METRIC AND TOPOLOGICAL ENTROPIES OF GEODESIC FLOWS

by

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1. Introduction

Let M be an n–dim. compact Riemannian manifold of negative sectional curvature. The geodesic flow ϕ^t acts on the unit tangent bundle T^1M of M preserving the Lebesgue–Liouville measure λ. This action is ergodic with respect to λ and the metric entropy h_λ, i.e. the entropy of ϕ^t with respect to λ, is not larger than the topological entropy h of ϕ^t. If M is locally symmetric then h and h_λ coincide. A well known conjecture of Katok ([18]) says that locally symmetric spaces are the only ones with this property.

The purpose of this paper is to derive a partial result related to this conjecture. We show the following:

Theorem A: If the metric and the topological entropy of the geodesic flow on T^1M coincide then the mean curvature of the horospheres in M is constant.

For surfaces of negative curvature (i.e. $\dim M = 2$), constant mean curvature for horospheres is equivalent to constant curvature. In this case our theorem is due to Katok ([18]). Ledrappier ([23]) observed that for 3–dim. manifolds of negative curvature constant mean curvature for horospheres implies constant curvature. We give a (different) easy proof of this fact in section 6. Together with theorem A we thus obtain:

Corollary B: If dim $M = 2$ or 3 and if the metric and the topological entropy of the geodesic flow on T^1M coincide then M has constant curvature.

Recall that every nontrivial conjugacy class $<\psi>$ in the fundamental group $\Gamma = \pi_1(M)$ of M can be represented by a unique closed geodesic in M of length $\ell<\psi>$. Let $\mathscr{C}$ be the set of all conjugacy classes in Γ. The marked length spectrum of M is defined to be the element $(\ell<\psi>)_{<\psi>\in\mathscr{C}}$ of the direct product $\mathbb{R}^{\mathscr{C}}$ indexed by e. It has been conjectured that the marked length spectrum determines M uniquely (up to isometry): If S is homotopy equivalent to M and if the marked length spectra of M and S coincide, then M and S are isometric. This is known to be true for surfaces (proved by Otal [29] and Croke [4]; for surfaces of constant curvature it was first derived by Katok [19]). In this paper we extend Katok's result to arbitrary dimensions:

Theorem C: Let S be a compact negatively curved locally symmetric space. Assume that M is compact, negatively curved, homotopy equivalent to S and that the marked length spectra of S and M coincide. Then M and S are isometric.

However we derive theorem C only for compact quotients of a real or complex hyperbolic space; the remaining cases (compact quotients of a quaternionic hyperbolic space or the Cayley plan) are contained in [12]. Also theorem C can be applied to improve an important result of Kanai ([17]), sharpened by Feres and Katok ([8], [9], [10]). The combination of these results can be expressed as follows:

Corollary D: Let M be a compact Riemannian manifold of negative curvature and assume either that

i) the dimension of M is odd or

ii) the curvature of M is strictly 1/4–pinched.

If the unstable foliation of M is of class C^∞ then the curvature of M is constant.

We refer to [17] and [8] for further results and references. The organization of the paper and some of the notations used throughout are as follows:

Let $\tilde{M}$ be the universal covering of M and recall that the geodesic flow on T^1M (resp. $T^1\tilde{M}$) admits continuous invariant foliations W^{su}, W^u, W^s, W^{ss} which are called the strong unstable, unstable, stable, strong stable foliations. Denote by $W^i(v)$ the leaf of W^i containing $v \in T^1M$ (resp. $v \in T^1\tilde{M}$). The flip $w \longrightarrow -w$ then maps $W^{su}(v)$ diffeomorphically onto $W^{ss}(-v)$, moreover
$W^u(v) = \bigcup_{t \in \mathbb{R}} \phi^t\, W^{su}(v) = -W^s(-v)$.

For $v \in T^1\tilde{M}$ let γ_v be the unique geodesic in $\tilde{M}$ with initial velocity $\gamma_v'(0) = v$ and denote by θ_v the Busemann function at the point $\gamma_v(-\infty)$ of the ideal boundary $\partial\tilde{M}$ of $\tilde{M}$ which is normalized by $\theta_v(\gamma_v(0)) = 0$. The canonical projection $P : T^1\tilde{M} \longrightarrow \tilde{M}$ maps $W^{su}(v)$ diffeomorphically onto the horosphere $\theta_v^{-1}(0)$ and maps $W^u(v)$ diffeomorphically onto $\tilde{M}$. Thus the Riemannian metric $<,>$ on $\tilde{M}$ lifts to a Riemannian metric g^i on $W^i(v)$ which induces a distance d^i and a Lebesgue measure $\lambda^i(i = su,u,s,ss)$. Let moreover dist be the distance on $\tilde{M}$ induced by the Riemannian metric and denote by $B(x,r)$ $(x \in \tilde{M}, r > 0)$ the open ball of radius r about x in $(\tilde{M},dist)$.

Recall that T^1M admits a unique ϕ^t–invariant Borel–probability measure of maximal entropy, the Bowen–Margulis measure μ . Then $h = h_\lambda$ is equivalent to $\lambda = \mu$. The measure μ admits a family of conditional measures μ^i on the leaves of W^i that are uniquely determined up to a universal constant. In section 2 we derive a representation of μ as a weak limit of the images under ϕ^t as $t \longrightarrow \infty$ of the restriction of μ^{su} to any open subset A of a leaf of W^{su} with $\mu^{su}(A) = 1$. Section 3 is devoted to a proof of the fact that for $\lambda = \mu$ the Radon–Nikodym derivative of the measures λ^u with respect to the measures μ^u on the leaves of W^u is a continuous function on T^1M (recall that the measures λ^i on the leaves of $W^i \subset T^1\tilde{M}$ project naturally to measures λ^i on the leaves of $W^i \subset T^1M$) . This function is used in section 4 to construct a stochastic process on T^1M that preserves a variant of λ . In section 5 we derive theorem A from the results of sections 2 – 4 and a result of Ledrappier ([22]). Section 6 is devoted to the proof of Corollary B (compare [23]). In section 7 we use theorem A and the results of [12] to show theorem C.

Appendix A contains some results on the existence and uniqueness for fundamental solutions of the Cauchy–problem $L - \frac{\partial}{\partial t} = 0$ for certain uniformly elliptic operators L on $\tilde{M}$ that are needed in section 4. In Appendix B we show that the shift transformations of the stochastic process on T^1M which was constructed in section 4 is ergodic.

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2. A description of the Bowen–Margulis measure

The Bowen–Margulis measure μ on T^1M is determined by its family of conditional measures μ^{su} on the leaves of W^{su} which transform under the geodesic flow via $\mu^{su} \circ \phi^{-t} = e^{ht}\mu^{su}$. The measures μ^u on the leaves of W^u , defined by $d\mu^u = d\mu^{su} \times dt$, are invariant under canonical maps ([26]). If μ^{ss} (resp. μ^s) denotes the image of the measures μ^{su} (resp. μ^u) under the flip $w \longrightarrow -w$ which maps W^{su} to W^{ss} (resp. W^u to W^s) then up to a constant $d\mu = d\mu^{su} \times d\mu^s = d\mu^u \times d\mu^{ss}$. The purpose of this section is to show that μ can be described as the weak limit of the images under the geodesic flow of suitably chosen probability measures on a leaf of W^u or W^{su} or on an arbitrarily chosen fibre T^1_pM of the unit tangent bundle $T^1M \longrightarrow M$.

Let $v \in T^1M$ and let A be an open relativ compact neighborhood of v in $W^u(v)$. The measure μ^u on $W^u(v)$ then induces a probability measure μ_A on A by defining for $B \subset A$ $\mu_A(B) = \mu^u(B)/\mu^u(A)$.

Lemma 2.1: For every continuous function φ on T^1M the limit $\lim\limits_{R \to \infty} \int_A \varphi \circ \phi^R d\mu_A$ exists and equals $\int \varphi d\mu$.

Proof: It suffices to show the lemma for nonnegative functions φ which do not vanish identically, i.e. which satisfy $\int \varphi d\mu = \beta > 0$. By subdividing A if necessary we may (as in chapter 4 of [25]) assume that there is an open subset C of $W^{ss}(v)$, an open neighborhood B of v in T^1M and a homeomorphism $\psi: B \longrightarrow A \times C$ with the following properties:

i) $\psi(w) = (w,v)$ for every $w \in A$.

ii) $\psi(w) = (v,w)$ for every $w \in C$.

iii) For every $w \in C$, $\psi^{-1}(A \times \{w\})$ is contained in a leaf of W^u .

iv) For every $w \in A$, $\psi^{-1}(\{w\} \times C)$ is contained in a leaf of W^{ss} .

Let $\varphi \geq 0$ be as above and let $\varepsilon \in (0,1)$. We have to find a number $R_0 > 0$ such that

$$(1 - \varepsilon)\beta < \int_A \varphi \circ \phi^R d\mu_A < (1 + \varepsilon)\beta \quad \text{for all} \quad R > R_0 .$$

Let $\delta \in (0,\varepsilon/4)$ be sufficiently small that $(1 + \delta)^{-3} > 1 - \varepsilon/2$ and $(1 + \delta)^3 < 1 + \varepsilon/2$. Since for every $w \in A$ the Jacobian at w of ψ as a map of (B,μ) into $(A \times C, \mu^u \times \mu^{ss})$ equals 1 we may assume by choosing C small enough that the Jacobian of ψ is contained in $((1 + \delta)^{-1}, 1 + \delta)$ for all $w \in B$. This means in particular

$$(*) \qquad (1 + \delta)^{-1}\mu^u(A)\mu^{ss}(C) < \mu(B) < (1 + \delta)\mu^u(A)\mu^{ss}(C) .$$

As $R \longrightarrow \infty$, the diameters of the sets $\phi^R\psi^{-1}(\{w\} \times C)$ tend to zero uniformly in $w \in A$. Since φ is continuous, hence uniformly continuous on T^1M this implies that there is a number $R_1 > 0$ such that for every $R > R_1$, $w \in A$ and

$\overline{w} \in \psi^{-1}(\{w\} \times C) \quad \varphi(\phi^R\overline{w}) > \varphi(\phi^R w) - \delta\beta$. By the choice of B we obtain for $R > R_1$

$$(**) \qquad (1+\delta)^{-1}\int_B \varphi \circ \phi^R d\mu - \delta\beta\mu(B) \le \mu^{ss}(C)\int_A \varphi \circ \phi^R d\mu^u$$

$$\le (1+\delta)\int_B \varphi \circ \phi^R d\mu + \delta\beta\mu(B)\ .$$

Now the geodesic flow on T^1M is mixing with respect to the Bowen–Margulis measure. Hence there is $R_0 > R_1$ such that for all $R > R_0$

$$(***) \qquad (1+\delta)^{-1}\beta\mu(B) \le \int_B \varphi \circ \phi^R d\mu \le (1+\delta)\beta\mu(B)\ .$$

Equations $(*) - (***)$ show that for $R > R_0$

$$((1+\delta)^{-3} - (1+\delta)\delta)\beta\mu^u(A) \le \int_A \varphi \circ \phi^R d\mu^u \le ((1+\delta)^3 + (1+\delta)\delta)\beta\mu^u(A)\ .$$

By the choice of δ this is the required inequality. □

Let again $v \in T^1M$ and let now $B \subset W^{su}(v)$ be an open relativ compact neighborhood of v in $W^{su}(v)$. As above the measure μ^{su} on $W^{su}(v)$ induces a Borel–probability measure μ_B on B .

Corollary 2.2: For every continuous function φ on T^1M the limit $\lim_{R\to\infty} \int_B \varphi \circ \phi^R d\mu_B$ exists and equals $\int \varphi d\mu$.

Proof: As in the proof of 2.1 we may assume that φ is nonnegative and that $\beta = \int \varphi d\mu > 0$. Let again $\varepsilon > 0$ and let $\delta < \varepsilon/2$ be sufficiently small that $(1+\delta)^{-3} > 1 - \varepsilon/2$ and $(1+\delta)^4 < 1 + \varepsilon$.

Let $t_0 = (\log(1+\delta))/h$. Since φ is continuous, hence uniformly continuous on T^1M we can find $\tau \le t_0$ in such a way that $|\varphi(\phi^t w) - \varphi(w)| < \delta\beta$ for all $w \in T^1M$ and $t \in (-\tau,\tau)$. Define $A = \bigcup_{-\tau<t<\tau} \phi^t B$; by the choice of τ and the definition of μ^u , μ^{su} we then have

$$(*) \qquad (1+\delta)^{-1}\Big[\int_A \varphi \circ \phi^R d\mu^u - \delta\beta\mu^u(A)\Big] \le 2\tau \int_B \varphi \circ \phi^R d\mu^{su}$$

$$\le (1+\delta)\Big[\int_A \varphi \circ \phi^R d\mu^u + \delta\beta\mu^u(A)\Big]$$

for all $R > 0$. By lemma 2.1 we can find a number $R_0 > 0$ such that for all $R > R_0$

$$(1+\delta)^{-1}\beta\mu^u(A) \le \int_A \varphi \circ \phi^R d\mu^u \le (1+\delta)\beta\mu^u(A) \quad \text{or}$$

$$(**) \qquad (1+\delta)^{-2} 2\tau\beta\mu^{su}(B) \le \int_A \varphi \circ \phi^R d\mu^u \le (1+\delta)^2 2\tau\beta\mu^{su}(B) .$$

By the choice of δ , insertion of $(**)$ into inequality $(*)$ yields for all $R > R_0$

$$(1-\varepsilon)\beta\mu^{su}(B) \le \int_B \varphi \circ \phi^R d\mu^{su} \le (1+\varepsilon)\beta\mu^{su}(B)$$

which is the required inequality. □

Define a projection $\pi : T^1\tilde{M} \longrightarrow \partial\tilde{M}$ by $\pi(v) = \gamma_v(\infty)$. For every $v \in T^1\tilde{M}$ the restriction π_v of π to $W^{su}(v)$ is a homeomorphism of $W^{su}(v)$ onto $\partial\tilde{M} - \pi(-v)$. Let $p = Pv$; then $T_p^1\tilde{M}$ is canonically homeomorphic to $\partial\tilde{M}$ and consequently the measure μ^{su} on $W^{su}(v)$ induces a measure $\bar{\mu}$ on T_p^1M via $\bar{\mu}(A) = \mu^{su}(\pi_v^{-1}\pi(A))$. Define a function $\sigma : T_p^1\tilde{M} - \{-v\} \longrightarrow \mathbb{R}$ by $\sigma(w) = \theta_{-w}(P\pi_v^{-1}\pi(w))$ and let $\tilde{\mu}_p$ be the measure on $T_p^1\tilde{M}$ whose Radon–Nikodym derivative with respect to $\tilde{\mu}$ equals $e^{-h\sigma}$. Then $\tilde{\mu}_p$ is a finite Borel–measure on $T_p^1\tilde{M}$ which does not depend on the choice of $v \in T_p^1\tilde{M}$; moreover the measures $\tilde{\mu}_p$ project to finite measures $\tilde{\mu}_q (q \in M)$ on the fibres T_q^1M of the fibration $T^1M \longrightarrow M$ that can be normalized to probability measures μ_q on T_q^1M .

