ENUMERATION OF RATIONAL CURVES VIA TORUS ACTIONS

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INTRODUCTION

This paper contains an attempt to formulate rigorously and to check predictions in enumerative geometry of curves following from Mirror Symmetry.

In a sense, we almost solved both problems. There are still certain gaps in foundations. Nevertheless, we obtain "closed" formulas for generating functions in topological sigma-model for a wide class of manifolds, covering many Calabi-Yau and Fano varieties. We reduced Mirror Symmetry in a basic example to certain complicated but explicit identity. We have made several computer checks. All results were as expected. In particular, we computed the "physical" number of rational curves of degree 4 on a quintic 3-folds (during 5 minutes on Sun), which was out of reach of previuos algebro-geometric methods.

The text consists of 5 parts. The first part contains the definition of stable maps used through all the paper. We establish several basic properties of moduli spaces of stable maps. Also, we give an outline of a contsruction of Gromov-Witten invariants for all algebraic projective or closed symplectic manifolds. For reader who is interested mainly in computations it is enough to look through 1.1 and to the statements of theorems in 1.3.1-1.3.2.

In section 2 we describe few examples of counting problems in enumerative geometry of curves. One of examples is rational curves on quintics. We give a simple algebro-geometric definition for the number of curves without assuming the validity of the Clemens conjecture or using symplectic methods.

The main body of computations is contained in section 3. Our strategy here is quite standard: we reduce problems to questions concerning Chern classes on a space of rational curves lying in projective spaces (A. Altman - S. Kleiman, S. Katz), and then use Bott's residue formula for the action of the group of diagonal matrices (G. Ellingsrud and S. A. Strømme). As a result we get in all our examples certain sums over trees.

In section 4 we develop a general scheme for summation over trees. By Feynman rules we know that such a sum should be equal to the critical value of some functional. Using a trick we obtain an equivalent functional which is a quadratic polynomial in infinitely many variables with coefficients depending on a finite number of variables. Thus, all our counting problems are reduced to the inversion of certain explicit square matrices with coeffcients of hypergeometric kind. This last

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step we were not able to accomplish. Presumably, there is here a hidden structure of an integrable system and Sato's grassmanians.

In section 5 we describe extensions of our computation scheme to other enumerative problems, including Calabi-Yau and Fano complete intersections (of arbitrary dimension) in projective spaces, toric varieties and generalized flag varieties.

1. Stable maps.

1.1. Definition.

Let V be a scheme of finite type over a field (or a smooth scheme, or a complex manifold, or an almost complex manifold).

Definition. *Stable map is a structure $(C; x_1, \dots, x_k; f)$ consisting of a connected compact reduced curve C with $k \geq 0$ pairwise distinct marked non-singular points x_i and at most ordinary double singular points, and a map $f : C \longrightarrow V$ having no non-trivial first order infinitesimal automorphisms, identical on V and $x_1, \dots, x_k$ (stability).*

The condition of stability means that every irreducible component of C of genus 0 (resp. 1) which maps to a point must have at least 3 (resp. 1) special (i.e. marked or singular) points on its normalization. Also, it means that the automorphism group of $(C; x_1, \dots, x_k; f)$ is finite.

For a curve C with at most ordinary double singular points its arithmetic genus $p_a(C) := dim\ H^1(C, \mathcal{O})$ can be computed from the formula

$$2 - 2p_a(C) = \chi(C \setminus C^{sing}) \ .$$

Let $\beta \in H^2(V, \mathbf{Z})$ be a homology class. (In algebro-geometric situation β should be an element of the group of 1-dimensional cycles modulo homological equivalence).

Notation. *$\overline{\mathcal{M}}_{g,k}(V, \beta)$ denotes the moduli stack of stable maps to V of curves of arithmetic genus $g \geq 0$ with $k \geq 0$ marked points such that $f_*[C] = \beta$.*

More precisely, in algebro-geometric setting one can define a family of flat maps to V as a flat proper morphism $\mathcal{C} \longrightarrow S$ to a scheme S of finite type over the ground field and a map $f : \mathcal{C} \longrightarrow V$ such that its restriction to each geometric fiber of $\mathcal{C}$ over S is a stable map.

In the setting of almost complex manifolds we consider $\overline{\mathcal{M}}_{g,k}(V, \beta)$ as a set of equivalence classes of stable maps endowed with a natural topology (see [P]) and an orbispace structure (see the next subsection).

Remark. For many reasons one has to consider curves not in a fixed manifold V but in manifolds V_λ varying in families. We have not developed the corresponding formalism yet. In subsection 5.4 we describe a simple example of algebraic $K3$-surfaces which shows the necessity of families. It is also clear from our example that one can consider non-compact V as well.

1.2. Orbispaces.

The notion of orbispace introduced here is a topological counterpart of

(1) algebraic stacks (from algebraic geometry), and
(2) orbifolds, or V-manifolds (from differential topology).

We define orbispace as a small topological category C (i.e. a category for which $\mathrm{Ob}\,C$ and $\mathrm{Mor}\,C$ carry topological structures) satisfying following axioms.

A.1. C is a groupoid (every morphism is invertible).

A.2. For each $X, Y \in \mathrm{Ob}\,C$ the set of morphisms $Mor_C(X,Y)$ is finite.

A.3. Two maps from $\mathrm{Mor}\,C$ to $\mathrm{Ob}\,C$, assigning to a morphism its source and its target respectively, are locally homeomorphisms (étale maps).

Functors between orbispaces which are continous, locally homeomorphisms and induce equivalence of categories we can call equivalences between orbispaces.

The set $|S|$ of equivalence classes of objects of C has natural induced topology. We can associate with each element $[X] \in |S|$ an equivalence class (modulo interior automorphisms) of finite groups, $Aut(X)$.

1.3. Properties of moduli spaces of stable maps.

The notion of a stable map is a mixture of the notion of a *stable curve* from algebraic geometry and of the notion of a *cusp-curve* from symplectic topology. By definition from [P], cusp-curve is a holomorphic map $\tilde{f}$ from a compact (not necessarily connected) smooth complex curve $\tilde{C}$ to an almost-complex manifold V and a finite collection $\mathcal{S}$ of non-intersecting 2-element subsets of $\tilde{C}$ such that, for each $S \in \mathcal{S}$, its image $\tilde{f}(S)$ is a 1-element set. Glueing points from pairs $S \in \mathcal{S}$ together we obtain a curve C with at most ordinary double singular points and a map $f : C \longrightarrow V$. P. Pansu claimed in [P] that if V is compact and endowed with a riemannian metric, then the space of equivalence classes of cusp-curves of bounded genus and area is compact and Hausdorff. His claim is wrong, exactly because the condition of stability on components which are mapped to a point was forgotten! It seems that, after appropriate corrections, the proof from [P] shows that the moduli space of stable maps of bounded genus and area is compact and Hausdorff.

Recall that in symplectic topology one considers usually almost-complex structures on symplectic manifolds compatible in an evident sense with the symplectic form. Such a structure defines a Riemannian metric on the underlying manifold, and the riemannian area of each holomorphic curve coincides with its symplectic area. The latter is a pure homological invariant. Hence $\overline{\mathcal{M}}_{g,k}(V,\beta)$ is compact and Hausdorff in such a situation.

In the next subsection, we prove analogous properties of $\overline{\mathcal{M}}_{g,k}(V,\beta)$ in algebro-geometric setting.

In 1.3.2, we describe a situation in which the moduli space of stable maps is smooth (as a stack).

1.3.1. Algebraicity and properness.

Theorem. *Let V be a projective scheme of finite type over a field. Then $\overline{\mathcal{M}}_{g,k}(V,\beta)$ is an algebraic proper stack of finite type.*

The proof uses results from [DM]. We refere to [DM] for definitions concerning properties of stacks, and for other technical details as well.

We want to realize $\overline{\mathcal{M}}_{g,n}(V,\beta)$ as a quotient stack of a scheme of finite type modulo étale equivalence relation.

From the boundness of the Hilbert scheme of 1-dimensional subschemes of V it follows that for $(C; x_*, \beta)$ with fixed $p_a(C)$ and β, the number of singular points on C and the number of irreducible components of C are bounded.

In the next step, we will realize $\overline{\mathcal{M}}_{g,n}(V,\beta)$ as a quotient space of a space of maps of stable curves into V. For this we can choose a finite collection of hypersurfaces D_i in V such that each non-stable component of any curve C from $\overline{\mathcal{M}}_{g,n}(V,\beta)$ intersects transversally some of D_i at least at three non-special points. Then we can consider such intersection points as new marked points on C. Finally, we can glue a fixed smooth curve of genus bigger than 0 with one marked point to each marked point on C obtaining a stable curve of a bounded genus. We map each glued component into a point of V.

This way our moduli space is realized locally in étale topology as a closed subspace of the space of maps from stable curves to V with a fixed image of the fundamental class. Such space can be realized inside the Hilbert scheme of 1-dimensional subschemes of the product of the universal curve times V via graphs of maps. Thus, $\overline{\mathcal{M}}_{g,n}(V,\beta)$ is an algebraic stack of finite type.

Separatedness and properness of moduli of stable maps follow from corresponding properties of V. Recall that the property of properness implies separatedness by definition.

If we have a family of stable maps $\mathcal{C}/K$, $f:\mathcal{C}\longrightarrow V$ over a discrete valuation field K with the ring of integers $\mathcal{O}_K$, then there exists a finite extension L/K and a family of stable curves over $\mathcal{O}_L$ extending the pull-back of $\mathcal{C}/K$. First of all, we can construct a proper two-dimensional scheme S over $Spec(\mathcal{O}_K)$ which maps to V by taking the closure of the graph of f into the product of V and an arbitrary model of $\mathcal{C}$ over $\mathcal{O}_K$. It follows from well-known facts about degenerations of curves that there exists such an extension L/K and a curve $\mathcal{C}'$ over $\mathcal{O}_L$ which maps to V with the property that the geometric fiber of $\mathcal{C}'$ over the closed point of $Spec(\mathcal{O}_L)$ is a connected reduced curve with pairwise distinct marked non-singular points and at most ordinary double singular points. We can contract consecutively non-stable components of this geometric fiber and obtain a stable map. This proves the existence part of the valuative criterion of properness.

Moreover, in such a situation components which we contract all have genus zero and form a subforest in the degeneration graph of the curve. One can see easily that this subforest does not depend on the order in which we contract components. From this uniqueness and separatedness of V one can conclude that the moduli stack of stable maps is separated. Hence, we have also the uniqueness part of the valuative criterion of properness. $\square$

1.3.2. Smoothness.

Theorem. *Let V be a smooth proper scheme of finite type over a field which is convex in the sense of [KM]. Then the stack $\overline{\mathcal{M}}_{0,k}(V,\beta)$ is smooth, and the complement to the open subset $\overline{\mathcal{M}}^0_{0,k}(V,\beta)$ consisting of smooth curves is a divisor with normal crossings.*

Recall that convex manifolds V (definition 2.4.2 in [KM]) are defined as a manifolds with vanishing $H^1(C,f^*T_V)$ for any stable map of degree zero. It is enough to check only for smooth curves. At the moment we now only one group of examples, namely, homogeneous projective varieties. In sections 2-4 we will consider only projective spaces.

Stable maps have the following important property. Let us consider a flat proper morphism $\mathcal{C} \longrightarrow S$, of relative dimension 1 to a scheme S of finite type over the ground field, sections x_i, $i = 1, \dots, k$ of $\mathcal{C}$ over S, and a map $f : \mathcal{C} \longrightarrow V$. We claim that the set of points p of S such that the restriction of f to the geometric fiber of $\mathcal{C}$ over p is a stable map, is an open subset of S. Hence, deformation theory of a given stable map is equivalent to the deformation theory of it as of a map from a compact (non-fixed) curve to V.

First of all, deformations of 1-st order of stable map $(C; x_1, \dots, x_k; f)$ which do not change the structure of singularities of C are given by

$$\mathbf{H}^1(C, \mathcal{T}'_C \to f^*\mathcal{T}_V),$$

where $\mathcal{T}'_C$ denotes the sheaf of vector fields on C vanishing at points x_i, $i = 1, \dots, k$. We put $\mathcal{T}'_C$ in the degree 0 and $f^*\mathcal{T}_V$ in the degree 1. The hyper-cohomology group in degree 0 vanishes by the stability condition.

