Polycommutators of vector fields

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Mathematische Arbeitstagung June 10-16, 2005
Max-Planck-Institut für Mathematik, Bonn, Germany

In our talk we discuss the following topics related to generalisations of commutators

- N-commutators
- nilpotency of odd derivations
- q-commutators, and
- algebras with skew-symmetric identities

Partially these results are published in [Dzh1], [Dzh4].

1 N-commmutators

Let X_1 and X_2 are vector fields (differential operators of first order). In general their composition is not a vector field. It is a differential operator of second order,

$$X_1 = u_i\partial_i, X_2 = v_j\partial_j \Rightarrow X_1X_2 = u_i\partial_i(v_j)\partial_j + u_iv_j\partial_i\partial_j.$$

To obtain vector field we need to calculate a commutator

$$[X_1, X_2] = X_1X_2 - X_2X_1 = u_i\partial_j(v_j)der_i - v_i\partial_i(u_j)\partial_j.$$

Now we consider k-ary generalisation of commutators. Let

$$s_k = \sum_{\sigma\in Sym_k} sign\,\sigma\,(\cdots((t_{\sigma(1)}t_{\sigma(2)})t_{\sigma(3)})\cdots)t_{\sigma(k)}$$

be standard skew-symmetric polynomial of degree k. Then

$$[X_1, X_2] = s_2(X_1, X_2).$$

In general $s_k(X_1, \ldots, X_k)$ are differential operators of order k.

Problem. *Is it possible to define on s space of vector fields $Vect(n)$ a new tensor operation induced by multiplication s_k.*

In other words, might it happen that for some $k = k(n)$ all higher differntial degrees are cancelled ?

As it turned out for some $k = k(n)$ such situation is possible. For example, if $n = 2$, then for any $X_1, \ldots, X_6 \in Vect(2)$ a differential operator $s_6(X_1, \ldots, X_6)$ is once again a vectoe field. All degree 2,3,4,5,6-parts are cancelled.Moreover, the number 6 here can not be improved. If one consider s_7 instead of s_6 here are cancelled all differential parts including linera part,

$$s_7(X_1, \ldots, X_7) = 0,$$

for any $X_1, \ldots, X_7 \in Vect(2)$, It is easy to see that

$$s_k(X_1, \ldots, X_k) = 0, \quad \forall X_1, \ldots, X_k \in Vect(2),$$

if $k > 6$. As far as s_5, it is not well defined operation on $Vect(2)$. For example,

$$s_5(\partial_1, \partial_2, x_1\partial_1, x_2\partial_1, x_2\partial_2) = \partial_1^2.$$

If one restricts consideration to divergenceless vector fields $Vect_0(2) \subset Vect(2)$, then s_5 will be well-defined operation in $Vect_0(2)$.

$$\forall X_1, \ldots, X_5 \in Vect_0(2) \Rightarrow s_5(X_1, \ldots, X_5) \in Vect_0(2).$$

Write divergenceless vector field X_i in a form

$$X_i = D_{1,2}(u_i) = \partial_1(u_i)\partial_2 - \partial_2(u_i)\partial_1,$$

where u_i is a potencial of X_i. Then 5-commutator in terms of potencials can be written as a determinant.

Let U be an associative commutative algebra with two commuting derivations ∂_1 and ∂_2. Then

$$s_5(D_{12}(u_1), D_{12}(u_2), D_{12}(u_3), D_{12}(u_4), D_{12}(u_5)) = -3D_{12}([u_1, u_2, u_3, u_4, u_5]),$$

