P}^2 \setminus C_2(k))$ is abelian. Thus we obtain an infinite series of Zariski pairs $C_1(k)$ and $C_2(k)$.

(2) *Pull-backs of the three cuspidal quartic.* Let $C_0 \subset \mathbb{P}^2$ be a curve of degree 4 with 3 cusps and no other singularities; for example, the curve defined by

$$x^2y^2 + y^2z^2 + z^2x^2 - 2xyz(x+y+z) = 0.$$

In fact, it is known that any three cuspidal quartic curve C_0 is projectively isomorphic to the curve defined by this equation [2]. This curve was discovered and studied by Zariski in [14]. (See also [3, Chapter 4, §4].) The remarkable property of C_0 is that $\pi_1(\mathbb{P}^2 \setminus C_0)$ is isomorphic to the binary dihedral group of order 12. Other than this three cuspidal quartic, there have been no examples of projective plane curves $C \subset \mathbb{P}^2$ such that $\pi_1(\mathbb{P}^2 \setminus C)$ is non-abelian and finite.

Let $C_0(k)$ be the pull-back of C_0 by $\varphi : \mathbb{P}^2 \to \mathbb{P}^2$ in Theorem 2. Then $C_0(k)$ is of degree $4k$ and $\operatorname{Sing} C_0(k)$ consists of $3k^2$ cusps. By Corollary 1, $\pi_1(\mathbb{P}^2 \setminus C_0(k))$ is non-abelian and finite of order $12k$.

In the forthcoming paper [11], we will construct other examples of such curves.

(3) *The fundamental group* $\pi_1(\mathbb{P}^2 \setminus C_{p,q,k})$. Let p and q be positive integers prime to each other, and let f and g be general homogeneous polynomials in (ξ_0, ξ_1, ξ_2) of degree pk and qk, respectively, with $k \geq 1$. We define $C_{p,q,k}$ to be the projective plane curve of degree pqk defined by $f^q + g^p = 0$ (cf. [5]). The fundamental group of $U := \mathbb{A}^2 \setminus \{x^q + y^p = 0\}$ is well-known (cf. [3, Chapter 4, §2]), and the homomorphism from $\pi_1(\mathbb{G}_m)$ to $\pi_1(U)$ induced by the action of $\mathbb{G}_m$ on U with weights (pk, qk) on variables (x, y) can be easily described. Then, from Theorem 1, we see that $\pi_1(\mathbb{P}^2 \setminus C_{p,q,k})$ is isomorphic to

$$\langle\ a, b, c \mid a^q = b^p = c,\ \ c^k = 1\ \rangle.$$

Some parts of this fact have been already proved; by Zariski [14] when $p = 2$, $q = 3$ and $k = 1$, by Turpin [13] when $p = 2$, $q = 3$ and $k > 1$, and by Oka [7] and Némethi [6] when p, q are arbitrary and $k = 1$.

As another corollary of the proof of Theorem 2, we will show the following:

Corollary 2. *Let C be a reduced projective plane curve of degree d, and let L_∞ be a general line. Then the fundamental group $\pi_1(\mathbb{P}^2 \setminus (C \cup L_\infty))$ of the affine part of $\mathbb{P}^2 \setminus C$ is isomorphic to $\operatorname{Ker} \tilde{\ell}$. In particular, $\pi_1(\mathbb{P}^2 \setminus (C \cup L_\infty))$ is abelian if and only if so is $\pi_1(\mathbb{P}^2 \setminus C)$.*

Oka raised the following problem:

Question. Let C_1 and C_2 be projective plane curves such that $\pi_1(\mathbb{P}^2 \setminus C_1)$ and $\pi_1(\mathbb{P}^2 \setminus C_2)$ are isomorphic. Let L be a general line. Then are the fundamental groups $\pi_1(\mathbb{P}^2 \setminus (C_1 \cup L))$ and $\pi_1(\mathbb{P}^2 \setminus (C_2 \cup L))$ isomorphic?

Suppose that C_1 and C_2 are reduced and irreducible. If the isomorphism $H_1(\mathbb{P}^2 \setminus C_1, \mathbb{Z}) \cong H_1(\mathbb{P}^2 \setminus C_2, \mathbb{Z})$, induced from the given isomorphism $\pi_1(\mathbb{P}^2 \setminus C_1) \cong \pi_1(\mathbb{P}^2 \setminus C_2)$ via the Hurwicz homomorphism, maps the positive generator of $H_1(\mathbb{P}^2 \setminus C_1, \mathbb{Z})$ to the positive generator of $H_1(\mathbb{P}^2 \setminus C_2, \mathbb{Z})$, then the answer is affirmative because of Corollary 2.

The main tool of the proof of Theorem 1 is [10, Theorem 1]. A similar idea was used to calculate $\pi_1(\mathbb{P}^2 \setminus C_{p,q,1})$ in [10, §4]. Theorem 2 is proved by applying Theorem 1 to the case when $n = 3$, $d_1 = d_2 = d_3 = k$, and F is the homogeneous polynomial defining $C \subset \mathbb{P}^2$.

Independently from us, Oka [9] also composed examples of projective plane curves as our examples (1) and (2) above. As our method, his construction makes use of the covering of the plane, but not of the projective plane as ours, but an affine part of it. The curves he constructed have singularity of different types from our examples. The method of proof is also quite different.

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§1. Proof of the weighted Zariski's hyperplane section theorem

We consider $\mathbb{P}^2$ as the space of 1-dimensional linear subspaces in a 3-dimensional linear space V with linear coordinates (ξ_0, ξ_1, ξ_2). Let

$$A := H^0(\mathbb{P}^2, \mathcal{O}(d_1)) \times \cdots \times H^0(\mathbb{P}^2, \mathcal{O}(d_n))$$

be the space of all n-tuples $f = (f_1, \ldots, f_n)$. Then we have a natural morphism

$$\begin{array}{cccccc} \Psi & : & V \times A & \longrightarrow & \mathbb{A}^n \\ & & (\ (\xi_0,\xi_1,\xi_2),\ (f_1,\ldots,f_n)\ ) & \longmapsto & (\ f_1(\xi_0,\xi_1,\xi_2),\ \ldots\ ,\ f_n(\xi_0,\xi_1,\xi_2)\ ). \end{array}$$

Let $\widetilde{W} \subset V \times A$ be the pull-back of $\Sigma \subset \mathbb{A}^n$ by Ψ. Since F is weighted homogeneous, we can make $\mathbb{G}_m$ act on the complement $(V \times A) \setminus \widetilde{W}$ by

$$\lambda \cdot ((\xi_0,\xi_1,\xi_2),(f_1,\ldots,f_n)) = (\ (\lambda\xi_0, \lambda\xi_1, \lambda\xi_2),\ (f_1,\ldots,f_n)\ ) \quad \text{where} \quad \lambda \in \mathbb{G}_m. \tag{1.1}$$