<u>Lemma</u> 2.3: For every continuous positive function ρ on T_p^1M and every continuous function φ on T^1M the limit $\lim_{R\to\infty} \int_{T_p^1M} (\varphi \circ \phi^R)\rho d\mu_p$ exists and equals $(\int\varphi d\mu)(\int\rho d\mu_p)$.

<u>Proof</u>: As before we may assume that the function φ on T^1M is nonnegative with $\beta = \int\varphi d\mu > 0$. Let $\varepsilon \in (0,1)$; again we have to find $R_0 > 0$ such that for all $R > R_0$ $(\int \rho d\mu_p)\beta(1-\varepsilon) \le \int(\varphi \circ \phi^R)\rho d\mu_p \le (\int \rho d\mu_p)\beta(1+\varepsilon)$.

Let $\delta \in (0,\varepsilon)$ be sufficiently small that $(1+\delta)^6 < 1+\varepsilon$ and $(1+\delta)^{-5}(1-\delta) > 1-\varepsilon$. Let d be the distance on M induced by the Riemannian metric. Then there is $\tau > 0$ such that $|\varphi(w) - \varphi(\bar{w})| < \rho\delta(1+\delta)^{-2}$ for all $w,\bar{w} \in T^1M$ with $d(w,\bar{w}) < \tau$. There are finitely many points $v_1, \dots ,v_\ell \in T_p^1M$ and open neighborhoods U_i of v_i in T_p^1M with the following properties:

i) $U_i \cap U_j = \phi$ for $i \neq j$ and $\mu_p(\bigcup_{i=1}^{\ell} U_i) = 1$.

ii) For every $i \in \{1, \dots, \ell\}$ and $w \in U_i$ $\rho(w)/\rho(v_i) \in ((1+\delta)^{-1}, 1+\delta)$.

iii) For every $i \in \{1, \dots, \ell\}$ there is a homeomorphism π_i of U_i onto a neighborhood V_i of v_i in $W^{su}(v_i)$ such that for every $w \in U_i$ $\pi_i(w) \in W^s(w)$ and $d(w, \pi_i w) < \tau$.

iv) For every $i \in \{1, \dots, \ell\}$ the Jacobian of π_i as a map of $(U_i, \tilde{\mu}_p)$ onto (V_i, μ^{su}) has its range in $((1+\delta)^{-1}, 1+\delta)$.

Let $i \in \{1, \dots, \ell\}$. By Corollary 2.2 there is a number $R_i > 0$ such that for all $R > R_i$

$$(1+\delta)^{-1}\beta\mu^{su}(V_i) \le \int_{V_i} \varphi \circ \phi^R d\mu^{su} \le (1+\delta)\beta\mu^{su}(V_i).$$

Since $d(\phi^R w, \phi^R \pi_i w) \le d(w, \pi_i w)$ for all $R > 0$ this together with iii) and iv) shows

$$(1+\delta)^{-3}(1-\delta)\tilde{\mu}_p(U_i)\beta \le \int_{U_i} \varphi \circ \phi^R d\tilde{\mu}_p \le (1+\delta)((1+\delta)^2+\delta)\beta\, \tilde{\mu}_p(U_i).$$

Now ii) implies

$$(1+\delta)^{-1}\rho(v_i)\int_{U_i} (\varphi \circ \phi^R) d\tilde{\mu}_p \le \int_{U_i} (\varphi \circ \phi^R)\rho d\tilde{\mu}_p \le (1+\delta)\rho(v_i)\int_{U_i} (\varphi \circ \phi^R) d\tilde{\mu}_p$$

and consequently by the choice of δ for $R > R_i$

$$(1-\varepsilon)\beta \int_{U_i} \rho d\tilde{\mu}_p \leq \int_{U_i} (\varphi \circ \phi^R)\rho d\tilde{\mu}_p \leq (1+\varepsilon)\beta \int_{U_i} \rho d\tilde{\mu}_p .$$

With $R_0 = \max\{R_i \mid i = 1, \ldots, \ell\}$ we obtain for $R > R_0$

$$(1-\varepsilon)\beta \int_{T_p^1 M} \rho d\tilde{\mu}_p \leq \int_{T_p^1 M} (\varphi \circ \phi^R)\rho d\tilde{\mu}_p \leq (1+\varepsilon)\beta \int_{T_p^1 M} \rho d\tilde{\mu}_p$$

as required. □

Remark 2.4: Let $p \in M$ and let λ be the Lebesgue measure on T^1M, λ_p be the normalized Lebesgue measure on T_p^1M. It follows as above that for every continuous function φ on T^1M the limit $\lim_{R\to\infty} \int_{T_p^1 M} \varphi \circ \phi^R d\lambda_p$ exists and equals $\int_{T^1M} \varphi d\lambda$. Notice that locally λ can be written as a product of the Lebesgue measure on the fibres of the fibration $T^1M \longrightarrow M$ and the Lebesgue measure on M, moreover the geodesic flow is mixing with respect to λ.

3. Continuity of the Radon–Nikodym derivative

Assume now that the Bowen–Margulis measure μ and the Lebesgue measure λ on T^1M coincide. For $v \in T^1M$ let $f(v)$ be the Radon–Nikodym derivative of λ^{su} with respect to μ^{su} at v whenever this exists and is contained in $(0,\infty)$ and let $f(v) = \infty$ otherwise. Since the measures λ^{su} and μ^{su} are transversals for the same measure on T^1M they define the same measure class. Thus $f(v) < \infty$ for μ-almost every $v \in T^1M$. Observe that we would obtain the same function f from the above definition applied to the measures ν^u and μ^u (see [14]).

Lemma 3.1: The function f is continuous and finite on all of T^1M.

Proof: Recall that the tangent bundles TW^u and TW^{ss} of the foliations W^u, W^{ss} define a continuous decomposition of the tangent bundle TT^1M of T^1M. Since the restriction of the canonical projection $T^1M \longrightarrow M$ to the leaves of W^u (resp. W^s) is a local diffeomorphism, the Riemannian metric on M induces continuous Riemannian metrics g^u resp. g^{ss} on the vector bundles TW^u resp. TW^{ss}. Let g be the Riemannian metric on T^1M inducing the Lebesgue–Liouville measure and for $v \in T^1M$ let $\beta(u)$ be the determinant at v of the identity $(TW^u \oplus TW^{ss}, g^u \times g^{ss}) \hookrightarrow (TT^1M, g)$. Since g is invariant under the flip $w \longrightarrow -w$, the definition of the metrics g^u, g^{ss} implies that β is a continuous flip–invariant function on T^1M.

Let λ^u (resp. λ^{ss}) be the Lebesgue measure on the leaves of W^u (resp. W^{ss}) induced by the metric g^u (resp. g^{ss}) and define a measure $\bar{\lambda}^{ss}$ on the leaves of W^{ss} by $\frac{d\bar{\lambda}^{ss}}{d\lambda^{ss}}(v) = \beta(v)^{-1}$. The Lebesgue–Liouville measure λ on T^1M then satisfies $d\lambda = d\lambda^u \times d\bar{\lambda}^{ss}$. Since the measures λ^u are absolutely continuous with respect to the canonical maps, with Hölder continuous Jacobian, the arguments of section 2 of [14] apply and show that f is finite and continuous on T^1M. □

For $v \in T^1\tilde{M}$ let $U(v)$ be the second fundamental form at Pv of the horosphere $\theta_v^{-1}(0)$, normalized in such a way that $U(v)$ is positive definite. The function $\mathrm{tr}\, U$ which assigns to $v \in T^1\tilde{M}$ the trace of $U(v)$ is continuous on $T^1\tilde{M}$ (compare [15], proof of 3.1) and moreover it is invariant under the action of the fundamental group Γ of M on $T^1\tilde{M}$, i.e. $\mathrm{tr}\, U$ can be viewed as a continuous function on T^1M.

Following [24] we say that a function $\varphi : T^1M \longrightarrow \mathbb{R}$ is <u>of class</u> C_u^j for some $j \in [0,\infty]$ if the restriction of φ to every unstable manifold is of class C^j and if the jets of order $\leq j$ of these restrictions are continuous on T^1M.

<u>Lemma</u> 3.2: tr U is of class C_u^∞.

<u>Proof</u>: For $v \in T^1\tilde{M}$ let Z_v be the gradient of the Busemann function θ_v on $\tilde{M}$. Z_v is the projection into $T^1\tilde{M}$ of the restriction of the geodesic spray to $W^u(v)$.

Every smooth function $\varphi : \tilde{M} \longrightarrow \mathbb{R}$ lifts to a smooth functions $\tilde{\varphi} : T^1\tilde{M} \longrightarrow \mathbb{R}$ and every smooth vector field X on $\tilde{M}$ lifts to a continuous section of TW^u which is smooth along the unstable manifolds. This applies to coordinate functions on $\tilde{M}$, the induced basis vector fields on $\tilde{M}$ and the corresponding Christoffel symbols of the Riemannian connection on $\tilde{M}$. Thus if $< , >$ denotes the Riemannian metric on $\tilde{M}$, then for any smooth vector fields X, Y on $\tilde{M}$ the assignment $W^u(v) \longrightarrow \mathbb{R}$, $w \longrightarrow < D_X Z_v, Y >(Pw)$ is the restriction to $W^u(v)$ of a function of class C_u^∞ on $T^1\tilde{M}$. But $D_X Z_v = U(v)X$ (see [15]) and consequently tr U is of class C_u^∞ as claimed.

Let X be the geodesic spray on T^1M.

<u>Lemma</u> 3.3: tr $U - h = X(\log f)$.

Proof: For $v \in T^1M$ and $t \geq 0$ let $\psi_t(v)$ be the Jacobian at v of the restriction of ϕ^t to $(W^u(v),\lambda^u)$. Then $f(\phi^t v)$ is the product of the Jacobian at $\phi^t v$ of ϕ^{-t} with respect to μ^u, which equals e^{-ht}, $f(v)$ and $\psi_t(v)$, i.e. $f(\phi^t v) = \psi_t(v)e^{-ht}f(v)$ for all $t \geq 0$. By the choice of the measure λ^u this implies $X(\log f)(v) = \frac{d}{dt}\psi_t(v)e^{-ht}\big|_{t=0} = \operatorname{tr} U(v) - h$ as claimed. □

3.3 shows in particular that our theorem is equivalent to f being constant.

Define $g = f^{-1}$; then

Corollary 3.4: $X(g) = g(h - \operatorname{tr} U)$ and $X(f) = f(\operatorname{tr} U - h)$.

Corollary 3.5: f and g are of class C_u^∞.

Proof: It suffices to show that $\log f$ is of class C_u^∞. But this follows from 3.1 – 3.3 and the smooth Livsic theorem (lemma 2.2) of [24]. □

4. A stochastic process on T^1M

In this section we use the function g on T^1M to construct a stochastic process on T^1M preserving the measure $f\lambda$. For this let Δ be the Laplacian on $\tilde{M}$ and denote as usual by ∇u the gradient of a C^1–function $u : M \longrightarrow \mathbb{R}$ and by $\operatorname{div}(Y)$ the divergence of a vector field Y of class C^1 on $\tilde{M}$. Lift g to a function of class C_u^∞ on $T^1\tilde{M}$ which we denote by the same symbol. For $v \in T^1\tilde{M}$ let g_v be the smooth function on $\tilde{M}$ which is induced from the restriction of g to $W^u(v)$. Denote by $\mathcal{L}_v$

the differential operator on $\tilde{M}$ which acts on functions u of class C^2 by $u \longrightarrow \mathcal{L}_v(u) = g_v^{-1}\mathrm{div}(g_v \nabla u)$. Clearly $\mathcal{L}_v$ is uniformly elliptic, with bounded coefficients and without terms of order zero.

A function u on $\tilde{M}$ is called ξ–harmonic for some $\xi \in \mathbb{R}$ if $(\mathcal{L}_v - \xi)u = 0$. Let θ be a Busemann function at $\gamma_v(-\infty)$.

Lemma 4.1: For $\xi \geq -\frac{h^2}{4}$ define $\alpha(\xi) = \frac{h}{2} + (\frac{h^2}{4} + \xi)^{1/2}$ and $\beta(\xi) = \frac{h}{2} - (\frac{h^2}{4} + \xi)^{1/2}$; then the functions $e^{-\alpha(\xi)\theta}$, $e^{-\beta(\xi)\theta}$ are ξ–harmonic .

Proof: Let $X_v = \nabla(\theta_v)$ and let $\mathrm{tr}\, U_v$ be the function on $\tilde{M}$ which is induced from the restriction of $\mathrm{tr}\, U$ to $W^u(v)$. Then $\mathrm{div}(X_v) = \mathrm{tr}\, U_v$ (see [14]) and 3.4 shows that for $\alpha > 0$ we have $g_v^{-1}\mathrm{div}(g_v \nabla e^{-\alpha\theta}) = -\alpha g_v^{-1}\mathrm{div}(g_v e^{-\alpha\theta} X_v) =$
$-\alpha e^{-\alpha\theta}(\mathrm{tr}\, U_v + (h - \mathrm{tr}\, U_v) - \alpha) = (\alpha^2 - h\alpha)e^{-\alpha\theta}$ which proves the lemma. □

Lemma 4.1 shows in particular that the function $\varphi_v = e^{-h\theta_v/2}$ is $(-\frac{h^2}{4})$ – harmonic .

Let now L_v be the differential operator on $\tilde{M}$ which is defined by $L_v(u) = \mathrm{div}(g_v \nabla u) + 2g_v < \nabla u, \nabla \log \varphi_v >$. Since the curvature on $\tilde{M}$ is uniformly bounded and since L_v is uniformly elliptic with bounded smooth coefficients and without term of order zero, the associated Cauchy problem $L_v - \frac{\partial}{\partial t} = 0$ admits a unique weak fundamental solution $p_v(x,y,t)$ (see appendix A). Here $p_v(x,y,t)$ is the density with respect to the Riemannian volume dy on $\tilde{M}$ of the 1–dim. distribution of the unique probability measure on the space $C^0(\mathbb{R}_+,\tilde{M})$ of continuous paths in $\tilde{M}$ which describes the unique diffusion in $\tilde{M}$ generated by L_v with initial distribution δ_x (see [16] and appendix A). Since the elliptic differential operator $g_v(\mathscr{L}_v + \frac{h^2}{4})$ is self–adjoint, every weak fundamental solution of its associated Cauchy–problem is symmetric in the space variables (compare appendix A and [1]). This operator is related to L_v as follows:

Lemma 4.2: i) The function $\tilde{M} \times \tilde{M} \times (0,\infty) \longrightarrow (0,\infty)$, $(x,y,t) \longrightarrow p_v(x,y,t)\varphi_v(x)/\varphi_v(y)$ is a weak fundamental solution of the $g_v(\mathscr{L}_v + \frac{h^2}{4})$ – Cauchy problem; in particular $p_v(x,y,t)\varphi_v(x)/\varphi_v(y) = p_v(x,y,t)\varphi_v(y)/\varphi_v(x)$.

ii) Let ν be a measure on $\tilde{M}$ which is equivalent to the Lebesgue measure and which is preserved by the diffusion process generated by L_v. Then ν is of the form $d\nu = \psi dy$ where $\psi : \tilde{M} \longrightarrow [0,\infty)$ is a smooth function such that $(\mathscr{L}_v + \frac{h^2}{4})(\varphi_v^{-1}\psi) = 0$.