for any $u_1, \ldots, u_5 \in U$, where

$$[u_1, u_2, u_3, u_4, u_5] = \begin{vmatrix} \partial_1 u_1 & \partial_1 u_2 & \partial_1 u_3 & \partial_1 u_4 & \partial_1 u_5 \\ \partial_2 u_1 & \partial_2 u_2 & \partial_2 u_3 & \partial_2 u_4 & \partial_2 u_5 \\ \partial_1^2 u_1 & \partial_1^2 u_2 & \partial_1^2 u_3 & \partial_1^2 u_4 & \partial_1^2 u_5 \\ \partial_1\partial_2 u_1 & \partial_1\partial_2 u_2 & \partial_1\partial_2 u_3 & \partial_1\partial_2 u_4 & \partial_1\partial_2 u_5 \\ \partial_2^2 u_1 & \partial_2^2 u_2 & \partial_2^2 u_3 & \partial_2^2 u_4 & \partial_2^2 u_5 \end{vmatrix}$$

and $D_{12}(u) = \partial_1(u)\partial_2 - \partial_2(u)\partial_1$.

Theorem 1.1 *Let $n > 1$ and $N = n^2 + 2n - 2$. Then s_N is well-defined operation on $Vect(n)$.*

An algebra A with a serie of operations $\omega = (\omega_1, \omega_2, \ldots)$, where ω_N is N-ary operation, is called sh-Lie [Stashef] if

$$\sum_{\sigma, i+j=k-1, i,j\geq 1} (-1)^{(j-1)i} sign\, \sigma\, \omega_j(\omega_i(a_{\sigma(1)}, \ldots, a_{\sigma(i)}), a_{\sigma(i+1)}, \ldots, a_{\sigma(i+j-1)}) = 0,$$

for any $k = 1, 2, \ldots,$ and any $a_1, \ldots, a_{i+j-1} \in A$.

Theorem 1.2 *Algebra* (W_n, s_2, s_{n^2+2n-2}) *is sh-Lie.*

Appears one more question. Is it possible for $N > 2$ to construct other N-commutators on $Vect(n)$? We have established that 5-commutator on $Vect_0(2)$ and 6-commutator on $Vect(2)$ are unique. But for $n = 3$ there are two nontrivial N-commutators: 10-commutator and 13-commutator. To construct these commutators easier to use super-derivations language.

2 Powers of odd derivations

Here we give reformulation of our problem in terms of super-derivations. It is well known that a square of odd derivation is a derivation:

$$\varepsilon(D) = 1, D \in Der\, U \Rightarrow D^2 \in Der\, U.$$

Now we pose the following question: *Is it possible to construct some power D^N of odd derivation D such that D^N will be derivation also ?*

We find that this question is equivalent to the question on N-commutators of vector fields and to the question on nilpotency of odd derivations.

Suppose that we have an associative algebra A and we need to calculate alternating sum $s_k(a_1, \ldots, a_k) = \sum_{\sigma \in Sym_k} sign\, \sigma\, a_{\sigma(1)} \cdots a_{\sigma(k)}$. Let us show how can appear super-algebras and odd derivations.

Let L be Grassman algebra with generators $\xi_1, \xi_2, \ldots$. It is an infinite-dimensional associative super-commutative algebra. Let $L(A) = A \otimes G$ be super-algebra with multiplication $(a \otimes \xi)(a' \otimes \xi') = (aa') \otimes (\xi\xi')$ and parity $\varepsilon(a \otimes \xi) = \varepsilon(\xi)$. Let $D = \sum_{i=1}^{k} a_i \otimes \xi_k \in L(A)$. Then

$$D^k = s_k(a_1, \ldots, a_k) \otimes (\xi_1 \wedge \cdots \wedge \xi_k).$$

So, calculating of $s_k(a_1, \ldots, a_k)$ and k-th power of D are equivalent problems. In particular, $D^k = 0$ if and only if $s_k(a_1, \ldots, a_k) = 0$. If $s_k = 0$ is identity on A, then $D^k = 0$ for any odd derivation D of the form $D = \sum_{i=1}^{k} a_i \otimes \xi_i$.