The morphism

$$\Phi \ : \ (V \times A) \setminus \widetilde{W} \ \longrightarrow \ \mathbb{A}^n \setminus \Sigma,$$

which is the restriction of Ψ, is equivariant under the actions (0.1) and (1.1) of $\mathbb{G}_m$ on each side. Let $W \subset \mathbb{P}^2 \times A$ be the divisor defined by

$$F(f_1(\xi_0,\xi_1,\xi_2), \ldots, f_n(\xi_0,\xi_1,\xi_2)) = 0,$$

which is the universal family of the subschemes $\{C_f\}_{f \in A}$ of $\mathbb{P}^2$ parameterized by A. The action (1.1) of $\mathbb{G}_m$ on $(V \times A) \setminus \widetilde{W}$ is fixed-point free, and the quotient space is nothing but the complement $(\mathbb{P}^2 \times A) \setminus W$. Hence we get a commutative diagram

$$\begin{array}{ccccccccc} \pi_1(\mathbb{G}_m) & \longrightarrow & \pi_1((V \times A) \setminus \widetilde{W}) & \longrightarrow & \pi_1((\mathbb{P}^2 \times A) \setminus W) & \longrightarrow & \{1\} & & \text{(exact)} \\ \| & & \downarrow \Phi_* & & & & & & \\ \pi_1(\mathbb{G}_m) & \longrightarrow & \pi_1(\mathbb{A}^n \setminus \Sigma) & \longrightarrow & \text{the cokernel of (0.2)} & \longrightarrow & \{1\} & & \text{(exact).} \end{array}$$

On the other hand, the complement $\mathbb{P}^2 \setminus C_f$ is the fiber over $f \in A$ of the second projection $(\mathbb{P}^2 \times A) \setminus W \to A$. Hence Theorem 1 follows from the following two claims.

Claim 1. *The homomorphism $\Phi_* : \pi_1((V \times A) \setminus \widetilde{W}) \to \pi_1(\mathbb{A}^n \setminus \Sigma)$ is an isomorphism.*

Claim 2. *The inclusion $\mathbb{P}^2 \setminus C_f \hookrightarrow (\mathbb{P}^2 \times A) \setminus W$ induces an isomorphism on the fundamental groups when $f \in A$ is general enough.*

Proof of Claim 1. Note that $\{o_V\} \times A \subset V \times A$ is contained in $\widetilde{W}$, where o_V is the origin of V. Using the homotopy exact sequence, we can verify Claim 1 by proving that the restriction of Ψ to $(V \setminus \{o_V\}) \times A$ gives $(V \setminus \{o_V\}) \times A$ a structure of the locally trivial fiber space over $\mathbb{A}^n$ with simply connected fibers. Note that the restriction of Ψ to $\{v\} \times A$ is a surjective affine linear map from $\{v\} \times A \cong A$ to $\mathbb{A}^n$ for any point $v \in V \setminus \{o_V\}$. For arbitrary points $u \in \mathbb{A}^n$ and $v \in V \setminus \{o_V\}$, the intersection $\Psi^{-1}(u) \cap (\{v\} \times A)$ is then isomorphic, via the second projection, to an affine subspace of A of codimension n. Thus every fiber of $\Psi|_{(V\setminus\{o_V\})\times A} : (V \setminus \{o_V\}) \times A \to \mathbb{A}^n$ is a fiber bundle over $V \setminus \{o_V\}$ with fibers isomorphic to an affine space of dimension dim $A - n$. $\square$

Proof of Claim 2. Let $\Xi_{nr} \subset A$ be the locus of all $f \in A$ such that C_f is *not* a reduced curve; that is, $f = (f_1, \dots, f_n)$ belongs to Ξ_{nr} if and only if $F_f(\xi_0, \xi_1, \xi_2)$ is constantly zero on $\mathbb{P}^2$, or $F_f(\xi_0, \xi_1, \xi_2) = 0$ defines a non-reduced curve. By [10, Theorem 1], the proof of Claim 2 is reduced to the verification of the following:

Claim 2'. *The locus $\Xi_{nr} \subset A$ is of codimension ≥ 2.*

Before proving this claim, we need two preparations.

Let $\Sigma_1, \dots, \Sigma_M$ be the irreducible components of $\Sigma \subset \mathbb{A}^n$, and let $(\mathrm{Sing}\,\Sigma)_1, \dots, (\mathrm{Sing}\,\Sigma)_{M'}$ be the irreducible components of $\mathrm{Sing}\,\Sigma$ of dimension $n-2$. We give them the reduced structure as subschemes of $\mathbb{A}^n$. For $f \in A$, let $\psi_f : V \setminus \{o_V\} \to \mathbb{A}^n$ be the restriction of $\Psi : V \times A \to \mathbb{A}^n$ to the subvariety $(V \setminus \{o_V\}) \times \{f\} \cong V \setminus \{o_V\}$.

Sub-claim. *Let $f \in A$ be generally chosen. Then, for $i = 1, \dots, M$ and $j = 1, \dots, M'$, the following conditions (i) and (j') hold.*

(i) There exists a point $P_i \in V \setminus \{o_V\}$ such that $\psi_f(P_i)$ is contained in $\Sigma_i \setminus (\mathrm{Sing}\,\Sigma)$, and the pull-back of Σ by ψ_f is non-singular and of codimension 1 locally around P_i.

(j') There exists a point $P'_j \in V \setminus \{o_V\}$ such that $\psi_f(P'_j)$ is contained in $(\mathrm{Sing}\,\Sigma)_j$, and the pull-back of $(\mathrm{Sing}\,\Sigma)_j$ by ψ_f is non-singular and of codimension 2 locally around P'_j.

