On the set of complex points of a 2-sphere

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1. Introduction

Let M be a 2-dimensional C^1-smooth manifold in $\mathbb{C}^2$. A point p on M is called a *complex point* if the tangent plane T_pM to M at p is a complex line. Denote by $\mathcal{E}$ the set of all complex points on M. If M is smooth enough and in a general position, then the set $\mathcal{E}$ consists of isolated points. In this case the topology of M can be described in terms of the local behaviour of M near the points of $\mathcal{E}$ (see [L]). The structure of the set M near the points in $\mathcal{E}$ plays a key roll in different questions of complex analysis (see, for example, [B], [BK], [K], [N], [Wi] and [J]). In this paper we study the structure of the set $\mathcal{E}$ in the case when M is a 2-dimensional sphere, denoted by S in what follows, embedded into the boundary ∂G of a C^∞-smooth strictly pseudoconvex domain G in $\mathbb{C}^2$ (this case is important for applications as was shown in [El] and [Er]). Our goal here is to describe the set $\mathcal{E}$ depending on the smoothness of S. Recall, that a manifold is said to be of class C^{2-} if it can be represented locally as the graph of a function that belongs to the class $\mathrm{Lip}^{1,\alpha}$ for each positive $\alpha < 1$. Our main result can now be formulated as follows.

Theorem. *Let G be a strictly pseudoconvex domain in $\mathbb{C}^2$ with C^∞-smooth boundary ∂G. Let S be a 2-dimensional sphere embedded into ∂G. Then, depending on the smoothness of S, the following holds:*

1) If S is of class C^2, then the set $\mathcal{E}$ of complex points of S is contained in a C^1-smooth nonclosed curve $\gamma \subset S$.

2) There exists a 2-dimensional sphere $S \subset \partial G$ of class C^{2-} such that the set $\mathcal{E}$ contains a Jordan curve of positive 2-dimensional measure.

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2. Proof of the first part of the theorem

We start with an argument which goes back to Bishop [B] (see also [J]). Namely, if p is a point of $\mathcal{E}$, then after a polynomial change of coordinates that moves p to the origin we can locally represent S as the disc

$$D = \{(z, f(z)) \in \mathbb{C}^2 : z \in \triangle\}$$

with $\triangle$ being a small disc centred at the origin and f being a complex valued C^2-smooth function. Moreover, after this change of coordinates the function f will have the special form

$$f(z) = \frac{1}{2}|z|^2 - \beta \operatorname{Re} z^2 + o(|z|^2), \quad \beta \geq 0$$

near zero. Recall, that zero is called an *elliptic point* if $0 \leq \beta < \frac{1}{2}$, a *hyperbolic point* if $\beta > \frac{1}{2}$ and a *parabolic point* if $\beta = \frac{1}{2}$. Elliptic and hyperbolic points are always isolated in $\mathcal{E}$. In the case of a parabolic point we can use the real coordinates $z = x + iy$ and represent f as $f(z) = y^2 + o(|z|^2)$. Hence $\partial_{\bar{z}} f(z) = iy + o(|z|)$ and then, by the implicit function theorem, we obtain that $\sigma = \{z \in \Delta : \operatorname{Im} \partial_{\bar{z}} f(z) = 0\}$ is a C^1-smooth curve and locally $\mathcal{E} = \{(z, f(z)) : z \in \sigma \text{ and } \operatorname{Re} \partial_{\bar{z}} f(z) = 0\}$. Therefore, locally the set $\mathcal{E}$ is a closed subset of a C^1-smooth curve.

Since the set $\mathcal{E}$ is compact, the only obstruction for $\mathcal{E}$ to be a subset of a nonclosed C^1-smooth curve $\gamma \subset S$ is that there is a closed C^1-smooth curve $\Gamma \subset \mathcal{E}$.