Let $\mathcal{L}_n$ be an associative super-commutative algebra with odd generators denoted (α, s), where $\alpha \in \mathbf{Z}_+^n$ and $i = 1, 2, \ldots$. Define a derivation $\partial_i : \mathcal{L}_n \to \mathcal{L}_n$ by

$$\partial_i(\alpha, s) = (\alpha + \epsilon_i, s).$$

Then ∂_i became even derivation, commuting each other,

$$[\partial_i, \partial_j] = 0, \quad \forall i, j \in \{1, \ldots, n\}$$

and

$$\cap_{i=1}^{n} \ker \partial_i =< 1 >\cong \mathbf{C}.$$

In particular, the dimension of a linear span $\mathcal{D}$ generated by commuting derivations $\partial_1, \ldots, \partial_n$ is $\dim < \mathcal{D} >= n$. Then $\mathcal{L}_n$ became a $\mathcal{D}$-differential super-algebra. It becomes a free algebra in the category of $\mathcal{D}$-differential algebras. Further we use the following notations. If otherwise are not stated $u_s = (0, s)$ and $\partial^\alpha(u_s) = (\alpha, s)$. So, $\mathcal{L}_n$ is a super-algebra generated by odd elements

$\partial^\alpha u_s, s = 1, \ldots, n$ and commuting even derivations $\partial_1, \ldots \partial_n$. We can interpet $\mathcal{L}_n$ as an algebra of super-lagrangians generated by odd elements.

Let us show some examples of calculations of differential operators powers in $Diff\,\mathcal{L}_n$.

Example. Let $n = 1$ and $D = u_1\partial_1 \in Der\mathcal{L}_1$. Then

$$D^2 = u_1\partial_1(u_1)\partial_1 + u_1^2\partial_1^2.$$

Since u_1 is odd element, $u_1^2 = 0$ and

$$D^2 = u_1\partial(u_1)\partial_1$$

is a derivation and nilpotency index of D is 3:

$$D^3 = 0.$$

Example. Let $n = 2$ and $D = u_1\partial_1 + u_2\partial_2 \in \mathcal{L}_2$. Since u_1, u_2 are odd elements, then

$$D^2 = D(u_1)\partial_1 + D(u_2)\partial_2$$

and

$$D^4 = (D^2)^2$$

$$= (D(u_1)\partial_1(D(u_1)) + D(u_2)\partial_2(D(u_1))\partial_1 + (D(u_1)\partial_1(D(u_2) + D(u_2)\partial_2(D(u_2))\partial_2$$

$$+D(u_1)^2\partial_1^2 + D(u_2)D(u_1)\partial_2\partial_1 + D(u_1)D(u_2)\partial_1\partial_2 + D(u_2)D(u_2)\partial_2^2.$$

Since u_1, u_2 are odd elemens and ∂_1, ∂_2 are even derivations, we see that $D(u_1) = u_1\partial_1(u_1) + u_2\partial_2(u_1)$ and $D(u_2) = u_1\partial_1(u_2) + u_2\partial_2(u_2)$ are even elements, and

$$D(u_1)^2 = u_1\partial_1(u_1)u_2\partial_2(u_1) + u_2\partial_2(u_1)u_1\partial_1(u_1) = -2u_1u_2\partial_1(u_1)\partial_2(u_1),$$

$$D(u_2)^2 = u_1\partial_1(u_2)u_2\partial_2(u_2) + u_2\partial_2(u_2)u_1\partial_1(u_2) = -2u_1u_2\partial_1(u_2)\partial_2(u_2),$$

$$D(u_1)D(u_2) = u_1\partial_1(u_1)u_2\partial_2(u_2) + u_2\partial_2(u_1)u_1\partial_1(u_2) = u_1u_2(-\partial_1(u_1)\partial_2(u_2) + \partial_2(u_1)\partial_1(u_2)).$$

We have

$$\text{quadratic part of } D^4$$

$$= u_1u_2(-2\partial_1(u_1)\partial_2(u_1)\partial_1^2 - (\partial_2(u_1)\partial_1(u_2) + \partial_1(u_1)\partial_2(u_2))\partial_1\partial_2 - \partial_1(u_2)\partial_2(u_2)\partial_2^2).$$

