INFINITESIMAL DEFORMATION OF PARABOLIC HIGGS SHEAVES

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INTRODUCTION

Let X be a compact Riemann surface of genus $g \geq 2$. The theorem of Narasimhan and Seshadri gave us the beautiful correspondence between stable vector bundles of degree 0 on X and irreducible unitary representations of the fundamental group $\pi_1(X)$. Several extensions of this theorem have been considered and yielded two algebraic objects: parabolic bundles and Higgs bundles. The notion of parabolic bundles was introduced by Seshadri [19]. Let $D = \{p_1, \ldots, p_n\}$ be a finite set of points on X. A parabolic bundle is a triple (E, F_*^*, α_*^*) consisting of a vector bundle E on X, filtrations $E_{p_i} = F_1^i \supset \cdots \supset F_{l_i+1}^i = 0$ at points of D and systems of weights $0 \leq \alpha_1^i < \cdots < \alpha_{l_i}^i < 1$. He gave a correspondence between stable parabolic bundles of parabolic degree 0 and irreducible unitary representations of $\pi_1(X - D)$ (cf. [15],[19]). The notion of Higgs bundles was introduced by Hitchin [7]. A Higgs bundle is a pair (E, φ) consisting of a vector bundle E on X and a homomorphism $\varphi : E \to E \otimes_X \Omega_X^1$. In this case, he gave a correspondence between stable Higgs bundle of degree 0 and irreducible representations of $\pi_1(X)$ (cf. [7]). In both cases, the correspondences induce homeomorphisms between the moduli spaces.

To generalize the correspondences to non-compact and non-unitary cases, Simpson [22] introduced the notion of filtered regular Higgs bundles and gave a correspondence between stable filtered regular Higgs bundles of degree zero and stable filtered local systems of degree zero. He regarded a parabolic bundle (E, F_*^*, α_*^*) as a filtered sheaf by setting

$$E_\alpha^i = \operatorname{Ker}(E \twoheadrightarrow E_{p_i}/F_{j+1}^i)$$

for real numbers $\alpha_j^i < \alpha \leq \alpha_{j+1}^i$, $(i = 1, \ldots, n, j = 0, \ldots l_j, \alpha_0^i = 1 - \alpha_{l_i}^i, \alpha_{l_i+1}^i = 1)$ and $E_{\alpha+1}^i = E_\alpha^i \otimes_X \mathcal{O}_X(-p_i)$. In this paper, we shall consider only one filtration by setting $E_\alpha = \bigcap_{i=1}^n E_\alpha^i$ because we can recover the original filtrations from this and those categories are equivalent. Now, a filtered regular Higgs bundle is a pair (E_*, φ) consisting of a filtered sheaf E_* with $E_{\alpha+1} = E_\alpha \otimes_X \mathcal{O}_X(-D)$ and a homomorphism $\varphi : E_* \to E_* \otimes_X \Omega_X^1(\log D)$ which preserves the filtrations. In this paper, we shall call this object a parabolic Higgs bundle.

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Let X be a non-singular projective variety over the complex numbers. These results also have been generalized to higher dimensional cases for usual case by Donaldson, Uhlenbeck and Yau and for the case of Higgs bundles by Donaldson, Corlette and Simpson. Simpson [20] constructed three moduli spaces: moduli of semi-simple representations of $\pi_1(X)$, moduli of coherent $\mathcal{D}_X$-modules and moduli of semi-stable Higgs bundles on X with vanishing Chern classes and proved that these moduli spaces are homeomorphic.

Though the correspondence is not yet generalized for parabolic cases on higher dimensional varieties, the notion of parabolic bundles and parabolic Higgs bundles is naturally generalized (cf. [14],[25]). Let D be an effective Cartier divisor on X and let Ω be a locally free $\mathcal{O}_X$-module. A parabolic sheaf is a filtered sheaf E_* which satisfies some finiteness conditions (cf. [14]) and the condition $E_{\alpha+1} = E_\alpha \otimes_X \mathcal{O}_X(-D)$ for all real numbers α. A parabolic Ω-pair is a pair (E_*, φ) consisting of a parabolic sheaf E_* and a homomorphism $\varphi : E_* \to E_* \otimes_X \Omega$ with $\varphi \wedge \varphi = 0$ and $\varphi(E_\alpha) \subseteq E_\alpha \otimes_X \Omega$. When $\Omega = \Omega_X^1(\log D)$, we shall call an Ω-pair a parabolic Higgs sheaf. In previous papers [14] and [25], we have constructed moduli schemes of semi-stable parabolic Ω-pairs as well as moduli of semi-stable parabolic sheaves (see Theorem 2.3). The purpose of this article is to analyze the infinitesimal properties of these moduli schemes. As in the case of usual stable sheaves, it is expected that the Zariski tangent space of them at a point corresponding to a parabolically stable Ω-pair (E_*, φ) is naturally isomorphic to some "Ext-group" $\mathrm{Ext}_X^1((E_*, \varphi), (E_*, \varphi))$ and the obstruction classes for smoothness at (E_*, φ) are in $\mathrm{Ext}_X^2((E_*, \varphi), (E_*, \varphi))$. Unfortunately, the categories of parabolic sheaves or parabolic Ω-pairs are not abelian categories. To avoid this difficulty, we shall change the definition of parabolic sheaves, i.e. we redefine a filtered sheaf E_* as a family of homomorphisms

$$\{i_E^{\alpha,\beta} : E_\alpha \to E_\beta | \alpha \geq \beta\}$$

with $i_E^{\beta,\gamma} \circ i_E^{\alpha,\beta} = i_E^{\alpha,\gamma}$. Using this new filtered sheaves, the notion of parabolic sheaves (or, parabolic Ω-pairs) is redefined. Then parabolic Ω-pairs forms an abelian category with enough injective objects (Proposition 1.1). Thus, we get Ext-groups for parabolic Ω-pairs. These Ext-groups are, in fact, what we need.

To calculate these Ext-groups for parabolic Ω-pairs, we shall introduce tensor products, Hom-sheaves and an operator $\hat{}$ for our new parabolic sheaves. Then the Serre duality theorem for parabolic sheaves is given by the following isomorphism (Proposition 3.7):

$$\theta^i : \mathrm{Ext}_X^i(E_*, F_* \otimes_X \omega_X(D)) \xrightarrow{\simeq} \mathrm{Ext}_X^{n-i}(F_*, \hat{E}_*)^\vee.$$

Here $\hat{E}_*$ is no longer a parabolic sheaf in the meaning of the original definition even if E_* is. Moreover, we shall give a spectral sequence

$$E_1^{pq} = \mathrm{Ext}_X^q(E_*, F_* \otimes_X \wedge^p \Omega) \Rightarrow \mathrm{Ext}_X^{p+q}((E_*, \varphi), (F_*, \psi)).$$

Using these tools, we can calculate Ext-groups for parabolic Ω-pairs in the case of curves. In particular, we show that the moduli scheme of semi-stable parabolic Higgs bundles is irreducible normal quasi-projective variety of dimension

$$2(1 - r^2(1-g) + \sum_i \dim F_i) + \sum_{i,j}(n_i^j - n_i^{j+1})^2 - 1$$

where g is the genus of curve, r is the rank of E, F_i is a flag variety of type $(n_i^1, \ldots, n_i^{l_i})$ which corresponds to a flag structure at E_{p_i} (Theorem 5.2).

In §1, we shall give new definitions of $\mathbb{R}$-filtered sheaves and parabolic sheaves and prove that the category of parabolic sheaves is an abelian category with enough injective objects. The Ext-groups for parabolic sheaves or pairs shall be defined. In §2, we shall prove that the Zariski tangent space of our moduli scheme at a parabolically stable Ω-pair (E_*, φ) is naturally isomorphic to $\mathrm{Ext}^1_X((E_*, \varphi), (E_*, \varphi))$ and the obstruction classes of smoothness at the point are in $\mathrm{Ext}^2_X((E_*, \varphi), (E_*, \varphi))$. In §3, we shall introduce tensor products, Hom-sheaves for parabolic sheaves and generalize the Serre duality theorem for parabolic sheaves. §4 is devoted to give above spectral sequence which joins the Ext-groups for parabolic sheaves with those for parabolic Ω-pairs. In the last section, we shall study the moduli schemes of parabolic Higgs sheaves on non-singular curves. We shall show that the moduli schemes are normal, quasi-projective, irreducible varieties and calculate its dimensions.

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Notation and Convention

Let $f : X \to S$ be a smooth, projective, geometrically integral morphism of locally noetherian schemes, let D be an effective relative Cartier divisor and let $\mathcal{O}_X(1)$ be an f-ample invertible sheaf. The category of locally noetherian schemes over S is denoted by (Sch/S). For an S-scheme T and an $\mathcal{O}_X$-module E, $X_T = X \times_S T$ and E_T is the pullback of E over X_T. We denote by $S^i(E)$ the i-th symmetric product, by $S^*(E)$ the symmetric $\mathcal{O}_X$-algebra. $E^\vee$ means a dual sheaf of E. Tensor product of $\mathcal{O}_X$-modules E and F is written by $E \otimes_X F$. $\mathrm{Hom}_X(E, F)$ is $\mathrm{Hom}_{\mathcal{O}_X}(E, F)$. Similarly, we use notations $\mathcal{H}om_X(E, F)$, $\mathrm{Ext}^i_X(E, F)$ and $\mathcal{E}xt^i_X(E, F)$.