Assume, to get a contradiction, that such a closed C^1-smooth curve $\Gamma \subset \mathcal{E}$ exists. Let Γ' be a complex tangential C^∞-smooth closed curve in ∂G (complex tangential here means that for each point $p \in \Gamma'$ the tangent line $T_p\Gamma'$ to Γ' at p is contained in the complex tangent plane $T_p^{\mathbb{C}}(\partial G)$ to ∂G at p) close enough to Γ in the C^1-metric. Then there is a small C^1-perturbation S' of S in ∂G such that $\Gamma' \subset S'$ and each point in Γ' is a complex point on S'. Moreover, one can choose S' to be C^∞-smooth in a neighbourhood of Γ'. For each point $p \in \Gamma'$, consider the unit vector $\vec{u}(p)$ tangent to Γ', the vector $i\vec{u}(p) \in T_p^{\mathbb{C}}(\partial G)$ and the unit vector $\vec{n}(p) \in T_p(\partial G)$ orthogonal to $T_p^{\mathbb{C}}(\partial G)$ and such that the vectors $(\vec{u}(p), i\vec{u}(p), \vec{n}(p))$ define the positive orientation of ∂G at the point p. Let $O(p)$ be the rotation of $T_p(\partial G)$ around the direction $\vec{u}(p)$ that transforms the vector $i\vec{u}(p)$ into the vector $\vec{n}(p)$. Using the tubular neighbourhood theorem (see, for example, Theorem 1.4 in [H]) we can change S' in a neighbourhood of Γ' to get a new 2-sphere, $S'' \subset \partial G$, C^∞-smooth near Γ' such that $\Gamma' \subset S''$ and $\vec{n}(p) \in T_p(S'')$ for each point $p \in \Gamma'$. It is easy to see that S'' is totally real near Γ'. Then we can perturb S'' slightly outside a small neighbourhood of Γ' to get a C^∞-smooth 2-sphere $\tilde{S} \subset \partial G$ in general position. To finish the proof of the first part of our theorem we use an argument of Gromov (see [G], p. 342). Namely, by the result of Bedford-Klingenberg [BK] and Kružilin [K], there is a smooth 3-ball $\mathcal{B}$ which is the disjoint union of holomorphic discs $\{D_\alpha\}$, such that $\partial\mathcal{B} = \tilde{S}$. By Chirka's theorem [C] we know that discs D_α are C^∞-smooth to the boundary ∂D_α near the totally real part of $\tilde{S}$ (i. e. outside of finitely many complex points of $\tilde{S}$) and, moreover, the boundary ∂D_α of each disc D_α is C^∞-smooth at this part of $\tilde{S}$ and transversal there to the distribution $\{T_p^{\mathbb{C}}(\partial G)\}$ of complex tangencies to ∂G. Consider a disc D_{α_0} from the family $\{D_\alpha\}$ such that its boundary ∂D_{α_0} "touches" the curve Γ' "for the first time" and let p be a point of $\partial D_{\alpha_0} \cap \Gamma'$. More precisely, let $D_{\alpha_0} \subset \mathcal{B}$ be a holomorphic disc with the property that $\partial D_{\alpha_0} \cap \Gamma' \neq \emptyset$ and such that for some connected component of the set $\mathcal{B}\backslash D_{\alpha_0}$, each holomorphic disc D_α, which is contained in this component, satisfies $\partial D_\alpha \cap \Gamma' = \emptyset$. Now we can see that, on the one hand, since the curves ∂D_{α_0} and Γ' are tangent to each other at the point p, and since the curve Γ' was chosen to be complex tangential, the curve ∂D_{α_0} is complex tangential at p. On the other hand, since the point p is contained in the totally real part of $\tilde{S}$, the curve ∂D_{α_0} has to be transversal to $T_p^{\mathbb{C}}(\partial G)$. This gives the desired contradiction and completes the proof of the first part of the theorem.

3. Proof of the second part of the theorem

We prove the second part of the theorem in several steps. First, we construct a special arc $E \subset \mathbb{R}^2_{x,y}$ of positive 2-dimensional measure. Then we define a function G on E such that $G \in C^{2-}(E)$ with the functions $G'_x(x,y) = y$ and $G'_y(x,y) = 0$ chosen to be the first derivatives of G on E. Next, following an idea of H. Whitney (see [Wh]), we contruct a nonconstant function $H \in C^{2-}(E)$ with $H'_x(x,y) = 0$ and $H'_y(x,y) = 0$. Using G and H we define a function $F \in C^{2-}(E)$ with $F'_x(x,y) = y$ and $F'_y(x,y) = 0$ which is zero at both endpoints of E. Then, using E, we contruct a Jordan curve $\tilde{E} \subset \mathbb{R}^2_{x,y}$ of positive 2-dimensional measure and, using F, we define a function $\tilde{F}$ on $\tilde{E}$ of class $C^{2-}(\tilde{E})$ with derivatives $\tilde{F}'_x(x,y) = y$ and $\tilde{F}'_y(x,y) = 0$. Next, applying Whitney's extension theorem to the function $\tilde{F}$, we contruct a 2-sphere $S^2 \subset \mathbb{R}^3_{x,y,z}$ which contains a Jordan curve of positive 2-dimensional measure such that at each point of this curve the tangent plane to S^2 coincides with the corresponding plane of the standard contact distribution in $\mathbb{R}^3_{x,y,z}$ and finally, using the Darboux theorem, we embed this sphere into the boundary bG of the given strictly pseudoconvex domain G.

1. Construction of the arc E. First, we define for each $\alpha \in (0,1)$ an auxiliary set

$$\mathbb{E}^\alpha = \left(\left[0, \frac{1-\alpha}{2}\right] \cup \left[\frac{1+\alpha}{2}, 1\right]\right) \times \left(\left[0, \frac{1-\alpha}{2}\right] \cup \left[\frac{1+\alpha}{2}, 1\right]\right) \cup$$

$$\left(\{0\} \times [0,1]\right) \cup \left([0,1] \times \left\{\frac{1+\alpha}{2}\right\}\right) \cup \left(\{1\} \times [0,1]\right).$$