Similar calculations show that quadratic part is disappear in D^6 and it has only differential linear part

$$D^6 = u_1u_2 \cdot \{$$

$$\{-\partial_1 u_1\, \partial_2 u_1\, \partial_1 u_2\, \partial_2^2 u_2\; - \partial_1 u_1\, \partial_2 u_1\, \partial_2 u_2\, \partial_1^2 u_1\; - \partial_2 u_1\, \partial_1 u_2\, \partial_2 u_2\, \partial_2^2 u_2$$

$$+2\,\partial_1 u_1\, \partial_2 u_1\, \partial_1 u_2\, \partial_{12} u_1 + 2\,\partial_1 u_1\, \partial_2 u_1\, \partial_2 u_2\, \partial_{12} u_2\; + 2\,\partial_2 u_1\, \partial_1 u_2\, \partial_2 u_2\, \partial_{12} u_1$$

$$-3\,\partial_1 u_1\, \partial_1 u_2\, \partial_2 u_2\, \partial_2^2 u_1\, \}\partial_1$$

$$\{-\partial_1 u_1\, \partial_1 u_2\, \partial_2 u_2\, \partial_2^2 u_2\; - \partial_1 u_1\, \partial_2 u_1\, \partial_1 u_2\, \partial_1^2 u_1\; - \partial_2 u_1\, \partial_1 u_2\, \partial_2 u_2\, \partial_1^2 u_1$$

$$+2\,\partial_1 u_1\, \partial_1 u_2\, \partial_2 u_2\, \partial_{12} u_1 + 2\,\partial_1 u_1\, \partial_2 u_1\, \partial_1 u_2\, \partial_{12} u_2\; + 2\,\partial_2 u_1\, \partial_1 u_2\, \partial_2 u_2\, \partial_{12} u_2$$

$$-3\,\partial_1 u_1\, \partial_2 u_1\, \partial_2 u_2\, \partial_1^2 u_2\, \}\partial_2\}$$

In other words D^6 is derivation for $n = 2$. It is not difficult to see that

$$D^7 = 0.$$

In terms of determinants this means that 6-commutator $s_6(X_1, \ldots, X_6)$ is a sum of fourteen 6×6-determinants and $s_k = 0$ is identity for $k > 6$.

Theorem 2.1 *Let $D = \sum_{i=1}^n u_i \partial_i \in \L_n$ with odd elements $u_1, \ldots, u_n$ and $n > 1$. Then*

$$D^{n^2+2n} = 0, \qquad D^{n^2+2n-2} \in Der\, \mathcal{L}_n.$$

Theorem 2.2 *Let $D = u_1\partial_1 + u_2\partial_2 + u_3\partial_3 \in \mathcal{L}_3$. Then*

$$D^{14} = 0, \qquad D^{10} \in Der\, \mathcal{L}_3, \qquad D^{13} \in Der L_3.$$

3 Escorts of N-commutators

Calculations of N-commutators or N-powers of odd derivations of super-Lagrangians algebra can be reduced to the problem on calculation invariants of sl_n-modules and sp_n-modules. Let us show how to do it for the case sl_n.

Let $W(n) = Der\, \mathbf{C}[x_1, \ldots, x_n]$ be Witt algebra. As a vector space $W(n)$ has a base $x^\alpha \partial_i$, where $\alpha = (\alpha_1, \ldots, \alpha_n), \alpha_i \in \mathbf{Z}_+$ is a multi-index,

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n},$$

and

$$\partial_i = \frac{\partial}{\partial x_i}$$

are partial derivations.Let Γ_n be a set of multi-indexes. Set for $\alpha \in \Gamma_n$

$$|\alpha| = \sum_{i=1}^n \alpha_i.$$

and

$$W(n)_s =< x^\alpha \partial_i | |\alpha| = s+1, \alpha \in \Gamma_n, i = 1, \ldots, n > .$$