Let s be a geometric point of S. For a coherent $\mathcal{O}_{X_s}$-module F, the degree of F with respect to $\mathcal{O}_X(1)$ is that of the first Chern class of F with respect to $\mathcal{O}_{X_s}(1) = \mathcal{O}_X(1)_{X_s}$ and it is denoted by $\deg_{\mathcal{O}_X(1)} F$ or simply $\deg F$. The rank of F is denoted by $\mathrm{rk}(F)$, $\mu(F) = \deg F / \mathrm{rk}(F)$, $\chi(F) = \sum_i (-1)^i \dim_{k(s)} H^i(X_s, F)$ and $F(m) = F \otimes_X \mathcal{O}_X(m)$. For a parabolic sheaf E_* on X_s, par-deg$(E_*) = \int_0^1 \deg E_\alpha d\alpha + \mathrm{rk}(E) \cdot \deg D$ and par-$\mu(E_*) = $ par-deg$(E_*)/\mathrm{rk}(E)$.

1. Categories of Parabolic Sheaves

In this section, we shall develop the sheaf cohomology theory for parabolic sheaves. Unfortunately, the category of parabolic sheaves defined in [14] is not abelian. So, we shall generalize the notion of parabolic sheaves.

Let D be an effective Cartier divisor on a scheme X. In this paper, we shall regard $\mathbb{R}$ as a category with

$$\mathrm{Ob}(\mathbb{R}) = \mathbb{R} \quad \text{(the set of all real numbers)},$$
$$\mathrm{Mor}(\alpha,\beta) = \begin{cases} \{i^{\alpha,\beta}\} & \text{if } \alpha \geq \beta \\ \emptyset & \text{otherwise.} \end{cases}$$

An $\mathbb{R}$-filtered $\mathcal{O}_X$-module is a covariant functor from $\mathbb{R}$ to the category $\mathcal{M}_X$ of all $\mathcal{O}_X$-modules. For an $\mathbb{R}$-filtered $\mathcal{O}_X$-module $E : \mathbb{R} \to \mathcal{M}_X$, we denote the $\mathcal{O}_X$-module $E(\alpha)$ by E_α and the $\mathcal{O}_X$-homomorphism $E(i^{\alpha,\beta})$ by $i_E^{\alpha,\beta}$ for each $\alpha \geq \beta$. We use a symbol E_* instead of E and sometimes E means the sheaf E_0. The category of $\mathbb{R}$-filtered $\mathcal{O}_X$-modules and their natural transformations is denoted by $\mathcal{C}_X$. For an $\mathcal{O}_X$-module F, the tensor product $E_* \otimes_X F$ is defined by setting $(E_* \otimes_X F)_\alpha = E_\alpha \otimes_X F$ and $i^{\alpha,\beta}_{E_*\otimes F} = i_E^{\alpha,\beta} \otimes id_F$. We can shift an $\mathbb{R}$-filtered $\mathcal{O}_X$-module by real numbers. A natural transformation $f : E \to F$ is often denoted by f_* and $f(\alpha) : E_\alpha \to F_\alpha$ is denoted by f_α.

Definition 1.1. For an $\mathbb{R}$-filtered $\mathcal{O}_X$-module E_* and a real number α, $E[\alpha]_*$ is an $\mathbb{R}$-filtered $\mathcal{O}_X$-module with $E[\alpha]_\beta = E_{\alpha+\beta}$ and $i^{\beta,\gamma}_{E[\alpha]} = i_E^{\beta+\alpha,\gamma+\alpha}$. For each pair $\alpha \geq \beta$, we have a natural transformation

$$i_E^{[\alpha,\beta]} : E[\alpha]_* \longrightarrow E[\beta]_*$$

defined by $i_E^{\alpha+\gamma,\beta+\gamma}$. Each $f : E_* \to F_*$ induces $f[\alpha] : E[\alpha]_* \longrightarrow F[\alpha]_*$.

In this paper, we shall change the definition of parabolic sheaves in [14] or [25] as follows.

Definition 1.2. A parabolic $\mathcal{O}_X$-module (with respect to D) is an $\mathbb{R}$-filtered $\mathcal{O}_X$-module E_* together with an isomorphism of $\mathbb{R}$-filtered $\mathcal{O}_X$-modules

$$j_E : E_* \otimes_X \mathcal{O}_X(-D) \xrightarrow{\simeq} E[1]_*$$

such that

$$i_E^{[1,0]} \circ j_E = id_{E_*} \otimes i_D : E_* \otimes_X \mathcal{O}_X(-D) \to E_* \tag{1.1}$$

where $i_D : \mathcal{O}_X(-D) \to \mathcal{O}_X$ is a natural map defined by D.

For parabolic $\mathcal{O}_X$-modules E_* and F_*, a natural transformation $f : E_* \to F_*$ is called a parabolic homomorphism if the following diagram is commutative.

$$\begin{array}{ccc} E_* \otimes_X \mathcal{O}_X(-D) & \xrightarrow{f \otimes id} & F_* \otimes_X \mathcal{O}_X(-D) \\ \simeq \big\downarrow j_E & & \simeq \big\downarrow j_F \\ E[1]_* & \xrightarrow{f[1]} & F[1]_* \end{array} \tag{1.2}$$

The module of all parabolic homomorphisms is denoted by $\mathrm{Hom}_X(E_*, F_*)$. By $\mathcal{H}om_X(E_*, F_*)$, we denotes a sheaf with $\mathcal{H}om_X(E_*, F_*)(U) = \mathrm{Hom}_U(E_*|_U, F_*|_U)$ for each open subset U. Let us denote by $\mathcal{P}_{X/D}$ the category of parabolic $\mathcal{O}_X$-modules and their parabolic homomorphisms. Every $\mathcal{O}_X$-module E can be regarded naturally as a parabolic $\mathcal{O}_X$-module by setting

$$E_\alpha = E \otimes_X \mathcal{O}_X(-mD) \qquad \text{for } \; m-1 < \alpha \leq m.$$

This structure is called the special structure. Note that the category $\mathcal{M}_X$ can be regarded as a full sub-category of $\mathcal{P}_{X/D}$ through the identification E and E_* with the special structure.

Proposition 1.1. *The category $\mathcal{P}_{X/D}$ is an abelian category with enough injective objects.*

Proof. It is easy to see that $\mathcal{P}_{X/D}$ is an abelian category. In $\mathcal{P}_{X/D}$, a sequence

$$0 \longrightarrow E'_* \xrightarrow{f_*} E_* \xrightarrow{g_*} E''_* \longrightarrow 0,$$

is exact if and only if for each $\alpha \in \mathbb{R}$, the following sequence is exact.

$$0 \longrightarrow E'_\alpha \xrightarrow{f_\alpha} E_\alpha \xrightarrow{g_\alpha} E''_\alpha \longrightarrow 0.$$

To prove that $\mathcal{P}_{X/D}$ has enough injective objects, by virtue of Théorème 1.10.1 in [4], it is enough to prove that $\mathcal{P}_{X/D}$ satisfies Grothendieck's axiom AB 5) [1] and has a generator. Verification of the condition AB 5) is easy. Recall that g is a generator if for each object a and each proper sub-object b, then there is a morphism of $g \to a$ which does not factor through b.

Let F_* be a proper sub-object of E_*. Then there are a real number α and an open subset U of X such that $F_\alpha(U) \subsetneqq E_\alpha(U)$. An element of $E_\alpha(U)\backslash F_\alpha(U)$ defines a homomorphism $f : (i_U)_!\mathcal{O}_U \to E_\alpha$ which does not factor through F_α where $i_U : U \hookrightarrow X$ is the natural inclusion and $(i_U)_!\mathcal{O}_U$ is the sheaf obtained by extending $\mathcal{O}_U$ by zero outside U. Let $I(U,\alpha)_*$ be a parabolic $\mathcal{O}_X$-module with

$$I(U,\alpha)_\beta = (i_U)_!\mathcal{O}_U \otimes_X \mathcal{O}_X(-mD)$$

[1]An abelian category $\mathcal{A}$ satisfies AB 5) if and only if $\mathcal{A}$ has infinite direct sums and for each object a, each sub-object b and each family of sub-objects $\{a_i \subseteq a \mid i \in I\}$ such that each pair a_i and a_j is contained in some a_k, $(\sum_{i \in I} a_i) \cap b = \sum_{i \in I}(a_i \cap b)$ (see [4]).

for $\alpha+m-1 < \beta \leq \alpha+m$. Then $f : (i_U)_!\mathcal{O}_U \to E_\alpha$ defines a parabolic homomorphism $\varphi : I(U,\alpha)_* \to E_*$ with $\varphi_\alpha = f$. Hence, a direct sum

$$\bigoplus_{U:\text{open},\alpha\in\mathbb{R}} I(U,\alpha)_*$$

is a generator of $\mathcal{P}_{X/D}$. $\square$

Now, we can define various derived functors as in the case of usual $\mathcal{O}_X$-modules. For a morphism $f : X \to Y$, we have the direct image functor $f_* : \mathcal{P}_{X/D} \to \mathcal{C}_Y$. Clearly, it is a left exact functor. Hence, we get a right derived functor R^*f_* of f_*.

For each $\mathcal{O}_X$-module E, let $I_\alpha(E) = I_\alpha(E)_*$ be a parabolic $\mathcal{O}_X$-module which is constructed from E as $I(U,\alpha)_*$ is constructed from $(i_U)_!\mathcal{O}_U$ in the above proof. Then I_α is a functor from $\mathcal{M}_X$ to $\mathcal{P}_{X/D}$. We can easily know the following.

Lemma 1.2. *The functor I_α is a left adjoint functor of a forgetful functor F_α : $\mathcal{P}_{X/D} \to \mathcal{M}_X$ with $F_\alpha(E_*) = E_\alpha$, i.e. we have a natural isomorphism;*

$$\mathrm{Hom}_X(I_\alpha(E), J_*) \simeq \mathrm{Hom}_X(E, J_\alpha).$$

Corollary 1.3. *For each injective object J_* in the category $\mathcal{P}_{X/D}$, all J_α are injective $\mathcal{O}_X$-modules.*

Proof. Note that I_α is an exact functor. Hence, for each injective homomorphism $E \to E'$, we obtain the desired surjection.

$$\begin{array}{ccccc} \mathrm{Hom}_X(I_\alpha(E'), J_*) & \to & \mathrm{Hom}_X(I_\alpha(E), J_*) & \to & 0 \\ \| & & \| & & \\ \mathrm{Hom}_X(E', J_\alpha) & & \mathrm{Hom}_X(E, J_\alpha) & & \end{array}$$

$\square$

By this corollary, for each parabolic sheaf E_*, $R^if_*E_*$ is an $\mathbb{R}$-filtered $\mathcal{O}_Y$-module F_* with $F_\alpha = R^if_*(E_\alpha)$ and $i_F^{\alpha,\beta} = R^if_*(i_E^{\alpha,\beta})$.