Proof: For a function u on $\tilde{M}$ let $[u]$ be multiplication by u. Then $[\varphi_v]^{-1}[g_v](\mathscr{L}_v + \frac{h^2}{4})[\varphi_v]u = \varphi_v^{-1}g_v(\mathscr{L}_v + \frac{h^2}{4})\varphi_v u = \varphi_v^{-1}[g_v\Delta(\varphi_v u) + < \nabla g_v, \varphi_v \nabla u + u\nabla\varphi_v > + \frac{h^2}{4} g_v\varphi_v u] = \operatorname{div}(g_v \nabla u) + 2g_v < \nabla \log \varphi_v, \nabla u > = L_v u$ for every smooth function u on $\tilde{M}$. From this the lemma follows. □

Lemma 4.2 shows in particular that the diffusion process on $\tilde{M}$ generated by L_v preserves the measure $\varphi_v^2(y)dy$ (compare [32]).

For $v \in T^1\tilde{M}$, $w \in W^u(v)$ and $t > 0$ define $\varphi_v(w) = \varphi_v(Pw)$ and $p(v,w,t) = \varphi_v(v)p_v(Pv,Pw,t)/\varphi_v(w)$. The lift of L_v to an operator on $W^u(v)$ defines a diffusion process with transition probabilities $\varphi_v(w)p(v,w,t)/\varphi_v(v)$ which we call the φ_v–process.

Lemma 4.3: For every $t > 0$ the function $(v,w) \longrightarrow p(v,w,t)$ is measurable and lower semi–continuous on $\{(v,w) \in T^1\tilde{M} \times T^1\tilde{M} \mid w \in W^u(v)\}$.

Proof: Let $B \subset \tilde{M}$ be an open ball of radius $r > 0$. For $x \in B$ and $v \in T^1_xB$ the coefficients of the operator L_v on B with all its derivatives depend continuously on v; hence the same is true for the fundamental solution q_v of the L_v–Cauchy problem on B (see [21]). The lemma now follows from the definition of p. □

Let Ω (resp. $\tilde{\Omega}$) be the space of paths $\xi : [0,\infty) \longrightarrow T^1M$ (resp. $\xi : [0,\infty) \longrightarrow T^1\tilde{M}$), equipped with the smallest σ–algebra $\mathfrak{A}$ (resp. $\tilde{\mathfrak{A}}$) for which the projections $\xi \longrightarrow \xi(t)$ $(t \in [0,\infty))$ are measurable. For $v \in T^1\tilde{M}$ the φ_v–process on $W^u(v)$ is given by a Markovian family $\{P^w\}_{w \in W^u(v)}$ of probability measures P^w on $\tilde{\Omega}$ with initial distribution δ_w. For every $t > 0$ and every Borel set $A \subset T^1\tilde{M}$ we have $P^v\{\xi \mid \xi(t) \in A\} = \int_{A \cap W^u(v)} (\varphi_v(w)/\varphi_v(v))p(v,w,t)d\lambda^u(w)$, moreover P^v–almost every path in $\tilde{\Omega}$ is continuous. The collection of probability measures

$\{P^v\}_{v \in T^1\tilde{M}}$ then defines a stochastic process on $T^1\tilde{M}$ (by 4.3) which we call the φ–process .

Lemma 4.4: The φ–process preserves the measure $f\lambda$.

Proof: For $v \in T^1\tilde{M}$ the φ_v–process is given by an action of the semi–group $[0,\infty)$ on functions on $W^u(v)$ by the kernel $(\varphi_v(w)/\varphi_v(v))p(v,w,t)$. Let $(t,u) \longrightarrow \Lambda^t u$ be the action of $[0,\infty)$ on functions u on $T^1\tilde{M}$ which describe the φ–process . Choose a continuous function u on $T^1\tilde{M}$ with compact support and recall $d(f\lambda) = d\lambda^u \times d\mu^{ss}$. Then

$$\begin{gathered}\int(\Lambda^t u)fd\lambda = \int(\Lambda^t u)d\lambda^u \times d\mu^{ss} = \\ \iint p(v,w,t)u(w)(\varphi_v(w)/\varphi_v(v))d\lambda^u(w)(d\lambda^u \times d\mu^{ss})(v)\end{gathered} \tag{$*$}$$

Let $v \in T^1\tilde{M}$, $t \in \mathbb{R}$, $w \in \phi^t W^{su}(v)$ and let B be a compact neighborhood of $\gamma_v(-\infty)$ in $\partial\tilde{M}$ such that $\{\pi(v),\pi(w)\} \subset \partial\tilde{M} - B$. Then the measures μ^{ss} on $W^{ss}(v)$ (resp. $W^{ss}(w)$) project to measures μ_1 (resp. μ_2) on B . Since the conditionals $\mu^s = \mu^{ss} \times dt$ of the Bowen–Margulis measure on the leaves of the stable foliation are invariant under canonical maps ([23]) the Radon–Nikodym derivative at $\gamma_v(-\infty)$ of μ_2 with respect to μ_1 equals $e^{-ht} = \varphi_v^2(w)/\varphi_v^2(v)$. By the above consideration the integrand of $(*)$ is measurable and lower semi–continuous; thus this integral is just $\iint p(v,w,t)(\varphi_v(v)/\varphi_v(w))d\lambda^u(v)u(w)(d\lambda^u \times d\mu^{ss})(w) = \int ud(f\lambda)$ which shows the lemma. □

Let $\Pi : T^1\tilde{M} \longrightarrow T^1M$ be the canonical projection. Π induces a measurable projection of $\tilde{\Omega}$ onto Ω, hence for every $w \in T^1\tilde{M}$ the measure P^w projects to a probability measure on Ω which only depends on $\Pi w = v$ and will be denoted by P^v. We then obtain a finite measure P on Ω by $P(B) = \int P^v(B)f(v)d\lambda(v)$.

Recall that the semi–group $[0,\infty)$ acts on Ω by the <u>shift–transformations</u> $(t,\xi) \longrightarrow T^t\xi$ where $T^t\xi(s) = \xi(s+t)$.

<u>Lemma</u> 4.5: P is invariant under the shift transformations.

<u>Proof</u>: For $t \in [0,\infty)$ let $R_t : \Omega \longrightarrow T^1M$ be the measurable projection $\xi \longrightarrow R_t(\xi) = \xi(t)$. This projection maps the measure P on Ω to a Borel–measure $R_t(P)$ on T^1M. Since the σ–algebra on Ω is generated by the sets $R_t^{-1}(A)(t \in [0,\infty), A \subset T^1M$ Borel) it suffices to show that $R_t(P) = R_0(P)$ for all $t \geq 0$. For this let $D \subset T^1\tilde{M}$ be a compact fundamental domain for the action of Γ on $T^1\tilde{M}$ and denote by $\tilde{P}$ the measure on $\tilde{\Omega}$ which describes the φ–process. Let $A \subset D$ be a Borel set; then $R_0(P)(\Pi A) = f\lambda(A)$. On the other hand, by the definition of P we have

$$R_t(P)(\Pi A) = \int_D P^v\{\xi \mid \Pi\xi(t) \in \Pi A\}f(v)d\lambda(v) = \sum_{\psi\in\Gamma} \tilde{P}\{\xi \mid \xi(0) \in D, \xi(t) \in d\psi(A)\}.$$

Since the φ–process preserves the measure $f\lambda$ it follows $\tilde{P}\{\xi \mid \xi(0) \in D, \xi(t) \in d\psi(A)\} = \tilde{P}\{\xi \mid \xi(0) \in d\psi(A), \xi(t) \in D\} =$ $\tilde{P}\{\xi \mid \xi(0) \in A, \xi(t) \in d\psi^{-1}(D)\}$ and consequently $R_t(P)(\Pi A) = \tilde{P}\{\xi \mid \xi(0) \in A\} =$ $R_0(P)(A)$. This shows the lemma. □

5. Proof of theorem A

We continue to use the assumptions and notations of sections 1 – 4. The Laplacian Δ on $\tilde{M}$ admits an extension to a linear self–adjoint endomorphism of $L^2(\tilde{M})$, the space of square integrable functions on $\tilde{M}$. The top of the spectrum of this extension equals the negative of the infimum of $\int \| \nabla\eta \|^2 dy / \int \eta^2 dy$ over smooth functions η on $\tilde{M}$ with compact support (compare [3], [32]). The following important result is due to Ledrappier ([22]):

Theorem 5.1: The top of the spectrum of the action of Δ on $L^2(\tilde{M})$ is not smaller than $-\frac{h^2}{4}$, with equality only if the mean curvature of the horospheres in $\tilde{M}$ is constant.

Let now $v \in T^1\tilde{M}$ and recall the definition of the operator $\mathscr{L}_v$ from section 4. For smooth functions ψ, η on $\tilde{M}$ with compact support we then have $\int (\mathscr{L}_v\psi)\eta\, g_v dy = \int \operatorname{div}(g_v \nabla\psi)\eta\, dy = -\int <\nabla\psi,\nabla\eta> g_v dy$, i.e. $\mathscr{L}_v$ admits an extension to a linear self–adjoint endomorphism of $L^2(\tilde{M})$, equipped with the scalar product $(\psi,\eta) \longrightarrow \int \psi\eta g_v dy$. In particular the top of the spectrum of this action equals the negative of the infimum of $\int \| \nabla\eta \|^2 g_v dy / \int \eta^2 g_v dy$ over smooth functions η on $\tilde{M}$ with compact support. Now φ_v is a positive function on $\tilde{M}$ which satisfies $(\mathscr{L}_v + \frac{h^2}{4})\varphi_v = 0$; consequently the arguments of Sullivan ([32]) imply:

Lemma 5.2: The top of the spectrum of the action of $\mathscr{L}_v$ on $L^2(\tilde{M})$ is not larger than $-\frac{h^2}{4}$.

The proof of theorem A now consists in combining 5.1 and 5.2 in a suitable way. For this we first derive from the results of section 4 an integral formula for functions of class C_u^2 on $T^1\tilde{M}$.

Via the projection P the Laplace operator for functions on $\tilde{M}$ lifts for every $v \in T^1\tilde{M}$ to an elliptic operator for functions on $W^u(v)$ which we denote by the same symbol. For a function u of class C_u^1 on $T^1\tilde{M}$ let ∇u be the gradient of the restriction of u to the leaves of W^u with respect to the Riemannian metric $g^u = \langle\, ,\, \rangle$; ∇u is a continuous section of TW^u. Let moreover X be the geodesic spray on T^1M and $T^1\tilde{M}$.

Lemma 5.3: $\int_{T^1\tilde{M}} [\Delta(\varphi) + \langle \nabla(\log g), \nabla\varphi \rangle - hX(\varphi)]\, d\lambda = 0$ for every function φ of class C_u^2 on $T^1\tilde{M}$ with compact support.

Proof: Let $\varphi : T^1\tilde{M} \longrightarrow \mathbb{R}$ be a function of class C_u^2 with compact support. Then $\varphi(\xi(t)) - \varphi(\xi(0)) - \int_0^t [g\Delta(\varphi) + \langle \nabla g, \nabla\varphi \rangle - ghX(\varphi)]\,(\xi(s))ds$ is a $(P^v, \mathfrak{B}_t)$–martingale for all $v \in T^1\tilde{M}$ where $\mathfrak{B}_t$ is the σ–algebra generated by the Borel cylinder sets up to time t (compare [16] p. 189), in particular $\int(\varphi(T^t\xi(0)) - \varphi(\xi(0)))dP^v(\xi) = \iint_0^t [g\Delta(\varphi) + \langle \nabla g, \nabla\varphi \rangle - ghX(\varphi)]\,(\xi(s))dsdP^v(\xi)$.

On the other hand, since the φ–process preserves the measure $f\lambda$ we have

$\iint u(\xi(t))dP^v(\xi)d(f\lambda)(v) = \int ud(f\lambda)$ for every continuous function u on $T^1\tilde{M}$ with compact support. Thus

$$0 = \iiint_0^t [g\Delta(\varphi) + < \nabla g,\nabla\varphi > - ghX(\varphi)]\,(\xi(s))dsdP^v(\xi)d(f\lambda)(v) =$$

$$t\int [\Delta(\varphi) + < \nabla(\log g),\nabla\varphi > - hX(\varphi)]\,d\lambda$$

for all $t > 0$ as claimed. □

Let D be a compact fundamental domain for the action of Γ on $\tilde{M}$. Assume that D is connected, with dense interior int D, and that the topological boundary ∂D of D is a compact set of vanishing Lebesgue measure. We denote by T^1D the restriction of the bundle $T^1\tilde{M}$ to D.

Lemma 5.4: If φ is the lift to $T^1\tilde{M}$ of a function of class C^1 on $\tilde{M}$ then $\int_{T^1D} X(\varphi)d\lambda = 0$.

Proof: Let $\psi: \tilde{M} \longrightarrow \mathbb{R}$ be a function of class C^1 and let $\varphi = \psi \circ P$. If $\nabla\psi$ denotes the gradient of ψ in $(\tilde{M},< , >)$ then $X(\varphi)(v) = < v,\nabla\psi >$ and consequently $\int_{T^1_x\tilde{M}} X(\varphi)(v)d\omega(v) = 0$ for every $x \in \tilde{M}$ where ω is the Lebesgue measure on the (n – 1)–dim. standard sphere $S^{n-1} \sim T^1_x\tilde{M}$. But $\int_{T^1D} X(\varphi)d\lambda =$ $\int_D\int_{T^1_x\tilde{M}} X(\varphi)(v)d\omega(v)dx$ from which the lemma follows. □

Every bounded function $\varphi : T^1(\text{int } D) \longrightarrow \mathbb{R}$ can uniquely be extended to a Γ–invariant bounded function on $T^1\tilde{M}$ vanishing on $\bigcup_{\psi\in\Gamma} T^1(\psi\partial D)$; we denote this extension again by φ . If φ is continuous then for every $\psi \in \Gamma$ the restriction of its extension is continuous on $T^1(\psi(\text{int } D))$.

Let $v \in T^1(\text{int } D)$ and let B be an open, relativ compact ball about v in $W^{su}(v)$ with $\mu^{su}(B) = 1$. The next lemma is a slight generalization of corollary 2.2.

<u>Lemma</u> 5.5.: If φ is continuous on $T^1(\text{int } D)$, then for every $\varepsilon > 0$ there is a number $t(\varepsilon) > 0$ such that $\left| \int_B \varphi(\phi^t v) d\mu^{su}(v) - \int_{T^1 D} \varphi d\lambda \right| < \varepsilon$ for all $t > t(\varepsilon)$.

<u>Proof</u>: By our assumption φ is bounded and hence $\sup\{ |\varphi(w)| \mid w \in T^1\tilde{M}\} = c < \infty$.