Denote by T the tangent space to $\overline{\mathcal{M}}_{g,k}(V, \beta)$ at the point $(C; x_1, \dots, x_k; f)$. One can show that we have the following exact sequence:

$$0 \longrightarrow \mathbf{H}^1(C, \mathcal{F}^\bullet) \longrightarrow T \longrightarrow \bigoplus_{y \in C^{sing}} T^1_y C \otimes T^2_y C \to \mathbf{H}^2(C, \mathcal{F}^\bullet) \ ,$$

where the fourth term comes from the deformations of C resolving double points y. Tangent spaces to two branches of C at y are denoted by $T^1_y C$ and $T^2_y C$ (in arbitrary order); $\mathcal{F}^\bullet$ denotes the complex of sheaves of length 2 used above.

We are ready now for the proof of the smoothness criterium. For arbitrary map f from a curve C of arithmetic genus zero to convex V we have $H^1(C, f^*\mathcal{T}_V) = 0$. Hence $\mathbf{H}^2(C, \mathcal{F}^\bullet) = 0$, and the dimension of tangent space to $\overline{\mathcal{M}}_{0,k}(V, \beta)$ is constant. One can elaborate the argument above for maps parametrized by spectra of Artin algebras and show that there is no obstructions for the deformation theory. Also, we see that maps from singular curves form a divisor with normal crossings. $\square$

From the proven properties of $\overline{\mathcal{M}}_{0,k}(V, \beta)$ one can easily deduce the tree level system of Gromov-Witten invariants on convex varieties (see [KM]).

1.4. The structure of an intersection of manifolds.

The last theorem shows that the moduli space of stable maps to V inherits the property of smoothness of V in some cases. Here we are trying to define for *all* smooth V certain structure on $\overline{\mathcal{M}}_{g,k}(V, \beta)$ which permits us to construct an analogue of the fundamental class. We will do it in the setting of almost-complex real-analytic manifolds and describe in 1.4.2 the situation in algebraic geometry.

Let Y_1, Y_2 be two submanifolds in a manifold X (manifolds are real-analytic, or complex, or algebraic). The intersection $Z := Y_1 \cap Y_2$ in general is not smooth. Nevertheless, we define its "virtual tangent bundle" $[\mathcal{T}_Z]^{virt} \in K^0(Z)$ by the formula

$$[\mathcal{T}_Z]^{virt} := [\mathcal{T}_{Y_1}]_{|Z} + [\mathcal{T}_{Y_2}]_{|Z} - [\mathcal{T}_X]_{|Z} \ .$$

Also, if X, Y_1, Y_2 are oriented then there is a canonical "virtual fundamental class" $[Z]^{virt}$ with values in homology with closed support

$$H_d^{closed}(Z) := \widetilde{H}_d(\widetilde{Z}, \mathbf{Z}) \ .$$

Here $\widetilde{Z}$ denotes one-point compactification of Z and $\widetilde{H}_d$ denotes d-th reduced homology group of a punctured space. Number $d = dim(Y_1) + dim(Y_2) - dim(X)$ is the virtual dimension of Z. More precisely, one can construct a fundamental class in the complex bordism group with closed support $\Omega_d^{closed}(Z)$, defined analogously, when X, Y_1, Y_2 are almost-complex. The idea is obvious: Z is homotopy equivalent to its sufficiently small tubular neighborhood UZ in X. In UZ one can perturbe generically Y_1, Y_2 and obtain a transversal intersection. Mention, that in smooth situation Z can have pathological topology and can be not homotopy equivalent to any of tubular neighborhoods. It is plausible that one can define a cobordism analogue of Borel-Moore homology and extend intersection theory to the smooth case.

Singular space Z can have several representations as an intersection of germs at Z of manifolds containing Z. For example, we can multiply X, Y_1, Y_2 locally by X', Y_1', Y_2' where Y_1' intersect transversally Y_2' in one point. Globally one can pass from X, Y_1, Y_2 to the total spaces of vector bundles $\mathcal{E}^X, \mathcal{E}^{Y_1}, \mathcal{E}^{Y_2}$ on corresponding spaces endowed with embeddings

$$\mathcal{E}^{Y_i} \hookrightarrow \mathcal{E}^X_{|Y_i}, \quad \mathcal{E}^X_{|Z} \simeq \mathcal{E}^{Y_1}_{|Z} \oplus \mathcal{E}^{Y_2}_{|Z} .$$

Such pairs of representations we call stably equivalent.

If Z is intersection of several submanifolds Y_i, $1 \leq i \leq n$ in X, then one can represent Z as an intersection of two submanifolds:

$$Z \simeq (Y_1 \times \cdots \times Y_n) \cap \text{ diagonal in } X^n .$$

Also, if we have two maps of manifolds $Y_i \xrightarrow{f_i} X$, $i = 1, 2$ then the fiber product $Z := Y_1 \times_X Y_2$ carries a structure of an intersection of two manifolds. It follows from the identification of Z with the intersection in $Y_1 \times Y_2 \times X$ of graphs of f_i multipled by Y_{3-i}.

Let space Z carry a system of representations of open subsets U_i of Z as intersection of a manifolds endowed with a system of stable equivalence between models for $U_i \cap U_j$ arising from U_i and U_j. Such a system should be associative up to a homotopy, homotopies between homotopies etc. Then we expect that Z has global virtual tangent bundle and virtual fundamental class. In a sense, all this should be a non-linear analogue of an element of $K^0(Z)$ represented locally as a formal difference of two vector bundles.

Let us return to the moduli space of stable maps. We claim that it has a canonical structure of an orbifoldic version of an intersection of almost-complex manifolds.

First of all, near each point $(C; x_1, \dots, x_k; f)$ we will represent $\overline{\mathcal{M}}_{g,k}(V, \beta)$ as an intersection of several infinite-dimensional Frechet submanifolds (forgetting temporarily the presence of the finite group of automorphisms of the stable map $(C; x_1, \dots, x_k; f)$). Let us choose several closed non-intersecting simple parametrized loops L_i on the surface C which divide it into pieces C_j each of which is either a smooth surface with a non-empty boundary and no marked points, or a smooth disc with one marked point in the interior of the disc, or two discs with glued centers.

We consider as the first approximation to the ambient manifold X, the space X' of smooth maps from $\coprod L_i$ into V which are sufficiently close to $f_{|\coprod L_i}$. This space as an infinite-dimensional almost complex manifold with the complex structure on the tangent bundle induced pointwise from the complex structure on $\mathcal{T}_V$.

For each piece C_j of the surface C we introduce the space Y'_j consisting of pairs (J', f'), where J' is a complex structure on C_j close to the initial one and f' is a J'-holomorphic map considered modulo diffeomorphisms of C_j close to the identity. For pieces C_j which are two intersecting discs we add small flat deformations:

$$\{(x,y) : x,y \in \mathbf{C},\ xy = 0,\ |x| + |y| \leq 1\}$$
$$\text{deform to } \{(x,y) : x,y \in \mathbf{C},\ xy = \epsilon,\ |x| + |y| \leq 1\}, \ |\epsilon| \ll 1 .$$

Spaces Y'_j are almost-complex and they are maped into X' by passing to the restriction of maps to boundaries. Their fiber product over X' consists of stable maps of curves close to the initial point endowed with parametrized loops. As usual, one can pass to the quotient of all the picture modulo the action of the product over loops L_i of the "complexified diffeomorphism group of a circle"(= replacing curves L_i by close curves). This action is free exactly due to the condition of stability. Finally, we can pass to the case of two submanifolds, as we already explained.

Thus, we will get germs of Frechet manifolds X and Y_i. The natural map of tangent spaces

$$T_z Y_1 \oplus T_z Y_2 \longrightarrow T_z X\ , \ \ z \in Z\ ,$$

is Fredholm. In the next subsection we develope a technique producing in such a situation finite-dimensional models.

Globally, we can cover $Z = \overline{\mathcal{M}}_{g,k}(V,\beta)$ by finitely many open sets: and on each of them we have an equivalence class of representations as intersections of manifolds. It is almost clear that different representations on intersections of open sets are equivalent modulo homotopy and higher homotopies between homotopies on multiple intersections. Unfortunately, we don't know how to formulate all this precisely. We have tried to avoid choices and use a Dolbeaut-type resolvent. The space Y_1 in this case should be a space of real-analytic maps from complex surfaces to V satisfying the same condition of stability as before. A natural candidate for X will be the total space of a vector bundle on X arising from the Cauchy-Riemann equation, Y_2 will be a section of X as of a vector bundle. We have met an unpleasant difficulty in considering deformations resolving double points. May be, nevertheless, it is possible to find a version of the Dolbeaut complex which form a complex of infinite-dimensional vector bundles over a neighborhood of Z in Y_1. Such a construction, if it exists, will give a really simple definition of the virtual fundamental class of Z.

1.4.1. Reduction to finite dimensional manifolds.

Suppose that we have Frechet manifolds X and Y_i with the Fredholm property as above and $Z := Y_1 \cap Y_2$ being compact.

We can choose smooth finite-dimensional sub-bundles $\mathcal{E}_i \subset (\mathcal{T}_X)_{|Z}$, $i = 1,2$ such that

$$\mathcal{E}_i \cap (\mathcal{T}_{Y_i})_{|Z} = 0\ , \ \ (\mathcal{T}_X)_{|Z} = \mathcal{E}_1 + \mathcal{E}_2 + (\mathcal{T}_{Y_1})_{|Z} + (\mathcal{T}_{Y_2})_{|Z}\ \ .$$

We can prolong $\mathcal{E}_i$ to neigborhoods of Z in Y_i. Then we can choose submanifolds $\widetilde{Y}_i$ in X containing Y_i such that

$$(\mathcal{T}_{\widetilde{Y}_i})_{|Y_i} = \mathcal{T}_{Y_i} \oplus \mathcal{E}_i\ .$$

Submanifolds $\widetilde{Y}_i \subset X$ intersect each other transversally near Z. From now on we can forget about the ambient manifold X and consider only the system of 5 manifolds and inclusions between them:

$$Y_1 \hookrightarrow \widetilde{Y}_1 \hookleftarrow \widetilde{Y}_1 \cap \widetilde{Y}_2 \hookrightarrow \widetilde{Y}_2 \hookleftarrow Y_2 \ .$$

We can choose a sub-bundles $\mathcal{F}_i$ of finite codimension in $(\mathcal{T}_{Y_1})_{|Z}$ such that

$$\mathcal{F}_i \cap (\mathcal{T}_{\widetilde{Y}_{3-i}})_{|Z} = 0 \ .$$

Then we can choose a foliation in Y_i with tangent spaces to fibers at points from Z equal to $\mathcal{F}_i$. We can prolong these foliations to foliations of $\widetilde{Y}_i$. Near Z these foliations are tangent to fibers of smooth fibrations, due to the transversality condition above. Passing to the spaces of fibers of these folitations in $Y_i, \widetilde{Y}_i$ we obtain germs of finite-dimensional manifolds $Y_i', \widetilde{Y}_i'$. We can take for the middle term the same finite-dimensional manifold as above, $\widetilde{Y}_1 \cap \widetilde{Y}_2$ and get a new finite-dimensional system of 5 manifolds and inclusions. We can construct a new ambient manifold X' in which $\widetilde{Y}_1'$ and $\widetilde{Y}_2'$ intersect transversally along $\widetilde{Y}_1' \cap \widetilde{Y}_2' \simeq \widetilde{Y}_1 \cap \widetilde{Y}_2$. Thus, Z is realized as an intersection of finite-dimensional manifolds. One can check that different procedures give stably equivalent representations.