Proof. The conditions (i) and (j') are open for the choice of $f \in A$. Therefore, it is enough to show that, for each condition, there exists at least one $f \in A$ which satisfies it. This can be verified from the following fact. Let R be the point $(1, 0, 0)$ on $V \setminus \{o_V\}$. For an arbitrary point $Q = (c_1, \dots, c_n) \in \mathbb{A}^n$ and arbitrary two tangent vectors

$$\mathbf{u} = u_1 \frac{\partial}{\partial x_1} + \cdots + u_n \frac{\partial}{\partial x_n} \in T_Q(\mathbb{A}^n), \quad \text{and} \quad \mathbf{v} = v_1 \frac{\partial}{\partial x_1} + \cdots + v_n \frac{\partial}{\partial x_n} \in T_Q(\mathbb{A}^n)$$

of $\mathbb{A}^n$ at Q, there exists an element $f \in A$ such that $\psi_f(R) = Q$, and the image of the tangential map $\psi_{f*} : T_R(V \setminus \{o_V\}) \to T_Q(\mathbb{A}^n)$ contains both of $\mathbf{u}$ and $\mathbf{v}$. Indeed, if the coefficients of $\xi_0^{d_i}$, $\xi_0^{d_i-1}\xi_1$ and $\xi_0^{d_i-1}\xi_2$ in f_i are c_i, u_i and v_i, respectively, then $f = (f_1, \dots, f_n)$ satisfies the required condition. $\square$

Next we consider the action of the general linear transformation group $GL(V)$ of V on the space A. Note that $H^0(\mathbb{P}^2, \mathcal{O}(1))$ can be identified with the dual space $V^* :=$ Hom $(V, \mathbb{C})$. We let $GL(V)$ act on V^* from left by

$$(g(l))(v) = l(g^{-1}(v)) \quad \text{for} \quad g \in GL(V),\ l \in V^* \text{ and } v \in V.$$

This action can be extended in the natural way to the action on $A = \mathrm{Sym}^{d_1} V^* \times \cdots \times \mathrm{Sym}^{d_n} V^*$. Thus $GL(V)$ acts on $(V \setminus \{o_V\}) \times A$, and $\mathbb{P}^2 \times A$. By definition, we have

$$\Psi(g(P), g(f)) = \Psi(P, f) \tag{1.2}$$

for every $P \in V \setminus \{o_V\}$, $f \in A$ and $g \in GL(V)$. In particular, the divisor $W \subset \mathbb{P}^2 \times A$ is invariant under the action of $GL(V)$, and $C_{g(f)} = g(C_f)$ for every $g \in GL(V)$ and $f \in A$. It follows that $\Xi_{nr} \subset A$ is also invariant under the action of $GL(V)$.

Now we start the proof of Claim 2'. For a line $L \subset \mathbb{P}^2$, we put

$$\Xi(L) := \{\ f \in A\ ;\quad L \cap C_f \text{ does } \textit{not} \text{ consist of distinct } d \text{ points}\}.$$

The locus Ξ_{nr} is contained in $\Xi(L)$ for every line L. We fix a line $L_0 \subset \mathbb{P}^2$, and let $\Xi(L_0)_1$, $\dots$, $\Xi(L_0)_N$ be the irreducible components of $\Xi(L_0)$. Note that $\Xi(g(L_0)) = g(\Xi(L_0))$ for every $g \in GL(V)$. Since the subgroup $G(L_0) \subset GL(V)$ of all $g \in GL(V)$ which leave L_0 invariant is connected, the action of $g \in G(L_0)$ on $\Xi(L_0)$ does not interchange irreducible components of $\Xi(L_0)$; that is, each irreducible component is invariant under the action of $G(L_0)$. Thus, for an arbitrary line L, we have a natural numbering $\Xi(L)_1$, $\dots$, $\Xi(L)_N$ of the irreducible components of $\Xi(L)$ such that $g(\Xi(L)_i) = \Xi(L_0)_i$ for every $g \in GL(V)$ with $g(L) = L_0$. We shall show the following:

Claim 2". *For each $i = 1, \dots, N$, $\Xi(L_0)_i \subset A$ is of codimension ≥ 1, and $\Xi(L_0)_i \setminus \Xi_{nr}$ is non-empty.*

Since Ξ_{nr} is a Zariski closed subset of $\Xi(L_0)$, this will prove Claim 2'. Roughly speaking, the idea of the proof of Claim 2" is to show that, if $f^0 \in A$ is general, then $f^0 \notin \Xi_{nr}$, while there exist lines L_i $(i = 1, \dots, N)$ such that $f^0 \in \Xi(L_i)_i$. Suppose that these are proved. Let $g_i \in GL(V)$ be a linear transformation such that $g_i(L_i) = L_0$. Then, we have $g_i(\Xi(L_i)_i \setminus \Xi_{nr}) = \Xi(L_0)_i \setminus \Xi_{nr}$, and this contains an element $g_i(f^0)$, so that $\Xi(L_0)_i \setminus \Xi_{nr}$ is non-empty.

To carry out this idea, we have to investigate the irreducible components of $\Xi(L_0)$ more closely. Let P be a point on a line L. We put

$$\Xi(L, P) := \left\{\ f \in A\ ;\ \begin{array}{l} \text{the restriction of } F_f(\xi_0, \xi_1, \xi_2) \in H^0(\mathbb{P}^2, \mathcal{O}(d)) \text{ to } L \text{ has} \\ \text{a zero of order} \geq 2 \text{ at } P \end{array} \right\}.$$

As before, we have

$$g(\Xi(L, P)) = \Xi(g(L), g(P)) \tag{1.3}$$

for every $g \in GL(V)$. We also have

$$\Xi(L_0) = \bigcup_{P \in L_0} \Xi(L_0, P) = \bigcup_{g(L_0)=L_0} g(\Xi(L_0, P_0)),$$

where $P_0 \in L_0$ is a fixed point. From these two, by the same argument as above, it follows that the set of irreducible components of $\Xi(L_0, P_0)$ corresponds bijectively and in a natural way to that of $\Xi(L_0)$. Let $\Xi(L_0, P_0)_1, \ldots, \Xi(L_0, P_0)_N$ be the numbering of the irreducible components of $\Xi(L_0, P_0)$ according to that of $\Xi(L_0)$; that is, we have $\Xi(L_0, P_0)_i \subset \Xi(L_0)_i$ for $i = 1, \ldots, N$. Since $\dim \Xi(L_0) \leq \dim \Xi(L_0, P_0) + 1$, it is enough to show that, for $i = 1, \ldots, N$, the locus $\Xi(L_0, P_0)_i \subset A$ is of codimension ≥ 2 and $\Xi(L_0, P_0)_i \setminus \Xi_{nr}$ is non-empty.