Then for any $k, s \geq -1$,

$$[W(n)_k, W(n)_s] \subseteq W(n)_{k+s}.$$

In other words, $W(n)$ has natural grading

$$W(n) = \oplus_{k \geq -1} W(n)_k.$$

In particular, $W(n)$ has a subalgebra

$$W(n)_0 \cong gl_n$$

and any homogeneous component $W(n)_k$ has a structure of gl_n-module. For highest weight γ denote $R(\gamma)$ an irreducible sl_n)-module with weight γ. Let $\pi_1, \ldots, \pi_{n-1}$ are fundamental weights of sl_n. Then for any $k \geq -1$,

$$W(n)_k \cong R((k+1)\pi_1) \otimes R(\pi_{n-1}).$$

For example,

$$W(n)_{-1} \cong R(\pi_{n-1}).$$

Let

$$Q(n) = W(n)_0 + W(n)_{-1}$$

be semi-direct sum of gl_n with standard module. Let $U = \mathbf{C}[x_1, \ldots, x_n]$ be standard $W(n)$-module. Call a $Q(n)$-module M as $(Q(n), U)$-*module,* if M has additional structure of unital U-module such that

$$X(um) = X(u)m + u(X(m),$$

for any $X \in W(n), u \in U, m \in M$.

Say that M is $(Q(n), U)$-*module with base* M_0 if M is $(Q(n), U)$-module and as U-module M is a free module with base M_0. Notice that base of $(Q(n), U)$-module has a structure $W(n)_0$-module.

Example. $W(n)$ is $(Q(n), U)$-module with base $R(\pi_{n-1})$.

Let $\psi : \wedge^N W(n) \to W(n)_{-1}$ be $\mathcal{D}$-invariant map. Call the following gl_0-module *support* of ψ

$$supp\, \psi = \oplus_{(i_{-1} i_0, \ldots i_k)} \wedge^{i_{-1}} (W(n)_{-1}) \otimes \wedge^{i_0}(W(n)_0) \otimes \cdots \otimes \wedge^{i_k}(W(n)_k),$$

where summations are made by $(i_{-1}, i_0, \ldots, i_k)$, such that $i_1 + \cdots + i_k = N$ and $-i_{-1} + i_1 + \cdots + k i_k = -1$. Induced map

$$supp\ \psi \to W(n)_{-1}, \quad supp\ \psi(X_1, \ldots, X_N) = pr_{-1}\psi(X_1, \ldots, X_N),$$

is called *escort* of ψ. Here $pr : W(n) \to W(n)_{-1}$ is a projection,

In [Dzh2] is proved that $\mathcal{D}$-invariant map can be uniquily restored by its escort. So, to calculate N-commutator it is enough to calculate its escort.

Example. Let us find escort for 5-commutator on $Vect_0(2)$. We have

$$supp\, s_5 = \wedge^2 R(\pi_1) \otimes \wedge^2(R(2\pi_1)) \otimes R(3\pi_1) \cong R(2\pi_1) \otimes R(3\pi_1),$$

and calculating of 5-commutator is reduced to calculation

$$escort(s_5) = Hom(R(2\pi_1) \otimes R(3\pi_1), R(\pi_1)).$$

Well known that $(R(2\pi_1) \otimes R(3\pi_1)$ contains $R(\pi_1)$ with multiplicity 1. This fact is enough to construct 5-commutator.

One more observation that makes easier calculations of N-commutators concerns right-symmetric sructure on $W(n)$. On $W(n)$, let $\circ$ be the multiplication

$$u\partial_i \circ v\partial_j = v\partial_j(u)\partial_i.$$

Recall that the multiplication $\circ$ is *right-symmetric* if it satisfies the right-symmetric identity

$$X_1 \circ (X_2 \circ X_3 - X_3 \circ X_2) = (X_1 \circ X_2) \circ X_3 - (X_1 \circ X_3) \circ X_2.$$

Right-symmetric algebras are called also *pre-Lie, Vinberg,* or *Vinberg-Koszul* [Gerst], [Koszul], [Vinberg].