We denote $A = (0,0), B = (1,0), Q_0 = [0, \frac{1-\alpha}{2}] \times [0, \frac{1-\alpha}{2}], Q_1 = [0, \frac{1-\alpha}{2}] \times [\frac{1+\alpha}{2}, 1], Q_2 = [\frac{1+\alpha}{2}, 1] \times [\frac{1+\alpha}{2}, 1]$ and $Q_3 = [\frac{1+\alpha}{2}, 1] \times [0, \frac{1-\alpha}{2}]$. Further, we denote $A_0 = A = (0,0), B_0 = (0, \frac{1-\alpha}{2}), A_1 = (0, \frac{1+\alpha}{2}), B_1 = (\frac{1-\alpha}{2}, \frac{1+\alpha}{2}), A_2 = (\frac{1+\alpha}{2}, \frac{1+\alpha}{2}), B_2 = (1, \frac{1+\alpha}{2}), A_3 = (1, \frac{1-\alpha}{2})$ and $B_3 = B = (1,0)$ (see the set $\mathbb{E}^\alpha$ on Fig. 1).

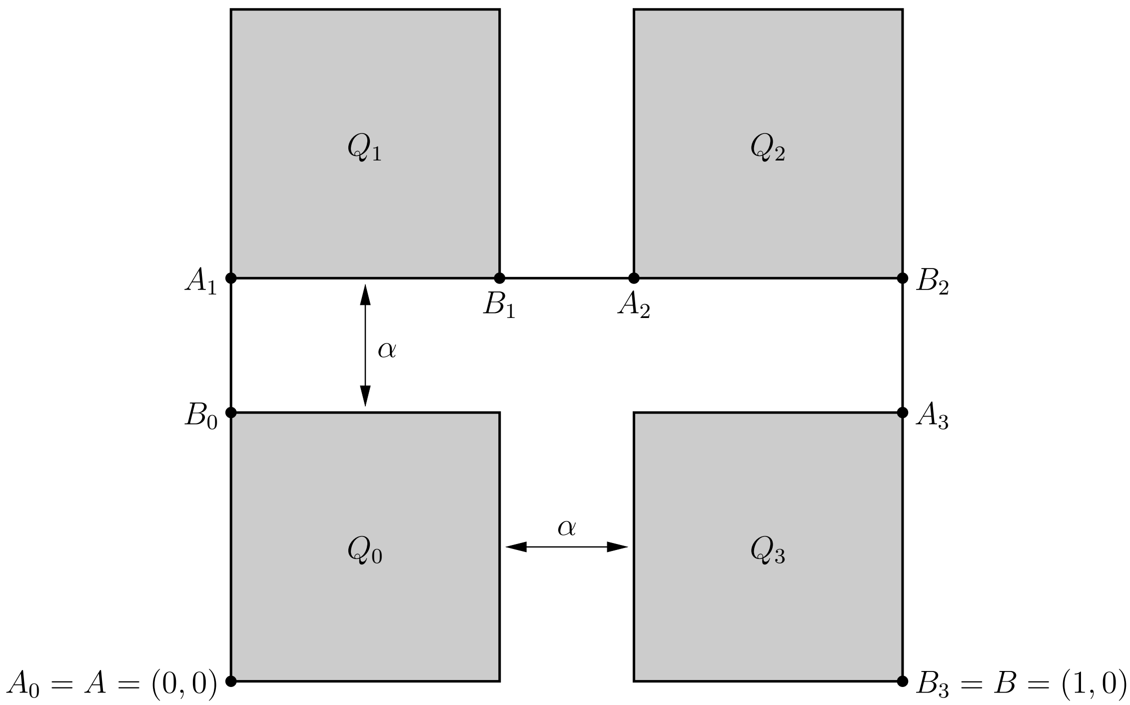

Figure 1: The set $\mathbb{E}^\alpha$

To define the set E we consider the sequence $\alpha_n = \frac{1}{(n+1)^2}, n = 1, 2, \ldots$. We construct the set E as the intersection of a decreasing sequence of compact sets E_n which will be defined inductively. We set $E_1 = \mathbb{E}^{\alpha_1}$. To define the set E_2 we consider the image $\tilde{\mathbb{E}}^{\alpha_2}$ of the set $\mathbb{E}^{\alpha_2}$ under the homothety with coefficient $\frac{1-\alpha_1}{2}$. Then for each $i = 0,1,2,3$ we consider the set $\mathbb{E}_i$ obtained from the set $\tilde{\mathbb{E}}^{\alpha_2}$ by an orthogonal transformation (if necessary) and translation in such a way that the image of the points A and B will coincide with the points A_i and B_i, respectively. It is easy to see that we need an orthogonal transformation only for $i = 0, 3$. The set E_2 is obtained from the set E_1 by substituting for each $i = 0,1,2,3$ the square Q_i by the corresponding set $\mathbb{E}_i$. For each $j = 0,1,2,3$ we denote by Q_{ij}, A_{ij} and

B_{ij} the images of the square Q_j and the points A_j, B_j in the corresponding set $\mathbb{E}_i$, respectively (the set E_2 is shown on Fig. 2).