For each parabolic $\mathcal{O}_X$-module E_*, $\mathrm{Ext}^i_X(E_*, -)$ (or, $\mathcal{E}xt^i_X(E_*, -)$) is the i-th right derived functor of $\mathrm{Hom}_X(E_*, -)$ (or, $\mathcal{H}om_X(E_*, -)$, respectively). By virtue of Lemma 1.2 and Corollary 1.3, for each $\mathcal{O}_X$-module E' and each α, we have $\mathrm{Hom}_X(I_\alpha(E'), E_*) = \mathrm{Hom}_X(E', E_\alpha)$ and hence, for all i,

$$\mathrm{Ext}^i_X(I_\alpha(E'), E_*) = \mathrm{Ext}^i_X(E', E_\alpha). \tag{1.3}$$

An extension of E''_* by E'_* is a short exact sequence of parabolic homomorphisms;

$$0 \longrightarrow E'_* \longrightarrow E_* \longrightarrow E''_* \longrightarrow 0.$$

Two extensions ξ and η of E''_* by E'_* are isomorphic if there is a commutative diagram of parabolic homomorphisms;

$$\begin{array}{ccccccccc} \xi : 0 & \longrightarrow & E'_* & \longrightarrow & E_* & \longrightarrow & E''_* & \longrightarrow & 0 \\ & & \| & & \downarrow & & \| & & \\ \eta : 0 & \longrightarrow & E'_* & \longrightarrow & F_* & \longrightarrow & E''_* & \longrightarrow & 0. \end{array}$$

By the same proof as in the case of usual "Ext"-groups (cf. [11]), we have

Lemma 1.4. *There is a one-to-one correspondence between isomorphism classes of extensions of E''_* by E'_* and elements of the group $\mathrm{Ext}^1_X(E''_*, E'_*)$ and the sum of two extensions is given by the Baer sum.*

Definition 1.3. A parabolic sheaf E_* is said to be coherent if the following conditions are satisfied.

(1.4) All E_α are coherent $\mathcal{O}_X$-modules.

(1.5) There is a sequence of real numbers $0 \le \alpha_1 < \alpha_2 < \cdots < \alpha_l < 1$ such that $i_E^{\alpha_i,\alpha} : E_{\alpha_i} \to E_\alpha$ are isomorphisms for $\alpha_{i-1} < \alpha \le \alpha_i$ ($\alpha_0 = 0$, $\alpha_{l+1} = 1$).

Clearly, all coherent parabolic sheaves form an abelian sub-category of $\mathcal{P}_{X/D}$. Moreover, when X is an integral scheme, a coherent parabolic sheaf E_* is said to be torsion free if all E_α are torsion free $\mathcal{O}_X$-modules. We call the above set $\{\alpha_1, \alpha_2, \ldots, \alpha_l\}$ a system of weights of E_* if every $i_E^{\alpha_{i+1},\alpha_i} : E_{\alpha_{i+1}} \to E_{\alpha_i}$ is not an isomorphism for $i = 1, 2, \ldots, l$.

Remark 1.1. (i) By the condition (1.1), if E_* is torsion free, then all $i_E^{\alpha,\beta}$ are injective. Hence, we get a parabolic sheaf in the meaning of the original definition (cf. [14])

$$E = E_0 \supseteq E_{\alpha_1} \supseteq \cdots \supseteq E_{\alpha_l} \supseteq E_1 \simeq E_0 \otimes_X \mathcal{O}_X(-D)$$

By this correspondence, the original notion of parabolic sheaves and parabolic homomorphisms is same as the notion of torsion free, parabolic sheaves and their parabolic homomorphisms.

(ii) If E_* is a torsion free, then the structure of parabolic sheaf of E_* is uniquely determined by its underlying structure of $\mathbb{R}$-filtered sheaf. In fact, the isomorphism $j_E : E_* \otimes_X \mathcal{O}_X(-D) \to E[1]_*$ is uniquely determined by the condition (1.1) and injectivity of $i_E^{[1,0]}$. Moreover, a morphism in the category $\mathcal{C}_X$ between two torsion free, parabolic sheaves automatically satisfies the condition (1.2). Hence, the category of all torsion free, parabolic sheaves and their parabolic homomorphisms consists a full (not abelian) sub-category of $\mathcal{C}_X$.

(iii) In a short exact sequence of parabolic $\mathcal{O}_X$-modules

$$0 \longrightarrow E'_* \longrightarrow E_* \longrightarrow E''_* \longrightarrow 0,$$

if E'_* and E''_* are coherent (or, torsion free), then so is E_*.

2. Infinitesimal property of moduli of stable parabolic Ω-pairs

In this section, let $f : X \to S$ be a quasi-projective morphism of noetherian schemes, let $\mathcal{O}_X(1)$ be an f-very ample invertible sheaf and let D be a relative effective Cartier divisor on X/S.

Definition 2.1. A coherent parabolic sheaf E_* is said to be flat over S or S-flat if all E_α are flat over S. The support of E_* is proper over S if the support of each E_α is. For each object T of (Sch/S), set

$$\mathrm{Par}_{X/D/S}(T) = \left\{ E_* \;\middle|\; \begin{array}{l} E_* \text{ is a } T\text{-flat, coherent} \\ \text{parabolic } \mathcal{O}_{X_T}\text{-module} \\ \text{with support proper over } T \end{array} \right\} \Big/ \sim$$

where $\sim$ is an equivalence relation such that $E_* \sim E'_*$ if and only if there is an invertible sheaf L on T and an isomorphism $E_* \simeq E'_* \otimes_T L$. Then $\mathrm{Par}_{X/D/S}$ defines a functor from (Sch/S) to (Sets) in a natural way.

Now let us consider a deformation situation $A' \twoheadrightarrow A = A'/I \twoheadrightarrow A_0 = A'/M$ with $E_* \in \mathrm{Par}_{X/D/S}(A)$ where M and I are nilpotent ideals of the noetherian $\mathcal{O}_S$-algebra A' and $MI = 0$ (cf. [1]). $\bar{E}_*$ denote the image of E_* in $\mathrm{Par}_{X/D/S}(A_0)$. We use this notation for other elements of $\mathrm{Par}_{X/D/S}(A)$ or $\mathrm{Par}_{X/D/S}(A')$.

Let us show that the deformation theory for $\mathrm{Par}_{X/D/S}$ is given by the modules

$$D(A_0, I, \bar{E}_*) = \mathrm{Ext}^1_{X_{A_0}}(\bar{E}_*, I \otimes_{A_0} \bar{E}_*).$$

and $D(A_0, I, \bar{E}_*)$ operates freely on $\mathrm{Par}_{X/D/S}(A')_{\bar{E}_*}$. Here, $\mathrm{Par}_{X/D/S}(A')_{\bar{E}_*}$ denotes the subset of elements of $\mathrm{Par}_{X/D/S}(A')$ whose image in $\mathrm{Par}_{X/D/S}(A_0)$ is $\bar{E}_*$.

Proposition 2.1. $\Phi : \mathrm{Par}_{X/D/S}(A')_{\bar{E}_*} \to \mathrm{Par}_{X/D/S}(A)_{\bar{E}_*}$ *is a principal homogeneous space for the group* $D(A_0, I, \bar{E}_*)$.

Proof. Let E_* be an element of $\mathrm{Par}_{X/D/S}(A)_{\bar{E}_*}$ such that $\Phi^{-1}(E_*) \neq \emptyset$. Then, each element E'_* of $\Phi^{-1}(E_*)$ determines the extension

$$e_{E'_*} : 0 \to I \otimes_{A_0} \bar{E}_* \to E'_* \to E_* \to 0$$

in $\mathrm{Ext}^1_{X_{A'}}(E_*, I \otimes_{A_0} \bar{E}_*)$. Since $e_{E'_*}$ determines the isomorphism class of E'_*, $\Phi^{-1}(E_*)$ can be regarded as a subset of $\mathrm{Ext}^1_{X_{A'}}(E_*, I \otimes_{A_0} \bar{E}_*)$. $\mathrm{Ext}^1_{X_{A'}}(E_*, I \otimes_{A_0} \bar{E}_*)$ contains a submodule $\mathrm{Ext}^1_{X_A}(E_*, I \otimes_{A_0} \bar{E}_*)$. Here, since E_* is A-flat, for each extension ζ in $\mathrm{Ext}^1_{X_A}(E_*, I \otimes_{A_0} \bar{E}_*)$, we have the following commutative diagram with exact rows.

$$\begin{array}{rccccccccc} \zeta : & 0 & \longrightarrow & I \otimes_A E_* & \longrightarrow & H_* & \longrightarrow & E_* & \longrightarrow & 0 \\ & & & \simeq\downarrow & & \downarrow & & \downarrow p & & \\ \zeta \otimes_A A_0 : & 0 & \longrightarrow & I \otimes_{A_0} \bar{E}_* & \longrightarrow & \bar{H}_* & \longrightarrow & \bar{E}_* & \longrightarrow & 0 \end{array}$$

This shows that the map $p^* : \mathrm{Ext}^1_{X_{A_0}}(\bar{E}_*, I \otimes_{A_0} \bar{E}_*) \to \mathrm{Ext}^1_{X_A}(E_*, I \otimes_{A_0} \bar{E}_*)$ is an isomorphism. Hence, the group $\mathrm{Ext}^1_{X_{A_0}}(\bar{E}_*, I \otimes_{A_0} \bar{E}_*)$ acts freely on $\mathrm{Ext}^1_{X_{A'}}(E_*, I \otimes_{A_0} \bar{E}_*)$ by translation. Then, elementary verifications using Baer sums (cf. Lemma 1.4) show that $\mathrm{Ext}^1_{X_{A_0}}(\bar{E}_*, I \otimes_{A_0} \bar{E}_*)$ acts on $\Phi^{-1}(E_*)$ freely and effectively. □

We shall prove the following along the proof of Proposition 6.7 in [13], but we don't use "locally free" resolutions of E_*.