Let L_0 be defined by $\xi_2 = 0$, and P_0 the point $(1 : 0 : 0)$ on L_0. Let $t = \xi_1/\xi_0$ be the affine coordinate on L_0 with the origin P_0. We put

$$G_f(t) := F_f(1, t, 0) = F(f_1(1, t, 0), \ldots, f_n(1, t, 0)).$$

Then f is contained in $\Xi(L_0, P_0)$ if and only if

$$G_f(0) = \frac{dG_f}{dt}(0) = 0 \tag{1.4}$$

holds. Let a_i be the coefficient of $\xi_0^{d_i}$ in f_i, and b_i the coefficient of $\xi_0^{d_i-1}\xi_1$ in f_i. Then $(a_1, \ldots, a_n, b_1, \ldots, b_n)$ form a subset of a linear coordinate system of A. Let $\mathbb{A}^n \times \mathbb{A}^n$ be the affine space with affine coordinates $(a_1, \ldots, a_n, b_1, \ldots, b_n)$. There exists a natural projection $q : A \to \mathbb{A}^n \times \mathbb{A}^n$. The condition (1.4) can be written as follows in terms of $(a_1, \ldots, a_n, b_1, \ldots, b_n)$.

$$\begin{gathered} F(a_1, \ldots, a_n) = 0, \quad \text{and} \\ \frac{\partial F}{\partial x_1}(a_1, \ldots, a_n) \cdot b_1 + \cdots + \frac{\partial F}{\partial x_n}(a_1, \ldots, a_n) \cdot b_n = 0. \end{gathered}$$

We consider $\mathbb{A}^n \times \mathbb{A}^n$ with the first projection $\mathbb{A}^n \times \mathbb{A}^n \to \mathbb{A}^n$ as the tangent bundle $T\mathbb{A}^n$ of the first factor $\mathbb{A}^n$ in which the zero section is given by $P \mapsto (P, O)$ (*not* by the diagonal $P \mapsto (P, P)$) where O is the origin $(b_1, \ldots, b_n) = (0, \ldots, 0)$ of the second factor. Under this identification, the equations above define in $\mathbb{A}^n \times \mathbb{A}^n$ the space

$$T\Sigma := \left\{ (P, Q) \in \mathbb{A}^n \times \mathbb{A}^n \; ; \; \begin{array}{l} Q \text{ is contained in the Zariski tangent} \\ \text{space } T_P(\Sigma) \subset T_P(\mathbb{A}^n) \text{ of } \Sigma \text{ at } P \end{array} \right\}.$$

Thus $\Xi(L_0, P_0)$ can be identified with $q^{-1}(T\Sigma)$, which is isomorphic to the product of $T\Sigma$ with an affine space of dimension $\dim A - 2n$. Since Σ is reduced by the assumption, $T\Sigma$ is of dimension $2n - 2$, and hence $\Xi(L_0, P_0) \subset A$ is of codimension 2. Thus $\Xi(L_0) \subset A$ is of codimension ≥ 1, and hence so is $\Xi_{nr} \subset A$.

Let $p : T\Sigma \to \Sigma$ be the projection induced by the natural projection $T\mathbb{A}^n \cong \mathbb{A}^n \times \mathbb{A}^n \to \mathbb{A}^n$. The fiber of p over $P \in \Sigma$ is a linear space of dimension $n-1$ (resp. n) if Σ is non-singular (resp. singular) at P. Thus the irreducible components of $T\Sigma$ are listed as follows; the closure of $p^{-1}(\Sigma_i \setminus (\operatorname{Sing}\Sigma))$ for $i = 1, \ldots, M$, and $p^{-1}((\operatorname{Sing}\Sigma)_j)$ for $j = 1, \ldots, M'$. Hence we get $N = M + M'$, and by changing the numbering, we have

$$\begin{array}{lcllll} \Xi(L_0, P_0)_i & = & q^{-1}(\text{ the closure of } p^{-1}(\Sigma_i \setminus (\operatorname{Sing}\Sigma))\ ) & \text{for} & i = 1, \ldots, M, & \text{and} \\ \Xi(L_0, P_0)_{M+j} & = & q^{-1}(p^{-1}((\operatorname{Sing}\Sigma)_j)) & \text{for} & j = 1, \ldots, M' & . \end{array}$$

Let $\widetilde{P}_0^{(1)}$ be the point $(1, 0, 0) \in V \setminus \{o_V\}$, which is located over $P_0 \in \mathbb{P}^2$. Note that $f \in \Xi(L_0, P_0)$ implies $P_0 \in C_f$, and hence $\psi_f(\widetilde{P}_0^{(1)}) \in \Sigma$. We identify two affine spaces $\mathbb{A}^n$ with coordinates $(a_1, \ldots, a_n)$ and $\mathbb{A}^n$ with coordinates $(x_1, \ldots, x_n)$ by putting $a_i = x_i$ for $i = 1, \ldots, n$. Then, recalling the definitions of the morphisms ψ_f, q and p, we can easily check that

$$\psi_f(\widetilde{P}_0^{(1)}) = p(q(f)) \quad \text{for every} \quad f \in \Xi(L_0, P_0).$$

Note that $\psi_f : V \setminus \{o_V\} \to \mathbb{A}^n$ is equivariant under the actions of $\mathbb{G}_m$ with weights $(1, 1, 1)$ on the left-hand side and with weights $(d_1, \ldots, d_n)$ on the right-hand side. Moreover, each of $\Sigma_i \subset \mathbb{A}^n$ and $(\operatorname{Sing}\Sigma)_j \subset \mathbb{A}^n$ are invariant under the action of $\mathbb{G}_m$, because $\mathbb{G}_m$ is connected. Therefore, given an element $f \in \Xi(L_0, P_0)$, we can tell which irreducible components of $\Xi(L_0, P_0)$ this f belongs to by looking at the point $\psi_f(\widetilde{P}_0) \in \Sigma$ where $\widetilde{P}_0 \in V \setminus \{o_V\}$ is an arbitrary point positioned over $P_0 = (1 : 0 : 0) \in \mathbb{P}^2$.

Criterion. If $\psi_f(\widetilde{P}_0) \in \Sigma$ is a non-singular point of Σ_i, then f belongs to $\Xi(L_0, P_0)_i$. If $\psi_f(\widetilde{P}_0) \in \Sigma$ is contained in $(\operatorname{Sing}\Sigma)_j$, then f belongs to $\Xi(L_0, P_0)_{M+j}$.

Recall that we have already proved that $\Xi_{nr} \subset A$ is of codimension ≥ 1. By Sub-claim, there exists $f^0 \in A \setminus \Xi_{nr}$ such that the image of ψ_{f^0} intersects $\Sigma_i \setminus (\operatorname{Sing}\Sigma)$ for $i = 1, \ldots, M$, and $(\operatorname{Sing}\Sigma)_j$ for $j = 1, \ldots, M'$. Let $\widetilde{P}_i$ and $\widetilde{P}'_j$ be points on $V \setminus \{o_V\}$ such that $\psi_{f^0}(\widetilde{P}_i) \in \Sigma_i \setminus (\operatorname{Sing}\Sigma)$ and $\psi_{f^0}(\widetilde{P}'_j) \in (\operatorname{Sing}\Sigma)_j$. We denote by P_i and P'_j the points on $\mathbb{P}^2$ corresponding to $\widetilde{P}_i$ and $\widetilde{P}'_j$, respectively. It is obvious that these points are on C_{f^0}. There exists a line $L_i \subset \mathbb{P}^2$ which intersects C_{f^0} at P_i with multiplicity ≥ 2. This means that $f^0 \in \Xi(L_i, P_i)$. Let $g_i \in GL(V)$ be a linear transformation such that $g_i(P_i) = P_0$ and $g_i(L_i) = L_0$. Then, by (1.3), we have $g_i(f^0) \in \Xi(L_0, P_0)$. By (1.2), we also have

$$\psi_{g_i(f^0)}(g_i(\widetilde{P}_i)) = \psi_{f^0}(\widetilde{P}_i) \in \Sigma_i \setminus (\operatorname{Sing}\Sigma).$$