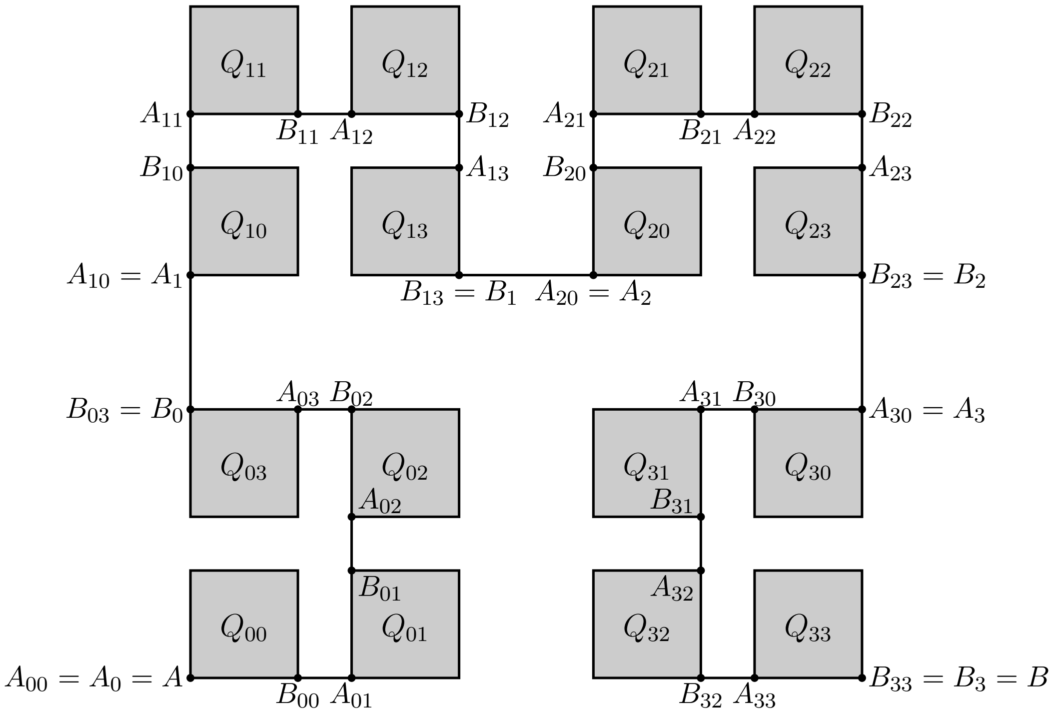

Figure 2: The set E_2

To describe the inductive step of our construction we assume that the set E_n is already constructed and define the set E_{n+1}. Consider the image $\tilde{\mathbb{E}}^{\alpha_{n+1}}$ of the set $\mathbb{E}^{\alpha_{n+1}}$ under the homothety with coefficient $\prod_{i=1}^{n} \left(\frac{1-\alpha_i}{2}\right)$. Then for each multiindex $(i_1, i_2, \ldots, i_n), i_j = 0, 1, 2, 3,\ j = 1, 2, \ldots, n$, consider the set $\mathbb{E}_{i_1 \ldots i_n}$ obtained from the set $\tilde{\mathbb{E}}^{\alpha_{n+1}}$ by an orthogonal transformation (if necessary) and translation in such a way that the image of the points A and B will coincide with the points $A_{i_1 \ldots i_n}$ and $B_{i_1 \ldots i_n}$, respectively. The set E_{n+1} is obtained from the set E_n by substituting each square $Q_{i_1 \ldots i_n}$ by the corresponding set $\mathbb{E}_{i_1 \ldots i_n}$. For each $i_j = 0, 1, 2, 3,\ j = 1, 2, \ldots, n+1$, we denote by $Q_{i_1 \ldots i_{n+1}}, A_{i_1 \ldots i_{n+1}}$ and $B_{i_1 \ldots i_{n+1}}$ the images of the square $Q_{i_{n+1}}$ and the points $A_{i_{n+1}}, B_{i_{n+1}}$ in the corresponding set $\mathbb{E}_{i_1 \ldots i_n}$, respectively. Note, that for each multiindex $(i_1, \ldots, i_n)$ one has $A_{i_1 \ldots i_n 0} = A_{i_1 \ldots i_n}$ and $B_{i_1 \ldots i_n 3} = B_{i_1 \ldots i_n}$.

Since $\{E_n\}$ is a decreasing sequence of compact sets, $E = \bigcap_{n=1}^{\infty} E_n$ is a nonempty compact subset of $\mathbb{R}^2_{x,y}$. It is easy to see that it is an arc (see the set E on Fig. 3).