1.4.2. Intersections of manifolds in algebraic geometry.

If Y_1, Y_2 are submanifolds of an algebraic manifold X then on $Z := Y_1 \cap Y_2$ we can construct a structure of *super-scheme*. This means that on Z we have a super-structure sheaf

$$\mathcal{O}_Z^* = \bigoplus_{n \leq 0} \mathcal{O}_Z^n$$

of $\mathbf{Z}_{\leq 0}$-graded super-commutative rings such that Z is a scheme with respect to $\mathcal{O}_Z^0$ and $\mathcal{O}_Z^n$ are coherent sheaves of $\mathcal{O}_Z^0$-modules. The formula for components of the higher structure sheaf is

$$\mathcal{O}_Z^n := (i_Z)^* \, Tor_{-n}^{\mathcal{O}_X}\left((i_{Y_1})_* \mathcal{O}_{Y_1}, (i_{Y_2})_* \mathcal{O}_{Y_2}\right) \ ,$$

where $i_?$ denotes the embedding map. Also we have a virtual tangent bundle in $K^0(Z)$ given by the same formula as in the almost complex setting.

Structures $(\mathcal{O}_Z^*, [\mathcal{T}_Z]^{virt})$ do not change if we pass to an equivalent representation of Z as an intersection of two manifolds. We call a pair of such structures a "quasi-manifold".

Our discussion leads to the prediction of the existence of the structure of quasi-manifold on $Z = \overline{\mathcal{M}}_{g,k}(V, \beta)$ defined in purely algebro-geometric terms. In fact, one can define $[\mathcal{T}_Z]^{virt}$ as the direct image of the deformation sheaf on the universal curve. Also, $\mathcal{O}_Z^0$ is the usual structure sheaf on algebraic stack Z, and $\mathcal{O}_Z^1$ is equivalent to the first obstruction sheaf. We are planning to write later more about definitions of higher structure sheaves and virtual tangent bundles arising ubiquitously in algebraic geometry. For example, various moduli spaces and Hilbert schemes should carry canonical structures of quasi-manifolds. Idea of introducing higher structure sheaves on moduli spaces existed implicitly quite a long time ago. It was recently spelled out clearly in a letter of P. Deligne to H. Esnault, together with a proposal to apply it to the algebro-geometric formulation of Mirror Symmetry.

We finish this subsection with a formula which produces a virtual fundamental class for a quasi-manifold Z with $\mathcal{O}_Z^n = 0$ for $n \ll 0$. Note that it is applicable to the quasi-manifold structure arising on the intersection of two manifolds.

First of all, for each separated scheme Z of finite type over a field and for any coherent sheaf $\mathcal{F}$ on Z a homological Chern class $\tau(\mathcal{F}) \in CH_*(Z) \otimes \mathbf{Q}$ is defined (see [BFM]). Here $CH_*(Z)$ denotes the Chow group of cycles on Z modulo rational equivalence and regarded as an algebraic counterpart of $H_*^{closed}(Z(\mathbf{C}), \mathbf{Z})$ for schemes over $\mathbf{C}$. For the definition of the virtual fundamental class, we will use, for the sake of simplicity, a smooth ambient manifold X. We can prolong the virtual tangent bundle $[\mathcal{T}_Z]^{virt}$ to an element of $K^0(X)$ after replacing X by a sufficiently small neighborhood of Z. The formula for the virtual fundamental class is

$$[Z]^{virt} := \left(\sum_k (-1)^k \tau(\mathcal{O}_Z^k) \right) \cap td([\mathcal{T}_Z]^{virt})^{-1} .$$

One can see that for quasi-manifolds arising as an intersection of two submanifolds this fromula gives the same class as the usual intersection theory of Fulton-MacPherson. For zero-dimensional Z our formula is equivalent to the Serre formula for multiplicities.

Note that we have for quasi-manifolds a refined fundamental class with values in $CH_* \otimes \mathbf{Q}[c_1, c_2, \dots]$ arising from the action of Chern classes of $[\mathcal{T}_Z]^{virt}$ on $[Z]^{virt}$. It can be considered as an algebraic version of the fundamental class with values in complex cobordism groups.

1.5. Gromov-Witten invariants.

We have a compact moduli orbi-space of stable curves and a virtual fundamental class of it in two situations:

(1) V is a smooth projective algebraic manifold over a field, or
(2) V is a compact real-analytic symplectic manifold endowed with a compatible real-analytic almost-complex structure.

The virtual fundamental class takes values in $CH_* \otimes \mathbf{Q}[c_1, c_2, \dots]$ and in $\Omega_* \otimes \mathbf{Q}$ respectively.

Suppose that $2 - 2g - k < 0$; so, that $\overline{\mathcal{M}}_{g,k}$ exists. We have an evident map:

$$\Phi : \overline{\mathcal{M}}_{g,k}(V, \beta) \longrightarrow V^k \times \overline{\mathcal{M}}_{g,k} ,$$

$$(C; x_1, \dots, x_; f) \mapsto \left(f(x_1), \dots, f(x_k); (\widetilde{C}, \tilde{x}_1, \dots, \tilde{x}_k) \right) .$$

Here $(\widetilde{C}, \tilde{x}_1, \dots, \tilde{x}_k)$ is the stable curve with marked points obtained from $(C, x_1, \dots, x_k)$ by cosecutive contractions of non-stable components.

The image under Φ of $[\overline{\mathcal{M}}_{g,k}(V, \beta)]^{virt}$ is a class in $V^k \times \overline{\mathcal{M}}_{g,k}$ which leads to Gromov-Witten invariants of V (see [KM]). It should not depend on the choice of an almost-complex structure in the symplectic case.

We expect that these classes satisfy all axioms postulated in [KM]. In fact, the definition of stable maps was designed specially for this purpose.

Here we get a refinement of the picture from [KM]: Gromov-Witten invariants take their values not just in cohomology groups, but in complex cobordism groups. Also, we have line bundles on $\overline{\mathcal{M}}_{g,k}(V,\beta)$ with fibers equal to $T_{x_i}C$, and we can take in account actions of their Chern classes on $[\overline{\mathcal{M}}_{g,k}(V,\beta)]^{virt}$. It is an essential additional data, because $T_{x_i}C$ are not isomorphic to the pullbacks of analogous bundles on $\overline{\mathcal{M}}_{g,k}$. In the deformation formula 6.4.c from [KM], one can use $T_{x_i}C$ instead of $T_{\tilde{x}_i}\widetilde{C}$.

1.6. Comparision with other definitions.

It was proposed earlier several times that for the definition of the topological sigma model (=Gromov-Witten invariants) in algebro-geometric terms one should use the Hilbert scheme of V and, possibly, modify it. Our moduli space of stable maps does it in a sense. Its advantage is smoothness in the case when V is a generalized flag variety. Also, our definition gives the same moduli space for complex projective V considered as an algebraic or as a symplectic manifold.

In symplectic geometry the most advanced construction was announced recently by Y. Ruan and G. Tian in [RT]. They construct a part of Gromov-Witten invariants (essentially, genus zero invariants) in the case of semi-positive symplectic manifolds. Their main idea is that in this case one can ignore curves with singularities, because the dimension of space of degenerate curves is strictly less than the dimension of the space of smooth curves for generic almost-complex structures. The advantage of approach of [RT] is a control on integrality of arising homology classes. In [RT] "numbers of rational curves" of fixed homology class passing through several submanifolds were defined in the case of the number of cycles greater than, or equal to 3. Without this condition "number of curves" should be fractional in examples of quintic 3-folds (see the next section).

Our pre-definition should work for all symplectic manifolds and, presumably, in the case of surfaces with boundaries, opening a way to extend Floer's proof of the Arnold conjecture to the case of non semi-positive symplectic manifolds.

As we already mentioned, Gromov-Witten invariants should be defined also for families of not necessarily compact symplectic or algebraic manifolds. It is not clear at the moment, in which generality such a theory can be developed. For example, we don't know should families be flat or only smooth, should the parameter space be smooth, etc. .

The general scheme described in 1.4 can be applied in other situations: moduli of vector bundles on algebraic curves and surfaces, moduli of complex structures on surfaces, moduli of vector bundles on Calabi-Yau 3-folds. Common property of all such examples is that the natural complex whose 1-st cohomology group is equivalent to the tangent space to the appropriate moduli space, has trivial cohomology in degrees greater than, or equal to 3. The main problem is to define good compactifications in other situations.

2. Three examples.

In this section and in the next one we will use simplified notations: $\overline{\mathcal{M}}_{g,k}(\mathbf{P}^n, d[\mathbf{P}^1])$ will be denoted by $\overline{\mathcal{M}}_{g,k}(\mathbf{P}^n, d)$ or, simply $\overline{\mathcal{M}}(\mathbf{P}^n, d)$ if $g = k = 0$.

2.1. Rational curves on $\mathbf{P}^2$.

Dimension of the space $\overline{\mathcal{M}}_{0,k}(\mathbf{P}^2, d)$ is equal to $3d - 1 + k$. For $k = 3d - 1$ it coincides with the dimension of $(\mathbf{P}^2)^k$. Hence the number P_d of rational curves on $\mathbf{P}^2$ of degree $d \geq 1$ passing through generic $k = 3d - 1$ points is finite and equal to the degree of the map

$$\phi : \overline{\mathcal{M}}_{0,k}(\mathbf{P}^2, d) \to (\mathbf{P}^2)^k, \qquad \phi(C; x_1, \ldots, x_k; f) = (f(x_1), \ldots, f(x_k)).$$

We can rewrite it as the integral:

$$P_d = \int_{\overline{\mathcal{M}}_{0,3d-1}(\mathbf{P}^2,d)} \prod_{i=1}^{3d-1} \phi^*(c_1(\mathcal{O}(1)_i)^2),$$

where $\mathcal{O}(1)_i$ denotes the pullback of the line bundle $\mathcal{O}(1)$ from the i-th factor $\mathbf{P}^2$ of $(\mathbf{P}^2)^{3d-1}$.

These numbers are known from the recursion relations following from the associativity equations (see [KM]). The first few values of P_d are:

d	1	2	3	4	5
P_d	1	1	12	620	87304

Our proof of the associativity relations is based (following Witten [W]) on a study of the boundary divisors of the moduli spaces of stable maps. Here we want to compute number of curves directly.

2.2. Rational curves on quintics.

Let a smooth quintic 3-fold V be given by an equation $Q(x_1, \ldots, x_5) = 0$ in homogeneous coordinates in $\mathbf{P}^4$. Polynomial Q of degree 5 can be considered as a section of the line bundle $\mathcal{O}(5)$ on $\mathbf{P}^4$.

The orbispace $\overline{\mathcal{M}}_{0,0}(V, d[\mathbf{P}^1])$ is a subspace of $\overline{\mathcal{M}}(\mathbf{P}^4, d)$. Let $\mathcal{E}_d$ be a coherent sheaf on $\overline{\mathcal{M}}(\mathbf{P}^4, d)$ equal to the direct image under the forgetful map $\overline{\mathcal{M}}_{0,1}(\mathbf{P}^4, d) \to \overline{\mathcal{M}}_{0,0}(\mathbf{P}^4, d) = \overline{\mathcal{M}}(\mathbf{P}^4, d)$ of $\phi^*(\mathcal{O}(5))$. Here again $\phi(C; x_1; f) = f(x_1) \in \mathbf{P}^4$. Sheaf $\mathcal{E}_d$ is actually a vector bundle: for any stable map $f : C \to \mathbf{P}^4$ from a curve of arithmetic genus zero $H^1(C, f^*(\mathcal{O}(5))) = 0$, because the line bundle $\mathcal{O}(5)$ is generated by its global section.

Section Q of $\mathcal{O}(5)$ defines sections $\widetilde{Q}_d$ of $\mathcal{E}_d$ for all d. It is clear that $\overline{\mathcal{M}}(V, d[\mathbf{P}^1])$ coincides with the scheme of zeroes of $\widetilde{Q}_d$. We claim that this identification is compatible with structures of an intersection of manifolds.

The orbifold $\overline{\mathcal{M}}(\mathbf{P}^4, d)$ has dimension $5d+1$, the same as the rank of $\mathcal{E}_d$. Hence, we get the algebro-geometric definition of the "number of rational curves on quintic". It should be equal to the integral of the Euler class of $\mathcal{E}_d$:

$$N_d := \int_{\overline{\mathcal{M}}(\mathbf{P}^4,d)} c_{5d+1}(\mathcal{E}_d).$$

Numbers N_d are not integers, because we use orbifolds. The table of first few N_d is the following:

d	1	2	3	4
N_d	$\frac{2875}{1}$	$\frac{4876875}{8}$	$\frac{8564575000}{27}$	$\frac{15517926796875}{64}$

N_d are related with integer numbers N_d^0 by the following formula:

$$N_d = \sum_{k:k|d} k^{-3} N^0_{d/k} \ .$$

N_d^0 is the number of geometric (unparametrized) rational curves on V with generically perturbed almost complex structure.