Since $g_i(\widetilde{P}_i) \in V \setminus \{o_V\}$ is a point over $g_i(P_i) = P_0$, we see that $g_i(f^0)$ belongs to $\Xi(L_0, P_0)_i$ by the above criterion. From $f^0 \notin \Xi_{nr}$, we have $g_i(f^0) \notin \Xi_{nr}$. Thus $\Xi(L_0, P_0)_i \setminus \Xi_{nr}$ is non-empty for $i = 1, \ldots, M$. Similarly, there exists a line $L'_j \subset \mathbb{P}^2$ which intersects C_{f^0} at P'_j with multiplicity ≥ 2, which means $f^0 \in \Xi(L'_j, P'_j)$. We choose $g'_j \in GL(V)$ such that $g'_j(P'_j) = P_0$ and $g'_j(L'_j) = L_0$. Then (1.3) implies $g'_j(f^0) \in \Xi(L_0, P_0)$. By (1.2), we have $\psi_{g'_j(f^0)}(g'_j(\widetilde{P}'_j)) = \psi_{f^0}(\widetilde{P}'_j) \in (\operatorname{Sing}\Sigma)_j$. This means $g'_j(f^0) \in \Xi(L_0, P_0)_{M+j}$ by the criterion above. Thus $\Xi(L_0, P_0)_{M+j} \setminus \Xi_{nr}$ is non-empty for $j = 1, \ldots, M'$. $\square$

§2. Proof of the comparison theorem

We shall prove Theorem 2 and Corollaries in this section. In fact, most parts of Corollaries can be proved directly without using Theorem 2.

Suppose that $C \subset \mathbb{P}^2$ is defined by the homogeneous equation $F(\xi_0, \xi_1, \xi_2) = 0$. Let $\Sigma \subset \mathbb{A}^3$ be the hypersurface defined by $F = 0$ in the affine space $\mathbb{A}^3$ with affine coordinates (ξ_0, ξ_1, ξ_2). For a positive integer ν, we denote by

$$\rho_\nu \quad : \quad \pi_1(\mathbb{G}_m) \quad \longrightarrow \quad \pi_1(\mathbb{A}^3 \setminus \Sigma)$$

the homomorphism induced by the action of $\mathbb{G}_m$ on $\mathbb{A}^3 \setminus \Sigma$ with weights (ν, ν, ν);

$$\lambda \cdot (\xi_0, \xi_1, \xi_2) = (\lambda^\nu \xi_0, \lambda^\nu \xi_1, \lambda^\nu \xi_2) \quad \text{where} \quad \lambda \in \mathbb{G}_m. \tag{2.1}$$

The image of ρ_ν is contained in the center of $\pi_1(\mathbb{A}^3 \setminus \Sigma)$. It is obvious that $\pi_1(\mathbb{P}^2 \setminus C)$ is isomorphic to the cokernel of ρ_1. Since C is reduced by the assumption, Σ is also reduced. Now, applying Theorem 1 with $n = 3$ and $d_1 = d_2 = d_3 = k$, we see that $\pi_1(\mathbb{P}^2 \setminus \varphi^{-1}(C))$ is isomorphic to the cokernel of ρ_k, because $\varphi^{-1}(C)$ is defined by $F(f_0, f_1, f_2) = 0$. Let

$$\sigma_\nu \quad : \quad \pi_1(\mathbb{G}_m) \quad \longrightarrow \quad \pi_1(\mathbb{G}_m)$$

be the homomorphism induced from the morphism $\mathbb{G}_m \to \mathbb{G}_m$ given by $\lambda \mapsto \lambda^\nu$. Then we get a commutative diagram

$$\begin{array}{ccccccccc}
\pi_1(\mathbb{G}_m) & \xrightarrow{\rho_k} & \pi_1(\mathbb{A}^3 \setminus \Sigma) & \longrightarrow & \pi_1(\mathbb{P}^2 \setminus \varphi^{-1}(C)) & \longrightarrow & \{1\} & & \text{(exact)} \\
\downarrow{\scriptstyle \sigma_k} & & \| & & & & & & \\
\pi_1(\mathbb{G}_m) & \xrightarrow{\rho_1} & \pi_1(\mathbb{A}^3 \setminus \Sigma) & \longrightarrow & \pi_1(\mathbb{P}^2 \setminus C) & \longrightarrow & \{1\} & & \text{(exact).}
\end{array} \tag{2.2}$$

From this diagram, we can naturally derive an exact sequence

$$\text{Coker}\, \sigma_k \xrightarrow{\delta} \pi_1(\mathbb{P}^2 \setminus \varphi^{-1}(C)) \longrightarrow \pi_1(\mathbb{P}^2 \setminus C) \longrightarrow \{1\} \quad \text{(exact).} \tag{2.3}$$

The cokernel of σ_k is isomorphic to $\mathbb{Z}/(k)$. Hence (2.3) proves the implications

$$\begin{array}{lcll}
\pi_1(\mathbb{P}^2 \setminus C) \text{ is non-abelian} & \Longrightarrow & \pi_1(\mathbb{P}^2 \setminus \varphi^{-1}(C)) \text{ is non-abelian,} & \text{and} \\
\pi_1(\mathbb{P}^2 \setminus C) \text{ is finite} & \Longleftrightarrow & \pi_1(\mathbb{P}^2 \setminus \varphi^{-1}(C)) \text{ is finite.} &
\end{array} \tag{2.4}$$

Thus half of Corollary 1 is already proved.

Since the image of ρ_1 is contained in the center of $\pi_1(\mathbb{A}^3 \setminus \Sigma)$, the image of δ in (2.3) is also contained in the center of $\pi_1(\mathbb{P}^2 \setminus \varphi^{-1}(C))$. We shall show that δ is injective, and the central extension (2.3) is the same as the one described in Theorem 2. For this purpose, we investigate the bottom exact sequence of (2.2) more closely.