Proposition 2.2. *There exists an element ξ of $\mathrm{Ext}^2_{X_{A_0}}(\bar{E}_*, I \otimes_{A_0} \bar{E}_*)$ such that $\xi = 0$ if and only if E_* is liftable to an element of $\mathrm{Par}_{X/D/S}(A')$.*

Proof. Clearly, we may replace E_* by $E_* \otimes_X \mathcal{O}_X(m)$ so that all E_α for $0 \le \alpha \le 1$ are generated by its global sections . Then, E_* is a quotient of a coherent parabolic sheaf $F_* = \oplus_i I_{\alpha_i}(\mathcal{O}_{X_A}) \in \mathrm{Par}_{X/D/S}(A)$ where I_{α_i} is the functor defined in §1.

$$0 \longrightarrow K_* \longrightarrow F_* \xrightarrow{\pi} E_* \longrightarrow 0 \tag{2.1}$$

Clearly, F_* is liftable to $F'_* = \oplus_i I_{\alpha_i}(\mathcal{O}_{X_{A'}})$ in $\mathrm{Par}_{X/D/S}(A')$. By virtue of (1.3), we may assume that

$$\mathrm{Ext}^1_{X_{A_0}}(\bar{F}_*, I \otimes_{A_0} \bar{E}_*) = \mathrm{Ext}^1_{X_{A'}}(F'_*, I \otimes_A E_*) = 0. \tag{2.2}$$

As in the case of Quot-schemes (cf. Lemma 6.7 in [1]), the obstruction class η to lifting the parabolic quotient $F_* \twoheadrightarrow E_*$ to $F'_* \twoheadrightarrow E'_*$ is in $\mathrm{Ext}^1_{X_{A'}}(K_*, I \otimes_A E_*)$. We claim that this class is, in fact, in the submodule $\mathrm{Ext}^1_{X_A}(K_*, I \otimes_A E_*)$. As in the proof of Lemma 6.7 in [1], η is determined by the following commutative diagram with exact rows.

$$\begin{array}{cccccccccc}
e: 0 & \longrightarrow & IF'_* & \xrightarrow{i} & F'_* & \xrightarrow{p} & F_* & \longrightarrow & 0 \\
 & & \downarrow q & & \downarrow & & \| & & \\
e': 0 & \longrightarrow & I \otimes_A E_* & \longrightarrow & G_* & \longrightarrow & F_* & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow j & & \\
\eta: 0 & \longrightarrow & I \otimes_A E_* & \longrightarrow & G'_* & \longrightarrow & K_* & \longrightarrow & 0
\end{array}$$

where e is the canonical extension, $G_* = \mathrm{Coker}(IF'_* \xrightarrow{(i,-q)} F'_* \oplus (I \otimes_A E_*))$ and $G'_* = G_* \times_{F_*} K_*$. Then easy calculations show that $IG'_* = 0$. This proves our claim.

Since K_* is A-flat, as in the proof of Proposition 2.1, we have a canonical isomorphism $\mathrm{Ext}^1_{X_{A_0}}(\bar{K}_*, I \otimes_{A_0} \bar{E}_*) \simeq \mathrm{Ext}^1_{X_A}(K_*, I \otimes_A E_*)$. Let $\bar{\eta}$ be the image of η in $\mathrm{Ext}^1_{X_{A_0}}(\bar{K}_*, I \otimes_{A_0} \bar{E}_*)$. Then, from (2.1) and (2.2), we get an injection

$$0 = \mathrm{Ext}^1_{X_{A_0}}(\bar{F}_*, I \otimes_{A_0} \bar{E}_*) \longrightarrow \mathrm{Ext}^1_{X_{A_0}}(\bar{K}_*, I \otimes_{A_0} \bar{E}_*) \xrightarrow{\delta} \mathrm{Ext}^2_{X_{A_0}}(\bar{E}_*, I \otimes_{A_0} \bar{E}_*).$$

Hence, if $\xi = \delta(\bar{\eta}) = 0$, then E_* is liftable to an A'-flat quotient of F'_*. Conversely, if E_* is liftable to $E'_* \in \mathrm{Par}_{X/D/S}(A')$, then, by virtue of (2.2), $\pi : F_* \twoheadrightarrow E_*$ can be extended to an A'-linear map $\pi' : F'_* \to E'_*$. Clearly, π' is surjective. Hence, $\xi = 0$. □

Remark 2.1. When A' contains a field, we can give the obstruction class more explicitly. For each element E_* in $\mathrm{Par}_{X/D/S}(A)$, let

$$\xi_0 : 0 \longrightarrow (M/I) \otimes_A E_* \xrightarrow{p} E_* \xrightarrow{q} \bar{E}_* \longrightarrow 0$$

be the canonical extension. We have a long exact sequence

$$\mathrm{Ext}^1_{X_{A_0}}(\bar{E}_*, I \otimes_A E_*) \xrightarrow{\rho} \mathrm{Ext}^1_{X_{A_0}}(\bar{E}_*, M \otimes_A E_*) \xrightarrow{\pi} \mathrm{Ext}^1_{X_{A_0}}(\bar{E}_*, (M/I) \otimes_A E_*) \xrightarrow{\delta} \mathrm{Ext}^2_{X_{A_0}}(\bar{E}_*, I \otimes_A E_*).$$

Then, we can show that $\delta(\xi_0)$ is the obstruction class for lifting E_* to $\mathrm{Par}_{X/D/S}(A')$. $\delta(\xi_0)$ is represented by the following exact sequence

$$0 \longrightarrow I \otimes_A E_* \longrightarrow M \otimes_A E_* \longrightarrow E_* \longrightarrow \bar{E}_* \longrightarrow 0.$$

Now, suppose that $f : X \to S$ is smooth, projective and geometrically integral and S is a scheme of finite type over a universally Japanese ring Ξ. Let Ω be a locally free $\mathcal{O}_X$-module of finite rank and let $\pi : Y = \mathrm{Spec}(S^*(\Omega^\vee)) \to X$ be the natural projection map.

Definition 2.2. A parabolic Ω-pair is a pair (E_*, φ) consisting of a parabolic $\mathcal{O}_X$-module E_* and a parabolic homomorphism $\varphi : E_* \to E_* \otimes_X \Omega$ with $\varphi \wedge \varphi = 0$. We say that (E_*, φ) is coherent (or, torsion free) if E_* is.

By Remark 1.1 (i), the original notion of parabolic Ω-pairs in [25] is same as that of torsion free, parabolic Ω-pairs.

By the condition $\varphi \wedge \varphi = 0$, E_* has a structure of parabolic $\mathcal{O}_Y$-module with respect to π^*D. This correspondence gives us an equivalence between the category of parabolic Ω-pairs and $\mathcal{P}_{Y/\pi^*D}$. By Proposition 1.1, $\mathcal{P}_{Y/\pi^*D}$ has enough injective objects. For parabolic Ω-pairs (E_*, φ) and (E'_*, φ'), we denote by $\mathrm{Ext}^i_X((E_*, \varphi), (E'_*, \varphi'))$ the Ext-group $\mathrm{Ext}^i_Y(E_*, E'_*)$. The deformation theory for parabolic Ω-pairs is given by $\mathrm{Ext}^1_X((E_*, \varphi), (E_*, \varphi))$ and $\mathrm{Ext}^2_X((E_*, \varphi), (E_*, \varphi))$.

For convenience of readers, let us recall some notion defined in [14] and [25].

Definition 2.3. A coherent parabolic Ω_s-pair (E_*, φ) on a geometric fibre X_s is said to be parabolically stable (or, parabolically semi-stable) if it is torsion free and for every coherent φ-invariant parabolic sub-sheaf F_* of E_* with $0 \neq F \neq E$,

$$\int_0^1 \frac{\chi(F_\alpha(m))}{\mathrm{rk}\, F} d\alpha < \int_0^1 \frac{\chi(E_\alpha(m))}{\mathrm{rk}\, E} d\alpha \quad (\text{or, } \leq, \text{ resp.})$$

for sufficiently large integers m. Here, "φ-invariant" means $\varphi(F_*) \subseteq F_* \otimes_X \Omega$.

Let α_* be a set of real numbers $\{\alpha_1, \alpha_2, \dots, \alpha_l\}$ with $0 \le \alpha_1 < \alpha_2 < \cdots < \alpha_l < 1$ and let H_* be a set of numerical polynomials $\{H, H_1, H_2, \dots, H_l\}$. Let us denote by $\mathrm{Par}^{H_*,\alpha_*,s}_{Y/\pi^*D/S}$ (or, $\mathrm{Par}^{H_*,\alpha_*,ss}_{Y/\pi^*D/S}$) a sub-functor of $\mathrm{Par}_{Y/\pi^*D/S}$ such that $(E_*, \varphi) \in \mathrm{Par}_{Y/\pi^*D/S}(T)$ is a T-valued point of it if and only if for all geometric points t of T, the system of weights of $E_*|_{X_t}$ is α_*, $(E_*, \varphi)_{X_t}$ is parabolically stable (or, parabolically semi-stable, resp.), $\chi(E(m)_{X_t}) = H(m)$ and $\chi((E/i_E^{\alpha_{i+1},0}(E_{\alpha_{i+1}}))(m)_{X_t}) = H_i(m)$ for $i = 1, 2, \dots, l$ ($\alpha_{l+1} = 1$). By virtue of Theorem 1.11 of [25], $\mathrm{Par}^{H_*,\alpha_*,s}_{Y/\pi^*D/S}$ and $\mathrm{Par}^{H_*,\alpha_*,ss}_{Y/\pi^*D/S}$ are open sub-functors of $\mathrm{Par}_{Y/\pi^*D/S}$.