Let $\varepsilon > 0$; by the choice of D there is then an open connected subset C of $\text{int } D$ with smooth boundary $\partial C \subset \text{int } D$ such that $\lambda(T^1 D - T^1 C) < \varepsilon/8c$.

Since ∂D and ∂C are compact disjoint subsets of $\tilde{M}$, their distance is strictly positive. This means that there is a continuous function $\bar{\alpha} : D \longrightarrow [0,1]$ with the following properties:

i) $\bar{\alpha}(X) = 1$ for every $X \in C$

ii) $\bar{\alpha}(X) = 0$ for every $X \in \partial D$.

The lift of $\bar{\alpha}$ to T^1D then extends to a continuous Γ–invariant function α on $T^1\tilde{M}$ vanishing on $\bigcup_{\psi\in\Gamma} T^1(\psi\partial D)$ and $\alpha\varphi$ is the lift to $T^1\tilde{M}$ of a continuous function on T^1M which satisfies

iii) $\quad \left|\int_{T^1D} \alpha\varphi \, d\lambda - \int_{T^1D} \varphi \, d\lambda\right| < \varepsilon/8 .$

By the choice of C there is a continuous, Γ–invariant nonnegative function ψ on $T^1\tilde{M}$ with the following properties:

iv) $\quad \psi(v) = c$ for every $v \in T^1D - T^1C$.

v) $\quad \int_{T^1D} \psi \, d\lambda < \varepsilon/4$.

Again ψ is the lift of a continuous function on T^1M . Thus an application of corollary 2.2 shows the existence of a number $t(\varepsilon) > 0$ such that for all $t > t(\varepsilon)$ we have

vi) $\quad \left|\int_B \alpha\varphi(\phi^t w) d\mu^{su}(w) - \int_{T^1D} \varphi \, d\lambda\right| < \varepsilon/2$ and

vii) $\quad \left|\int_B \psi(\phi^t w) d\mu^{su}(w)\right| < \varepsilon/2$.

Since $|\varphi(w) - \alpha\varphi(w)| \le \psi(w)$ for all $w \in T^1\tilde{M}$ the lemma follows. □

Now we are ready for the proof of theorem A. Using theorem 5.1 above we argue by contradiction and assume that the top of the spectrum of the Laplacian Δ on $L^2(\tilde{M})$ is strictly larger than $-\frac{h^2}{4}$. Then there is a number $\varepsilon > 0$, a compact ball $K \supset D$ of radius $r > 0$ and a continuous function u on K with the following properties:

a) u vanishes on the boundary ∂K of K.

b) u is smooth on $K - \partial K$ and satisfies $\Delta u = (-\frac{h^2}{4} + \varepsilon)u$ on $K - \partial K$.

For the existence of such a function see [32].

Extend u by zero to a continuous function on $\tilde{M}$ and denote this extension again by u. Let φ be the lift of u to $T^1\tilde{M}$; since u^2 is of class C^1 lemma 5.4 shows $\int_{T^1\tilde{M}} X(\varphi^2)d\lambda = 0$. On the other hand, Green's formula implies $\int_{\tilde{M}} \Delta(u^2)dy = 0$ and consequently also $\int_{T^1\tilde{M}} \Delta(\varphi^2)d\lambda = 0$ (recall that $u = P \circ \varphi$ and hence $\Delta(\varphi^2)(v) = \Delta(u^2)(Pv)$ for all $v \in T^1(K - \partial K)$. Via approximation of φ^2 by functions of class C^2 we then obtain from lemma 5.3 that $\int_{T^1\tilde{M}} f < \nabla g, \nabla(\varphi^2) > d\lambda = 0$ as well.

Define

$$L\varphi(v) = \begin{cases} g(v)\Delta\varphi(v) + < \nabla g, \nabla\varphi >(v) & \text{if} \quad v \in T^1(K - \partial D - \partial K) \\ 0 & \text{otherwise}. \end{cases}$$

The above identity then yields

$$\int (f\varphi L\varphi + \frac{h^2}{4}\varphi^2)d\lambda = \int \varepsilon\varphi^2 d\lambda + \frac{1}{2}\int f < \nabla g, \nabla(\varphi^2) > d\lambda = \int \varepsilon\varphi^2 d\lambda = \alpha > 0 .$$

Now K is compact and hence there are finitely many isometries in Γ, say $\psi_1, \dots, \psi_k \in \Gamma$ for some $k > 0$, such that $E = \bigcup_{i=1}^{k} \psi_i(D)$ is a compact connected neighborhood of K in $\tilde{M}$. For $i \in \{1, \ldots, k\}$ we then obtain a continuous function u_i on D by $u_i(x) = u(\psi_i(x))$. The lift of u_i to T^1D then induces as before a bounded Γ–invariant function φ_i on $T^1\tilde{M}$. Similarly we obtain a bounded Γ–invariant function $L\varphi_i$ on $T^1\tilde{M}$ which satisfies $L\varphi_i(v) = L\varphi(d\psi_i(v))$ for every $v \in T^1(\text{int } D)$. For every $\psi \in \Gamma$ the restriction of φ_i and $\varphi_i L\varphi_i$ to $\psi(T^1(\text{int } D))$ is continuous.

Let $c = \sup\{(|f\varphi L\varphi + \frac{h^2}{4}\varphi^2| + \frac{h}{4}|X(\varphi^2)|)(v) \mid v \in T^1\tilde{M}\} \geq \alpha = \int \varepsilon\varphi^2 d\lambda > 0$ and let $R_0 > 0$ be sufficiently large that $R_0\alpha > 32\, c\beta$ where $\beta > 0$ is the diameter of E in $(\tilde{M},\text{dist})$. Let $v \in T^1D$ and let B be a compact ball about v in $W^{su}(v)$ with $\mu^{su}(B) = 1$ (i.e. the interior of B is dense in B and the boundary ∂B of B has vanishing Lebesgue measure). Choose a compact neighborhood C of B in $W^{su}(v)$ such that $\mu^{su}(C) \leq 1 + \alpha/16c \leq 2$. By standard comparison ([15]) there is then a number $R_1 > 0$ such that for every $t \geq R_1 - \beta$ the intersection with $PW^{su}(\phi^t v)$ of the β–neighborhood about $P\phi^t B$ in $(\tilde{M},\text{dist})$ is contained in $P\phi^t C$.

Recall that lemma 5.5 can be applied to the functions φ_i^2 and $\varphi_i L\varphi_i$; consequently there is a number $R_2 \geq R_1$ with the following properties:

i) $\quad |e^{-ht}\sum_{i=1}^{k}\int_{\phi^t B}(\varphi_i L\varphi_i + g\frac{h^2}{4}\varphi_i^2)d\lambda^{su} - \alpha| < \alpha/16$ for all $t \geq R_2$ (recall that $\sum_{i=1}^{k}\int_{T^1D}(f\varphi_i L\varphi_i + \frac{h^2}{4}\varphi_i^2)d\lambda = \alpha$ by the definition of the functions φ_i and $L\varphi_i$).

ii) $\quad |e^{-ht}\sum_{i=1}^{k}\int_{\phi^t B} g\,\frac{h}{4}\,X(\varphi_i^2)d\lambda^{su}\ | < \alpha/16$ for all $t \geq R_2$ (recall that $\int_{T^1 D} X(\varphi_i^2)d\lambda = 0$ for every $i \in \{1, \dots ,k\}$ by lemma 5.4).

Choose $R > R_2$ and define $\Omega_0 = \bigcup_{R \leq t \leq R_0 + R} \phi^t B$ and $\Gamma_0 = \{\psi \in \Gamma \mid \psi E \cap P\Omega_0 \neq \phi\}$.

For $i \in \{1, \dots ,k\}$ let moreover $\Omega_i = \{w \in W^u(v) \mid Pw \in \psi\psi^j D$ for some $\psi \in \Gamma_0\}$. Since $\psi_i D \subset E$ we have $\Omega_0 \subset \Omega_i$ and $\Omega_i \subset \bigcup_{R-\beta \leq t \leq R+R_0+\beta} \phi^t C$. Denote by θ the lift of the Busemann function θ_v to a function on $W^u(v)$. By the choice of R , R_0 and C we then have

iii) $\quad |\frac{1}{R_0}\sum_{i=1}^{k}\int_{\Omega_i} e^{-h\theta}(\varphi_i L\varphi_i + \frac{h^2}{4} g\varphi_i^2)d\lambda^u - \alpha| < \alpha/2$.

On the other hand, lemma 5.2 shows $\int_{\tilde{M}} (\eta \mathscr{L}_v \eta + \frac{h^2}{4}\eta^2)g_v dy \leq 0$ for every smooth function η on $\tilde{M}$ with compact support.

For $\psi \in \Gamma_0$ define now a continuous function $\zeta(\psi)$ on $\tilde{M}$ with support in ψE by $\zeta(\psi)(\psi x) = u(x)$. Write $L_v = g_v \mathscr{L}_v$ and apply the above inequality to $\zeta(\psi)e^{-h\theta_v/2}$; we obtain

$$\int [\zeta(\psi)e^{-h\theta_v/2}(L_v\zeta(\psi)e^{-h\theta_v/2}) + g_v e^{-h\theta_v}\zeta(\psi)^2]\,dy \leq 0 .$$

On the other hand, let $w \in W^u(v)$ be such that $Pw \in \psi\psi_i(D)$. Then $\varphi_i e^{-h\theta} L\varphi_i(w) = \zeta(\psi) e^{-h\theta_v} \operatorname{div}(g_v \nabla\zeta(\psi))(Pw) = \zeta(\psi) e^{-h\theta_v/2} \operatorname{div}(e^{-h\theta_v/2} g_v \nabla\zeta(\psi))(Pw) - \frac{h}{4} e^{-h\theta} gX(\varphi_i^2)(w)$ and consequently by the choice of Ω_0 and Ω_i and the above estimates we have

$$\frac{1}{R_0} \sum_{i=1}^{k} \int_{\Omega_i} e^{-h\theta}(\varphi_i L\varphi_i + g\frac{h^2}{4}\varphi_i^2)d\lambda^u \leq -\frac{1}{R_0} \sum_{i=1}^{k} \int_{\Omega_i} \frac{h}{4} e^{-h\theta} gX(\varphi_i^2) d\lambda^u < \alpha/4 .$$

But this is a contradiction to iii) and hence to the assumption that the top of the L^2–spectrum of $\tilde{M}$ is strictly larger than $-\frac{h^2}{4}$. Thus theorem A now follows from theorem 5.1 of Ledrappier.

Remark: The above arguments would simplify considerably if we could assume that M admits finite covers of arbitrarily large injectivity radius. This is for example true if M is homotopy equivalent to a compact locally symmetric space of negative curvature (then the fundamental group $\Gamma = \pi_1(M)$ of M is residually finite). We do not know any examples of compact negatively curved manifolds that violate the above property; however Gromov's work suggests (see for example [11]) that this property should only hold in very special cases.

6. Proof of corollary B

For $p \in M$ and $v \in T_p^1M$ let as before $U(v)$ be the automorphism of the orthogonal complement $v^\perp$ of v in T_p^1M which lifts to the second fundamental form of the horosphere $\theta_{\bar{v}}^{-1}(0)$ where $\bar{v}$ is a lift of v in $T^1\tilde{M}$. $U(v)$ is symmetric with positive eigenvalues; hence the same is true for $U(v) + U(-v)$.

Assume from now on that the mean curvature of the horopheres in $\tilde{M}$ is constant.

Lemma 6.1: The determinant of $U(v) + U(-v)$ is independent of $v \in T^1M$.

Proof: Let g^{ss} (resp. g^{su}) be the Riemannian metric on the leaves of W^{ss} (resp. W^{su}) which is the lift of the Riemannian metric on M. Denote by g the restriction of the Riemannian metric on T^1M to the bundle $E = TW^{ss} \oplus TW^{su}$. Since by our assumption g^{ss} (resp. g^{su}) induces (up to a constant) the measure μ^{ss} (resp. μ^{su}) on the leaves of W^{ss} (resp. W^{su}) the determinant of the identity $(E, g^{ss} + g^{su}) \longrightarrow (E,g)$ is constant.

Recall that E admits a smooth g–orthogonal decomposition $E = T^h \oplus T^v$ where T^v is tangent to the fibres of the fibration $T^1M \longrightarrow M$. For each $v \in T^1M$ the fibre T^h_v of T^h at v (resp. T^v_v of T^v at v) is canonically isomorphic to $v^\perp$ and with respect to this identification the metric g is just the product of the Riemannian metric on T^v and the Riemannian metric on $v^\perp$ (see [20]). Let $X_1, \dots, X_{n-1}$ be an orthonormal basis of $v^\perp$ of eigenvectors with respect to $U(-v)$. Then $(X_1, -U(-v)X_1), \dots, (X_{n-1}, -U(-v)X_{n-1})$, $(X_1, U(v)X_1), \dots, (X_{n-1}, U(v)X_{n-1})$ is a $g^{ss} + g^{su}$ – orthonormal basis of E_v, hence the determinant of the identity $(E, g^{ss} + g^{su}) \longrightarrow (E,g)$ at v equals the determinant of the matrix $\left[\begin{array}{c|c} Id & Id \\ \hline -U(-v) & U(v) \end{array}\right]$ where $-U(-v)$ is in diagonal form. But this determinant is just the determinant of $U(v) + U(-v)$.

Corollary 6.2: If $\dim M = 3$ then there is a constant $\alpha > 0$ such that $U(v) + U(-v) = \alpha\, Id$ for all $v \in T^1M$.

Proof: By 6.1 the determinant of $U(v) + U(-v)$ as well as its trace does not depend on $v \in T^1M$. Since $U(v) + U(-v)$ is a symmetric automorphism of the 2–dim. vector space $v^\perp$ the eigenvalues $\alpha_1(v)$, $\alpha_2(v)$ of $U(v) + U(-v)$ do not depend on $v \in T^1M$. Thus $\alpha_1(v) = \alpha_1$, $\alpha_2(v) = \alpha_2$ for some $\alpha_1 > 0$, $\alpha_2 > 0$ and $U(v) + U(-v) = \alpha\,\mathrm{Id}$ if and only if $\alpha_1 = \alpha_2 = \alpha$.

Recall the decomposition $TW^{su} \oplus TW^{ss} = T^h \oplus T^v$ from the proof of 6.1. Assume $\alpha_1 \neq \alpha_2$; then TW^{su} is a direct sum of two continuous line bundles T^1, T^2 on T^1M where T^i is spanned by elements of the form $(Y, U(v)Y)$ for eigenvectors Y of $U(v) + U(-v)$ with respect to the eigenvalue $\alpha_i (i = 1,2)$. Since the restriction to W^{su} of the canonical projection $T^h \oplus T^v \longrightarrow T^v$ is isomorphism this direct decomposition induces a decomposition of T^v as well. But T^v is the tangent bundle of the fibres of the 2–sphere bundle $T^1M \longrightarrow M$, hence such a decomposition is impossible. This implies the lemma. □

Using corollary 6.2, our corollary B from the introduction now follows from the following:

Lemma 6.3: Let M be a compact Riemannian manifold of negative curvature. If there is a number $\alpha > 0$ such that $U(v) + U(-v) = \alpha\,\mathrm{Id}$ for all $v \in T^1M$ then the curvature of M is constant.