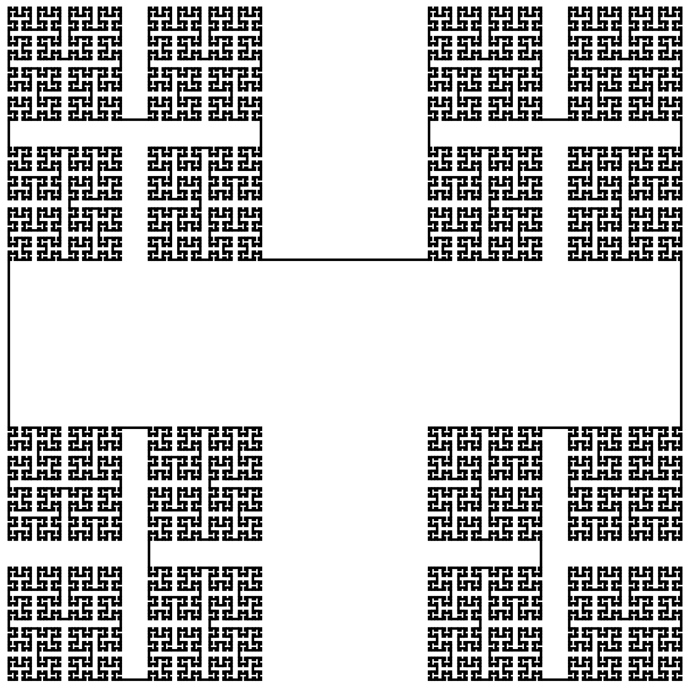

Figure 3: The set E

To estimate the area of the set E we observe that

$$\begin{aligned}\text{Area }(E_n) = (1-\alpha_n)^2 \text{ Area }(E_{n-1}) &= \prod_{k=1}^{n}(1-\alpha_k)^2 = \prod_{k=1}^{n}\left(1-\frac{1}{(k+1)^2}\right)^2 \\ &= \left(\frac{1}{2}\left(1+\frac{1}{n+1}\right)\right)^2 > \frac{1}{4}\end{aligned}$$

for every $n = 1, 2, \ldots$. Hence, the set E has a positive 2-dimensional measure (and, moreover, Area $(E) = \frac{1}{4}$).

2. Definition and properties of the function G. For each $n = 1, 2, \ldots$ let Ω_n be the connected component of the set $(0,1) \times (-1,1) \setminus E_n$ containing the square $(0,1) \times (-1,0)$ and let $J_n = \partial\Omega_n \cap ([0,1] \times [0,1])$. On each curve J_n we define a function G_n in the following way. For a point $p \in J_n$ we denote by J_n^p a part of J_n with initial point A and endpoint p and then we set $G_n(p) = \int\limits_{J_n^p} y dx$.

Now we define the function G on the arc E. Let p be a point of E. Consider a sequence of points $p_n \in J_n$, $n = 1, 2, \ldots$, such that $p_n \to p$, as $n \to \infty$ and set $G(p) = \lim\limits_{n\to\infty} G_n(p_n)$. Since, by Green's theorem, for a piece-wise smooth Jordan curve γ the integral $\int\limits_{\gamma} y dx$ represents the area of the domain bounded by γ, and since for each $k \geq 0$ one has Area $(\Omega_{n+k} \setminus \bar{\Omega}_n) \to 0$ as $n \to \infty$, closing up the curve $J_{n+k}^{p_{n+k}} - J_n^{p_n}$ by the segment $[p_{n+k}, p_n]$ we see that for each $k \geq 0$:

$$G_{n+k}(p_{n+k}) - G_n(p_n) - \int_{[p_n, p_{n+k}]} y dx \to 0, \text{ as } n \to \infty,$$

and therefore, by smoothness of the form ydx, we conclude that for each $k \geq 0$ we have $G_{n+k}(p_{n+k}) - G_n(p_n) \to 0$, as $n \to \infty$. This means that the function G is well defined on the arc E.

Next, we prove that $G \in C^{2-}(E)$ with $G'_x(x,y) = y$ and $G'_y(x,y) = 0$ chosen to be the first derivatives of G at each point $(x,y) \in E$. To do this rigorously we recall the definition of a function belonging to the class $C^{2-}(E)$ (this definition is due to H. Whitney. Further details can be found, for example, in [S]).

Definition. *Let E be a compact subset of $\mathbb{R}^2_{x,y}$ and let f be a function defined on E. We say that f belongs to the class $C^{2-}(E)$ if there exist bounded functions f'_x and f'_y defined on E with the property that for each $\varepsilon > 0$ there is a constant M such that*

$$|f(x+\Delta x, y+\Delta y) - f(x,y) - f'_x(x,y)\Delta x - f'_y(x,y)\Delta y| \leq M(|\Delta x| + |\Delta y|)^{2-\varepsilon} \quad (1)$$

for all $(x,y), (x+\Delta x, y+\Delta y) \in E$.

To prove that $G \in C^{2-}(E)$ we consider two points $p, p+\Delta p \in E$. Since the function G obviously is smooth on each of the segments $[B_{i_1 \dots i_n}, A_{i_1 \dots (i_n+1)}]$, $i_n = 0, 1, 2$, and satisfies condition (1) with $G'_x(x,y) = y$ and $G'_y(x,y) = 0$ there, it is enough to verify that the condition (1) holds true for points p and $p + \Delta p$ of the Cantor set $\mathbb{Q} \overset{def}{=} \bigcap_{n=1}^{\infty} \bigcup_{(i_1, \dots, i_n)} Q_{i_1 \dots i_n}$. Consider a number m such that $p, p+\Delta p$ belong to a square $Q_{i_1 \dots i_m}$ for some indexes $(i_1, \dots, i_m)$, but not to a smaller square $Q_{i_1, \dots i_m i_{m+1}}$, $i_{m+1} = 0, 1, 2, 3$. Since p and $p+\Delta p$ belong to different squares $Q_{i_1 \dots i_m i_{m+1}}$ and $Q_{i_1 \dots i_m i'_{m+1}}$, it follows that the distance between these points is not less than the minimal distance between $Q_{i_1 \dots i_m i_{m+1}}$ and $Q_{i_1 \dots i_m i'_{m+1}}$, that is,