In the previous paper [14] and [25], we have constructed moduli schemes of Seshadri equivalence classes of parabolically semi-stable Ω-pairs.

Theorem 2.3 (Theorem 4.6 of [25]**).** *Assume that all α_i are rational numbers. Then there exist an S-scheme $\bar{M}^{H_*,\alpha_*}_{\Omega/X/D/S}$ and a morphism $\Upsilon : \mathrm{Par}^{H_*,\alpha_*,ss}_{Y/\pi^*D/S} \to \bar{M}^{H_*,\alpha_*}_{\Omega/X/D/S}$ such that*

(i) *$\bar{M}^{H_*,\alpha_*}_{\Omega/X/D/S}$ is locally of finite type and separated over S,*

(ii) *$\bar{M}^{H_*,\alpha_*}_{\Omega/X/D/S}$ contains an open sub-scheme $M^{H_*,\alpha_*}_{\Omega/X/D/S}$ which is a coarse moduli scheme for the functor $\mathrm{Par}^{H_*,\alpha_*,s}_{Y/\pi^*D/S}$ and*

(iii) *for each geometric point s of S, $\Upsilon(k(s))$ induces a bijection*

$$\mathrm{Par}^{H_*,\alpha_*,ss}_{Y/\pi^*D/S}(k(s))/\sim \xrightarrow{\simeq} \bar{M}^{H_*,\alpha_*}_{\Omega/X/D/S}(k(s)),$$

where $\sim$ means the Seshadri equivalence relation (cf. Definition 1.12 of [25]*).*

If S is a noetherian scheme over a field of characteristic zero or $\dim X/S \le 2$, then $\bar{M}^{H_*,\alpha_*}_{\Omega/X/D/S}$ is quasi-projective over S. When $\Omega = 0$, $\bar{M}^{H_*,\alpha_*}_{\Omega/X/D/S}$ is a moduli scheme of Seshadri equivalence classes of parabolically semi-stable sheaves $\bar{M}^{H_*,\alpha_*}_{X/D/S}$.

Remark 2.2. If $(E_*, \varphi) \in \mathrm{Par}^{H_*,\alpha_*,ss}_{Y/\pi^*D/S}(T)$, then (E_*, φ) is a flat family of parabolic Ω-pairs i.e. all $i_E^{\alpha,\beta}$ are injective and all $E/i_E^{\alpha_{i+1},0}(E_{\alpha_{i+1}})$ are T-flat. In fact, the first assertion follows from Remark 1.10 of [25]. Since all E_α are T-flat and $i_E^{\alpha,\beta}|_{X_t} : E_\alpha|_{X_t} \to E_\beta|_{X_t}$ are injective, the local criterion of flatness implies that $E_\beta/i_E^{\alpha,\beta}(E_\alpha)$ are T-flat. Hence, the morphism Υ is defined by the condition (4.6.4) in Theorem 4.6 of [25].

Theorem 2.4. *Let s be a k-valued geometric point of S and let (E_*, φ) be an Ω-pair corresponding to a k-valued geometric point x of $(M^{H_*,\alpha_*}_{\Omega/X/D/S})_s$. Then the Zariski tangent space of $(M^{H_*,\alpha_*}_{\Omega/X/D/S})_s$ at x is naturally isomorphic to $\mathrm{Ext}^1_{X_s}((E_*, \varphi), (E_*, \varphi))$. If $\mathrm{Ext}^2_{X_s}((E_*, \varphi), (E_*, \varphi)) = 0$, then $M^{H_*,\alpha_*}_{\Omega/X/D/S}$ is smooth at x over S. Moreover, if $S = \mathrm{Spec}(k)$ for a field k and if $\mathrm{Ext}^2_{X_s}((E_*, \varphi), (E_*, \varphi)) = 0$ for all parabolically semi-stable Ω-pairs (E_*, φ) on a geometric fibre X_s, then $\bar{M}^{H_*,\alpha_*}_{\Omega/X/D/S}$ is normal.*

Proof. The moduli $\bar{M}^{H_\bullet,\alpha_\bullet}_{\Omega/X/D/S}$ was constructed as an inductive limit of a family of open immersions $\bar{M}^1 \to \bar{M}^2 \to \cdots$ parametrized by positive integers. Each $\bar{M}^e$ is a good quotient of a scheme R^e by a $\mathrm{PGL}(V_e)$-action where V_e is a free Ξ-module. On the scheme $X_e = X \times_S R^e$, we constructed an R^e-flat parabolic Ω-pair (E^e_*, φ^e) and a surjection $\psi^e : V_e \otimes_\Xi \mathcal{O}_{X_e} \twoheadrightarrow E^e(m_e)$ for some integer m_e. This (E^e_*, φ^e) determines a map $q^e : R^e \to \mathrm{Par}^{H_\bullet,\alpha_\bullet,ss}_{Y/\pi^*D/S}$. The map $\Upsilon \circ q^e$ is just the quotient map $\xi : R^e \to \bar{M}^e$. By the definition of R^e, it is easy to see that the smoothness of R^e over S at a point x is equivalent to the formal smoothness of the functor $\mathrm{Par}^{H_\bullet,\alpha_\bullet,ss}_{Y/\pi^*D/S}$ over S at $q^e(x)$. To prove normality of the moduli scheme, it is enough to show that all R^e are smooth. So, the last assertion follows from Proposition 2.2. Moreover, by the completely same proof as that of Proposition 6.4 of [13], the quotient map $\xi : \xi^{-1}(M^e) \to M^e$ is a principal fibre bundle with group $\mathrm{PGL}(V_e)$ where $M^e = \bar{M}^e \cap M^{H_\bullet,\alpha_\bullet}_{\Omega/X/D/S}$. Hence, vanishing of $\mathrm{Ext}^2_{X_\bullet}((E_*, \varphi), (E_*, \varphi))$ for a parabolically stable Ω-pair (E_*, φ) implies the smoothness of the moduli scheme over S at the corresponding point.

For tangent spaces, by Corollary 6.4.1 of [13], we may assume that $S = \mathrm{Spec}(k)$ with algebraically closed field k. Let $k[t]$ be a k-algebra with $t^2 = 0$. Since $\xi : \xi^{-1}(M^e) \to M^e$ is a principal fibre bundle, it is easy to see that the natural map

$$\mathrm{Par}^{H_\bullet,\alpha_\bullet,ss}_{Y/\pi^*D/S}(k[t])_{(E_*,\varphi)} \longrightarrow M^{H_\bullet,\alpha_\bullet}_{\Omega/X/D/S}(k[t])_{(E_*,\varphi)}.$$

is bijective. Since the action defined in Proposition 2.1 is compatible with any maps of deformation situations, this bijection induces the desired isomorphism of k-vector spaces. □

Remark 2.3. If we set $\Omega = 0$ in Theorem 2.4, we get the corresponding results for the moduli scheme $\bar{M}^{H_\bullet,\alpha_\bullet}_{X/D/S}$.

3. Tensor products, Hom-sheaves and Serre duality theorem for parabolic sheaves

Let us define parabolic tensor products for parabolic sheaves E_* and F_*. For each real number α, let us set

$$(E_* \otimes_X F_*)_\alpha = \Big(\bigoplus_{\alpha_1+\alpha_2=\alpha} (E_{\alpha_1} \otimes_X F_{\alpha_2})\Big)/R_\alpha$$

where R_α is a sub-$\mathcal{O}_X$-module of $\bigoplus_{\alpha_1+\alpha_2=\alpha}(E_{\alpha_1} \otimes_X F_{\alpha_2})$ generated locally by sections of types

(i) $i_E^{\alpha_1,\alpha_1'}(x) \otimes y - x \otimes i_F^{\alpha_2',\alpha_2}(y) \quad (x \in E_{\alpha_1},\ y \in F_{\alpha_2'},\ \alpha_1 + \alpha_2 = \alpha_1' + \alpha_2' = \alpha)$

(ii) $x - j^{\beta,\gamma}(x) \quad (x \in E_\beta \otimes_X F_\gamma,\ \beta + \gamma = \alpha)$

where $j^{\beta,\gamma}$ is the isomorphism

$$(1 \otimes j_F(\gamma)) \circ ((j_E(\beta - 1))^{-1} \otimes 1) : E_\beta \otimes_X F_\gamma \to E_{\beta-1} \otimes_X \mathcal{O}_X(-D) \otimes_X F_\gamma \to E_{\beta-1} \otimes_X F_{\gamma+1}.$$

For each pair of real numbers $\alpha \geq \beta$, a homomorphism

$$i^{\alpha,\beta}_{E_*\otimes F_*} : (E_* \otimes_X F_*)_\alpha \longrightarrow (E_* \otimes_X F_*)_\beta$$

is defined by setting for local sections $x \in E_{\alpha_1}$ and $y \in F_{\alpha_2}$ $(\alpha_1 + \alpha_2 = \alpha)$,

$$\begin{aligned} i^{\alpha,\beta}_{E_*\otimes F_*}(x \otimes y \mod R_\alpha) &= i_E^{\alpha_1,\beta-\alpha_2}(x) \otimes y \mod R_\beta \\ &= x \otimes i_F^{\alpha_2,\beta-\alpha_1}(y) \mod R_\beta. \end{aligned}$$

It is easy to see that this definition is well-defined and these homomorphisms $\{i^{\alpha,\beta}_{E_*\otimes F_*}\}$ make $(E_* \otimes_X F_*)_*$ an $\mathbb{R}$-filtered $\mathcal{O}_X$-module. Moreover, to define an isomorphism $j_{E_*\otimes F_*}$, let us consider isomorphisms

$$J_\alpha = \oplus_\gamma(1 \otimes j_F(\gamma)) : \bigoplus_\gamma (E_{\alpha-\gamma} \otimes_X F_\gamma \otimes_X \mathcal{O}_X(-D)) \longrightarrow \bigoplus_\gamma (E_{\alpha-\gamma} \otimes_X F_{\gamma+1}).$$

Then it is easy to see that $J_\alpha(R_\alpha \otimes_X \mathcal{O}_X(-D)) = R_{\alpha+1}$ and that

$$j_{E_*\otimes_X F_*} = J_* \mod R_* \otimes_X \mathcal{O}_X(-D)$$

is an isomorphism $(E_* \otimes_X F_*)_* \otimes_X \mathcal{O}_X(-D) \simeq (E_* \otimes_X F_*)[1]_*$ which satisfies the condition (1.1). Thus, we get a parabolic sheaf $(E_* \otimes_X F_*)_*$.