Proof: Write $A(v) = U(v) + U(-v)$; if $A(v) = \alpha\,\mathrm{Id}$ then $U(v)$ and $U(-v)$ commute and

$$A^2(v) = \alpha^2\mathrm{Id} = U^2(v) + 2U(v)U(-v) + U^2(-v) = U^2(-v) - U^2(v) + 2\alpha\, U(v)\,.$$

Choose an orthonormal basis $E_1, \dots ,E_{n-1}$ of $v^{\perp}$ and extend $E_1, \dots ,E_{n-1}$ to a system of parallel vector fields $E_1(s), \dots ,E_{n-1}(s) (s \in \mathbb{R})$ along the geodesic γ_v. With respect to the basis $E_1(s), \dots ,E_{n-1}(s)$ of $(\phi^s v)^{\perp}$ the map $U(\phi^s v)$ (resp. $U(-\phi^s v)$) is represented by a symmetric matrix $B(s)$ (resp. $C(s)$). Clearly
$\frac{d}{ds}(B(s) + C(s)) = 0$.

Let R be the curvature tensor on M; for $v \in T^1M$ we obtain a symmetric automorphism R_v of $v^{\perp}$ by defining $R_v(X) = R(v,X)v$. Clearly $R_v = R_{-v}$. Now the Riccati equation ([28]) shows

$$\frac{d}{ds} C(-s)\Big|_{s=0} + C^2(0) + R_{-v} = 0 = \frac{d}{ds} B(s)\Big|_{s=0} + B^2(0) + R_v$$

or equivalently $C^2(0) - B^2(0) = \frac{d}{ds}(B(s) + C(s))\Big|_{s=0} = 0$. By the above equation this means $U(v) = \frac{\alpha}{2}\,\mathrm{Id}$ for all $v \in T^1M$ and consequently again by the Riccati equation the curvature on M is constant. □

7. Proof of theorem C

Let S, M be homotopy equivalent compact Riemannian manifolds of negative curvature. Assume that the marked length spectra of M and S coincide; then there is a C^1–time preserving conjugacy $\Lambda : T^1S \longrightarrow T^1M$ ([12]). The map Λ preserves the strong stable and the strong unstable foliations on T^1S resp. T^1M, moreover the strong stable foliation is the image of the strong unstable foliation under the flip $\mathcal{F}: w \longrightarrow -w$. Thus Λ can be composed with the time–t–map of the geodesic flow on

T^1M for a suitable $t \in \mathbb{R}$ in such a way that the resulting map, again denoted by Λ, commutes with the flips $\mathcal{F}$ on T^1S and T^1M.

Assume now that the metric and the topological entropy of the geodesic flow on T^1S coincide. Then the same is true for the metric and the topological entropy of the geodesic flow on T^1M; in particular by theorem A the mean curvature of the horospheres in $\tilde{S}$ and $\tilde{M}$ is constant.

Recall that the Riemannian metric on M and S lifts to Riemannian metrics g^i on the leaves of $W^i (i = su,u,s,ss)$. These metrics induce a set of conditionals for the Bowen–Margulis measure which are preserved by Λ up to a constant ([12]). Since the flip $w \longrightarrow -w$ maps g^{su} to g^{ss} and g^u to g^s and Λ commutes with the flips there is a number $\beta > 0$ such that for every $v \in T^1S$ the determinant at v of the restriction of $d\Lambda$ to $(TW^i(v),g^i)$ as a linear map onto $(TW^i(\Lambda(v)),g^i)$ equals $\beta (i = su,u,s,ss)$.

Recall from 6.1 that the determinant of $U(v) + U(-v)$ does not depend on $v \in T^1S$ (resp. $v \in T^1M$); we denote this constant value by α_S (resp. α_M). Since Λ preserves the volume forms on T^1S and T^1M (see [12]), the computation in the proof of 6.1 shows $\alpha_M \beta^2 \alpha_S^{-1} = 1$. We formulate this as a lemma:

<u>Lemma</u> 7.1: $\alpha_S = \beta^2 \alpha_M$.

For $x \in \tilde{S}$ and $v \in T^1\tilde{S}$ there is a unique $\psi_x(v) \in T^1_x\tilde{S}$ with $\pi(\psi_x(v)) = \pi(v)$. The map $\psi_x : v \longrightarrow \psi_x(v)$ is continuous and for every $w \in T^1\tilde{S}$ its restriction to $W^{su}(w)$

is a homeomorphism onto $T_x^1\tilde{S} - \{\psi_x(-w)\}$ that is absolutely continuous with respect to the Lebesgue measure classes. Its Jacobian with respect to the measures λ^{su} induced by g^{su} and the Lebesgue measure λ_x on the standard sphere $T_x^1\tilde{S} \sim S^{n-1}$ can be computed using lemma 3.4 of [12]:

Lemma 7.2: For every $x \in \tilde{S}$ and $v \in T_x^1\tilde{S}$ the Jacobian of $\psi_x|_{W^{su}(v)}$ at v with respect to the measures λ^{su}, λ_x equals α_S.

Similarly we obtain maps ψ_y and measures λ_y on $T_y^1\tilde{M}$ for $y \in \tilde{M}$.

Let $\tilde{\Lambda}$ be the lift of Λ to a time–preserving conjugacy $T^1\tilde{S} \longrightarrow T^1\tilde{M}$. For $x \in \tilde{S}$ and $y \in \tilde{M}$ define a continuous function $\kappa(x,y) : T_x^1\tilde{S} \longrightarrow \mathbb{R}$ by $\kappa(x,y)(v) = \theta_{-\tilde{\Lambda}(v)}(y)$. Then

Lemma 7.3: $\beta\,\lambda_x(T_x^1\tilde{S}) = \int e^{-h\kappa(x,y)} d\lambda_x$.

Proof: The map $f : T_x^1\tilde{S} \longrightarrow T_y^1\tilde{M}$ which is defined by $f(w) = \psi_y \circ \tilde{\Lambda}(w)$ is a homeomorphism; by 7.2 its Jacobian at v with respect to the measures λ_x and λ_y equals $\alpha_M e^{-h\kappa(x,y)(v)}\beta\alpha_S^{-1} = \beta^{-1}e^{-h\kappa(v)}$. But $\lambda_x(T_x^1\tilde{S}) = \lambda_y(T_y^1\tilde{M})$ whence the lemma. □

Corollary 7.4: $\beta = 1$, in particular $\alpha_S = \alpha_M$.

Proof: For $x \in \tilde{S}$ and $R > 0$ let $S(x,R)$ be the distance sphere of radius R about x and let $\lambda(x,R)$ be the Lebesgue measure on $S(x,R)$. For $v \in T_x^1\tilde{S}$ we obtain a homeomorphism $\nu(v,R)$ of $W^{su}(\phi^R v)$ into $S(x,R)$ by defining $\nu(v,R)(w) = P\phi^R\psi_x(w)$. For every $v \in T_x^1\tilde{S}$ the Jacobian of $\nu(v,R)$ at $\phi^R v$ with respect to the measures λ^{su} on $W^{su}(\phi^R v)$ and the measures $\lambda(x,R)$ on $S(x,R)$ converges to 1 as $R \longrightarrow \infty$, and this uniformly in v (see the discussion in section 1 of [13]). By lemma 7.2 this implies that the limit $\lim_{R\to\infty} e^{-hR}\lambda(x,R)S(x,R)$ exists and equals $\alpha_S^{-1}\lambda_x(T_x^1\tilde{S})$.

Using the above notations as well for $\tilde{M}$, choose $y \in \tilde{M}$ and define for every $R > 0$ a homeomorphism $F(R) : S(x,R) \longrightarrow S(y,R)$ by $F(R)(P\phi^R v) = P\phi^R\psi_y(\tilde{\Lambda}(\phi^R v))(v \in T_x^1\tilde{S})$. The above considerations then show that the Jacobian of $F(R)$ at $\phi^R v$ converges as $R \longrightarrow \infty$ to $e^{-h\kappa(x,y)(v)}$, and this uniformly in $v \in T_x^1\tilde{S}$.

Thus $\lim_{R\to\infty} \lambda(y,R)S(y,R)e^{-hR} = \alpha_M^{-1}\lambda_y(T_y^1\tilde{M}) =$
$\lim_{R\to\infty} e^{-hR} \int e^{-h\kappa(x,y)(v)}d\lambda(x,R)(P\phi^R v) = \alpha_S^{-1} \int e^{-h\kappa(x,y)}d\lambda_x = \alpha_S^{-1}\beta\lambda_x(T_x^1\tilde{S})$.
Since $\lambda_x(T_x^1\tilde{S}) = \lambda_y(T_y^1\tilde{M})$ this shows $\alpha_M^{-1} = \alpha_S^{-1}\beta$; on the other hand $\alpha_S = \beta^2\alpha_M$ by lemma 7.1 and consequently $\beta = \beta^2$, i.e. $\beta = 1$ as claimed. □

To finish the proof of theorem C we need the following result from linear algebra:

Lemma 7.5: Let A be a symmetric positive definite real (n,n)–matrix with $\det A = 1$. Then $\operatorname{tr} A \geq n$ with equality only for $A = \mathrm{Id}$.

Proof: The matrix A has n real positive (not necessarily distinct) eigenvalues $0 < \alpha_1 \leq \dots \leq \alpha_n$. Then $\det A = \prod_{i=1}^{n} \alpha_i = 1$, i.e. $\sum_{i=1}^{n} \log \alpha_i = 0$, and $\operatorname{tr} A = \sum_{i=1}^{n} \alpha_i = n$. Thus it suffices to show the following:

(*) For $k > 0$ let $b_1, \dots, b_k \in \mathbb{R}$ be such that $\sum_{i=1}^{k} b_i = 0$. Then $\sum_{i=1}^{k} e^{b_i} \geq k$, with equality only if $b_i = 0$ for all $i \in \{1, \dots, k\}$.

But (*) is an easy consequence of the convexity of the exponential function whose proof will be omitted. □

As a corollary we obtain theorem C for manifolds of constant curvature:

Corollary 7.6: Let S be an n–dim. compact Riemannian manifold of constant negative curvature. If the marked length spectra of M and S coincide, then M and S are isometric.

Proof: We assume that the curvature of S equals -1 and we show that the same is true for M; the corollary then follows from Mostow's rigidity theorem ([27]). By 7.4, for every $v \in T^1M$ the determinant of the self–adjoint automorphism $U(v) + U(-v)$ of $v^{\perp}$ equals 2^{n-1} and its trace is $2(n-1)$. Thus 7.5 shows $U(v) + U(-v) = 2\,\mathrm{Id}$ for all $v \in T^1M$ and hence by 6.3 the curvature of M is constant -1. □

In the remainder of this section we show how the above argument can be modified to treat compact quotients of the complex hyperbolic space $\mathbb{C}H^m$.

Thus let S be a compact quotient of $\mathbb{C}H^m(\dim S = n = 2m)$, normalized in such a way that the maximum of the curvature of S equals -1 . Let Q_1 (resp. Q_2) be the restriction of the differential of the canonical projection $T^1S \longrightarrow S$ to TW^{su} (resp. TW^{ss}) .For every $v \in T^1S$ the restriction of Q_1 to T_vW^{su} (resp. Q_2 to T_vW^{ss}) is an isomorphism onto $v^\perp$ whose inverse we denote by Q_1^{-1} (resp. Q_2^{-1}) . Write $J = Q_2^{-1} \circ Q_1$.

For every $v \in T^1S$ there is an $n-2 = k-\dim.$ subspace E_v of T_vW^{su} whose nonvanishing elements are precisely those vectors X for which the sectional curvature of the plane in TS spanned by v and Q_1X equals -1 . The orthogonal complement V_v of E_v in T_vW^{su} contains all vectors X for which the sectional curvature of the plane spanned by v and Q_1X equals -4 (compare [12] and the references there). Then $TW^{su} = E \oplus V$ is a smooth g^{su}–orthogonal decomposition which is invariant under the action of the geodesic flow.

Let T^0 be the 1–dim. subbundle of $\mathbb{C}H^m$ which is spanned by the geodesic spray. Then $T^0 \oplus V \oplus JV$ is an integrable distribution in $TT^1\mathbb{C}H^m$ whose maximal integral manifolds are just the unit tangent bundles of the totally geodesic embedded hyperbolic planes of constant curvature -4 in $\mathbb{C}H^m$, the so called <u>$\mathbb{C}$–lines</u> (see [27]). The boundary of a $\mathbb{C}$–line in $\partial\,\mathbb{C}\,H^m$ is a smooth embedded circle, a so called <u>$\mathbb{C}$–circle</u> ([27]).

Lemma 7.7: Let $L \subset \mathbb{C}H^m$ be a $\mathbb{C}$–line ; then $\widetilde{\lambda}(T^1L)$ is the unit tangent bundle of a totally geodesic embedded plane in $\tilde{M}$.

Proof: Recall that there is a homeomorphism $f : \partial\tilde{S} \longrightarrow \partial\tilde{M}$ such that $\pi \circ \widetilde{\lambda} = f \circ \pi$. Since $\widetilde{\lambda}$ is of class C^1 , the smooth structure on $\partial\tilde{S}$ induces via f a C^1 – structure on $\partial\tilde{M}$ (see [12]). Thus the $\mathbb{C}$–circle ∂L that is the boundary of L in $\partial\mathbb{C}H^m$ is mapped via f to a C^1–embedded circle $f\partial L$ in $\partial\tilde{M}$ and $\widetilde{\lambda}(T^1L) = \{w \in T^1\tilde{M} \mid \pi(w) \in f\partial L, \pi(-w) \in f\partial L\}$.

Recall that the tangent bundle $TT^1\tilde{M}$ of $\tilde{M}$ admits a smooth direct orthogonal decomposition $TT^1\tilde{M} = T^0 \oplus T^h \oplus T^v$; here the <u>vertical bundle</u> T^v is tangent to the fibres of the fibration $T^1\tilde{M} \longrightarrow \tilde{M}$ and T^0 is spanned by the geodesic spray (see [12], [20]). Using these notations as well for $\mathbb{C}H^m$ we then have $d\widetilde{\lambda}(T^h \oplus T^v) = T^h \oplus T^v$ (compare [12]). The restriction of the canonical projection $Q^h : T^h \oplus T^v \longrightarrow T^h$ to TW^{su} is an isomorphism; since V is a 1–dim. subbundle of TW^{su} this means that for every $w \in T^1\mathbb{C}H^m$ the dimension of the intersection $d\widetilde{\lambda}((V \oplus JV)_w) \cap T^v_{\widetilde{\lambda}(w)}$ is at most one.

Now $Jd\widetilde{\lambda}(V) = d\widetilde{\lambda}(JV)$ by 4.2 of [12]; consequently since V is a 1–dim. subbundle of TW^{su} there is for $0 \neq X \in V$ a number $\alpha(X) \in \mathbb{R}$ such that $Jd\widetilde{\lambda}(X) = \alpha(X)d\widetilde{\lambda}(JX)$. Together with the above considerations this shows that $d\widetilde{\lambda}(V \oplus JV) \cap T^v$ is a 1–dim. continuous subbundle of T^v which we denote by Z .