Mirror Symmetry (see [Y]) gives the following description of the sequence N_d:

Let us introduce a function defined in a domain $\{t : \, Re(t) \ll 0, \, |Im(t)| < \pi\}$ in the complex affine line $\mathbf{C}$:

$$F(t) := \frac{5}{6} t^3 + \sum_{d \geq 1} N_d e^{dt} \ .$$

We denote by $G(q_1, q_2)$ corresponding function of homogeneity degree 2 in a domain of the vector space $\mathbf{C}^2$:

$$G(q_1, q_2) = F(q_1/q_2)\, q_2^2 \ .$$

Function G generates a Lagrangean cone $\mathcal{L}$ in the symplectic vector space $\mathbf{C}^4$:

$$\mathcal{L} := \{(p_1, p_2, q_1, q_2) : \, p_i = \partial G / \partial q_i\} \ .$$

On the other hand,

$$I(z) = \sum_{n \geq 0} \frac{(5n)!}{(n!)^5} z^n$$

is one of the periods of one-dimensional variation of Hodge structures $\mathcal{H}_z$ with Hodge numbers $h^{0,3} = h^{1,2} = h^{2,1} = h^{3,0} = 1$ arising from a mirror family of Calabi-Yau 3-folds. Poincaré pairing defines a covariantly constant symplectic structure on 4-dimensional vector bundle $\mathcal{H}$ with flat Gauss-Manin connection. We can trivialize the flat bundle $\mathcal{H}$ in the domain $\{z : |z| \ll 1, \, |Arg(z)| < \pi\}$. The union $\mathcal{U}$ of 1-dimensional terms of the Hodge filtration F_z^3 form a Lagrangean cone in $\mathbf{C}^4$.

Mirror Symmetry predicts that $\mathcal{L} = \mathcal{U}$. The same kind of correspondence is expected for other Calabi-Yau 3-folds.

2.3. Multiple coverings of rational curves on Calabi-Yau 3-folds.

Let $C_0 \simeq \mathbf{P}^1$ be a smooth rational curve in a complex 3-fold V with the normal bundle $\mathcal{T}_V/\mathcal{T}_{C_0}$ equivalent to $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. In such a situation $\overline{\mathcal{M}}_{0,0}(V, d[C_0])$ has a connected component $\overline{\mathcal{M}}_{0,0}(C_0, d[C_0])$ consisting of stable maps $C \longrightarrow C_0$ of degree d. This component is isomorphic to $\overline{\mathcal{M}}(\mathbf{P}^1, d)$ and has dimension $2d-2$. The virtual dimension of this component is zero. The obstruction sheaf $\mathcal{F}_d$ is a vector bundle of rank $2d-2$ with the fiber at each point $f : C \longrightarrow C_0$ equal to

$$H^1(C, f^*(\mathcal{T}_V/\mathcal{T}_{C_0})) \simeq \mathbf{C}^2 \otimes H^1(C, f^*(\mathcal{O}(-1))) \, .$$

Our definition of the contribution of $\overline{\mathcal{M}}_{C_0}$ will be the integral over it of the Euler class of the obstruction sheaf:

$$M_d := \int_{\overline{\mathcal{M}}(\mathbf{P}^1,d)} c_{2d-2}(\mathcal{F}_d) \ .$$

After Aspinwall and Morrison we expect that $M_d = d^{-3}$.

Actually, it is not clear why computations from [AM] give the same answer as we get with the stable curves. In [AM] authors consider the space of maps of rational curves with 3 marked points on it into $\mathbf{P}^1$ and compactified it by means of the Hilbert scheme of 1-dimensional subschemes in $\mathbf{P}^1 \times \mathbf{P}^1$ (they associate with a map $f : \mathbf{P}^1 \longrightarrow \mathbf{P}^1$ its graph). Then they used the Euler class of a natural candidate to the obstruction bundle and intersect it with a class of codimension 3. The answer which they get is 1. Of course, we can use $\overline{\mathcal{M}}_{0,3}(\mathbf{P}^1, d)$ instead of $\overline{\mathcal{M}}_{0,0}(\mathbf{P}^1, d)$ and modify the definition of "the number of curves" following the sample 2.1. One can see easily that the result will be $d^3 M_d$. At the moment we don't know how to relate the compactification from [AM] and the moduli spaces of stable maps.

3. Fixed points formulas.

3.1. Bott fixed points formula.

Let X be a smooth compact complex projective manifold, $\mathcal{E}$ be a holomorphic vector bundle on X. We suppose that a complex torus $\mathbf{T} \simeq \mathbf{C}^\times \times \cdots \times \mathbf{C}^\times$ acts algebraically on $(X, \mathcal{E})$.

Bott's formula reduces the computation of integrals of characteristic classes of $\mathcal{E}$ over X to the computations on the subspace of fixed points $X^{\mathbf{T}}$. This space is always a union of subvarieties, because the real subgroup $\mathbf{T}^{real} = U(1) \times \cdots \times U(1)$ is compact. We denote connected components of $X^{\mathbf{T}}$ by X^γ.

On each component X^γ, the vector bundle $\mathcal{E}$ splits into the direct sum of bundles $E^{\gamma,\lambda}$ over characters $\lambda : \mathbf{T} \longrightarrow \mathbf{C}^\times$, $\lambda \in \mathbf{T}^\vee \simeq \mathbf{Z} \oplus \cdots \oplus \mathbf{Z}$. Also, the normal bundle $\mathcal{N}^\gamma = \mathcal{T}_X / \mathcal{T}_{X^\gamma}$ splits into the direct sum of bundles $\mathcal{N}^{\gamma,\lambda}$, $\lambda \in \mathbf{T}^\vee \setminus \{0\}$.

We add to $H^{even}(X, \mathbf{Q})$ extra generators e_i, $i = 1, \ldots, rk(\mathcal{E})$ of degree 2 obeying relations

$$\sum_{k \geq 0} c_k(\mathcal{E}) = \prod_i (1 + e_i) \ .$$

Analogously, we add generators $e_i^{\gamma,\lambda}$ and $n_i^{\gamma,\lambda}$ to $H^{ev}(X^\gamma, \mathbf{Q})$.

Let P be a homogeneous symmetric polynomial (in sufficiently large number of variables) of degree $dim_{\mathbf{C}}(X)$. Bott's formula reads:

$$\int_X P(e_i) = \sum_\gamma \int_{X^\gamma} \frac{P(e_i^{\gamma,\lambda} + \lambda)}{\prod (n_i^{\gamma,\lambda} + \lambda)} \ .$$

Here the r.h.s. is considered as a rational function on $Lie(\mathbf{T})$ (each character λ defines a linear form on $Lie(\mathbf{T})$). In the numerator and the denominator we use all generators $e_i^{\gamma,\lambda}$ and $n^{\gamma,\lambda}$ with fixed index γ. Analogous formula is valid for finite collections $(\mathcal{E}^{(i)})_{i=1,N}$ of equivariant vector bundles and homogeneous polynomials P in N groups of variables symmetric inside each group.

This formula is valid for orbifolds too, because the original proof in [B] uses only the language of differential forms and transfers immediately to the more general setting of orbifolds.

3.2. Fixed points on moduli spaces of stable maps.

The action of the group $\mathbf{T} \simeq (\mathbf{C}^\times)^{n+1}$ of diagonal matrices on $\mathbf{P}^n$ induces an action of $\mathbf{T}$ on $\overline{\mathcal{M}}_{g,k}(\mathbf{P}^n, d)$. We will describe the set of fixed points in this subsection.

Denote by p_i, $i = 1, \ldots, n+1$ fixed points of $\mathbf{T}$ acting on $\mathbf{P}^n$. The point p_i is the projectivization of i-th coordinate line in $\mathbf{C}^{n+1}$. Also, denote by $l_{ij} = l_{ji}$, $i \neq j$ the line in $\mathbf{P}^n$ passing through p_i and p_j.

Let a stable map $f : C \longrightarrow \mathbf{P}^n$ represent a point of $\overline{\mathcal{M}}_{g,k}(\mathbf{P}^n, d)^{\mathbf{T}}$. First of all, the geometric image of f should be invariant under the $\mathbf{T}$-action. One can see easily that it means that $f(C)$ is a union of lines l_{ij}. Secondly, images of all marked and singular points, as well as of components of C contracted by f, should lie among points p_i. Thirdly, each irreducible component C^α of C which does not map to a point has genus zero and maps onto one of the lines l_{ij}. In some homogeneous coordinates it is given by

$$f(z_1 : z_2) = (0 : \ \ldots \ : 0 : z_1^{d_\alpha} : 0 : \ \ldots \ : 0 : z_2^{d_\alpha} : 0 : \ \ldots \ : 0), \ d_\alpha \geq 1 \ .$$

We will associate with each point $(C; x_1, \ldots, x_k; f) \in \overline{\mathcal{M}}_{g,k}(\mathbf{P}^n, d)^{\mathbf{T}}$ a graph Γ. By *graph* we mean a finite 1-dimensional CW-complex. Vertices $v \in Vert(\Gamma)$ correspond to connected components C_v of $f^{-1}(p_1, \ldots, p_{n+1})$. Note that each C_v can be either a point of C or a non-empty union of irreducible components of C. Edges $\alpha \in Edge(\Gamma)$ correspond to irreducible components C^α of genus zero mapping to lines l_{ij}. We endow Γ with additional specifications: vertices v will be labeled by numbers f_v from 1 to $n+1$ defined by the formula $f(C_v) = p_{f_v}$. Edges will be labeled by degrees $d_\alpha \in \mathbf{N}$. Also, we associate with each vertex $v \in Vert(\Gamma)$ its interior genus g_v (=arithmetic genus of the 1-dimensional part of $C_v \subset C$) and the set $S_v \subset \{1, \ldots, k\}$ of indices of marked points lying on C_v.

Our claim is that connected components of $\overline{\mathcal{M}}_{g,k}(\mathbf{P}^n, d)^{\mathbf{T}}$ are naturally labeled by equivalence classes of connected graphs Γ with specifications obeying the following conditions:

(1) if $\alpha \in Edge(\Gamma)$ connects vertices $v, u \in Vert(\Gamma)$ then $f_v \neq f_u$,

(2) $1 - \chi(\Gamma) + \sum_{v \in Vert(\Gamma)} g_v = g$,

(3) $\sum_{\alpha \in Edge(\Gamma)} d_\alpha = d$,

(4) $\{1, \ldots, k\} = \coprod_{v \in Vert(\Gamma)} S_v$.

Mention that from the condition (1) it follows that Γ has no simple loops.

Each component $\overline{\mathcal{M}}_{g,k}(\mathbf{P}^n, d)^\Gamma$ is isomorphic to the quotient space of the product of moduli spaces of stable curves over the set of vertices of Γ modulo action of the automorphism group of Γ. We will forget about $Aut(\Gamma)$ till 3.4.

3.3. Contributions of connected components.

From now on we assume that our curves have arithmetic genus zero. Graphs Γ in our description will be trees and interior genera of all vertices will be zero.

In 3.3.1-3.3.4, we will assume for the sake of simplicity that there is *no marked points* on curves. We will restore marked points in 3.3.5.

We denote $\overline{\mathcal{M}}(\mathbf{P}^n, d)$ simply by $\overline{\mathcal{M}}$ (numbers n and d are supposed to be fixed in this section).

For $\mathbf{T}$-equivariant vector bundle $\mathcal{E}$ on orbifold $\overline{\mathcal{M}}^\Gamma$ we denote by $[\mathcal{E}]$ the corresponding element of the equivariant K-group with rational coefficients:

$$K^0_{\mathbf{T}}(\overline{\mathcal{M}}^\Gamma) \otimes \mathbf{Q} \simeq K^0(\overline{\mathcal{M}}^\Gamma) \otimes \mathbf{T}^\vee \otimes \mathbf{Q} \; .$$

In 3.3.3-3.3.4, we will denote by $[\chi]$ the element of $K^0_{\mathbf{T}}(\overline{\mathcal{M}}^\Gamma) \otimes \mathbf{Q}$ corresponding to the trivial 1-dimensional bundle endowed with the action of $\mathbf{T}$ by the (orbi-)character $\chi \in \mathbf{T}^\vee \otimes \mathbf{Q}$.