$$\begin{aligned} |\Delta p| \geq \alpha_{m+1} \left(\frac{1-\alpha_1}{2}\right) \cdots \left(\frac{1-\alpha_m}{2}\right) &= \frac{1}{(m+2)^2} \cdot \frac{1}{2^m} \prod_{k=1}^{m} \left(1 - \frac{1}{(k+1)^2}\right) \\ &= \frac{1}{2^{m+1}} \frac{1}{(m+1)(m+2)} \end{aligned} \quad (2)$$

Now we estimate the left hand side of the condition (1) for our function G

$$\mathcal{L}_G(p, p+\Delta p) \overset{def}{=} G(p+\Delta p) - G(p) - G'_x(p)\Delta x - G'_y(p)\Delta y = G(p+\Delta p) - G(p) - y\Delta x,$$

where $p = (x, y)$ and $\Delta p = (\Delta x, \Delta y)$. It is easy to see that

$$\int_{[p,p+\Delta p]} y dx = y\Delta x + \frac{1}{2}\Delta x \Delta y,$$

hence

$$|\mathcal{L}_G(p, p + \Delta p)| \leq |G(p + \Delta p) - G(p) - \int_{[p,p+\Delta p]} y dx| + \frac{1}{2}|\Delta x||\Delta y| \qquad (3)$$

It follows from the definition of function G and Green's theorem that $G(p + \Delta p) - G(p) - \int_{[p,p+\Delta p]} y dx$ represents the sum (with signes) of the areas of domains bounded by the part of the arc E from the point p to the point $p + \Delta p$ and by the segment $[p, p + \Delta p]$. Since all these domains are contained in the square $Q_{i_1 \ldots i_m}$, we conclude that

$$\left| G(p + \Delta p) - G(p) - \int_{[p,p+\Delta p]} y dx \right| \leq \text{Area } (Q_{i_1 \ldots i_m})$$
$$= \left(\frac{1 - \alpha_1}{2}\right)^2 \ldots \left(\frac{1 - \alpha_m}{2}\right)^2 = \frac{1}{2^{2m+2}} \left(\frac{m+2}{m+1}\right)^2 \qquad (4)$$

Since $p, p + \Delta p \in Q_{i_1 \ldots i_m}$, it follows that $|\Delta x|$ and $|\Delta y|$ can be estimated from above by the length of the side of $Q_{i_1 \ldots i_m}$, that is, by $\frac{1}{2^{m+1}} \left(\frac{m+2}{m+1}\right)$. Therefore, we have by (3) and (4) that

$$|\mathcal{L}_G(p, p + \Delta p)| \leq \frac{3}{2} \; \frac{1}{2^{2m+2}} \left(\frac{m+2}{m+1}\right)^2 \qquad (5)$$

Finally, we conclude from the estimates (2) and (5) that to prove that G satisfies condition (1) we only need to verify that for each $\varepsilon > 0$ there is a constant M such that

$$\frac{3}{2} \; \frac{1}{2^{2m+2}} \left(\frac{m+2}{m+1}\right)^2 \leq M \left(\frac{1}{2^{m+1}} \cdot \frac{1}{(m+1)(m+2)}\right)^{2-\varepsilon} \quad \text{as } m \to \infty$$

which is equivalent to the inequality

$$\frac{1}{(2^\varepsilon)^{m+1}} \leq M\frac{2}{3} \left(\frac{1}{(m+1)(m+2)}\right)^{2-\varepsilon} \left(\frac{m+1}{m+2}\right)^2 \quad \text{as } m \to \infty.$$

The last inequality is obviously satisfied, since the left hand side tends to zero much faster than the right hand side, as $m \to \infty$. This proves that the function G belongs to the class $C^{2-}(E)$.

3. Definition and properties of the functions H and F. First, we define the function H on the Cantor set $\mathbb{Q}$. Each point p in this set is uniquely determined as the intersection of the decreasing sequence $Q_{i_1} \supset Q_{i_1 i_2} \supset Q_{i_1 i_2 i_3} \supset \dots$ of the squares $Q_{i_1 \dots i_n}$. Then, we define the value of H at the point p as $H(p) = \sum\limits_{n=1}^{\infty} \frac{i_n}{4^n}$. It is easy to see that for each $i_n = 0, 1, 2$ one has $H(A_{i_1 \dots i_{n-1}(i_n+1)}) = \sum\limits_{k=1}^{n} \frac{i_k}{4^k} + \frac{1}{4^n}$ and $H(B_{i_1 \dots i_{n-1} i_n}) = \sum\limits_{k=1}^{n} \frac{i_k}{4^k} + \sum\limits_{k=n+1}^{\infty} \frac{3}{4^k} = \sum\limits_{k=1}^{n} \frac{i_k}{4^k} + \frac{1}{4^n}$, therefore, we can extend the function H as a constant to each segment $[B_{i_1 \dots i_{n-1} i_n}, A_{i_1 \dots i_{n-1}(i_n+1)}]$, $i_n = 0, 1, 2$, with the value $\sum\limits_{k=1}^{n} \frac{i_k}{4^k} + \frac{1}{4^n}$ there. This defines the function H on the whole set E.