But $T_wT^1L \cap T_wW^{su}(w) = V_w$ and $T_wT^1L \cap T_wW^{ss}(w) = JV_w$ for every $w \in T^1L$ (see [12]) and hence via integration of a vector field which is tangent to Z we obtain the following: Whenever $v \in \tilde{\Lambda}(T^1L)$ then $\{w \in T^1_{Pv}\tilde{M} \mid \pi(w) \in f\partial L\}$ is contained in $\tilde{\Lambda}(T^1L)$. On the other hand, $\tilde{\Lambda}(T^1L \cap W^u(v))$ is a C^1–embedded plane in $T^1\tilde{M}$ which is mapped via the projection P to a C^1–embedded plane H in $\tilde{M}$. By the above considerations, $\tilde{\Lambda}(T^1L)$ contains a C^1–circle bundle $B \subset T^1\tilde{M}|_H$. But $v \in \tilde{\Lambda}(T^1L)$ was arbitrary and $\tilde{\Lambda}(T^1L)$ is diffeomorphic to B; thus $\tilde{\Lambda}(T^1L) = B$. On the other hand, $\tilde{\Lambda}(T^1L)$ is invariant under the action of the geodesic flow and consequently $P\phi^t w \in H$ for all $w \in B$ and all $t \in \mathbb{R}$. This means $B = T^1H$ and moreover that H is totally geodesic embedded as claimed. □

For $v \in T^1\tilde{M}$ write now $\mathcal{E}_v = (d\tilde{\Lambda}(E))_v$ and $\mathcal{V}_v = (d\tilde{\Lambda}(V))_v$.

<u>Corollary</u> 7.8: For every $v \in T^1\tilde{M}$ the subspace $Q_1(\mathcal{V}_v)$ of $v^\perp$ is invariant under $U(v)$ and $U(-v)$.

<u>Proof</u>: Let L be a $\mathbb{C}$–line in $\mathbb{C}H^m$, let $w \in T^1L$, $v = \tilde{\Lambda}(w)$ and let $X \in \mathcal{V}_v$. Then the Jacobi field $t \longrightarrow A(t) = Q_1\phi^t(X)$ is tangent to the totally geodesic embedded plane $H = Pd\tilde{\Lambda}(T^1L) \subset \tilde{M}$ and the same is true for its covariant derivative $A'(t)$. Thus $A'(t) = \alpha(t)A(t)$ for some function $\alpha : \mathbb{R} \longrightarrow \mathbb{R}$; but also $A'(0) = U(v)A(0) = U(v)X$ which means that X is an eigenvector with respect to $U(v)$. Similarly we obtain the invariance of $Q_1\,\mathcal{V}_v$ under $U(-v)$. □

<u>Corollary</u> 7.9: The direct decomposition $v^\perp = Q_1(\mathcal{E}_v) \oplus Q_1(\mathcal{V}_v)$ is $<,>$ – orthogonal and invariant under $U(v)$ and $U(-v)$ as well as under

parallel transport along the geodesic γ_v .

Proof: By 7.7 and 7.8 we only have to show that $Q_1(\mathcal{X}_v)$ equals the $<,>$ – orthogonal complement $Q_1(\mathcal{V}_v)^{\perp}$ of $Q_1(\mathcal{V}_v)$ in $v^{\perp}$. But this follows from 7.8 and the considerations in [12]. □

By 7.8 there is for every $v \in T^1\tilde{M}$ a number $\alpha(v) > 0$ such that $Q_1 \mathcal{V}_v$ is an eigenspace of $U(v) + U(-v)$ with respect to the eigenvalue $\alpha(v)$.

Lemma 7.10: $\alpha(v) \leq 4$ for all $v \in T^1\tilde{M}$, with equaltiy only if
$U(v) + U(-v)\big|_{Q_1 \mathcal{X}_v} = 2\mathrm{Id}$.

Proof: Let $v \in T^1\tilde{M}$ be such that $b = \frac{1}{2}\alpha(v) \geq 2$, i.e. $b = 2(1+\varepsilon)$ for some $\varepsilon \geq 0$. Let $0 < b_1 \leq \dots \leq b_{n-2}$ be the remaining $n-2$ eigenvalues of $\frac{1}{2}(U(v) + U(-v))$; by 7.4 and properties of $\mathbb{C}H^m$ we then have $\prod_{i=1}^{n-2} b_i = (1+\varepsilon)^{-1}$, in particular $b_1 \leq 1$. Write $c_1 = b_1(1+\varepsilon)$ and $c_i = b_i$ for $i \geq 2$. Since the trace of $\frac{1}{2}(U(v) + U(-v))$ equals n we obtain $\sum_{i=1}^{n-2} c_i = (1+\varepsilon)b_1 + \sum_{i=2}^{n-2} b_i \leq \varepsilon + \sum_{i=1}^{n-2} b_i =$ $\varepsilon + (n-2) - 2\varepsilon = n-2-\varepsilon$. On the other hand $\sum_{i=1}^{n-2} c_i = 1$ and consequently $\varepsilon = 0$ and $b_1 = \dots = b_{n-2} = 1$ by lemma 7.5. Since $U(v) + U(-v)$ is symmetric with respect to the scalar product $<,>$ on $v^{\perp}$ this means that the $<,>$ – orthogonal complement $Q_1 \mathcal{X}_v = (Q_1 \mathcal{V}_v)^{\perp}$ of $Q_1 \mathcal{V}_v$ in $v^{\perp}$ equals the eigenspace of $U(v) + U(-v)$ with respect to the eigenvalue 2 , whence the lemma. □

With the above assumptions, let now $v \in T^1M$ be a periodic element of ϕ^t of period say $\tau > 0$. Let $X \in \mathscr{V}_v$ and for $t \in [0,\tau)$ let $Y(t) = Q_1 d\phi^t(X)$. Then $Y(t)$ is an eigenvector of $U(\phi^t v)$ with respect to the eigenvalue $\xi(t)$ and an eigenvector of $U(-\phi^t v)$ with respect to the eigenvalue $\zeta(t)$; clearly $\xi(t) + \zeta(t) = \alpha(\phi^t v)$. Now $d\phi^\tau(X) = d\mathfrak{X} \circ d\phi^\tau \circ d\,\mathfrak{X}^{-1}(X) = e^{2\tau}X$ and consequently $\int_0^\tau \frac{d}{dt} \log ||Y(t)||^2 \, dt = 4\tau$. But $\frac{d}{dt} \log ||Y(t)||^2 = 2\zeta(t)$ shows $\int_0^\tau \zeta(t)dt = 2\tau$, similarly we obtain $\int_0^\tau \xi(t)dt = 2\tau$. Thus $\int_0^\tau \alpha(\phi^t v) = 4\tau$ which implies by lemma 7.9 that $\alpha(\phi^t v) = 4$ for every $t \in [0,\tau]$, moreover $Q_1 \mathscr{E}_{\phi^t v}$ is an eigenspace of $U(\phi^t v) + U(-\phi^t v)$ with respect to the eigenvalue 2. But periodic orbits of ϕ^t are dense in T^1M and thus by continuity $v^\perp = Q_1(\mathscr{E}_v) \oplus Q_1(\mathscr{V}_v)$ is a decomposition into the eigenspaces of $U(v) + U(-v)$ with respect to the eigenvalues 2 and 4 for every $v \in T^1M$. By 7.9 the arguments in the proof of 6.3 can now be applied separately to the distribution $\mathscr{E}$ and $\mathscr{V}$ and show that for every $v \in T^1M$ and every $0 \neq X \in \mathscr{E}_v$ (resp. $0 \neq X \in \mathscr{V}_v$) the curvature of the plane in TM spanned by v and X equals -1 (resp. -4). But this means that M is locally symmetric (see [28]) and hence theorem C for quotients of $\mathbb{C}H^m$ now follows from the Mostow rigidity theorem.

Appendix A

In this appendix we denote by $\tilde{M}$ an arbitrary simply connected Riemannian manifold of bounded negative sectional curvature. Let Δ be the Laplacian on $\tilde{M}$ and consider a differential operator $\mathscr{L}$ on $\tilde{M}$ which acts on functions u of class C^2 via

$$(*) \qquad u \longrightarrow \mathscr{L}(u) = \psi\Delta(u) + \langle X, \nabla u \rangle$$

where as usual ∇u is the gradient of u and

i) ψ is a smooth function on $\tilde{M}$ with range in a compact subinterval of $(0,\infty)$.

ii) X is a vector field on $\tilde{M}$ of uniformly bounded norm.

The following lemma is a consequence of 2.3 of [5]:

Lemma A.1: Let $x_0 \in \tilde{M}$ and define $r(x) = \mathrm{dist}(x_0,x)(x \in \tilde{M})$. Then there is a constant $c > 0$ independent of x_0 such that $\mathscr{L}(\varphi \circ r) \leq \psi\varphi''(r) + c\varphi'(r)$ for every nondecreasing function $\varphi : \mathbb{R} \longrightarrow [0,\infty)$ vanishing identically on $(-\infty,1/2]$.

Proof: Recall that $\tilde{M}$ has bounded geometry; hence by 2.3 of [5] there is $\bar{c} > 0$ such that $\Delta(\varphi \circ r) \leq \varphi''(r) + \bar{c}\varphi'(r)$. Since $\mathscr{L}(\varphi \circ r) = \psi\Delta(\varphi \circ r) + \langle \nabla(\varphi \circ r), X \rangle$ the lemma follows from the choice of X. □

Denote by $\mathscr{L}^*$ the differential operator on $\tilde{M}$ that is formally adjoint to $\mathscr{L}$ with respect to the L^2–scalar product on the space of smooth functions on $\tilde{M}$ with compact support.

Let $u_0 : \tilde{M} \longrightarrow \mathbb{R}$ be continuous. A continuous function $u : \tilde{M} \times [0,T) \longrightarrow \mathbb{R}$ with $u(x,0) = u_0(x)$ for all $x \in \tilde{M}$ is a solution of the $\mathscr{L}$– Cauchy problem with initial data u_0 if

1) $u\big|_{\tilde{M} \times (0,T)}$ is of class C^2 in the space variable, of class C^1 in the time variable.

2) $\mathscr{L}u - \frac{\partial}{\partial t} u = 0$ on $\tilde{M} \times (0,T)$.

We call u a weak solution of the $\mathscr{L}$– Cauchy problem with intial data u_0 if

1′) $\frac{d}{dt} \int \varphi(x)u(x,t)dx = \int (\mathscr{L}^* \varphi)(x)u(x,t)dx$ for every smooth function φ on $\tilde{M}$ with compact support and for every $t \in (0,T)$.

A nonnegative measurable map $p : \tilde{M} \times \tilde{M} \times (0,\infty) \longrightarrow \mathbb{R}$ is called a (weak) fundamental solution of the $\mathscr{L}$– Cauchy problem if for every bounded continuous function u_0 on $\tilde{M}$ the function

$$u(x,t) = \begin{cases} \int_{\tilde{M}} p(x,y,t)u_0(y)dy & \text{for } t > 0 \\ u_0(x) & \text{for } t = 0 \end{cases}$$

is a (weak) solution of the $\mathscr{L}$– Cauchy problem with initial data u_0.

Recall from corollary 6.2 of [16] that the operator $\mathscr{L}$ induces a unique diffusion on $\tilde{M}$. This diffusion is a stochastic process which can be described as follows: Compactify $\tilde{M}$ by adding a point ω at infinity; $\overline{M} = \tilde{M} \cup \{\omega\}$ is naturally a topological space. Let $W(\tilde{M})$ be the set of all continuous maps $w : [0,\infty) \longrightarrow \overline{M}$ with $w(t) = \omega$ for all $t \geq \inf\{s \geq 0 \mid w(s) = \omega\} = \zeta(w)$.

Denote by $\mathfrak{B}$ (resp. $\mathfrak{B}_t$) the σ–algebra on $W(\tilde{M})$ generated by the Borel–cylinder sets (resp. the Borel cylinder sets up to time t) (compare [16] p. 189). The $\mathscr{L}$– diffusion is then determined by the unique family $\{P_x\}_{x \in \tilde{M}}$ of probability measures on $(W(\tilde{M}),\mathfrak{B})$ with the following properties:

i) $P_x\{w \mid w(0) = x\} = 1$ for all $x \in \tilde{M}$.

ii) $f(w(t)) - f(w(0)) - \int_0^t (\mathscr{L} f)(w(s))ds$ is a $(P_x,\mathfrak{B}_t)$ – martingale for every smooth function f on $\tilde{M}$ with compact support and every $x \in \tilde{M}$.

Let $x_0 \in \tilde{M}$ and let B be an open ball of radius $r \in (0,\infty)$ about x_0 in $\tilde{M}$. Then there is a unique fundamental solution q_B of the equation $\mathscr{L}_x - \frac{\partial}{\partial t} = 0$ on $B \times (0,\infty)$ vanishing on the boundary ∂B of B ([21] chapter IV).

Let $B_1, B_2, \ldots$ be an exhaustion of $\tilde{M}$ by open balls such that $\overline{B}_j \subset B_{j+1}$ and $\bigcup_{j=1}^{\infty} B_j = \tilde{M}$. Define

$$q_i(x,y,t) = \begin{cases} q_{B_i}(x,y,t) & \text{for } x,y \in B_i \\ 0 & \text{otherwise} \end{cases} .$$

By the maximum principle for parabolic differential equations ([30] section III) we have $q_i \geq 0$ and $q_{i+1} \geq q_i$ for all $i > 0$. Define $p(x,y,t) = \sup_i q_i(x,y,t)$.

Lemma A.2: For every $x \in \tilde{M}$ and every Borel set $A \subset \tilde{M}$, $t > 0$ we have
$$P_x\{w \mid w(t) \in A\} = \int_A p(x,y,t)dy .$$

Proof: For every $t > 0$, every $i > 0$ the function q_i induces an operator Q_t^i on $L^2(B_i)$ by $(Q_t^i f)(x) = \int q_i(x,y,t)f(y)dy$. If $f : B_i \longrightarrow \mathbb{R}$ is a continuous function vanishing near ∂B_i then the function $u : (x,t) \longrightarrow (Q_t^i f)(x)$ is a solution of the equation $\mathscr{L}_x - \frac{\partial}{\partial t} = 0$ on $B_i \times (0,\infty)$ vanishing on $\partial B \times (0,\infty)$ which satisfies $\lim_{t\to 0} u(x,t) = f(x)$. Since such a solution is unique ([21] chapter IV) we have in particular $q_i(x,y,t+s) = \int_{B_i} q_i(x,z,t)q_i(z,y,s)dz$ for all $x,y \in B_i$, $t,s > 0$. It follows from the maximum principle for parabolic differential equations ([30] secion III) that $q_i(x,y,t) > 0$ for all $x,y \in B_i$, $t > 0$ and also $\int q_i(x,y,t)dy \leq 1$.