From now on, we denote by U the complement $\mathbb{P}^2 \setminus C$. For a positive integer m, let $\mathcal{L}_m \to \mathbb{P}^2$ be the line bundle corresponding to the invertible sheaf $\mathcal{O}(-m)$ on $\mathbb{P}^2$, and let

$\mathcal{L}_m^\times \subset \mathcal{L}_m$ denote the complement of the zero section. Let O be the origin of $\mathbb{A}^3$, and let $\mathbb{A}^3 \setminus \{O\} \to \mathbb{P}^2$ be the natural projection; that is, the quotient map by the $\mathbb{G}_m$-action (2.1) with $\nu = 1$. There exists an isomorphism between $\mathcal{L}_1^\times$ and $\mathbb{A}^3 \setminus \{O\}$ over $\mathbb{P}^2$. We fix such an isomorphism. From this, we get an isomorphism

$$\begin{array}{ccc} \mathcal{L}_1^\times|_U & \cong & \mathbb{A}^3 \setminus \Sigma \\ & \searrow \quad \swarrow & \\ & U & \end{array}$$

over U, where $\mathcal{L}_1^\times|_U$ is the restriction of $\mathcal{L}_1^\times \to \mathbb{P}^2$ to $U \subset \mathbb{P}^2$, and $\mathbb{A}^3 \setminus \Sigma \to U$ is the quotient map by the $\mathbb{G}_m$-action (2.1) with $\nu = 1$. Hence we have an isomorphism between the homotopy exact sequences;

$$\begin{array}{ccccccccc} \longrightarrow & \pi_1(\mathbb{G}_m) & \xrightarrow{\rho_1} & \pi_1(\mathbb{A}^3 \setminus \Sigma) & \longrightarrow & \pi_1(U) & \longrightarrow & \{1\} & \text{(exact)} \\ & \downarrow\wr & & \downarrow\wr & & \| & & & \\ \longrightarrow & \pi_1(\mathbb{C}^\times) & \xrightarrow{\iota_1} & \pi_1(\mathcal{L}_1^\times|_U) & \longrightarrow & \pi_1(U) & \longrightarrow & \{1\} & \text{(exact).} \end{array} \tag{2.5}$$

Here we denote the fiber of $\mathcal{L}_1^\times \to \mathbb{P}^2$ over the base point of U by $\mathbb{C}^\times$.

Since the line bundle $\mathcal{L}_1^{\otimes d}$ is isomorphic to $\mathcal{L}_d$, there exists a morphism

$$\mu \quad : \quad \mathcal{L}_1^\times \quad \longrightarrow \quad \mathcal{L}_d^\times \tag{2.6}$$

over $\mathbb{P}^2$ which induces $\zeta_1 \mapsto \zeta_d = \zeta_1^d$ on the fibers $\mathbb{C}^\times$ of $\mathcal{L}_1^\times$ and $\mathcal{L}_d^\times$ over the base point of U with appropriate fiber coordinates ζ_1 and ζ_d. From this morphism, we get a homomorphism between the homotopy exact sequences of $\mathcal{L}_1^\times|_U \to U$ and $\mathcal{L}_d^\times|_U \to U$;

$$\begin{array}{ccccccccc} \longrightarrow & \pi_1(\mathbb{C}^\times) & \xrightarrow{\iota_1} & \pi_1(\mathcal{L}_1^\times|_U) & \xrightarrow{\tau_1} & \pi_1(U) & \longrightarrow & \{1\} & \text{(exact)} \\ & \downarrow{\scriptstyle \mu_\#} & & \downarrow{\scriptstyle \mu_*} & & \| & & & \\ \longrightarrow & \pi_1(\mathbb{C}^\times) & \xrightarrow{\iota_d} & \pi_1(\mathcal{L}_d^\times|_U) & \xrightarrow{\tau_d} & \pi_1(U) & \longrightarrow & \{1\} & \text{(exact),} \end{array} \tag{2.7}$$

where $\mu_\#$ is the multiplication by d on $\pi_1(\mathbb{C}^\times) \cong \mathbb{Z}$. Note again that the images of ι_1 and ι_d are contained in the centers of $\pi_1(\mathcal{L}_1^\times|_U)$ and $\pi_1(\mathcal{L}_d^\times|_U)$, respectively. Since $\mathcal{L}_d \to \mathbb{P}^2$ is a line bundle corresponding to the invertible sheaf $\mathcal{O}(-d) \cong \mathcal{O}(-C)$, there exists a meromorphic section $s : \mathbb{P}^2 \cdots \to \mathcal{L}_d$ which has no zeros on $\mathbb{P}^2$ and is holomorphic outside C. Restricting s to U, we get a holomorphic section $s : U \to \mathcal{L}_d^\times|_U$. Hence the homotopy exact sequence of $\mathcal{L}_d^\times|_U \to U$ splits. In particular, the homomorphism ι_d in (2.7) is injective, and we have an isomorphism

$$(\iota_d, s_*) \quad : \quad \pi_1(\mathbb{C}^\times) \times \pi_1(U) \quad \xrightarrow{\sim} \quad \pi_1(\mathcal{L}_d^\times|_U). \tag{2.8}$$

Because $\mu_\#$ is also injective, we see by diagram chasing that ι_1 and μ_* are injective, too. Since ι_1 is injective, ρ_1 in (2.5) is also injective and hence δ in (2.3) is injective. Thus the proof of Corollary 1 (1) is completed.

We shall show that the image of the injective homomorphism μ_* in (2.7) is a normal subgroup of $\pi_1(\mathcal{L}_d^\times|_U)$. Let $[\alpha]$ and $[\gamma]$ be arbitrary elements of $\pi_1(\mathcal{L}_1^\times|_U)$ and $\pi_1(\mathcal{L}_d^\times|_U)$, respectively, and we put $[\beta] := [\gamma]^{-1}\cdot\mu_*([\alpha])\cdot[\gamma]$. Let $[\gamma'] \in \pi_1(\mathcal{L}_1^\times|_U)$ be an element such that $\tau_1([\gamma']) = \tau_d([\gamma])$. We put $[\alpha'] := [\gamma']^{-1}\cdot[\alpha]\cdot[\gamma']$ and $[\delta] := [\gamma]^{-1}\cdot\mu_*([\gamma'])$. Then we have $\mu_*([\alpha']) = [\delta]^{-1}[\beta][\delta]$. Since $\tau_d([\delta]) = 1$ by the definition, $[\delta]$ is contained in the image of ι_d, which is in the center of $\pi_1(\mathcal{L}_d^\times|_U)$. Hence we have $[\beta] = \mu_*([\alpha'])$.