We have a family of canonical maps

$$f_{\alpha,\beta} : E_\alpha \otimes_X F_\beta \longrightarrow (E_* \otimes_X F_*)_{\alpha+\beta}.$$

These maps defines a canonical parabolic bilinear map

$$f_{*,*} : E_* \times F_* \longrightarrow (E_* \otimes_X F_*)_*$$

i.e. each local section $b \in F_\beta$ (or, $a \in E_\alpha$) induces a parabolic homomorphism $f_{*,\beta}(-,b) : E_* \to (E_* \otimes_X F_*)[\beta]_*$ (or, $f_{\alpha,*}(a,-) : F_* \to (E_* \otimes_X F_*)[\alpha]_*$, resp.). It is easy to see that this parabolic bilinear map has a universal property as in the case of usual tensor products.

Example 3.1. Let E_* and F_* be parabolic sheaves. Assume that F_* has the special structure. Then $(E_* \otimes_X F_*)_\alpha \simeq E_\alpha \otimes_X F$ and $i^{\alpha,\beta}_{E_*\otimes F_*} = i_E^{\alpha,\beta} \otimes id_F$. In particular, for objects in $\mathcal{M}_X \subset \mathcal{P}_{X/D}$, parabolic tensor products of them are same as usual tensor products for $\mathcal{O}_X$-modules.

Clearly, two operations "tensor" and "shift" are commutative.

Lemma 3.1. *For each* $\alpha \in \mathbb{R}$, $(E[\alpha]_* \otimes_X F_*)_* \simeq (E_* \otimes_X F[\alpha]_*)_* \simeq (E_* \otimes_X F_*)[\alpha]_*$.

Proof. Clear. □

Definition 3.1. A coherent parabolic sheaf E_* is said to be locally free, if E_α are locally free $\mathcal{O}_X$-modules for all α and $E_\alpha / i_E^{\beta,\alpha}(E_\beta)$ are locally free $\mathcal{O}_D$-modules for all $\alpha \leq \beta < \alpha + 1$. If X is integral, the ranks of E_α have a constant value. It is called the rank of E_*.

Locally free parabolic sheaves of lank one (call them parabolic invertible sheaves) are obtained locally by shifting invertible sheaves with special structure. More generally, every locally free parabolic sheaf E_* is locally a direct sum of parabolic invertible sheaves.

Example 3.2. Let E_* and F_* be parabolic sheaves. Assume that E_* is locally free. Set $U = X - D$. Let $i : U \to X$ be an inclusion map. Since $E_\alpha \otimes_X F_\beta \subset E_\alpha \otimes_X i_*i^*(F) = i_*i^*(E \otimes_X F)$, all $E_\alpha \otimes_X F_\beta$ are regarded as sub-modules of $i_*i^*(E \otimes_X F)$. In this case,

$$(E_* \otimes_X F_*)_\alpha = \sum_{\alpha_1+\alpha_2=\alpha} (E_{\alpha_1} \otimes_X F_{\alpha_2})$$

as a sub-module of $i_*i^*(E \otimes_X F)$ and $i^{\alpha,\beta}_{E_* \otimes F_*}$ is the natural inclusion map.

Tensoring a parabolic sheaf induces a right exact functor.

Proposition 3.2. *For any exact sequence of parabolic sheaves*

$$0 \longrightarrow E'_* \xrightarrow{f} E_* \xrightarrow{g} E''_* \longrightarrow 0,$$

the sequence

$$E'_* \otimes_X F_* \xrightarrow{f \otimes id} E_* \otimes_X F_* \xrightarrow{g \otimes id} E''_* \otimes_X F_* \longrightarrow 0$$

is exact.

Proof. We shall prove in Lemma 3.5 that the functor $\otimes_X F_*$ has a right adjoint functor $\mathcal{H}om(F_*, -)_*$. For the functor $\mathcal{H}om(F_*, -)_*$, left exactness is clear. Then our claim follows from standard arguments. □

In the above proposition, if $f \otimes id$ is injective for any short exact sequences, F_* is said to be parabolically flat.

Proposition 3.3. *A coherent parabolic sheaf E_* is parabolically flat if and only if it is locally free.*

Proof. If E_* is locally free, it is locally a direct sum of parabolic invertible sheaves. Hence, it is clearly parabolically flat.

Conversely, assume that E_* is parabolically flat. The question is local, so we may assume that X is an affine scheme. Since the category of $\mathcal{O}_X$-modules $\mathcal{M}_X$ is a full sub-category of $\mathcal{P}_{X/D}$, all E_α must be locally free and of finite rank. Then all $i_E^{\alpha,\beta}$ are injective. Without loss of generality, we may identify E_α with a sub-module of E_β. If E_* is not locally free, we may assume that for some $0 < \alpha < 1$, $E_{-\alpha}/E$ is not a locally free $\mathcal{O}_D$-module. Then for some ideal I of $\mathcal{O}_X$ containing $\mathcal{O}_X(-D)$, $\mathrm{Tor}_1^{\mathcal{O}_X}(E_{-\alpha}/E, \mathcal{O}_X/I) \neq 0$. Let J_* be a parabolic sheaf such that $J_0 = \mathcal{O}_X$, $J_\beta = I$ for $0 < \beta \leq \alpha$ and $J_\beta = \mathcal{O}_X(-D)$ for $\alpha < \beta \leq 1$. Then we have a natural injection $\phi : J_* \to \mathcal{O}_X[-\alpha]_*$. It is easy to see that $(E_* \otimes_X J_*)_0$ is given by the exact sequence

$$0 \longrightarrow E \otimes_X I \xrightarrow{f} E \oplus (E_{-\alpha} \otimes_X I) \xrightarrow{g} (E_* \otimes_X J_*)_0 \longrightarrow 0$$

where $f(e\otimes a) = (ea, -i_E^{0,-\alpha}(e)\otimes a)$ and g is the composite map of the natural injection $E \oplus (E_{-\alpha} \otimes_X I) \simeq (E_0 \otimes_X J_0) \oplus (E_{-\alpha} \otimes_X J_\alpha) \to \oplus_\beta (E_{-\beta} \otimes_X J_\beta)$ and the quotient map $\oplus_\beta(E_{-\beta} \otimes_X J_\beta) \to (E_* \otimes_X J_*)_0$. On the other hand, $(E_* \otimes_X \mathcal{O}_X[-\alpha]_*)_0 \simeq E_{-\alpha}$. Hence, we obtain the following exact commutative diagram.

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & E_{-\alpha} \otimes I & \xrightarrow{g'} & \operatorname{Ker}\pi & & \\
 & & \downarrow & & \downarrow{\scriptstyle \binom{0}{id}} & & \downarrow & & \\
0 & \longrightarrow & E \otimes I & \xrightarrow{f} & E \oplus (E_{-\alpha} \otimes I) & \xrightarrow{g} & (E_* \otimes J_*)_0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i_E^{0,-\alpha}\otimes 1} & & \downarrow{\scriptstyle (i_E^{0,-\alpha}\ 0)} & & \downarrow{\scriptstyle \pi} & & \\
0 & \longrightarrow & E_{-\alpha} \otimes I & \longrightarrow & E_{-\alpha} & \longrightarrow & E_{-\alpha} \otimes (\mathcal{O}_X/I) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & (E_{-\alpha}/E) \otimes I & \longrightarrow & E_{-\alpha}/E & \longrightarrow & (E_{-\alpha}/E) \otimes (\mathcal{O}_X/I) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The connecting homomorphism $\operatorname{Ker}\pi \to (E_{-\alpha}/E) \otimes_X I$ induces an isomorphism

$$\operatorname{Ker}\pi / g'(E_{-\alpha} \otimes_X I) \simeq \operatorname{Tor}_1^{\mathcal{O}_X}(E_{-\alpha}/E, \mathcal{O}_X/I).$$

On the other hand, we have the following exact commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E_{-\alpha} \otimes I & \xrightarrow{g'} & (E_* \otimes J_*)_0 & \xrightarrow{h} & (E_* \otimes J_*)_0 / g'(E_{-\alpha} \otimes I) & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle (1\otimes\phi)_0} & & \downarrow{\scriptstyle (\overline{1\otimes\phi})_0} & & \\
0 & \longrightarrow & E_{-\alpha} \otimes I & \longrightarrow & E_{-\alpha} & \xrightarrow{h'} & E_{-\alpha} \otimes (\mathcal{O}_X/I) & \longrightarrow & 0.
\end{array}$$

Note that $h' \circ (1 \otimes \phi)_0 = (\overline{1 \otimes \phi})_0 \circ h = \pi$. Hence, by the snake lemma,

$$\operatorname{Ker}(1 \otimes \phi)_0 \simeq \operatorname{Ker}(\overline{1 \otimes \phi})_0 = \operatorname{Ker}\pi / g'(E_{-\alpha} \otimes_X I) \simeq \operatorname{Tor}_1^{\mathcal{O}_X}(E_{-\alpha}/E, \mathcal{O}_X/I).$$