Compactify B_i by adding a point β at infinity and define $W(B_i)$ as before. We then obtain a Markovian system of probability measures $\{\tilde{P}_x^i\}_{x\in B_i}$ on $W(B_i)$ by defining $\tilde{P}_x^i\{w \mid w(t) \in A\} = \int_A q_i(x,y,t)dy$. The measures $\{\tilde{P}_x^i\}_{x\in\tilde{M}}$ then describe the unique $\mathscr{L}$– diffusion on B_i ([16] chapter V, section 3).

For a path $w \in W(\tilde{M})$ with $w(0) = x \in B_i$ and $t > 0$ let
$\tau_i = \inf\{s \geq 0 \mid w(s) \in \tilde{M} - B_i\}$ and $t \wedge \tau_i(w) = \inf\{t, \tau_i(w)\}$. Then τ_i is a stopping time for $(W(\tilde{M}), \mathfrak{B})$ and consequently

$$f(w(t \wedge \tau_i(w))) - f(w(0)) - \int_0^{t \wedge \tau_i(w)} (\mathscr{L} f)(w(s))ds$$

is a $(P_x, \mathfrak{B}_t)$ – martingale for every $x \in B_i$ and every smooth function f with compact support in B_i .

Let $\{P_x^i\}_{x \in B_i}$ be the unique family of probability measures on $W(\tilde{M})$ which is defined by $P_x^i\{w \mid w(t) \in A\} = P_x\{w \mid w(t) \in A, t \leq \tau_i(w)\}$ where $x \in B_i$, $t > 0$ and $A \subset B_i$ is a Borel–set; by the above consideration these measures describe the $\mathscr{L}$– diffusion on B_i . Thus $P_x^i = \tilde{P}_x^i$ for all $x \in B_i$ and $i > 0$; since on the other hand clearly $P_x\{w \mid w(t) \in A\} = \sup_i P_x^i\{w \mid w(t) \in A\}$ we obtain $P_x\{w \mid w(t) \in A\} =$
$\sup_i \int_A q_i(x,y,t)dy = \int_A p(x,y,t)dy$ by Lebesgue's theorem of monotone convergence. This shows the lemma. □

<u>Remark</u>: As an increasing limit of continuous functions the function $p : \tilde{M} \times \tilde{M} \times (0,\infty) \longrightarrow (0,\infty)$ is measurable and lower semi–continuous.

<u>Lemma</u> A.3: The function p is a weak fundamental solution of the $\mathscr{L}$– Cauchy problem with the following properties:

i) $p(x,y,t) > 0$ for all $x,y \in \tilde{M}$, $t > 0$.

ii) $p(x,y,t+s) = \int_{\tilde{M}} p(x,z,t)p(z,y,s)dz$ for all $x,y \in \tilde{M}$, all $t,s > 0$.

iii) If $u : \tilde{M} \times [0,T) \longrightarrow \mathbb{R}$ is a bounded solution of the $\mathscr{L}$– Cauchy problem then $u(x,t) = \int p(x,y,t)u(y,0)dy$ for all $x \in \tilde{M}$, $t > 0$, in particulary $\int p(x,y,t)dy = 1$.

iv) If the unique extension of $\mathscr{L}$ to a linear operator on the Hilbert space $L^2(\tilde{M})$ is self–adjoint then $p(y,x,t) = p(x,y,t)$ for all $x,y \in \tilde{M}$, $t > 0$.

<u>Proof</u>: Since $\int q_i(x,y,t)dy \le 1$ for every $x \in \tilde{M}$ the functions $q_i(x,\cdot,t)$ converge to $p(x,\cdot,t)$ in $L^1(\tilde{M})$ by Lebesgue's theorem of monotone convergence. Let f (resp. φ) be a continuous (resp. smooth) function on $\tilde{M}$ with compact support contained in some ball B_i. Then $f,\varphi \in L^2(B_j)$ for all $j > i$ and consequently by Lebesgue's theorem, applied to the positive and negative part of f and φ, we have $\iint \varphi(x)q_j(x,y,t)f(y)dydx \longrightarrow \iint \varphi(x)p(x,y,t)f(y)dydx (j \longrightarrow \infty)$ in $L^2(\tilde{M} \times \tilde{M})$. Now for $j > i$ the function $u_j(x,t) = (Q_t^j f)(x)$ is a solution of the equation $\mathscr{L} - \frac{\partial}{\partial t} = 0$ on $B_j \times (0,\infty)$ and consequently $\frac{d}{dt}\int \varphi(x)(Q_t^j f)(x)dx = \int (\mathscr{L}^* \varphi)(x)(Q_t^j f)(x)dx$ $\longrightarrow \iint (\mathscr{L}^* \varphi)(x)p(x,y,t)f(y)dydx$. But this just means that the function $u(x,t) = \int p(x,y,t)f(y)dy$ is a weak solution of the $\mathscr{L}$– Cauchy problem. Moreover since p is defined by the $\mathscr{L}$– diffusion on $\tilde{M}$ the function u is continuous (see [16] chapter V).

Let $x \in B_i$ and let U be an open neighborhood of x in B_i. For $j > i$ we then have $1 \le \lim_{t \to 0} \int_U q_j(x,y,t)dy \le \lim_{t \to 0} \sup \int_U p(x,y,t)dy$; but $\int p(x,y,t)dy \le 1$ for all $x \in \tilde{M}$

and consequently $\lim_{t \to 0} \int_{\tilde{M}-U} p(x,y,t)dy = 0$. Since U was an arbitrary neighborhood of x it follows easily $\lim_{t \to 0} \int p(x,y,t)f(y)dy = f(x)$ for every continuous bounded function f on $\tilde{M}$. This shows that p is a weak fundamental solution of the $\mathscr{L}$– Cauchy problem. ii) and iv) follow from corresponding properties of the functions q_i.

For the verification of iii) we use the arguments of [5] (thm. 2.2).

Let $u : \tilde{M} \times [0,T) \longrightarrow \mathbb{R}$ be a bounded solution of the $\mathscr{L}$– Cauchy problem and define for $x \in \tilde{M}$ and $t > 0$ $\bar{u}(x,t) = \int p(x,y,t)u(y,0)dy$ and $\bar{u}(x,0) = u(x,0)$. Then $u - \bar{u}$ is a bounded continuous function on $\tilde{M} \times [0,T)$. Assume for simplicity that the function $u_0 : y \longrightarrow u(y,0)$ is nonnegative. Choose a nondecreasing function φ of class C^2 on $(0,\infty)$ such that $\varphi(s) = 0$ for $s \in (0,1/2)$ and $\varphi(s) = s$ for $s \geq 1$. Let $x_0 \in \tilde{M}$ and define $\rho(x) = \varphi(\mathrm{dist}(x_0,x))$. By A.1 there is $K > 0$ such that $\mathscr{L}\rho < K$. Let $N = \sup\{\,|(u-\bar{u})(x,t)|\;|\,(u,t) \in \tilde{M} \times [0,T)\}$ and let $R > 0$ be a large positive constant and choose $i > 0$ sufficiently large that $B(x_0,2R) \subset B_i$.

For $j > i$ let $\chi_j : B_j \longrightarrow [0,1]$ be a continuous function with compact support which satisfies $\chi_j(x) = 1$ for $x \in B_{j-1}$. Define a bounded function $u_j :\ B_j \times [0,\infty) \longrightarrow \mathbb{R}$ by $u_j(x,t) = \int q_j(x,y,t)\chi_j(y)u(y,0)dy$ for $t > 0$ and $u_j(x,0) = \chi_j(x)u(x,0)$. Then $u_j \longrightarrow \bar{u}$ pointwise on $B(x_0,R) \times [0,\infty)$.

Let $\varepsilon > 0$, let $x \in \bar{B}(x_0,R)$ and let $t \in [0,T]$. There is a number $j(x,t) > i$ such that $|\bar{u}(x,t) - u_j(x,t)| < \varepsilon/2$ for all $j \geq j(x,t)$. Then $|u_j(x,t) - u(x,t)| < N + \varepsilon/2$ and hence by continuity of u_j and u there is a neighborhood $U(x,t)$ of (x,t) in

$\tilde{M} \times [0,T]$ such that $|u_{j(x,t)}(y,s) - u(y,s)| < N + \varepsilon$ for all $(y,s) \in U(x,t)$. Now for $(y,s) \in U(x,t)$ the sequence of numbers $a_j = u_j(y,s)$ is monotonously increasing and consequently for every $j \geq j(x,t)$ we have $|a_j - u(y,s)| \leq \max\{|a_{j(x,t)} - u(y,s)|, |\bar{u}(y,s) - u(y,s)|\} < N + \varepsilon$. But this means $|u_j(y,s) - u(y,s)| < N + \varepsilon$ for all $(y,s) \in U(x,t)$ and all $j \geq j(x,t)$. By compactness of $\bar{B}(x_0,R) \times [0,T]$ there is then a number $j(\varepsilon) > 0$ such that $|u_j(x,t) - u(x,t)| < \varepsilon + N$ for all $(x,t) \in \bar{B}(x_0,R) \times [0,T]$ and all $j \geq j(\varepsilon)$.

Let $j \geq j(\varepsilon)$ and define $\nu(x,t) = u(x,t) - u_j(x,t) - \frac{N+\varepsilon}{R}(\rho + Kt)$; then $\nu \leq 0$ on $B(x_0,R) \times \{0\} \cup \partial B(x_0,R) \times [0,T)$ and consequently (see [5]) $|u(x,t) - u_j(x,t)| \leq \frac{N+\varepsilon}{R}(\rho(x) + Kt)$ for all $(x,t) \in B(x_0,R) \times [0,T)$ by the maximum principle. Since $\varepsilon > 0$ and $j \geq j(\varepsilon)$ was arbitrary this implies $|u(x,t) - \bar{u}(x,t)| \leq \frac{N}{R}(\rho(x) + K(t))$. Now $R > 0$ was arbitrary as well, hence $u = \bar{u}$ follows (compare [5]). This finishes the proof of the lemma. □

Remark: iii) shows in particular that we have $u(x) = \int p(x,y,t)u(y)dy$ for every bounded function u on $\tilde{M}$ which satisfies $\mathscr{L}u = 0$.

Appendix B

Resume the assumptions and notations of sections 1 – 4. Recall in particular from section 4 the definition of the operators $\mathcal{L}_v (v \in T^1\tilde{M})$ and of the probabiltiy measure P on the space Ω of continuous paths in T^1M that is invariant under the shift transformations. The purpose of this appendix is to show that the shift is ergodic with respect to P.

For this recall from appendix A that the $\mathcal{L}_v$ – Cauchy problem on $\tilde{M}$ admits a unique weak fundamental solution $\rho_v(x,y,t)$. Following Sullivan ([32]) we say that $\zeta \in \mathbb{R}$ belongs to the Green's region of $\tilde{M}$ if $\int_0^\infty e^{-\zeta t}\rho_v(x,y,t)dt < \infty$ for some pair $x,y \in \tilde{M}$, $x \neq y$. If ζ belongs to the Green's region then the function $G_\zeta : (x,y) \longrightarrow \int_0^\infty e^{-\zeta t}\rho_v(x,y,t)dt$ is finite for $x \neq y$ and defines a Green's function for the operator $\mathcal{L}_v - \zeta$. Recall from section 4 that the function φ_v is a positive $(-\frac{h^2}{4})$ – harmonic function on $\tilde{M}$, i.e. φ_v satisfies $(\mathcal{L}_v + \frac{h^2}{4})\varphi_v = 0$. This implies that the Green's region of $\tilde{M}$ contains the set $(-\frac{h^2}{4}, \infty) \subset \mathbb{R}$ (compare [32]). Moreover for $\zeta > -\frac{h^2}{4}$ the operator $\mathcal{L}_v - \zeta$ is weakly coercive (and of the class considered in [1], compare [1]) and consequently the Martin–boundary of $\mathcal{L}_v - \zeta$, i.e. the space of minimal ζ – harmonic functions on $\tilde{M}$, can naturally be identified with $\partial\tilde{M}$ ([1]).

The next lemma identifies one of these minimal ζ – harmonic functions.

Lemma B.1: Let $\zeta > -\frac{h^2}{4}$ and let $\alpha(\zeta) > 0$ be as in 4.4. Then $e^{-\alpha(\zeta)\theta}$ is a minimal ζ–harmonic function.

Proof: Let $K_y(y \in \tilde{M} \cup \partial\tilde{M})$ be the Martin kernel of $\tilde{M}$, normalized at a point $x_0 \in \tilde{M}$. Then there are unique positive measures ν_α, ν_β on $\partial\tilde{M}$ such that $e^{-\alpha(\zeta)\theta}(x) = \int_{\partial\tilde{M}} K_\xi(x)d\nu_\alpha(\xi)$, $e^{-\beta(\zeta)\theta}(x) = \int_{\partial\tilde{M}} K_\xi(x)d\nu_\beta(\xi)$ for all $x \in \tilde{M}$. Moreover for ν_α – almost every $\xi \in \partial\tilde{M}$ the ratio $e^{-\beta(\zeta)\theta}(x)/e^{-\alpha(\zeta)\theta}(x)$ converges as $x \longrightarrow \xi$ to the Radon–Nikodym derivative of ν_β with respect to ν_α at ξ (see [1]). On the other hand $e^{-\beta(\zeta)\theta}(x)/e^{-\alpha(\zeta)\theta}(x) \longrightarrow \infty$ whenever $x \longrightarrow \xi \in \partial\tilde{M} - \gamma_v(-\infty)$ which means that the measure ν_α is supported on $\gamma_v(-\infty)$. Thus $e^{-\alpha(\zeta)\theta}$ is minimal. □

Next we consider positive $(-\frac{h^2}{4})$ – harmonic functions on $\tilde{M}$. We want to show that $\varphi_v = e^{-h\theta/2}$ is minimal. Due to the following result of Sullivan ([32]) this is clear if $-\frac{h^2}{4}$ does not belong to the Green's region of $\tilde{M}$:

Lemma B.2: If there are two linearly independent nonnegative $(-\frac{h^2}{4})$ – harmonic functions on $\tilde{M}$ then $-\frac{h^2}{4}$ belongs to the Green's region.

Assume from now on that $-\frac{h^2}{4}$ belongs to the Green's region. Call a function $u : \tilde{M} \longrightarrow (-\infty,\infty]$ ζ – superharmonic $(\zeta \geq -\frac{h^2}{4})$ if

a) u is lower semicontinuous.

b) u is not identically ∞.

c) $u(x) \geq \int e^{\zeta t} p_v(x,y,t) u(y) dy$ for all $t \geq 0$, $x \in \tilde{M}$.

If u is a positive ζ – superharmonic function on $\tilde{M}$ and if $A \subset \partial \tilde{M}$ is any set then the <u>reduction</u> R_u^A of u on A is defined to be the infinum of the class Λ of all positive superharmonic functions v on $\tilde{M}$ which majorize u on a neighborhood A_0 of A in $\tilde{M} \cup \partial \tilde{M}$. If $u_1, u_2 \in \Lambda$ then so is $u_1 \wedge u_2 = \min\{u_1, u_2\}$ and consequently $R_u^A \leq u$. A point $\xi \in \partial \tilde{M}$ is called a <u>pole</u> for u if $u = R_u^{\{\xi\}}$ (see [6]).