Now we can derive the following commutative diagram from the diagram (2.7);

$$\begin{array}{ccccccccc}
 & & \{1\} & & \{1\} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
\{1\} & \longrightarrow & \pi_1(\mathbb{C}^\times) & \xrightarrow{\iota_1} & \pi_1(\mathcal{L}_1^\times|_U) & \xrightarrow{\tau_1} & \pi_1(U) & \longrightarrow & \{1\} \quad \text{(exact)} \\
 & & \downarrow{\scriptstyle \mu_\#} & & \downarrow{\scriptstyle \mu_*} & & \| & & \\
\{1\} & \longrightarrow & \pi_1(\mathbb{C}^\times) & \xrightarrow{\iota_d} & \pi_1(\mathcal{L}_d^\times|_U) & \xrightarrow{\tau_d} & \pi_1(U) & \longrightarrow & \{1\} \quad \text{(exact)}, \\
 & & \downarrow & & \downarrow{\scriptstyle \psi} & & & & \\
 & & \mathbb{Z}/(d) & = & \mathbb{Z}/(d) & & & & \\
 & & \downarrow & & \downarrow & & & & (2.9) \\
 & & \{1\} & & \{1\} & & & & \\
 & & \text{(exact)} & & \text{(exact)}. & & & &
\end{array}$$

We write the definition of ψ in this diagram explicitly. For $[\gamma] \in \pi_1(\mathcal{L}_d^\times|_U)$, let $[\gamma'] \in \pi_1(\mathcal{L}_1^\times|_U)$ be an element such that $\tau_1([\gamma']) = \tau_d([\gamma])$. There exists a unique element $[\epsilon] \in \pi_1(\mathbb{C}^\times)$ such that $\iota_d([\epsilon]) = [\gamma]\cdot\mu_*([\gamma'])^{-1}$. We define

$$\psi([\gamma]) \quad := \quad [\epsilon] \bmod \operatorname{im}\mu_\# \quad \in \quad \mathbb{Z}/(d).$$

The independence of $\psi([\gamma])$ on the choice of $[\gamma']$ can be checked easily. It is in showing that ψ is a group homomorphism that we have to use the fact that the image of ι_d is contained in the center. Let $[\gamma_1]$ and $[\gamma_2]$ be two elements in $\pi_1(\mathcal{L}_d^\times|_U)$. Because $[\gamma_2]\cdot\mu_*([\gamma_2'])^{-1} \in \operatorname{im}\iota_d$, we have

$$[\gamma_1]\mu_*([\gamma_1'])^{-1}\cdot[\gamma_2]\mu_*([\gamma_2'])^{-1} = [\gamma_1][\gamma_2]\cdot\mu_*([\gamma_2'])^{-1}\mu_*([\gamma_1'])^{-1} = ([\gamma_1][\gamma_2])\mu_*([\gamma_1'][\gamma_2'])^{-1}.$$

This implies $\psi([\gamma_1])\psi([\gamma_2]) = \psi([\gamma_1][\gamma_2])$. Now the surjectivity of ψ and $\operatorname{Ker}\psi = \operatorname{Im}\mu_*$ can be checked immediately.

The diagram (2.9) shows that $\pi_1(\mathbb{A}^3 \setminus \Sigma)$, which is isomorphic to $\pi_1(\mathcal{L}_1^\times|_U)$ by (2.5), is isomorphic to the kernel of

$$\psi \quad : \quad \pi_1(\mathcal{L}_d^\times|_U) \underset{\text{by (2.8)}}{\cong} \mathbb{Z}\times\pi_1(U) \longrightarrow \mathbb{Z}/(d).$$

Because the image of $\iota_d \circ \mu_\#$ is equal to $(d) \times \{1\}$ in $\pi_1(\mathcal{L}_d^\times|_U) \cong \mathbb{Z} \times \pi_1(U)$, the image of $\rho_1 : \pi_1(\mathbb{G}_m) \to \pi_1(\mathbb{A}^3 \setminus \Sigma)$, which corresponds to the image of $\iota_1 : \pi_1(\mathbb{C}^\times) \to \pi_1(\mathcal{L}_1^\times|_U)$ via the isomorphism (2.5), is given by

$$(d) \times \{1\} \quad \subset \quad \mathrm{Ker}\,\psi \quad \cong \quad \pi_1(\mathbb{A}^3 \setminus \Sigma).$$

By the diagram (2.2), the image of $\rho_k : \pi_1(\mathbb{G}_m) \to \pi_1(\mathbb{A}^3 \setminus \Sigma)$ is given by

$$(kd) \times \{1\} \quad \subset \quad \mathrm{Ker}\,\psi \quad \cong \quad \pi_1(\mathbb{A}^3 \setminus \Sigma).$$

Thus we obtain isomorphisms

$$\pi_1(\mathbb{P}^2 \setminus C) \;\cong\; \mathrm{Ker}\,\psi/((d)\times\{1\}), \quad \text{and} \quad \pi_1(\mathbb{P}^2 \setminus \varphi^{-1}(C)) \;\cong\; \mathrm{Ker}\,\psi/((kd)\times\{1\}). \tag{2.10}$$

Now we can complete the proof of Corollary 1. The only remaining part is the implication that, if $\pi_1(\mathbb{P}^2 \setminus C)$ is abelian, then so is $\pi_1(\mathbb{P}^2 \setminus \varphi^{-1}(C))$. If $\pi_1(\mathbb{P}^2 \setminus C) = \pi_1(U)$ is abelian, then $\mathrm{Ker}\,\psi \subset \mathbb{Z} \times \pi_1(U)$ is also abelian, and hence so is $\pi_1(\mathbb{P}^2 \setminus \varphi^{-1}(C)) \cong \mathrm{Ker}\,\psi/((kd) \times \{1\})$. Note that we have also proved that $\pi_1(\mathbb{A}^3 \setminus \Sigma)$ is abelian if and only if so is $\pi_1(\mathbb{P}^2 \setminus C)$.

In order to prove Theorem 2, it is enough to show that ψ coincides with -1 times the extended linking number map $\tilde{\ell}$ defined in Introduction, and this is equivalent to show that the homomorphism

$$\psi \circ s_* \quad : \quad \pi_1(U) \quad \longrightarrow \quad \mathbb{Z}/(d)$$

coincides with $-\ell$. It is obvious that $\psi \circ s_*$ factors as follows;

$$\pi_1(U) \quad \xrightarrow{\eta} \quad H_1(U, \mathbb{Z}) \cong \mathrm{Coker}\,(\, H^2(\mathbb{P}^2, \mathbb{Z}) \to \bigoplus_{i=1}^{s} H^2(C_i, \mathbb{Z})\,) \quad \longrightarrow \quad \mathbb{Z}/(d),$$

where η is the Hurwicz map, and $C_1, \ldots, C_s$ are the irreducible components of C. By the definition of ℓ, it is enough to show that, if $[\alpha] \in \pi_1(U)$ is mapped to the i-th positive generator

$$\epsilon_i := (0, \ldots, 0, e_i, 0, \ldots, 0) \bmod H^2(\mathbb{P}^2, \mathbb{Z}) \quad \in \quad \mathrm{Coker}\,(\, H^2(\mathbb{P}^2, \mathbb{Z}) \to \bigoplus_{i=1}^{s} H^2(C_i, \mathbb{Z})\,)$$

by the Hurwicz map, then $\psi \circ s_*([\alpha]) = -1 \bmod d$. Here $e_i \in H^2(C_i, \mathbb{Z})$ is the Poincaré dual of a point on C_i.