Example. $(W(n), \circ)$ is right-symmetric.

Theorem 3.1 *Let $N = n^2 + 2n - 2$ and $n > 1$. Then N-commutator can be calculated in terms of right-symmetric multiplication:*

$$s_N(X_1, \ldots, X_N) = \sum_{\sigma \in Sym_N} sign\, \sigma\, (\cdots((X_{\sigma(1)} \circ X_{\sigma(2)}) \circ X_{\sigma(3)}) \cdots) \circ X_{\sigma(N)},$$

for any $X_1, \ldots, X_N \in Vect(n)$.

4 q-commutators

Let $(A, \circ)$ be an algebra with a vector space A and a multiplication $\circ$. A q-commutator $\circ_q$ on A is defined by $a \circ_q b = a \circ b + q\, b \circ a$. Let $(a, b, c) = a \circ (b \circ c) - (a \circ b) \circ c$ be an associator. For some category of PI-algebras $\mathfrak{L}$ denote by $\mathfrak{L}^{(q)}$ a category of algebras $A^{(q)} = (A, \circ_q)$, where $A \in \mathfrak{L}$. Let $[a, b] = a \circ b - b \circ a$, and $\{a, b\} = a \circ b - b \circ a$.

Theorem 4.1 *Let $q^2 \neq 1$. Let $\mathfrak{Ass}$ be a category of associative algebras. Then the category $\mathfrak{Ass}^{(q)}$ is defined by the identity*

$$(q-1)^2(a, c, b) + q[c, [a, b]] = 0$$

if $q^2 - 4q + 1 \neq 0$. If $q^2 - 4q + 1 \neq 0$ then $\mathfrak{Ass}^{(q)}$ is equivalent to $\mathfrak{Ass}$. If $q^2 - 4q + 1 = 0$, then $\mathfrak{Ass}^{(q)}$ is equivalent to the category of alternative algebras.

Theorem 4.2 *Let $q^2 \neq 1, q^3 \neq -1$. Let $\mathfrak{Alt}$ be a category of alternative algebras. Then $\mathfrak{Alt}^{(q)}$ is defined by the identities*

$$(a, b, c) + (c, b, a) = 0,$$
$$(1 - q + q^2)c \circ \{a, b\} - q\{a, b\} \circ c - (q^2 + 1)((c \circ a) \circ b + (c \circ b) \circ a)$$
$$+2q((a \circ c) \circ b + (b \circ c) \circ a) = 0.$$

Theorem 4.3 *Let $q^2 \neq -1$. Let $\mathfrak{Rsym}$ be a category of right-symmetric algebras, i.e., algebras with identity $(a, b, c) + (a, c, b) = 0$. Then $\mathfrak{Rsym}$ satisfies the identity*

$$(q-1)(-(a, c, b) + (b, c, a)) + (q^2 - 1)((a, b, c) - (b, a, c)) - q[[a, b], c] = 0.$$

If $q^2 + 2q - 2 \neq 0$, then this identity is basic identity for $\mathfrak{Rsym}^{(q)}$. If $q^2 + 2q - 2 = 0$, then this identity and Lie-admissible identity form basis for identities of $\mathfrak{Rsym}^{(q)}$.

Theorem 4.4 *Let $\mathfrak{Lei}$ be the category of Leibniz algebras, i.e., algebras with the identity $(a \circ b) \circ c = a \circ (b \circ c) - b \circ (a \circ c)$. Then $\mathfrak{Lei}^{(-1)}$ is defined by skew-symmetric identity and by three identities of degree 5 (they are too huge to be presented here: they have 9, 60 and 62 terms). T-Ideal of identities for $\mathfrak{Lei}^{(1)}$ is generated by the identity $(a \circ b) \circ (c \circ d) = 0$. If $q^2 \neq 1$, then $\mathfrak{Lei}^{(q)}$ is generated by identities*

$$(1-q)a \circ (b \circ c) + (q^3 - q + 1)a \circ (c \circ b) - q^2 b \circ (a \circ c) - (q^3 - q)c \circ (a \circ b)$$
$$-q(b \circ c) \circ a + (q^3 + q^2 - q)(c \circ a) \circ b = 0,$$
$$-a \circ \{b, c\} + q\{b, c\} \circ a = 0.$$

If $(q+2)(q^4 + 2q^3 - q + 1) = 0$, then these identities are independent.