We will denote the restriction of any vector bundle $\mathcal{E}$ on $\overline{\mathcal{M}}$ to $\overline{\mathcal{M}}^\Gamma$ by the same symbol $\mathcal{E}$. Often, we will denote a vector bundle on $\overline{\mathcal{M}}^\Gamma$ by its geometric fiber at a point (C, f). In intermediate computations in 3.3.1 we will use decomposition of fibers of vector bundles into formal linear combinations of some other vector spaces arising from short exact sequences. These auxilary vector spaces will not form vector bundles, because their dimensions will be not constant. Nevertheless, we will use these vector spaces as "vector bundles" putting corresponding symbols. One can check that the final result after all cancellations is a class of a virtual equivariant vector bundle, and our formal computations give the correct answer.

3.3.1. Normal bundle.

The class of the normal bundle to $\overline{\mathcal{M}}^\Gamma$ is

$$[\mathcal{N}_{\overline{\mathcal{M}}^\Gamma}] = [\mathcal{T}_{\overline{\mathcal{M}}}] - [\mathcal{T}_{\overline{\mathcal{M}}^\Gamma}] \; .$$

In 1.3.2 we computed the tangent space to $\overline{\mathcal{M}}$:

$$[\mathcal{T}_{\overline{\mathcal{M}}}] = [H^0(C, f^*(\mathcal{T}_{\mathbf{P}^n}))] + \sum_{y \in C^\alpha \cap C^\beta : \alpha \neq \beta} [T_y(C^\alpha) \otimes T_y(C^\beta)] +$$

$$+ \left(\sum_{y \in C^\alpha \cap C^\beta : \alpha \neq \beta} ([T_y(C^\alpha)] + [T_y(C^\beta)]) - \sum_\alpha [H^0(C^\alpha, \mathcal{T}_{C^\alpha})] \right) \; .$$

The first summand corresponds to infinitesimal deformations of the map f of a fixed curve C. The second summand corresponds to flat deformations of C resolving double singular points. The third summand comes from deformations of C preserving singular points. Its first part comes from deformations of singular points. We retract from it classes of 3-dimensional spaces of vector fields on irreducible components C^α.

The class of the tangent space to $\overline{\mathcal{M}}^\Gamma$ is by analogous reasons equal to

$$[\mathcal{T}_{\overline{\mathcal{M}}^\Gamma}] = \sum_{y \in C^\alpha \cap C^\beta : \alpha \neq \beta ; \alpha, \beta \notin Edge(\Gamma)} [T_y(C^\alpha) \otimes T_y(C^\beta)] +$$

$$+\sum_{y\in C^\alpha\cap C^\beta:\alpha\neq\beta,\alpha\notin Edge(\Gamma)}([T_y(C^\alpha)] - \sum_{\alpha:\alpha\notin Edge(\Gamma)}[H^0(C^\alpha,\mathcal{T}_{C^\alpha})] .$$

Here the first summand corresponds to resolutions of double singular points which are intersection points of two contracted components, the second summand comes from deformations of singular points on contracted components. Again, we retract classes of spaces of vector fields on contracted components.

Combinig all the formulas above we get:

$$[\mathcal{N}_{\overline{\mathcal{M}}^\Gamma}] = [H^0(C, f^*(\mathcal{T}_{\mathbf{P}^n}))] + [\mathcal{N}^{abs}_{\overline{\mathcal{M}}^\Gamma}] ,$$

where the "absolute" part of the normal bundle is

$$[\mathcal{N}^{abs}_{\overline{\mathcal{M}}^\Gamma}] := \sum_{y\in C^\alpha\cap C^\beta:\alpha\neq\beta;\alpha,\beta\in Edge(\Gamma)}[T_y(C^\alpha)\otimes T_y(C^\beta)] +$$

$$+\sum_{y\in C^\alpha\cap C^\beta:\alpha\in Edge(\Gamma),\beta\notin Edge(\Gamma)}[T_y(C^\alpha)\otimes T_y(C^\beta)] +$$

$$+\left(\sum_{y\in C^\alpha\cap C^\beta:\alpha\neq\beta,\alpha\in Edge(\Gamma)}[T_y(C^\alpha)] - \sum_{\alpha:\alpha\in Edge(\Gamma)}[H^0(C^\alpha,\mathcal{T}_{C^\alpha})]\right) .$$

Note that the first and the third summands in the formula for $[\mathcal{N}^{abs}_{\overline{\mathcal{M}}^\Gamma}]$ above are trivial vector bundles on $\overline{\mathcal{M}}^\Gamma$ twisted with some characters of the torus $\mathbf{T}$. Also, the term $[H^0(C, f^*(\mathcal{T}_{\mathbf{P}^n}))]$ has the same nature. Later on we will see that in all our examples all equivariant components of the vector bundle $\mathcal{E}$ will be trivial too ($e_i^{\Gamma,\lambda} = 0$ in notations of 3.1).

Hence in the Bott formula applied to $\overline{\mathcal{M}}$ we have only one term which is not just a mupltiplicative factor with values in the field of rational functions on $Lie(\mathbf{T})$. This term is

$$\sum_{y\in C^\alpha\cap C^\beta:\alpha\in Edge(\Gamma),\beta\notin Edge(\Gamma)}[T_y(C^\alpha)\otimes T_y(C^\beta)] .$$

We will compute corresponding integrals over $\overline{\mathcal{M}}^\Gamma$ in the next subsection. Actually, we will compute some integrals over $\overline{\mathcal{M}}_{0,k}$ such that the integral over $\overline{\mathcal{M}}^\Gamma$ will be equal to their product.

For an arbitrary graph, we define a flag as an edge endowed with an orientation (an arrow). We denote it by a pair (vertex, edge) of adjacent cells, where the vertex is the sourse of the arrow on the edge. In general, this notation is ambiguous for graphs with simple loops. Nevertheless, we will use it, because all graphs in our computations will be trees.

Notation. *for a flag $F = (v,\alpha)$ of Γ we denote by w_F expression $(\lambda_{f_v} - \lambda_{f_u})/d_\alpha$ where $u \in Vert(\Gamma), u \neq v$ is the second vertex of the edge α.*

We consider w_F as a linear finction on $Lie(\mathbf{T})$. The geometric meaning of w_F is the following: it is the character of the action of $\mathbf{T}$ on the tangent space to C^α at the point $C_v \cup C^\alpha$. The flag $F = (v,\alpha)$ has a canonical dual $\overline{F} = (u,\alpha)$ and weights of dual flags are related as $w_{\overline{F}} = -w_F$.

Our nearest goal is the computation of the contribution of $[\mathcal{N}^{abs}_{\overline{\mathcal{M}}^\Gamma}]$ in terms of w_F.

3.3.2. Intersection theory on $\overline{\mathcal{M}}_{0,k}$.
In this subsection k is an arbitrary integer bigger than, or equal to 3. Let w_i, $i = 1, \dots, k$ be a sequence of formal variables.
We compute in this subsection the following integral:

$$I(w_1, \dots, w_k) := \int\limits_{\overline{\mathcal{M}}_{0,k}} \prod_{i=1}^{k} \frac{1}{(w_i + c_1(T_{x_i}(C)))} .$$

Recall that $\overline{\mathcal{M}}_{0,k} = \overline{\mathcal{M}}_{0,k}(\text{point}, 0)$ denotes the moduli space of stable curves $(C; x_1, \dots, x_k)$ of genus zero with marked points.
The value of the integral I is a rational symmetric function in variables w_i. We can expand it as a finite Laurent series:

$$I(w_1, \dots, w_k) = \sum_{d_1, \dots, d_k \geq 0 : \sum d_i = k-3} \prod_{i=1}^{k} w_i^{-d_i - 1} \langle \tau_{d_1} \dots \tau_{d_k} \rangle_0 ,$$

where, following Witten [W] we denote by $\langle \tau_{d_1} \dots \tau_{d_k} \rangle_0$ the rational constant

$$\int\limits_{\overline{\mathcal{M}}_{0,k}} \prod_{i=1}^{k} c_1(T^*_{x_i}(C))^{d_i} .$$

The generating function for these numbers and analogous numbers for higher genera was predicted in [W] and computed rigorously in [K]. The result is quite complicated. However, for genus zero case the formulas for intersection numbers are very simple. Physicists new it already for a long time.

Lemma. $\langle \tau_{d_1} \dots \tau_{d_k} \rangle_0 = \frac{(k-3)!}{d_1! \dots d_k!}$.

Proof: intersection numbers for $\overline{\mathcal{M}}_{0,k}$ are uniquely defined by the following properties (see [W]):

(1) $\langle \tau_0 \tau_0 \tau_0 \rangle_0 = 1$,
(2) $\langle \tau_{d_1} \dots \tau_{d_k} \rangle_0$ is invariant under permutations of d_i,
(3) if $d_1 = 0$ then

$$\langle \tau_{d_1} \dots \tau_{d_k} \rangle_0 = \sum_{j \geq 2 : d_j \geq 1} \langle \tau_{d_2} \dots \tau_{d_j - 1} \dots \tau_{d_k} \rangle_0 .$$

One can check easily that $\frac{(k-3)!}{d_1! \dots d_k!}$ satisfies all the conditions above. □

Corollary. $I(w_1, \dots, w_k) = \prod_{i=1}^{k} w_i^{-1} \times \left(\sum_{i=1}^{k} w_i^{-1} \right)^{k-3}$.

3.3.3. Contribution of $\mathcal{N}^{abs}_{\overline{\mathcal{M}}^{\Gamma}}$.

The space $\overline{\mathcal{M}}^{\Gamma}$ is isomorphic to the product of $\overline{\mathcal{M}}_{0,val(v)}$ over vertices $v \in Vert(\Gamma)$ such that their valency $val(v) := \#\{\text{flags } (v,\alpha)\}$ is at least 3. (Recall that we omit the action of $\mathrm{Aut}(\Gamma)$ temporarily).

The contribution of

$$\sum_{y \in C^{\alpha} \cap C^{\beta}: \alpha \in Edge(\Gamma), \beta \notin Edge(\Gamma)} [T_y(C^{\alpha}) \otimes T_y(C^{\beta})]$$

in the multiplicative form is equal to

$$\prod_{v \in Vert(\Gamma),\, val(v) \geq 3} \left(\prod_{\text{flags } F=(v,\alpha)} w_F^{-1} \left(\sum_{\text{flags } F=(v,\alpha)} w_F^{-1} \right)^{val(v)-3} \right) .$$

This formula follows from 3.3.2 and the fact that $T_y(C^{\alpha})$ is trivial as a line bundle and $\mathbf{T}$ acts trivially on $T_y(C^{\beta})$.

Terms

$$\sum_{y \in C^{\alpha} \cap C^{\beta}: \alpha \neq \beta; \alpha, \beta \in Edge(\Gamma)} [T_y(C^{\alpha}) \otimes T_y(C^{\beta})]$$

correspond to vertices of Γ of valency 2. Their contribution is

$$\prod_{v \in Vert(\Gamma): val(v)=2} \left(w_{F_1(v)} + w_{F_2(v)} \right)^{-1} ,$$

where $F_i(v)$, $i = 1,2$ are two flags containing v. Note that one can rewrite this expression as

$$\prod_{v \in Vert(\Gamma): val(v)=2} \left(\prod_{\text{flags } F=(v,\alpha)} w_F^{-1} \left(\sum_{\text{flags } F=(v,\alpha)} w_F^{-1} \right)^{val(v)-3} \right) .$$

The contribution of terms

$$- \sum_{\alpha \in Edge(\Gamma)} [H^0(C^{\alpha}, T_{C^{\alpha}})]$$

in equivariant K-group is equal to

$$- \sum_{\alpha \in Edge(\Gamma)} \left([-w_{F(\alpha)}] + [0] + [w_{F(\alpha)}] \right) ,$$

where $F(\alpha)$ is any of two flags containing edge α. We rewrite this as

$$- \sum_{\text{flags } F} [w_F] - \sum_{\alpha \in Edge(\Gamma)} [0] .$$

The contribution of

$$\sum_{y\in C^\alpha\cap C^\beta:\alpha\neq\beta,\alpha\in Vert(\Gamma)} [T_y(C^\alpha)]$$

we rewrite as

$$\sum_{\text{flags } F=(v,\alpha):val(v)\geq 2} [w_F]\,.$$

Hence, the contribution of last two terms in the formula for $\mathcal{N}^{abs}_{\overline{\mathcal{M}}^\Gamma}$ is equal to

$$\sum_{\text{flags } F=(v,\alpha):val(v)=1} [w_F] + \sum_{\alpha\in Edge(\Gamma)} [0]\,.$$

Let us forget for a moment about the sum of [0] over edges. Then the contribution above can be expressed in the multiplicative form as the product of w_F over tails (i.e. flags $F=(v,\alpha)$ with $val(v)=1$). We replace w_F by $(w_F)^{-1}\left((w_F)^{-1}\right)^{-2}$ and note that the exponent -2 is equal to $val(v)-3$ again.