Before showing this, we remark a property of the Hurwicz map. The Hurwicz map does not depend on the choice of the base point. Namely, if we connect two base points b_1 and b_2 in U by a path, we get an isomorphism $\pi_1(U, b_1) \cong \pi_1(U, b_2)$, which depends on the homotopy class of the connecting path. But the diagram

$$\begin{tikzcd} \pi_1(U, b_1) \arrow[rr, "\cong"] \arrow[dr, "\eta"'] & & \pi_1(U, b_2) \arrow[dl, "\eta"] \\ & H_1(U, \mathbb{Z}) & \end{tikzcd}$$

is always commutative, whichever homotopy class of paths we may have chosen.

Let $L \subset \mathbb{P}^2$ be a general line, and let t be an affine coordinate on L such that $t = \infty$ is not on the intersection $L \cap C$. Note that $\mathcal{L}_m|_{L\setminus\{\infty\}}$ is a trivial bundle over $L \setminus \{\infty\}$ for any m. Let ζ_1 and ζ_d be the fiber coordinates of $\mathcal{L}_1|_{L\setminus\{\infty\}}$ and $\mathcal{L}_d|_{L\setminus\{\infty\}}$, respectively, such that the morphism μ in (2.6) is given by $\zeta_1 \mapsto \zeta_d = \zeta_1^d$ over $L \setminus \{\infty\}$. Suppose that $t = a_1, a_2, \dots, a_d$ are the intersection points of L and C. Then the section $s : U \to \mathcal{L}_d^\times|_U$, restricted to $L \setminus (C \cup \{\infty\})$, is given by, for example,

$$t \quad \longmapsto \quad (t, \zeta_d) = (t, f(t)) \quad \text{where} \quad f(t) = \frac{1}{(t-a_1)\cdots(t-a_d)}.$$

Let us consider the section

$$\begin{array}{ccccc} s_1 & : & L \setminus \{\infty\} & \longrightarrow & \mathcal{L}_1^\times|_{L\setminus\{\infty\}} \\ & & t & \longmapsto & (t, \zeta_1) = (t, B) \ , \end{array}$$

where B is a non-zero constant. (Of course, this section s_1 cannot extend over the whole L.) We fix a base point b of U. By the remark above, we may assume that b is on L and is close enough to an intersection point $t = a_\nu$ of L and an irreducible component C_i of C. Let $t = t_b$ be the coordinate of the base point. We choose the non-zero constant B in such a way that $f(t_b) = B^d$; that is, the images of s and $\mu \circ s_1$ on the base point $b \in L$ coincide. Let α be the loop

$$\begin{array}{ccccc} \alpha & : & [0,1] & \longrightarrow & L \setminus C \\ & & \theta & \longmapsto & t = a_\nu + \varepsilon e^{2\pi i \theta}, \end{array}$$

where $\varepsilon = t_b - a_\nu$. It is easy to see that $\eta([\alpha])$ is the i-th positive generator ϵ_i corresponding to C_i. We calculate $\psi \circ s_*([\alpha]) = \psi([s \circ \alpha])$. It is obvious that $[s_1 \circ \alpha] \in \pi_1(\mathcal{L}_1^\times|_U, s_1(b))$ satisfies $\tau_1([s_1 \circ \alpha]) = [\alpha]$. Hence $\psi([s \circ \alpha])$ is represented by $[s \circ \alpha] \cdot (\mu_*([s_1 \circ \alpha]))^{-1}$, which is in the image of ι_d. Recall that $\mathcal{L}_d^\times \to \mathbb{P}^2$ is trivial on $L \setminus (C \cup \{\infty\})$; that is, we have an isomorphism

$$\mathcal{L}_d^\times|_{L\setminus(C\cup\{\infty\})} \quad \cong \quad (L \setminus (C \cup \{\infty\})) \times \mathbb{C}^\times \tag{2.11}$$

given by the coordinates (t, ζ_d). Hence we have

$$\pi_1(\mathcal{L}_d^\times|_{L\setminus(C\cup\{\infty\})}, s(b)) \quad \cong \quad \pi_1(L \setminus (C \cup \{\infty\}), b) \times \pi_1(\mathbb{C}^\times, B^d).$$

In this direct product, we have

$$[s \circ \alpha] = ([\alpha], [\beta]) \qquad \text{and} \qquad \mu_*([s_1 \circ \alpha]) = ([\alpha], 0),$$

where β is the loop on $\mathbb{C}^\times$ obtained from the loop $s \circ \alpha$ on $\mathcal{L}_d^\times|_{L\setminus(C\cup\{\infty\})}$ by the second projection $\mathcal{L}_d^\times|_{L\setminus(C\cup\{\infty\})} \to \mathbb{C}^\times$ in (2.11), which can be written explicitly as follows;

$$\theta \quad \longmapsto \quad f(a_\nu + \varepsilon e^{2\pi i \theta}) \in \mathbb{C}^\times.$$

Their difference $[s \circ \alpha] \cdot (\mu_*([s_1 \circ \alpha]))^{-1}$ is then given by $[\beta] \in \pi_1(\mathbb{C}^\times)$. Since $|\varepsilon| = |t_b - a_\nu|$ is small enough, β is homotopically equivalent in $\mathbb{C}^\times$ to the loop

$$\theta \quad \longmapsto \quad B^d e^{-2\pi i \theta},$$

which corresponds to $-1 \in \mathbb{Z} \cong \pi_1(\mathbb{C}^\times)$. Thus $\psi \circ s_*([\alpha]) = -1 \mod d$, and Theorem 2 is proved.

Now we shall prove Corollary 2. We have already shown that $\pi_1(\mathbb{A}^3 \setminus \Sigma)$ is isomorphic to Ker $\tilde{\ell}$, and it is abelian if and only if so is $\pi_1(\mathbb{P}^2 \setminus C)$. Hence it is enough to show that, when L is a general line, $\pi_1(\mathbb{P}^2 \setminus (C \cup L))$ is isomorphic to $\pi_1(\mathbb{A}^3 \setminus \Sigma)$. We consider $\mathbb{A}^3$ as the complement of a hyperplane H in $\mathbb{P}^3$. Let $\mathbb{P}^2 \subset \mathbb{P}^3$ be a general plane. Then $\mathbb{P}^2 \cap (\Sigma \cup H)$ is isomorphic to $C \cup L$. By the classical Zariski's hyperplane section theorem, we have $\pi_1(\mathbb{P}^2 \setminus (C \cup L)) \cong \pi_1(\mathbb{P}^3 \setminus (\Sigma \cup H)) \cong \pi_1(\mathbb{A}^3 \setminus \Sigma)$. $\square$

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