5 Algebras with skew-symmetric identities

Define non-commutative non-associative polynomials

$$alia^{(q)}(t_1,t_2,t_3) = [[t_1,t_2],t_3]_q + [[t_2,t_3],t_1]_q + [[t_3,t_1],t_2]_q,$$

where

$$[t_1,t_2]_q = t_1t_2 + q\,t_2t_1.$$

Recall also that an algebra $(A,\circ)$ is called skew-symmetric if $a\circ b = -b\circ a$, for any $a,b\in A$. Call skew-symmetric algebra s_k-Lie if it satisfies the identity $s_k = 0$. Notice that s_3-Lie algebras are nothing else Lie algebras.

Algebras with identity $alia^{(q)} = 0$ are called q-Anti-Lie-Admissible (shortly q-Alia). Notice that a class of -1-Alia algebras coincides with a class of Lie-admissible algebras.

Theorem 5.1 *Let A be an algebra with skew-symmetric identity of degree 3. Then A is either*

- 0-*Alia*
- 1-*Alia*
- -1-*Alia*
- *q-algebra of some* 0-*Alia algebra, where $q^3 \neq q$.*

Standard construction of 0-Alia algebras. Let $(A,\cdot)$ be associative commutative algebra and $f,g : A\to A$ linear maps. Then $(A,\circ)$ is 0-Alia if $a\circ b = a\cdot f(b) + g(a\cdot b)$. Denote such algebra as $A(\cdot,f,g)$. Call 0-Alia algebra L special if there exists some standard 0-Alia algebra $A(\cdot,f,g)$ such that L is isomorphic to a subalgebra of $A(\cdot,f,g)$.

One more example of 0-Alia algebra. Let U be an associative commutative algebra with derivation $\partial : U\to U$. Then for any $u\in U$ the algebra $(U,\odot)$, where $\odot = \odot_u$ is defined by

$$a\odot b = (\partial^3(a)\cdot b + 4\partial^2(a)\cdot\partial(b) + 5\partial(a)\cdot\partial^2(b) + 2a\cdot\partial^3(b))\cdot u$$
$$+(\partial^2(a)\cdot b + 3\partial(a)\cdot\partial(b) + 2\,a\cdot\partial^2(b))\cdot\partial(u),$$

is 0-Alia. We do not know whether this algebra is special.

For linear maps $f,g : U\to U$ define a bilinear map $f\cdot g : U\times U\to U$ and a skew-symmetric bilinear map $f\wedge g : \wedge^2 U\to U$ by

$$f\cdot g(u,v) = f(u)\cdot f(v),$$

$$f\wedge g = f\cdot g - g\cdot f.$$

Let $id : U\to U$ be identity map.

Recall that an algebra with identities $rsym = 0$ $lcom = 0$ is called Novikov, where

$$rsym(t_1,t_2,t_3) = t_1(t_2t_3 - t_3t_2) - (t_1t_2)t_3 + (t_1t_3)t_2,$$

$$lcom(t_1,t_2,t_3) = t_1(t_2t_3) - t_2(t_1t_3).$$

Example of Novikov algebra: $(\mathbf{C},\circ)$, where $a\circ b = \partial(a)b, \quad \partial = \frac{\partial}{\partial x}$.