Conclusion: the contribution of

$$\mathcal{N}^{abs}_{\overline{\mathcal{M}}^\Gamma} - \sum_{\alpha\in Edge(\Gamma)} [0]$$

in the multiplicative form is equal to

$$\prod_{v\in Vert(\Gamma)}\left(\prod_{\text{flags } F=(v,\alpha)} w_F^{-1}\left(\sum_{\text{flags } F=(v,\alpha)} w_F^{-1}\right)^{val(v)-3}\right).$$

3.3.4. Contribution of $[H^0(C, f^*(\mathcal{T}_{\mathbf{P}^n}))]$.

The space of global sections of the vector bundle $f^*(\mathcal{T}_{\mathbf{P}^n})$ is equal to the subspace of

$$\bigoplus_{\alpha\in Edge(\Gamma)} H^0(C^\alpha, f^*(\mathcal{T}_{\mathbf{P}^n}))$$

given by the condition that the values of sections at each vertex v will be the same for all edges α adjacent to v. More precisely, we have the following short exact sequence of equivariant vector bundles on $\overline{\mathcal{M}}^\Gamma$:

$$0\to H^0(C, f^*(\mathcal{T}_{\mathbf{P}^n}))\to \bigoplus_{\alpha\in Edge(\Gamma)} H^0(C^\alpha, f^*(\mathcal{T}_{\mathbf{P}^n}))\to \bigoplus_{v\in Vert(\Gamma)}\left(T_{p_{f_v}}\mathbf{P}^n\otimes \mathbf{C}^{val(v)-1}\right)\to 0$$

First, we study contributions of $[H^0(C^\alpha, f^*(\mathcal{T}_{\mathbf{P}^n})]$. Edge α passes through two points $p_i, p_j\in(\mathbf{P}^n)^{\mathbf{T}}$. In some coordinate $z=(z_1:z_2)$ on $\mathbf{P}^1\simeq C^\alpha$, the map f is given by

$$X_i(f(z)) = z_1^d,\ X_j(f(z)) = z_2^d,\ \ X_k(f(z)) = 0 \text{ for } k\neq i,j\,.$$

Here X_k, $k = 1, n+1$ are homogeneous coordinates on $\mathbf{P}^n$. We have a short exact sequence of vector bundles on l_{ij}

$$0 \longrightarrow \mathcal{T}_{l_{ij}} \longrightarrow \mathcal{T}_{\mathbf{P}^n} \longrightarrow \mathcal{N}_{l_{ij}} \longrightarrow 0$$

inducing a corresponding exact sequence of vector bundles on $\overline{\mathcal{M}}^{\Gamma}$.

One can check using this exact sequence that the following elements form a base of $H^0(C^\alpha, f^*(\mathcal{T}_{\mathbf{P}^n}))$:

(1) $z^a X_i \partial/\partial X_i$, $-d_\alpha \le a \le d_\alpha$,
(2) $z^a X_i \partial/\partial X_k$, $-d_\alpha \le a \le 0$, $k \ne i, j$,
(3) $z^a X_j \partial/\partial X_k$, $0 \le a \le d_\alpha$, $k \ne i, j$.

Note that there is exactly one base element ($a = 0$ in the first group) on which T acts trivially. Thus, in all we have $\#Edge(\Gamma)$ terms $[0]$ cancelled with analogous terms in 3.3.3.

Homogeneity degree of z under the action of $\mathbf{T}$ is equal to $w_F = (\lambda_i - \lambda_j)/d_\alpha$, where F is a flag of Γ containing α and a vertex projecting to p_i. Degree of coordinate X_k is equal to λ_k.

The contribution of $T_{p_{f_v}}\mathbf{P}^n \otimes \mathbf{C}^{val(v)-1}$ where $f_v = i$ is equal to

$$(1 - val(v)) \sum_{j: j \ne i} [\lambda_i - \lambda_j] ,$$

because vectors $X_i \partial/\partial X_j$, $j \ne i$ form a base of $T_{p_i}\mathbf{P}^n$.

Putting all terms together we get the formula for the contribution of $[H^0(C, f^*(\mathcal{T}_{\mathbf{P}^n}))] + \#Edge(\Gamma)[0]$ in the multiplicative form:

$$\prod_{\text{flags } F=(v,\alpha)} \left((w_F)^{d_\alpha} \prod_{k \ne f_v, f_u : \overline{F}=(u,\alpha)} \prod_{a=0}^{d_\alpha} (a w_F + \lambda_{f_v} - \lambda_k) \right) \times \prod_{\alpha \in Edge(\Gamma)} (d_\alpha!)^2 \times$$
$$\times \prod_{v \in Vert(\Gamma)} \left(\prod_{j: j \ne f_v} (\lambda_{f_v} - \lambda_j) \right)^{1 - val(v)} .$$

3.3.5. Marked points.

The only result of the introducing of marked points is that one has to replace the exponent $(val(v) - 3)$ in the last formula in 3.3.3 by $(val(v) - 3 + \#S_v)$. We leave all checking to the reader.

It is reasonable to change a little bit graphs associated with connected components of the fixed point set. Namely, we replace any graph Γ with specifications as in 3.2 by a new graph $\widehat{\Gamma}$. Vertices of $\widehat{\Gamma}$ are vertices of Γ together with a k-element set $T = Tail(\widehat{\Gamma})$ (tails of $\widehat{\Gamma}$). Elements of T are numbered from 1 to k, tail t has number $i(t)$. Edges of $\widehat{\Gamma}$ are edges of Γ together with one edge α_t for each tail $t \in T$ connecting t with the unique vertex v of Γ such that $t \in S_v$. Also we define f_t to be equal f_v for $t \in S_v$. We pose d_{α_t} to be equal to 0 for all $t \in T$. Then, for any flag F of $\widehat{\Gamma}$ containing a tail as an edge, formally $w_F^{-1} = 0$.

In the sequel we will denote $\overline{\mathcal{M}}^{\Gamma}$ by $\overline{\mathcal{M}}^{\widehat{\Gamma}}$.

3.3.6. Contributions of vector bundles in examples.

As we already mentioned in 3.3.1, in all three examples from the section 2 vector bundles arising on $\overline{\mathcal{M}}_{0,*}(\mathbf{P}^n,d)^\Gamma$ split into direct sums of trivial bundles twisted with characters of $\mathbf{T}$. Thus, their contributions will be rational functions on $Lie(\mathbf{T})$ depending on Γ. Computation of these contributions is easier than that of the previous ones.

In the example 2.1 the contribution of line bundles $\mathcal{O}(1)_i$ is

$$\prod_{v\in Vert(\Gamma)} (\lambda_{f_v})^{2\# S_v} ,$$

or, using the modified graph $\widehat{\Gamma}$,

$$\prod_{t\in Tail(\Gamma)} \lambda_{f_t}^2 .$$

In the example 2.2 the contribution of $\mathcal{E}_d$ is

$$\prod_{\alpha\in Edge(\Gamma):(v_1,v_2)\text{ - vertices of }\alpha} \left(\prod_{a,b\geq 0:a+b=5d_\alpha} (a\lambda_{f_{v_1}} + b\lambda_{f_{v_2}}) \right) \times \prod_{v\in Vert(\Gamma)} (5\lambda_{f_v})^{1-val(v)} .$$

Here we use the short exact sequence (omitting zeroes from the left and the right):

$$H^0(C, f^*(\mathcal{O}(5))) \to \bigoplus_{\alpha\in Edge(\Gamma)} H^0(C^\alpha, f^*(\mathcal{O}(5))) \to \bigoplus_{v\in Vert(\Gamma)} \left(\mathcal{O}(5)_{p_{f_v}}) \otimes \mathbf{C}^{val(v)-1}\right) .$$

In the example 2.3 the contribution of $\mathcal{F}_d$ is

$$\prod_{\alpha\in Edge(\Gamma):(v1,v2)\text{ - vertices of }\alpha} \left(\prod_{a,b<0:a+b=1-d_\alpha} (a\lambda_{f_{v_1}} + b\lambda_{f_{v_2}}) \right) \times \prod_{v\in Vert(\Gamma)} (-\lambda_{f_v})^{val(v)-1} .$$

It follows from the short exact sequence

$$\bigoplus_{v\in Vert(\Gamma)} \left(\mathcal{O}(-1)_{p_{f_v}}) \otimes \mathbf{C}^{val(v)-1}\right) \to H^1(C, f^*(\mathcal{O}(-1))) \to \bigoplus_{\alpha\in Edge(\Gamma)} H^1(C^\alpha, f^*(\mathcal{O}(-1))) .$$

3.4. Final sum.

In each example the integral over the corresponding moduli space of stable maps is equal to the sum over equivalence classes of appropriate graphs of

$$\frac{1}{\#Aut(\widehat{\Gamma})}(\text{ formula from 3.3.3})(\text{ formula from 3.3.4})(\text{ formula from 3.3.6}) .$$

The last formula from 3.3.3 is corrected in the first example according to 3.3.5.

4. Critical values.

4.1. Feynman rules and summation over trees.

Here we will describe a general formula well known in physics combinatorics which gives values of certian infinite sums over trees. For an additional information on summation over trees and graphs one can look to the chapter 7 in [ID].

The initial data consist of a finite or countable set of indices A, symmetric non-degenerate matrix $g = (g^{ab})$, $g^{ab} = g^{ba}$, $a, b \in A$, and an infinite sequence of symmetric tensors with lower indices:

$$C_{a_1 \dots a_k}, \ \ a_i \in A, \ \ k \geq 0 \ .$$

Here coefficients of tensor g^*, C_* are complex numbers or elements of a topological field of characteristic zero (for example, a field of formal power series in auxilary variables with coefficients in $\mathbf{C}$). We assume that all inifnite series appearing later are convergent in an appropriate topology.

These data defines a function on the set of equivalence classes of finite graphs. Let Γ be a graph and $Flags(\Gamma)$ be the set of its flags. The weight of the graph is defined as

$$w(\Gamma) := \sum_{\substack{\text{maps} \\ f: Flags(\Gamma) \longrightarrow A}} \left(\prod_{\substack{\alpha \in Edge(\Gamma) \\ F_1, F_2 \text{ - flags} \\ \text{containing } \alpha}} g^{f(F_1) f(F_2)} \prod_{\substack{v \in Vert(\Gamma) \\ F_1, \dots, F_k \text{ - flags} \\ \text{containing } v}} C_{f(F_1) \dots f(F_k)} \right) .$$

Note that in all our examples one can choose an appropriate set of indices and tensors g, C such that the sum of contributions of connected components of $\overline{\mathcal{M}}^{\widehat{\Gamma}}$ corresponding to any abstract tree (without specifications d_α, f_v, S_v) will be equal to the weight of this tree. We will show in 4.1.1 how to choose A, g, and C in all our examples.