Now we show that $H \in C^{2-}(E)$ with the functions $H'_x(x, y) = 0$ and $H'_y(x, y) = 0$ chosen to be the first derivatives of H on E. We proceed in the same way as in the case of the function G, namely, we consider two points $p, p + \Delta p \in E$. Since, by definition, H is a constant on each of the intervals constituting the set $E \setminus \mathbb{Q}$, we only need to verify that the function H satisfies condition (1) for points $p, p + \Delta p \in \mathbb{Q}$. Let, as above, m be a number such that $p, p + \Delta p \in Q_{i_1 \dots i_m}$, but $p, p + \Delta p \notin Q_{i_1 \dots i_m i_{m+1}}$ for any $i_{m+1} = 0, 1, 2, 3$. Then the definition of H gives us that $|H(p + \Delta p) - H(p)| \leq \frac{1}{4^m}$. Hence, by estimate (2) , it is enough to show that for each $\varepsilon > 0$ there is M such that

$$\frac{1}{4^m} \leq M \left(\frac{1}{2^{m+1}} \cdot \frac{1}{(m+1)(m+2)} \right)^{2-\varepsilon} \quad \text{as } m \to \infty,$$

which is obviously true with the same argument as above for the function G.

To define the function F on the set E we first note that by definition of H one has $H(A) = 0$ and $H(B) = 1$. Then, since by definition of G we have $G(A) = 0$, there is a constant C such that for the function $F = G + CH$ one has $F(A) = 0$ and $F(B) = 0$. Finally, we observe that since $G \in C^{2-}(E)$ with $G'_x(x, y) = y$ and $G'_y(x, y) = 0$, and since $H \in C^{2-}(E)$ with $H'_x(x, y) = 0$ and $H'_y(x, y) = 0$, it follows that $F \in C^{2-}(E)$ with $F'_x(x, y) = y$ and $F'_y(x, y) = 0$ at each point $(x, y) \in E$.

4. Construction of the sphere $\mathbf{S} \subset \partial \mathbf{G}$. Let $\mathbb{A}$ be the linear transformation of $\mathbb{R}^2_{x,y}$ represented by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Consider the sets $E^1 = E + \vec{e}_y$, $E^2 = -\mathbb{A}E + \vec{e}_x + \vec{e}_y$, $E^3 = -E + \vec{e}_x$ and $E^4 = \mathbb{A}E$, where $\vec{e}_x$ and $\vec{e}_y$ are the unit vectors in the coordinate directions x and y, respectively, and then define the set $\tilde{E}$ as

$\tilde{E} = \bigcup_{i=1}^{4} E^i$ (the set $\tilde{E}$ is shown in Fig. 4). It is easy to see that $\tilde{E}$ is a Jordan