Theorem 5.2 *Let* $(A, \circ)$ *be a Novikov algebra and* $u, v \in A$. *Define on* A *a new multiplication* $\star$ *by* $a \star b = (a \circ b) \circ u + v \circ (a \circ b)$. *Then* $(A, \star)$ *is* 1*-Alia and satisfies the identity* $s_4 = 0$. *In particular,* $(A, \star)$ s_4*-Lie-admissible.*

Corollary 5.3 *Let* $(U, \cdot)$ *be an associative commutative algebra with* $D \in Der\, U$ *and* $u, v, w \in U$. *Then* (U, μ) *is* s_4*-Lie-admissible, where* $\mu = id \cdot (u \cdot D^2 + v \cdot D + w \cdot D^0)$. *Here we set* $D^0 = id$.

Corollary 5.4 *Let* $(U, \cdot)$ *be an associative commutative algebra with* $D \in Der\, U$ *and* $u \in U$. *Then the following algebras are* s_4*-Lie:*

- $(U, u \cdot (id \wedge D^2))$
- $(U, u \cdot D \wedge D^2)$

Notice that the map $D : U \to U$ induces a homomorphism

$$D : (U, id \wedge D^2) \to (U, D \wedge D^2).$$

It is easy to check:

$$D(a) \cdot D^3(b) - D(b) \cdot D^3(a) = D(D(a) \cdot D^2(b) - D(b) \cdot D^2(a)).$$

So, we obtain a central extension

$$0 \to Ker\, D \to (U, id \wedge D^2) \to (U, D \wedge D^2) \to 0.$$

Theorem 5.5 *Let* A *be skew-symmetric algebra with skew-symmetric identity of degree* k. *Then* A *is* s_k*-Lie.*

Example of 1-Alia algebra. Let U be associative commutative algebra with derivation $\partial : U \to U$. Then for any $u \in U$ the algebra $(U, *_u)$, where

$$a *_u b = (\partial^2(a) \cdot b + \partial(a) \cdot \partial(b)) \cdot u,$$

is 1-Alia. Notice that if $u = 1 \in \ker \partial$, then this identity is not minimal. The algebra $(U, *_1)$ satisfies the identity

$$(t_1, t_2, t_3) - (t_1, t_3, t_2) + 2t_2(t_1 t_3) - 2t_3(t_1 t_2) = 0.$$

This identity for $(U, *_u)$, $u = 1$, is minimal. In particular, algebras $(U, *_1)$ and $(U, *_u)$, where $u \notin \ker \partial$, are not isomorphic.

Theorem 5.6 *Let* $(U, \cdot)$ *be associative commutative algebra with derivations* $D_1, D_2, D_3 \in Der\, U$ *and* $u_1, u_2, u_3 \in U$. *Let* $\omega = u_1\, D_2 \wedge D_3 + u_2\, D_1 \wedge D_3 + u_3\, D_1 \wedge D_2$. *Then the algebra* (U, ω) *is* s_4*-Lie.*

Corollary 5.7 *Let* $D_1, D_2 \in Der\, U$. *Then* $(U, D_1 \wedge D_2)$ *is* s_4*-Lie.*

Following V.T. Filippov [Filipov] say that $(A, \circ)$ is J -Lie if A is s_4-Lie and (A, jac) under 3-map (jacobian) $jac(a,b,c) = (a \circ b) \circ c + (b \circ c) \circ a + (c \circ a) \circ b$ is 3-Lie:

$$jac(a, b, jac(c, d, e)) =$$

$$jac(jac(a, b, c), d, e) + jac(c, jac(a, b, d), e) + jac(c, d, jac(a, b, e)),$$

for any $a, b, c, d, e \in A$.

Theorem 5.8 *Let $(U, \cdot)$ be an associative commutative algebra, $u_1, u_2, u_3 \in U$, and $D_1, D_2, D_3 \in Der\, U$, such that $[D_1, D_2] = [D_1, D_3] = [D_2, D_3] = 0$. Let $\omega = u_1\, D_2 \wedge D_3 + u_2\, D_1 \wedge D_3 + u_3\, D_1 \wedge D_2$. Then the algebra (U, ω) is J-algebra.*

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