Define the "tree-level partition function" (just an element of the ground field) by formula

$$Z^{tree} := \sum_{\substack{\Gamma: \text{ equivalence classes of} \\ \text{finite nonempty trees } \Gamma}} \frac{1}{\#Aut(\Gamma)} w(\Gamma) \ .$$

Let us introduce auxilary formal variables ϕ_a, $a \in A$ and a series

$$S(\phi_*) := -\sum_{a,b \in A} \frac{g_{ab} \phi_a \phi_b}{2} + \sum_{k \geq 0} \frac{1}{k!} \sum_{a_1, \dots, a_k \in A} C_{a_1 \dots a_k} \phi_{a_1} \dots \phi_{a_k} \ .$$

Here g_{ab} denote matrix coefficients of the inverse matrix $(g)^{-1}$:

$$(g_{ab}) = (g^{ab})^{-1} \ .$$

Later it will be convenient to refer to the first summand in the formula for S as to "kinetic" part and to the second summand as to the "potential" part of the "action" functional S, in analogy with the classical mechanics.

Formula. $Z^{tree} = Crit\, S(\phi_*)$, *where the l.h.s. denotes the critical value of function* S.

This formula follows from a more general formula

$$log\left((det(2\pi\hbar g))^{-1/2}\int e^{\frac{S(\phi_*)}{\hbar}}\prod_{a\in A} d\phi_a\right) = \sum_{\substack{\Gamma:\text{ equivalence classes of}\\ \text{connected nonempty graphs }\Gamma}} \frac{\hbar^{-\chi(\Gamma)}}{\#Aut(\Gamma)}w(\Gamma)\ ,$$

where both sides of the formula are considered as formal power series expansions at $\hbar \to 0$.

The last formula is the usual expansion of integrals over Feynman diagrams.

The argument above is valid only in the case of coefficients with values in $\mathbf{C}$ when A is finite, g^* is real-valued and positive-definite, C_* are sufficiently small and the integral above is convergent.

A direct proof of the formula $Z^{tree} = Crit\, S(\phi_*)$ can be obtained by a formal inversion of the map $(\phi_*) \mapsto dS_{|(\phi_*)}$ and evaluating S to the critical point.

4.1.1. Action functionals in three examples.

Let us first describe the situation without marked points on curves (and tails on graphs).

We denote variables by $\phi_{ij,d}$ where $i \neq j,\ \ i,j \in \{1,\dots,n+1\},\ \ d \geq 1$. Hence, the set of indices is

$$A = \left(\{1,\dots,n+1\}^2 \setminus diagonal\right) \times \mathbf{N}\ .$$

For any graph Γ, we put on each flag $F = (v,\alpha)$ the composite index ij,d , where $d = d_\alpha$, $i = f_v$, $j = f_u$ (here ,as usual, u denotes the second vertex of α).

Potential part of the action is standard:

$$S^{pot}(\phi_*) = \sum_{i:1\leq i\leq n+1} \mu_i c_i \sum_{k\geq 1}\frac{1}{k!}\sum_{\substack{j_1,\dots,j_k:j_*\neq i\\ d_1,\dots,d_k:d_*\geq 1}} (v_{ij_1,d_1}+\dots+v_{ij_kd_k})^{k-3}\phi_{ij_1,d_1}\dots\phi_{ij_kd_k}\ ,$$

where $v_{ij,d} := d/(\lambda_i - \lambda_j)$ is equal to w_F^{-1} for corresponding flags F and

$$\mu_i = \prod_{j:j\neq i}(\lambda_i - \lambda_j)\ ,$$

and some constant c_i depending on the situation. Potential part consists of the contribution of $\mathcal{N}^{abs}_{\overline{\mathcal{M}}\Gamma}$ divided by the product of w_F^{-1} and multiplied by one factor μ_i in the contribution of $[H^0(C, f^*(\mathcal{T}_{\mathbf{P}^n}))]$, and c_i coming from the vector bundle on $\overline{\mathcal{M}}$. We will find another formula for $S^{pot}(\phi_*)$ in the next subsection.

Coefficients $g^{ij,d;i'j',d'}$ will be non-zero only for $i = j'$, $j = i'$, $d = d'$. This garantees that graphs with indices on flags which have non-zero weight will be in one-to-one correspondence with graphs with specifications of the type introduced in 3.2. We will denote $g^{ij,d;ji,d}$ simply by $g^{ij,d}$. Inversion of the matrix g is reduced to the inversions of numbers $g^{ij,d}$.

Contribution of $[H^0(C, f^*(\mathcal{T}_{\mathbf{P}^n}))]$ in $g^{ij,d}$ (with removed factor μ_i and added factors w_F^{-1}) is equal to

$$-w_F^{2d-2}(d!)^2\mu_i^{-1}\mu_j^{-1}\prod_{k\neq i,j}\ \prod_{a,b\geq 0:a+b=d}\left(\frac{a\lambda_i+b\lambda_j}{d}-\lambda_k\right)^2 .$$

The contribution of $\mathcal{E}_d$ in the example 2.2 is equal to

$$(25\lambda_i\lambda_j)^{-1}\prod_{a,b\geq 0:a+b=5d}(a\lambda_i+b\lambda_j) .$$

Constant c_i in the potential part is $5\lambda_i$.

In the example 2.3, the contribution of $\mathcal{F}_d$ is equal to

$$\lambda_i\lambda_j\prod_{a,b<0:a+b=1-d}(a\lambda_i+b\lambda_j) .$$

Constant c_i in the potential part is $(-\lambda_i)^{-1}$.

In both examples 2.2 and 2.3, we multiply $g^{ij,d}$ by z^d, where z is a new formal variable. Thus the resulting critical value will be a series in z with coefficients equal to N_d and M_d respectively.

In the example 2.1, we add variables $\tilde{\phi}_i$, $1\leq i\leq n+1=3$. Matrix g is the same as above for indices from the set A (no contributions from vector bundles), and

$$g^{i;ij,d}=0,\ \ g^{i;j}=\delta_{ij}\,\lambda_i^2 .$$

Here δ_{ij} is the Kronecker symbol. The potential part is equal to

$$S^{pot}(\phi_*,\tilde{\phi}_*)=\sum_{i:1\leq i\leq n+1}\mu_i\sum_{k\geq 1}\frac{1}{k!}\sum_{l\geq 0}\frac{1}{l!}$$

$$\left(\sum_{\substack{j_1,\dots,j_k:j_\bullet\neq i\\ d_1,\dots,d_k:d_\bullet\geq 1}}\ \sum_{\tilde{j}_1,\dots,\tilde{j}_l}(v_{ij_1,d_1}+\dots+v_{ij_kd_k})^{k+l-3}(\phi_{ij_1,d_1}\dots\phi_{ij_kd_k})(\tilde{\phi}_{\tilde{j}_1}\dots\tilde{\phi}_{\tilde{j}_l})\right) .$$

As before, we multiply matrix coeficients of g by extra variables:

$$g^{ij,d}\mapsto z_1^d\,g^{ij,d},\ \ g^{i;i}\mapsto z_2\,g^{i,i} .$$

Exponents of variables z_1,z_2 count the total degree of curves and the number of marked points respectively. Then in the resulting sum over trees considered as a series in z_1,z_2 we have to extract monomials of the form $z_1^dz_2^{3d-1}$. It can be done by a contour integration.

4.2. Potential part of the action functional.

$S^{pot}(\phi_*)$ comes essentially from the intersection numbers of $\overline{\mathcal{M}}_{0,k}$.

In the sequel of this subsection we use some auxilary set of indices I and two sequences of variables $v_i,\ \phi_i,\ \ i \in I$. Later we will set $v_i := w_*^{-1}$. In the notations of the previous subsection, we consider only a summand of S^{pot} corresponding to a fixed point $p_* \in \mathbf{P}^n$ and omit multiples μ_*, c_*.

Define a function S by the formula

$$S(v_*, \phi_*) := \sum_{k \geq 1} \frac{1}{k!} \sum_{i_1, \dots, i_k \in I} (v_{i_1} + \dots + v_{i_k})^{k-3} \phi_{i_1} \dots \phi_{i_k} \ .$$

Theorem. *$S(v_*, \phi_*) = Crit\, B(\xi)$, where*

$$B(\xi) = \frac{\xi^3}{6} + \frac{1}{2} \sum_{i,j \in I} \phi_i \phi_j \frac{exp(\xi v_i + \xi v_j)}{v_i + v_j} + \sum_{i \in I} \phi_i \frac{exp(\xi v_i)}{v_i^2} - \sum_{i \in I} \phi_i \frac{\xi\ exp(\xi v_i)}{v_i} \ .$$

Note that in the final form we will have $n+1$ distinct variables ξ_i corresponding to different parts of S^{pot}.

The rest of this subsection will be devoted to the proof of the theorem.

First of all, we dismantle the definition of S into simple pieces, doing opposite to what was done in 3.3.3. May be it is not the most economical way to prove our formula.

Denote by S_k the k-th summand in the definition of S. For $k \geq 3$, we have:

$$S_k(v_*, \phi_*) = \frac{1}{k!} \sum_{i_1, \dots, i_k \in I} (v_{i_1} + \dots + v_{i_k})^{k-3} \phi_{i_1} \dots \phi_{i_k} =$$

$$= \frac{(k-3)!}{k!} \sum_{i_1, \dots, i_k \in I} \ \sum_{d_1, \dots, d_k \geq 0 : \sum d_j = k-3} \ \prod_{j=1}^{k} \frac{\phi_{i_j} v_{i_j}^{d_j}}{d_j!} \ .$$

Let us introduce more notations:

$$\mathcal{Z}_d := \sum_{i \in I} \phi_i \frac{v_i^d}{d!}, \quad \mathcal{Z} := \sum_{d=0}^{\infty} \mathcal{Z}_d \xi^{d-1} = \xi^{-1} \sum_{i \in I} \phi_i\, exp(\xi v_i) \ .$$

We can rewrite the formula for S_k above as

$$S_k(v_*, \phi_*) = \frac{(k-3)!}{k!}\ Coeff_{\xi^{-3}} \left(\sum_{d_1, \dots, d_k \geq 0} (\mathcal{Z}_{d_1} \xi^{d_1 - 1}) \dots (\mathcal{Z}_{d_k} \xi^{d_k - 1}) \right) =$$

$$= \frac{(k-3)!}{k!}\ Coeff_{\xi^{-3}} \mathcal{Z}^k \ .$$

Thus,

$$\sum_{k \geq 3} S_k = Coeff_{\xi^{-3}} \left(\sum_{k \geq 3} \frac{(k-3)!}{k!} \mathcal{Z}^k \right) = Coeff_{\xi^{-3}} \left(\widetilde{\Psi}(\mathcal{Z}) \right) ,$$

where

$$\widetilde{\Psi}(\mathcal{Z}) := \frac{\mathcal{Z}^3}{1\cdot 2\cdot 3} + \frac{\mathcal{Z}^4}{2\cdot 3\cdot 4} + \cdots = \frac{(1-\mathcal{Z})^2}{2} log \frac{1}{1-\mathcal{Z}} + \frac{3}{4}\mathcal{Z}^2 - \frac{\mathcal{Z}}{2}\,.$$

We can replace $\widetilde{\Psi}(\mathcal{Z})$ by $\Psi(\mathcal{Z}) = \frac{(1-\mathcal{Z})^2}{2} log \frac{1}{1-\mathcal{Z}}$, because we take the coefficient of the monomial ξ^{-3} and $\mathcal{Z}$ has pole of the first order at $\xi = 0$.

$$Coeff_{\xi^{-3}}(\Psi(\mathcal{Z})) = \frac{1}{2\pi i} \oint\limits_{|\xi|=1} \frac{(1-\mathcal{Z})^2}{2} log \frac{1}{1-\mathcal{Z}}\ \xi^2 d\xi =$$

$$= \frac{1}{2\pi i} \oint\limits_{|\xi|=1} A'(\xi)\, log\, \frac{1}{1-\mathcal{Z}(\xi)} d\xi\,,$$

where regular function $A(\xi)$ is defined by conditions

$$A'(\xi) = \frac{(\xi - \xi\mathcal{Z}(\xi))^2}{2}\,, \quad A(0) = 0\,.$$

Now we can integrate by parts:

$$Coeff_{\xi^{-3}}(\Psi(\mathcal{Z})) = -\frac{1}{2\pi i} \oint\limits_{|\xi|=1} A(\xi) \left(log \frac{1}{1-\mathcal{Z}(\xi)} \right)' d\xi =$$

$$= -\frac{1}{2\pi i} \oint\limits_{|\xi|=1} A(\xi) \frac{\mathcal{Z}'(\xi)}{1-\mathcal{Z}(\xi)} d\xi = res_{\xi_0}\left(A(\xi)\, d\, log(1-\mathcal{Z}(\xi))\right) = A(\xi_0)\,.$$

Here ξ_0 is the root of equation $\mathcal{Z}(\xi) = 1$. Note that by the definition of $A(\xi)$ its derivative at ξ_0 vanishes. Hence,

$$Coeff_{\xi^{-3}}(\Psi(\mathcal{Z})) = Crit\ A(\xi)\,.$$

We can compute $A(\xi)$ explicitly:

$$A'(\xi) = \frac{(\xi - \xi\mathcal{Z}(\xi))^2}{2} = \frac{\xi^2}{2} + \frac{1}{2}\left(\sum_{i\in I} \phi_i\, exp(\xi v_i)\right)^2 - \xi \sum_{i\in I} \phi_i\, exp(\xi v_i)\,,$$

$$A(\xi) = \frac{\xi^3}{6} + \frac{1}{2}\sum_{i,j\in I} \phi_i\phi_j\, \frac{exp(\xi v_i + \xi v_j) - 1}{v_i + v_j} +$$

$$+ \sum_{i\in I} \phi_i \frac{exp(\xi v_i) - 1}{v_i^2} - \sum_{i\in I} \phi_i \frac{\xi\, exp(\xi v_i)}{v_i}\,.$$

After adding to the formula above two terms $S_1(v_*, \phi_*)$ and $S_2(v_*, \phi_*)$ which do not depend on ξ we obtain $B(\xi)$. $\square$

4.2.1. Effect of marked points.

We need a generalization of previous computations to the case when some v_i are equal to zero (for curves with marked points).