<u>Lemma</u> B.3: $\gamma_v(-\infty)$ is a pole for φ_v.

<u>Proof</u>: By the definition of the topology of $\tilde{M} \cup \partial \tilde{M}$ there is for every neighborhood A_0 of ξ in $\tilde{M} \cup \partial \tilde{M}$ a number $c \in \mathbb{R}$ such that $A_0 \supset \bigcup_{s \leq c} \theta^{-1}(s)$. Since $R_u^A \geq R_u^B$ for $A \supset B$ it suffices (via renormalization) to show the following: If u is a positive $(-\frac{h^2}{4})$ – harmonic function on $\Omega = \tilde{M} - \bigcup_{s \leq 0} \theta^{-1}(s)$ such that $u(x) \longrightarrow 1$ for $x \longrightarrow \xi \in \theta^{-1}(0)$ then $u \geq \varphi_v$.

Choose $x_0 \in \theta^{-1}(0)$ and define for $R \geq 1$ $\Omega_R = \Omega \cap B(x_0, R)$. Via a small deformation of Ω_R we may assume that the boundary $\partial\Omega_R$ of Ω_R is smooth and contains $\theta^{-1}(0) \cap B(x_0, R)$ and that moreover $\bigcup_{R>1} \Omega_R = \Omega$, $\Omega_r \subset \Omega_R$ for $r \leq R$.

For $\zeta > -\frac{h^2}{4}$ the operator $\mathcal{L}_v - \zeta$ admits a Green's function on Ω_R, and the Martin boundary of Ω_R can naturally be identified with the topological boundary $\partial\Omega_R$. Fix a point $x_1 \in \Omega_1$ and let $K_y^{\zeta,R}$ be the Martin kernel of $\mathcal{L}_V - \zeta$ on Ω_R, normalized at $x_1 (y \in \Omega_R \cup \partial\Omega_R)$. Then there is a unique measure $\nu_{\zeta,R}$ on $\partial\Omega_R$ of total mass $e^{-\alpha(\zeta)\theta}(x_1)$ such that $e^{-\alpha(\zeta)\theta}(x) = \int K_\xi^{\zeta,R}(x) d\nu_{\zeta,R}(\xi)$ for all $x \in \Omega_R$. Define a measure $\bar{\nu}^{\zeta,R}$ on $\partial\Omega_R$ by

$$\frac{d\bar{\nu}^{\zeta,R}}{d\nu^{\zeta,R}}(\xi) = \begin{cases} 1 & \text{if} \quad \xi \in \partial\Omega_R \cap \theta^{-1}(0) \\ 0 & \text{otherwise} \end{cases}$$

and let $u_{\zeta,R}$ be the positive ζ-harmonic function on Ω_R corresponding to the measure $\bar{\nu}^{\zeta,R}$. Then $u_{\zeta,R}(x) \longrightarrow 1 = e^{-\alpha(\zeta)\theta}(\xi)$ if $x \longrightarrow \xi \in \partial\Omega_R \cap \theta^{-1}(0)$ and $u_{\zeta,R}(x) \longrightarrow 0$ if $x \longrightarrow \xi \in \partial\Omega_R - \theta^{-1}(0)$. By the maximum principle we have moreover $u_{\zeta,R} \le e^{-\alpha(\zeta)\theta}$.

For a fixed number $\zeta > -h^2/4$ the function $u_{\zeta,R} - u_{-h^2/4,R}$ is ζ-subharmonic on Ω_R and vanishes on $\partial\Omega_R$. Thus the maximum principle implies $u_{-h^2/4,R} \ge u_{\zeta,R}$.

Choose a sequence $R_i \subset (0,\infty)$ such that $R_i \longrightarrow \infty \ (i \longrightarrow \infty)$ and that the measures ν_{ζ,R_i} on $\partial\Omega_R \subset \tilde{M}$ converge weakly to a measure ν on the Martin boundary $\partial\Omega$ of Ω (compare [1] and [6] and recall that for $\zeta > -\frac{h^2}{4}$ the Martin boundary of Ω equals the topological boundary in $\tilde{M} \cup \partial\tilde{M}$). Then the functions u_{ζ,R_i} converge uniformly on compact subsets of Ω to a positive ζ-harmonic function u_ζ on Ω. Clearly $e^{-\alpha(\zeta)\theta} \ge u_\zeta$ and moreover $u_\zeta(x) \longrightarrow 1$ whenever $x \longrightarrow \xi \in \theta^{-1}(0)$. But $e^{-\alpha(\zeta)\theta}$ is minimal and consequently $u = e^{-\alpha(\zeta)\theta}$.

If u is any positive $(-\frac{h^2}{4})$ – harmonic function on Ω with $u(x) \longrightarrow 1$ for $x \longrightarrow \xi \in \theta^{-1}(0)$ then by the maximum principle $u(x) \geq u_{-h^2/4,R}(x)$ for all $x \in \Omega_R$, $R > 1$, in particular also $u(x) \geq u_{\zeta,R}(x)$. But this means $u \geq \lim_{i\to\infty} u_{\zeta,R_i} = e^{-\alpha(\zeta)\theta}$ and since $\zeta > -\frac{h^2}{4}$ was arbitrary also $u \geq \sup_{\zeta > -h^2/4} e^{-\alpha(\zeta)\theta} = \varphi_v$ as claimed. □

<u>Remark</u>: Since $R_u^A \geq R_u^B$ for $A \supset B$ B.3 also shows the following: If $\Omega \subset \tilde{M}$ is a domain with smooth boundary whose closure in $\tilde{M} \cup \partial \tilde{M}$ does not contain $\gamma_v(-\infty)$, then φ_v is smaller than every positive $(-\frac{h^2}{4})$ – harmonic function u on Ω which satisfies $u(x) \longrightarrow \varphi_v(x)$ for $x \longrightarrow \xi \in \partial\Omega \cap \tilde{M}$.

By our assumption $\mathscr{L} + \frac{h^2}{4}$ admits a Green's function on $\tilde{M}$; thus $\tilde{M}$ can be compactified by adding the <u>Martin boundary</u> $\partial_{\mathscr{L}} \tilde{M}$. Choose $x_0 \in \tilde{M}$ and let $K_\xi (\xi \in \tilde{M} \cup \partial_{\mathscr{L}} \tilde{M})$ be the Martin kernel at ξ, normalized at x_0.

<u>Lemma</u> B.4: Let U be an open neighborhood of $\xi \in \partial_{\mathscr{L}} \tilde{M}$ in $\tilde{M} \cup \partial_{\mathscr{L}} \tilde{M}$. Then the closure of $\tilde{M} - U$ in $\tilde{M} \cup \partial \tilde{M}$ does not contain a pole of K_ξ.

<u>Proof</u>: Assume to the contrary that there is an open neighborhood U of ξ in $\partial_{\mathscr{L}} \tilde{M}$ such that the closure of $\tilde{M} - U$ in $\tilde{M} \cup \partial\tilde{M}$ contains a pole z of K_ξ. Since the sets $U_\xi(\varepsilon,r) = \{y \in \tilde{M} \mid y \neq x_0, |K_\xi(x) - K_y(x)| < \varepsilon$ for all $x \in \tilde{M}$ with $\mathrm{dist}(x,x_0) < r\}$ form the basis for the filter induced on $\tilde{M}$ by open neighborhoods of ξ in $\tilde{M} \cup \partial_{\mathscr{L}} \tilde{M}$ there is an open subset V of $\tilde{M}$ with the following properties:

i) The closure of V in $\tilde{M} \cup \partial_{\mathcal{L}} \tilde{M}$ does not contain ξ.

ii) The closure of V in $\tilde{M} \cup \partial \tilde{M}$ contains z.

Let θ be a Busemann function at z and define for an integer $j > 0$ $V_j = V \cap \bigcup_{t<-j} \theta^{-1}(t)$.

V_j is an open subset of $\tilde{M}$ whose closure in $\tilde{M} \cup \partial \tilde{M}$ contains z. Since z is a pole for K_ξ the reduction of K_ξ on V_j equals K_ξ; thus there is a probability measure ν_i on the boundary ∂V_i of V_i in $\tilde{M}$ such that $K_\xi = \int_{\partial V_j} K_y d\nu_j(y)$ on $\tilde{M} - \bar{V}_i$ (see [1], [6]). Let ν be a cluster value in the weak topology of the measures ν_j as $j \longrightarrow \infty$ (we may assume $x_0 \notin V_j$ for all j). Then ν is supported on the intersection of $\partial_{\mathcal{L}} \tilde{M}$ with the boundary $\partial_{\mathcal{L}} V$ of V in $\tilde{M} \cup \partial_{\mathcal{L}} \tilde{M}$. But $\partial_{\mathcal{L}} \tilde{M} \cap \partial_{\mathcal{L}} V \subset \partial_{\mathcal{L}} \tilde{M} - \xi$ and $K_\xi = \int K_\zeta d\nu(\zeta)$, a contradiction which shows the lemma. □

<u>Corollary</u> B.5: φ_v is a minimal $(-\frac{h^2}{4})$ – harmonic function.

<u>Proof</u>: Let ν be the unique measure on $\partial_{\mathcal{L}} \tilde{M}$ such that $\varphi_v = \int K_\xi d\nu(\xi)$; we have to show that ν is supported at a single point ([6], [1]). By 4.6 φ_v has a pole at $\gamma_v(-\infty)$; thus for ν – almost every $\xi \in \partial_{\mathcal{L}} \tilde{M}$ the same is true for K_ξ. Let $\xi \in \partial_{\mathcal{L}} \tilde{M}$ be such a point and assume that φ_v is not a constant multiple of K_ξ. Then there is an open neighborhood U of ξ in $\tilde{M} \cup \partial_{\mathcal{L}} \tilde{M}$ such that the reduction of

φ_v on U does not coincide with φ_v .

By lemma B.3 the closure of $\tilde{M} - U$ in $\tilde{M} \cup \partial\tilde{M}$ does not contain $\gamma_v(-\infty)$; thus the closure of U in $\tilde{M} \cup \partial\tilde{M}$ contains an open neighborhood of $\gamma_v(-\infty)$. By the definition of the topology of $\tilde{M} \cup \partial\tilde{M}$ this means that U contains a cone $C = \{\gamma_{\overline{w}}(t) \mid P\overline{w} = p \in U , \measuredangle(\overline{w},w) < \varepsilon , \pi(w) = \gamma_v(-\infty) , t \in [0,\infty)\}$ and hence the reduction of φ_v on C does not coincide with φ_v . But the boundary of C is smooth except at the vertex p and consequently this contradicts the remark following lemma B.2 and shows the lemma. $\square$

<u>Remark</u>: Let $\tilde{M}$ be a simply connected Riemannian manifold of bounded negative curvature and assume that the mean curvature of the horospheres in $\tilde{M}$ is constant. Let $h > 0$ be this constant and let θ be a Busemann function in $\tilde{M}$. Then $e^{-h\theta/2}$ is a positive $(-\frac{h^2}{4})$ – harmonic function on $\tilde{M}$ (with respect to the Laplace operator Δ) . Thus $-\frac{h^2}{4}$ is contained in the Green's region; our arguments above show that the Martin boundary for the operator $\Delta + \frac{h^2}{4}$ has a natural identification with the ideal boundary $\partial\tilde{M}$. This generalizes a result of Sullivan ([31]).

Recall now from 4.5 that the Borel measure P on the space Ω of continuous paths $\xi : [0,\infty) \longrightarrow T^1M$ is invariant under the shift transformations $T^t : \xi \longrightarrow T^t\xi$ where $T^t\xi(s) = \xi(s + t)$. Using the notations of section 4 we are now ready to show:

<u>Proposition B.6</u>: The shift is ergodic with respect to P .

Proof: Let $A \subset \Omega$ be a measurable set which is invariant under the transformations $T^t(t \geq 0)$. Assume that P is normalized, i.e. $P(\Omega) = 1$; we then have to show that $\alpha = P(A)$ equals 0 or 1 .

Define a function $\psi : T^1M \longrightarrow [0,1]$ by $\psi(v) = P^v(A)$. This function is measurable and lifts to a function on $T^1\tilde{M}$ which we denote again by ψ. By the definition of P and the T^t – invariance of A we have for every $v \in T^1\tilde{M}$ and every $t \geq 0$

$$\psi(v) = P^v\{\xi \mid \Pi T^t \xi \in A\} = \int (\varphi_v(w)/\varphi_v(v))p(v,w,t)\psi(w)d\lambda^u(w) . \qquad (*)$$

Let ψ_v be the projection to $\tilde{M}$ of the restriction of ψ to the unstable manifold $W^u(v)$. By (*) the function $\psi_v\varphi_v$ satisfies $(\mathcal{L}_v + \frac{h^2}{4})(\varphi_v\psi_v) = 0$. Thus by the maximum principle either ψ_v vanishes identically or $\psi_v > 0$. Assume $\psi_v > 0$; then $\varphi_v\psi_v$ is a positive $(-\frac{h^2}{4})$ – harmonic function for $\mathcal{L}_v$ which satisfies $\varphi_v\psi_v \leq \varphi_v$. But by B.5 φ_v is a minimal $(-\frac{h^2}{4})$ – harmonic function and consequently ψ_v is constant, i.e. ψ is constant along the leaves of the unstable foliation. But ψ is measurable and the unstable foliation of T^1M is ergodic, hence ψ is constant almost everywhere on T^1M with respect to the Lebesgue measure. Clearly this constant equals $\alpha = P(A)$.

Let $R_t : \Omega \longrightarrow T^1M$ be as in the proof of 4.5 $(t \in (0,\infty))$. Then the finite intersections of sets of the form $R_t^{-1}(B)(B \subset T^1M$ Borel, $t \in (0,\infty))$ form a $\cap$ – stable generator for the σ–algebra on Ω. Thus under the assumption $\alpha \in (0,1)$ there are for every $\varepsilon > 0$ Borel sets $B_1^i, \dots, B_k^i \subset T^1M$ and numbers $t_1^i, \dots, t_k^i \in (0,\infty)$ $(k > 0$ and $i = 1, \dots, \ell)$ with the following properties:

i) The sets $B_i = \bigcap_{j=1}^{k} R_{t_j^i}^{-1}(B_j^i)$ are pairwise disjoint.

ii) $P(\bigcup_{i=1}^{\ell} B_i) > 1 - \alpha - \varepsilon$

iii) $P(A \cap (\bigcup_{i=1}^{\ell} B_i)) < \varepsilon$.

But since ψ is constant on T^1M we have by the Markov property and the definition of P that $P(A \cap B_i) = \alpha P(B_i)$ for every $i \in \{1, \dots, \ell\}$, i.e. $P(A \cap (\bigcup_{i=1}^{\ell} B_i)) = \alpha P(\bigcup_{i=1}^{\ell} B_i)$ .If we choose $\varepsilon < \alpha(1-\alpha)/(1+\alpha)$, this is a contradiction and hence the proposition follows. $\square$

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