Thus, $(1 \otimes \phi)_0 : (E_* \otimes J_*)_0 \to (E_* \otimes \mathcal{O}_X[-\alpha]_*)_0$ is not injective. □

Definition 3.2. Let E_* and F_* be parabolic sheaves. For each $\alpha \in \mathbb{R}$, let us set

$$\mathcal{H}om_X(E_*, F_*)_\alpha = \mathcal{H}om_X(E_*, F[\alpha]_*)$$

For each $\alpha \geq \beta$, the homomorphism $i_F^{[\alpha,\beta]}$ induces the natural homomorphism

$$i^{\alpha,\beta}_{\mathcal{H}om_X(E_*,F_*)} : \mathcal{H}om_X(E_*, F_*)_\alpha \longrightarrow \mathcal{H}om_X(E_*, F_*)_\beta.$$

Moreover, the canonical isomorphism

$$(\prod_\beta \mathcal{H}om_X(E_\beta, F_{\alpha+\beta})) \otimes_X \mathcal{O}_X(-D) \simeq \prod_\beta \mathcal{H}om_X(E_\beta, F_{\alpha+\beta+1})$$

induces an isomorphism

$$j_{\mathcal{H}om_X(E_*,F_*)}(\alpha) : \mathcal{H}om_X(E_*, F_*)_\alpha \otimes_X \mathcal{O}_X(-D) \to \mathcal{H}om_X(E_*, F_*)_{\alpha+1}.$$

It is easy to see that these isomorphisms make $\mathcal{H}om_X(E_*, F_*)_*$ a parabolic sheaf. It is called a parabolic Hom-sheaf.

Lemma 3.4. *For each $\alpha \in \mathbb{R}$, there are natural isomorphisms*

$$\mathcal{H}om_X(E_*, F_*)[\alpha]_* \simeq \mathcal{H}om_X(E[-\alpha]_*, F_*)_* \simeq \mathcal{H}om_X(E_*, F[\alpha]_*)_*.$$

Proof. Clear. □

Example 3.3. If E_* has the special structure, then $\mathcal{H}om_X(E_*, F_*)_\alpha \simeq \mathcal{H}om_X(E, F_\alpha)$ and $i^{\alpha,\beta}_{\mathcal{H}om_X(E_*,F_*)} \simeq (i^{\alpha,\beta}_F)_*$. In particular, $\mathcal{H}om_X((\mathcal{O}_X)_*, F_*)_* \simeq F_*$ if $(\mathcal{O}_X)_*$ has the special structure. If, furthermore, F_* has the special structure, $\mathcal{H}om_X(E_*, F_*)_*$ also has the special structure.

Lemma 3.5. *There is a natural isomorphism of parabolic sheaves*

$$\mathcal{H}om_X((E_* \otimes_X F_*)_*, G_*)_* \simeq \mathcal{H}om_X(E_*, \mathcal{H}om_X(F_*, G_*)_*)_*.$$

Proof. Clear by the universal property of the canonical parabolic bilinear map $f_{*,*} : E_* \times F_* \to (E_* \otimes_X F_*)_*$. □

Let us introduce two more operations for parabolic sheaves E_*. Let $\hat{E}_\alpha$ be an inductive limit $\varinjlim_{\beta>\alpha} E_\beta$. For each $\alpha \geq \beta$, we have a natural homomorphism $i^{\alpha,\beta}_{\hat{E}} : \hat{E}_\alpha \to \hat{E}_\beta$ induced by $i^{*,*}_E$ and have an isomorphism $j_{\hat{E}}(\alpha) : \hat{E}_\alpha \otimes_X \mathcal{O}_X(-D) \to \hat{E}_{\alpha+1}$. Thus, we obtain a parabolic sheaf $\hat{E}_*$. If E_* is a coherent parabolic sheaf corresponding to a filtration $E = F_1(E) \supset F_2(E) \supset \cdots \supset F_{l+1}(E) = E \otimes_X \mathcal{O}_X(-D)$ and weights $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_l < 1$, then $\hat{E}_*$ is a parabolic sheaf with $E_\alpha = F_i(E)$ for all $\alpha_{i-1} \leq \alpha < \alpha_i$ $(i = 1, \ldots, l+1, \alpha_0 = \alpha_l - 1, \alpha_{l+1} = 1 + \alpha_1)$.

Secondly, let us set $\vee(E_*)_\alpha = E^\vee_{-\alpha}$ (the dual sheaf of $E_{-\alpha}$), $i^{\alpha,\beta}_{\vee(E)} = (i^{-\beta,-\alpha}_E)^\vee$ for $\alpha \geq \beta$ and $j_{\vee(E)}(\alpha) = (j_E(-\alpha-1) \otimes id_{\mathcal{O}_X(D)})^\vee$. Then $\vee(E_*)_*$ is also a parabolic sheaf.

Now let F be a parabolic sheaf with special structure. Then we have

$$\mathcal{H}om_X(E_*, F)_\alpha \simeq \mathcal{H}om_X(\hat{E}[-1]_{-\alpha}, F) \simeq \mathcal{H}om_X(\hat{E}_{-\alpha} \otimes_X \mathcal{O}_X(D), F).$$

In particular, we have a natural isomorphism of parabolic sheaves.

$$E^\vee_* \simeq \vee(\hat{E}_* \otimes_X \mathcal{O}_X(D))_* \simeq \vee(\hat{E}_*)_* \otimes_X \mathcal{O}_X(-D). \tag{3.1}$$

where $E^\vee_* = \mathcal{H}om_X(E_*, \mathcal{O}_X)_*$. We have a canonical homomorphism

$$E_* \longrightarrow E^{\vee\vee}_*.$$

If this is an isomorphism, E_* is said to be reflexive. A coherent parabolic sheaf E_* is reflexive if and only if all E_α are reflexive, since $(E_*^{\vee\vee})_\alpha = (E_\alpha)^{\vee\vee}$. In particular, locally free parabolic sheaves are reflexive.

Lemma 3.6. *If E_* is locally free, then there are canonical isomorphisms*

$$(E_*^\vee \otimes_X F_*)_* \simeq \mathcal{H}om_X(E_*, F_*)_*, \tag{3.2}$$

$$\mathrm{Ext}^i_X(E_*, F_*) \simeq H^i(X, (E_*^\vee \otimes_X F_*)_0) \simeq H^i(X, \mathcal{H}om_X(E_*, F_*)). \tag{3.3}$$

Proof. For (3.2), since we have a canonical map, it is enough to prove it locally. Then (3.2) is clear since we may assume that E_* is a direct sum of parabolic invertible sheaves. For (3.3), the result follows from parabolically flatness of $E_*^\vee$, (3.2) and Corollary 1.3. □

The Serre duality theorem can be generalized for parabolic sheaves as follows.

Proposition 3.7. *Let X be a non-singular projective variety of dimension n over an algebraically closed field k. Let ω_X be a dualizing sheaf on X. Then for all locally free parabolic sheaves E_* and F_*, there are natural isomorphisms*

$$\theta^i : \mathrm{Ext}^i_X(E_*, F_* \otimes_X \omega_X(D)) \xrightarrow{\simeq} \mathrm{Ext}^{n-i}_X(F_*, \hat{E}_*)^\vee \tag{3.4}$$

If, moreover, D is non-singular, then this formula holds for all coherent parabolic sheaves E_* and F_*.

Proof. If $P^\bullet_* \to E_*$ and $Q^\bullet_* \to F_*$ are resolutions of finite length by locally free parabolic sheaves of finite rank, then $\mathrm{Ext}^i_X(E_*, F_* \otimes_X \omega_X(D))$ is canonically isomorphic to hypercohomology $\mathbb{H}^i(X, ((P^\bullet_*)^\vee \otimes_X Q^\bullet_*)^\bullet_* \otimes_X \omega_X(D))$. By Corollary 1.3, this is isomorphic to $\mathbb{H}^i(X, ((P^\bullet_*)^\vee \otimes_X Q^\bullet_*)^\bullet_0 \otimes_X \omega_X(D))$. Since for locally free parabolic sheaves P_* and Q_*,

$$(P_*^\vee \otimes_X Q_*)_0 = ((Q_*^\vee \otimes_X P_*)\hat{}_0)^\vee \otimes_X \mathcal{O}_X(-D) = ((Q_*^\vee \otimes_X \hat{P}_*)_0)^\vee \otimes_X \mathcal{O}_X(-D),$$

by the Serre duality theorem for complexes, we obtain (3.4). □

Remark 3.1. 1) If D is not smooth, a coherent parabolic sheaf E_* does not always have a locally free resolution of finite length.

2) The theorem can be generalized to isomorphisms of $\mathbb{R}$-filtered k-modules

$$\begin{aligned} \theta^i_* : \mathrm{Ext}^i_X(E_*, F_* \otimes_X \omega_X(D))_* &\simeq \vee(\mathrm{Ext}^{n-i}_X(F_*, \hat{E}_*)_*)_* \\ &\simeq \vee(H^{n-i}(X, \mathrm{Hom}_X(F_*, \hat{E}_*)_*)_*)_*. \end{aligned}$$

3) Even for locally free parabolic sheaves E_*, in general,

$$\mathcal{H}om_X(E_*, E_*)^\vee \not\simeq \mathcal{H}om_X(E_*, E_*)$$

though we have $\mathcal{H}om_X(E_*, E_*)^\vee_* \simeq \mathcal{H}om_X(E_*, E_*)_*$.