Now we have two groups of indices I, J, and variables v_i, ϕ_i, $i \in I$, and $\tilde{\phi}_j$, $j \in J$. Potential $S(v_*, \phi_*, \tilde{\phi}_j)$ is defined by the formula

$$S := \sum_{k \geq 1} \frac{1}{k!} \sum_{l \geq 0} \frac{1}{l!} \sum_{i_1, \ldots, i_k \in I} \sum_{j_1, \ldots, j_l \in J} (v_{i_1} + \cdots + v_{i_k})^{k+l-3} (\phi_{i_1} \ldots \phi_{i_k})(\tilde{\phi}_{j_1} \ldots \tilde{\phi}_{j_l}) .$$

One can rewrite it as

$$\sum_{k \geq 1} \frac{1}{k!} \sum_{i_1, \ldots, i_k \in I} (v_{i_1} + \cdots + v_{i_k})^{k-3} \phi_{i_1} \ldots \phi_{i_k} \, exp \left(\widetilde{\Phi} (v_{i_1} + \cdots + v_{i_k}) \right) ,$$

where $\widetilde{\Phi} = \sum_{j \in J} \tilde{\phi}_j$. Thus, we can use previous formulas from 4.2 with ϕ_i replaced by $\phi_i \, exp(\widetilde{\Phi} \, v_i)$.

4.3. The structure of the resulting functional.

Note that we have the following general scheme in all examples: the sum over trees is equal to the extremal values of a function in infinitely many variables ϕ_* and finitely many variables ξ_* and $\tilde{\phi}_*$ (variables λ are considered as constant and the result should not depend on them). The resulting functional is *quadratic* in ϕ. Thus, we can, in principle, find its extremal value by solving a system of linear equations. This system is infinite and it is not easy to solve it. At the moment we don't know how to proceed.

4.4. Examples.

We will not write the formula for the example 2.1.

It is convenient to rescale variables ϕ_* as

$$\phi_{ij,d} \mapsto \phi_{ij,d} \, exp(\phi_i v_{ij,d}) .$$

For 2.2 (curves on quintics) the functional is

$$\frac{1}{2} \sum_{\substack{i,j;d \\ i \neq j}} \frac{d^3 (\lambda_i - \lambda_j)^2 \prod_{a+b=d: a,b \geq 1} \prod_{k=1}^{5} (a\lambda_i + b\lambda_j - d\lambda_k)}{\prod_{a+b=5d: a,b \geq 1} (a\lambda_i + b\lambda_j)} exp \left(-td - \frac{\xi_i - \xi_j}{\lambda_i - \lambda_j} \right) \phi_{ij,d} \phi_{ji,d} +$$

$$+ \frac{1}{2} \sum_{\substack{i,j,d,j',d' \\ j,j' \neq i}} \nu_i \frac{\phi_{ij,d} \phi_{ij',d'}}{\frac{d}{\lambda_i - \lambda_j} + \frac{d'}{\lambda_i - \lambda_{j'}}} -$$

$$- \sum_{\substack{i,j,d \\ i \neq j}} \nu_i \frac{\lambda_i - \lambda_j}{d} \xi_i \phi_{ij,d} + \sum_{\substack{i,j,d \\ i \neq j}} \nu_i \frac{(\lambda_i - \lambda_j)^2}{d^2} \phi_{ij,d} + \sum_{i=1}^{5} \nu_i \frac{\xi_i^3}{6} .$$

Here ν_i denotes $\frac{5\lambda_i}{\prod_{j:j\neq i}(\lambda_i-\lambda_j)}$. The extremum is taken over variables ϕ_* and ξ_*. The result does not depend on λ_* and is equal (conjecturally) to the function $F(t)$ from 2.2 minus $5t^3/6$.

If we specify λ to be equal to roots of 1: $\lambda_j := exp(2\pi\sqrt{-1}j/5)$, then by homogeneity at the extremal point we have $\xi_j = \Xi\lambda_j$ for some $\Xi = \Xi(t)$. The last summand in the formula for the functional is equal to $5\Xi^3/6$. Note that Ξ is in a sense opposite to t because in the first term we have the exponent

$$exp\ (-(t+\Xi)d)\ .$$

In general we expect that the cubic term from the main formula of 4.2 corresponds to the contribution of maps of degree zero in the potential (see [KM]).

For 2.3 (multiple coverings), we will write only a formula for the functional with fixed λ-s: $\lambda_1 = -\lambda_2 = 1$. After some simplifications and changings of notations we get the following formula:

$$\sum_{k,l}\frac{\phi_k\phi_l}{k+l} - \sum_k \frac{2^{4k}}{k}\binom{2k}{k}^{-2} exp\ (-k(t+X))\ \phi_k^2 - \frac{X}{2}\sum_k\frac{\phi_k}{k} + \sum_k\frac{\phi_k}{k^2} + \frac{X^3}{6}\ .$$

Indices k,l take values in the set $\{1,3,5,\dots\}$. Conjecturally, the critical value of this functional (with fixed t) is equal to $Li_3(e^t)$.

5. Generalizations.

5.1. Higher genera.

We don't know at the moment how to treat higher genus curves in the same way as we do it with rational curves. One basic problem is that the moduli space of stable maps is never smooth for higher genus. May be, this is not a serious obstruction if one adopts the general phylosophy of hidden smoothness presented in 1.4, and one can apply Bott's formula ignoring singularities. If all these will work, we will use intersection numbers on $\overline{\mathcal{M}}_{g,k}$ including numbers computed in [K] and, may be, something else.

We hope, that the generating function over all genera ("string partition function") will be equal to a sum over graphs and reduces finally to a Feynman integral with auxilary matrices among fields (as in [K]). Also, we hope that a trick from 4.2 will work in quantum case too, reducing the final integral to a finite-dimensional one via an elimination of free fields.

The simplest case from which it is reasonable to start is the case of the projective plane. How many curves of genus g and degree d in $\mathbf{P}^2$ pass through generic $3d-1+g$ points?

5.2. Flag spaces and toric varieties.

All our computational scheme works well for any generalized flag space G/P, where G is a semi-simple algebraic group and P is a parabolic subgroup.

All we need is that the moduli spaces of genus zero stable maps are smooth and the Cartan subgroup T has isolated fixed points and isolated 1-dimensional orbits on G/P.

The first problem is to compute genus zero Gromov-Witten invariants on flag varieties. As was explained in [KM], heuristic arguments of [GK] don't give the whole information encoded in the potential.

Moduli spaces of genus zero stable maps to a toric variety are not smooth in general. Nevertheless, it is possible that the Bott formula can be modified and applicable again.

5.3. Complete intersections.

We can treat any smooth complete intersection of hypersurfaces in projective space in the same manner as quintic 3-folds. In generalizations to other varieties endowed with a torus action, we should consider equivariant vector bundles generated by global sections. If we have realized V as a zero set of a section of such a vector bundle transversal to zero section, then all the machinery applies.

5.4. Families.

Let us consider counting problems of genus zero curves on $K3$-surfaces. The first impression is that it should be trivial, because there are no non-trivial curves on non-algebraic Kähler surfaces and Gromov-Witten classes are invariant under deformations.

Let us consider now a 1-parameter holomorphic family of $K3$-surfaces S_t such that S_0 is algebraic and, for almost all t, S_t is not algebraic. The union $\mathcal{S} := \bigcup_t S_t$ is a non-compact 3-dimensional complex variety with the trivial first Chern class. Hence, we expect that compact rational curves on a generic small almost-complex perturbation of $\mathcal{S}$ are isolated and there will be finitely many of them sitting on S_0 in the limit as the perturbation tends to zero. D. Morrison (private communication) proposed to consider a particular 1-parameter deformation of $K3$-surfaces, namely, the twistor family of Kähler structures with a fixed Ricci-flat metric.

It seems that considering Gromov-Witten invariants of total spaces of families is reasonable only for genus zero curves, otherwise parasitic contributions to the virtual tangent bundle appear.

For V being a generic quartic surface in $\mathbf{P}^3$ the Picard group $Pic(V) \simeq \mathbf{Z}$ is generated by the plane section. Thus, degrees of curves on V are divisible by 4. The dimension of the space of curves of degree $4d$ on V is equal to $2d^2 + 1$ and the genus of the generic curve of degree $4d$ is also equal to $2d^2 + 1$. Hence, we expect a finite number of rational curves of degree $4d$ with $2d^2 + 1$ nodes. We beleive that these numbers fit into the picture above, because generic quartic has a canonical 1-st order non-algebraic deformation. Unfortunately, we were not able to define numbers of rational curves on quartics following pattern of section 2. Presumably, there should be a Mirror relation between these numbers and a variation of Hodge structures with one of periods equal to

$$I(z) = \sum_{n \geq 0} \frac{(4n)!}{(n!)^4} z^n \ .$$

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REFERENCES

[AM] P. S. Aspinwall, D. R. Morrison, *Topological field theory and rational curves*, Commun. Math. Phys. **151** (1993), 245–262.

[BFM] P. Baum, W. Fulton, R. MacPherson, *Riemann-Roch for singular varieties*, Publ. Math. I. H. E. S. **45** (1975), 101–145.

[B] R. Bott, *A residue formula for holomorphic vector fields*, Jour. Diff. Geom. 1 (1967), 311-330.

[DM] P. Deligne, D. Mumford, *The irreducibility of the space of curves of given genus*, Publ. Math. I. H. E. S. **36** (1969), 75-110.

[GK] A. Givental, B. Kim, *Quantum cohomology of flag manifolds and Toda lattices*, preprint (1993).

[ID] C. Itzykson, J.-M. Drouffe, *Statistical field theory*, Cambridge University Press, 1989.

[K] M. Kontsevich, *Intersection theory on the moduli space of curves and the matrix Airy function*, Commun. Math. Phys. **147** (1992), 1–23.

[KM] M. Kontsevich, Yu. Manin, *Gromov-Witten classes, quantum cohomology and enumerative geometry*, MPI preprint and hep-th/9402147 (1994).

[P] P. Pansu, *Chapter VIII, Compactness*, Holomorphic curves in symplectic geometry, eds. M. Audin, J. Lafontaine, Progress in Mathematics, vol. 117, Birkhäuser, 1994.

[RT] Y. Ruan, G. Tian, *Mathematical theory of quantum cohomology*, preprint (1993).

[W] E. Witten, *Two-dimensional gravity and intersection theory on moduli space*, Surveys in Diff. Geom. **1** (1991), 243–310.

[Y] S. T. Yau, ed., *Essays on Mirror Manifolds*, Iternational Press Co. , Hong Kong, 1992.

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