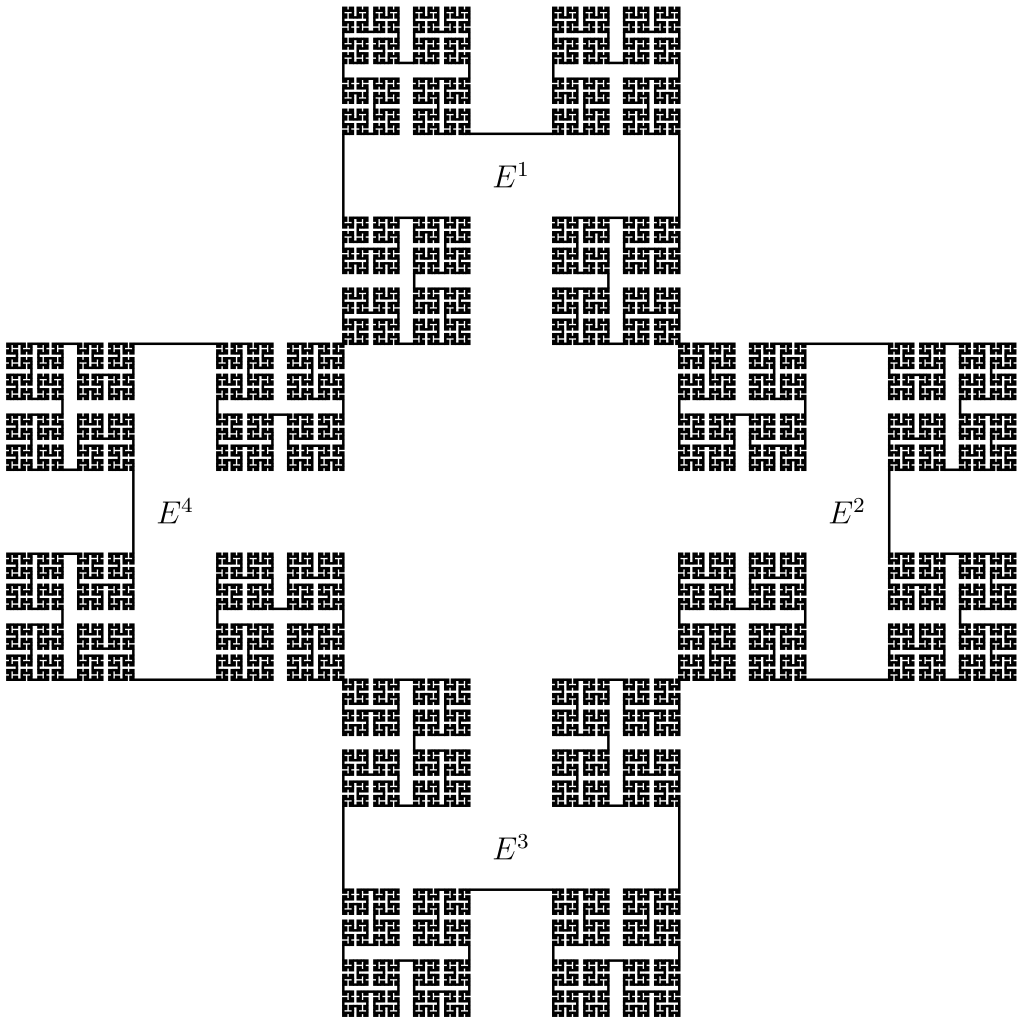

Figure 4: The set $\tilde{E}$

curve of positive 2-dimensional measure in $\mathbb{R}^2_{x,y}$. Applying to each of the sets $E^i, i = 1, 2, 3, 4$, a construction similar to the one that we had above for the function F on the set E, we will get functions F^i defined on the corresponding sets E^i with the properties:

1) $F^i \in C^{2-}(E^i)$,

2) $\frac{\partial F^i}{\partial x}(x, y) = y$ and $\frac{\partial F^i}{\partial y}(x, y) = 0$ for each $(x, y) \in E^i$,

3) F^i has zero values at the endpoints of the arc E^i.

Hence, we can define a function $\tilde{F}$ on the set $\tilde{E}$ as $\tilde{F}(p) = F^i(p)$ for $p \in E^i, i = 1,2,3,4$, and for this function we will obviously have that $\tilde{F} \in C^{2-}(\tilde{E})$ with $\frac{\partial \tilde{F}}{\partial x}(x,y) = y$ and $\frac{\partial \tilde{F}}{\partial y}(x,y) = 0$ at each point $(x,y) \in \tilde{E}$. Then, by the classical extension theorem of Whitney (see, for example, Theorem 4 on p. 177 in [S]), there is a function $\tilde{\tilde{F}} \in C^{2-}(\mathbb{R}^2_{x,y})$ such that $\tilde{\tilde{F}} \in C^{\infty}(\mathbb{R}^2_{x,y} \setminus \tilde{E})$ and $\tilde{\tilde{F}}(p) = \tilde{F}(p)$ for each $p \in \tilde{E}$. If we restrict the function $\tilde{\tilde{F}}$ to a disc $\mathbb{D} \subset \mathbb{R}^2_{x,y}$ such that $\tilde{E} \subset \mathbb{D}$ and consider a smooth extension of the graph of this restriction to a 2-dimensional sphere S^2 embedded into $\mathbb{R}^3_{x,y,z}$, then the set $\tilde{\tilde{F}}(\tilde{E})$ will be a Jordan curve in S^2 of positive 2-dimensional measure and at each point of this curve the tangent plane to S^2 will coincide with the corresponding plane of the standard contact distribution $\{dz - ydx = 0\}$.

Now let G be a given strictly pseudoconvex domain in $\mathbb{C}^2$ with C^{∞}-smooth boundary and let q be a point of ∂G. Then, by the theorem of Darboux, there is a neighbourhood U of q in ∂G and a C^{∞}-smooth diffeomorphism Φ of U onto a neighbourhood V of the origin in $\mathbb{R}^3_{x,y,z}$ such that the distribution of complex tangencies $\{T^{\mathbb{C}}_p(\partial G)\}$ will be transformed by Φ to the standard contact distribution in $\mathbb{R}^3_{x,y,z}$. We can assume without loss of generality that $S^2 \subset V$ (if not, we consider a linear transformation $x \to cx, y \to cy, z \to c^2 z$ of $\mathbb{R}^3_{x,y,z}$ which preserves the standard contact structure and use the image of S^2 under this transformation with $c > 0$ sufficiently small instead of S^2). Then $S = \Phi^{-1}(S^2)$ will be a 2-dimensional sphere in ∂G of class C^{2-} and the set $\mathcal{E} = \Phi^{-1}(\tilde{\tilde{F}}(\tilde{E})) \subset S$ will be a Jordan curve of positive 2-dimensional measure such that at each point $p \in \mathcal{E}$ the tangent plane T_pS to S is a complex line. This proves the second part of the theorem. $\square$

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