4. Ext-groups for parabolic Ω-pairs

Let (E_*, φ) be a parabolic Ω-pair. Let r be the rank of Ω and let $B = S^*(\Omega^\vee)$. Then we have the following Koszul exact sequence of B-modules.

$$\xi : 0 \longrightarrow \wedge^r(\Omega^\vee) \otimes_X B \xrightarrow{d^r} \cdots \xrightarrow{d^3} \wedge^2(\Omega^\vee) \otimes_X B \xrightarrow{d^2} \Omega^\vee \otimes_X B \xrightarrow{d^1} B \xrightarrow{d^0} \mathcal{O}_X \longrightarrow 0$$

Here d^i is given locally by

$$d^i((w_1 \wedge \cdots \wedge w_i) \otimes b) = \sum_{j=1}^{i} (-1)^{j+1} (w_1 \wedge \cdots \wedge \check{w}_j \wedge \cdots \wedge w_i) \otimes w_j b.$$

Since all $\wedge^i(\Omega^\vee) \otimes_X B$ are flat over X, $\xi \otimes_X E_*$ is also an exact sequence. Let us denote $E_*^i = E_* \otimes_X \wedge^i(\Omega^\vee) \otimes_X B$ and $d_E^i = id_{E_*} \otimes d^i$.

Let us define $\delta^i : E_*^i \to E_*^{i-1}$ by

$$\delta^i(e \otimes (w_1 \wedge \cdots \wedge w_i) \otimes b) = \sum_{j=1}^{i} (-1)^{j+1} w_j e \otimes (w_1 \wedge \cdots \wedge \check{w}_j \wedge \cdots \wedge w_i) \otimes b.$$

Then it is easy to see that $\delta^i \delta^{i+1} = 0$ and $\delta^i d^{i+1} + d^i \delta^{i+1} = 0$. Let us set $E_*^{i,k} = E_* \otimes_X \wedge^i(\Omega^\vee) \otimes_X S^k(\Omega^\vee)$. Then $E_*^i = \oplus_{k \geq 0} E_*^{i,k}$. Note that $d^i(E_*^{i,k}) \subset E_*^{i-1,k+1}$ and $\delta^i(E_*^{i,k}) \subset E_*^{i-1,k}$. Let us set $\partial^i = d^i - \delta^i$ for $i \geq 1$. Then $\partial^i \partial^{i+1} = 0$.

Lemma 4.1. *The sequence*

$$\zeta : 0 \longrightarrow E_*^r \xrightarrow{\partial^r} \cdots \xrightarrow{\partial^3} E_*^2 \xrightarrow{\partial^2} E_*^1 \xrightarrow{\partial^1} E_*^0 \xrightarrow{\partial^0} E_* \longrightarrow 0$$

is exact, where $\partial^0(e \otimes b) = be$.

Proof. Since all homomorphisms preserve filtrations, we may ignore parabolic structures. Note that the question is local. Clearly, $\partial^0 \partial^1 = 0$ and $\mathrm{Coker}(\partial^1) = E \otimes_B B = E$. Hence, ζ is exact at E^0 and ∂^0 is surjective. Now, let us take an element $x \in E^i$ with $\partial^i(x) = 0$. Set $x = \sum_{k=0}^{m} x_k$ $(x_k \in E_k^i)$. Since

$$\partial^i(x) = -\delta^i(x_0) + \sum_{k=0}^{m-1} (d^i(x_k) - \delta^i(x_{k+1})) + d^i(x_m) = 0,$$

we have $\delta^i(x_0) = 0$, $d^i(x_k) = \delta^i(x_{k+1})$ $(i = 0, \ldots, m-1)$ and $d^i(x_m) = 0$. Since $\xi \otimes E$ is exact, there exists an element $x'_{m-1} \in E_{m-1}^{i-1}$ such that $d^{i-1}(x'_{m-1}) = x_m$. Then

$$d^i(x_{m-1}) = \delta^i(d^{i+1}(x'_{m-1})) = -d^i(\delta^{i+1}(x'_{m-1})).$$

Hence, we get an element $x'_{m-2} \in E_{m-2}^{i+1}$ such that $x_{m-1} = d^{i+1}(x'_{m-2}) - \delta^{i+1}(x'_{m-1})$. Continuing this procedure, we get elements $x'_k \in E_k^{i+1}$ such that $x_k = d^{i+1}(x'_{k-1}) - \delta^{i+1}(x'_k)$ for $k = 0, \ldots, m$ with $x'_{-1} = 0$. Then $\partial^{i+1}(\sum_{k=0}^{m-1} x'_k) = x$. $\square$

Let (F_*, ψ) be another parabolic Ω-pair. For each i, we have the canonical isomorphism

$$\phi^i : \mathrm{Hom}_B(E^i_*, F_*) \xrightarrow{\simeq} \mathrm{Hom}_X(E_*, F_* \otimes_X \wedge^i\Omega).$$

We can easily verify the commutativity of the following diagram.

$$\begin{array}{ccc} \mathrm{Hom}_B(E^i_*, F_*) & \xrightarrow{(\partial^{i+1})^*} & \mathrm{Hom}_B(E^{i+1}_*, F_*) \\ \downarrow{\phi^i} & & \downarrow{\phi^{i+1}} \\ \mathrm{Hom}_X(E_*, F_* \otimes_X \wedge^i\Omega) & \xrightarrow{\lambda^i_{(\varphi,\psi)}} & \mathrm{Hom}_X(E_*, F_* \otimes_X \wedge^{i+1}\Omega) \end{array} \tag{4.1}$$

Here, $\lambda^i_{(\varphi,\psi)}(f) = \pi_1(\psi \otimes 1)f - \pi_2(f \otimes 1)\varphi$.

$$\begin{array}{ccccc} E_* & \xrightarrow{\varphi} & E_* \otimes_X \Omega & \xrightarrow{f\otimes 1} & F_* \otimes_X \wedge^i\Omega \otimes_X \Omega \\ \downarrow{f} & & & & \downarrow{\pi_2} \\ F_* \otimes_X \wedge^i\Omega & \xrightarrow{\psi\otimes 1} & F_* \otimes_X \Omega \otimes_X \wedge^i\Omega & \xrightarrow{\pi_1} & F_* \otimes_X \wedge^{i+1}\Omega \end{array}$$

$(\pi_1(e \otimes w \otimes v) = e \otimes (w \wedge v),\ \pi_1(e \otimes w \otimes v) = e \otimes (v \wedge w))$

Proposition 4.2. *For parabolic Ω-pairs (E_*, φ) and (F_*, ψ), there is a spectral sequence*

$$E_1^{pq} = \mathrm{Ext}^q_X(E_*, F_* \otimes_X \wedge^p\Omega) \Rightarrow \mathrm{Ext}^{p+q}_X((E_*, \varphi), (F_*, \psi)) \tag{4.2}$$

Moreover, if E_* is $\mathcal{O}_X$-locally free, then there is a canonical isomorphism

$$\mathrm{Ext}^i_X((E_*, \varphi), (F_*, \psi)) \simeq \mathbb{H}^i(X, C^\bullet(\varphi, \psi)). \tag{4.3}$$

where $C^\bullet(\varphi, \psi)$ is a complex of $\mathcal{O}_X$-modules with $C^i(\varphi, \psi) = \mathcal{H}om_X(E_*, F_* \otimes_X \wedge^i\Omega)$ and $d^i_{C^\bullet(\varphi,\psi)} = \lambda^i_{(\varphi,\psi)}$. In particular, there is another spectral sequence

$$E_2^{pq} = H^p(X, H^q(C^\bullet(\varphi, \psi))) \Rightarrow \mathrm{Ext}^{p+q}_X((E_*, \varphi), (F_*, \psi)). \tag{4.4}$$

Proof. Let us take an injective resolution $0 \to F_* \to I^\bullet_*$ of F_* as a parabolic B-module. Let $\psi^i : I^i_* \to I^i_* \otimes_X \Omega$ be the Ω-pair structure induced by its B-module structure. Then we obtain a double complex $C^{\bullet\bullet} = C^\bullet(\varphi, \psi^\bullet)$. By Lemma 4.1 and the commutativity of the diagram (4.1), for each i, $0 \longrightarrow \mathcal{H}om_B(E_*, I^i_*) \longrightarrow C^\bullet(\varphi, \psi^i)$ is exact. Hence, we get a resolution

$$0 \longrightarrow \mathcal{H}om_B(E_*, I^\bullet_*) \longrightarrow C^{\bullet\bullet}.$$

This induces a quasi-isomorphism of complexes $\mathcal{H}om_B(E_*, I^\bullet_*) \simeq \mathrm{tot}\, C^{\bullet\bullet}$, where $\mathrm{tot}\, C^{\bullet\bullet}$ is the total complex of $C^{\bullet\bullet}$. The first spectral sequence is given by the spectral sequence of the double complex $C^{\bullet\bullet}$.

If E_* is locally free as a parabolic $\mathcal{O}_X$-module, then for each j,

$$0 \longrightarrow \mathcal{H}om_X(E_*, F_* \otimes_X \wedge^j\Omega) \longrightarrow C^j(\varphi, \psi^\bullet)$$

gives us an injective resolution of $\mathcal{H}om_X(E_*, F_* \otimes_X \wedge^i\Omega)$. Hence, we have quasi-isomorphisms

$$\mathcal{H}om_B(E_*, I_*^\cdot) \simeq \operatorname{tot} C^{\cdot\cdot} \simeq C^\cdot(\varphi, \psi).$$

Hence, $\operatorname{Ext}_X^\cdot((E_*, \varphi), (F_*, \psi))$ is isomorphic to the hypercohomology $\mathbb{H}^\cdot(X, C^\cdot(\varphi, \psi))$. Finally, (4.4) is usual spectral sequence of the hypercohomology. □

Remark 4.1. The map $\lambda^0_{(\varphi,\psi)} : \mathcal{H}om_X(E_*, F_*) \to \mathcal{H}om_X(E_*, F_* \otimes_X \Omega)$ gives us an Ω-pair structure on $\mathcal{H}om_X(E_*, F_*)$. If $\Omega = \Omega^1_X$, the hypercohomology $\mathbb{H}^i(X, C^\cdot(\varphi, \psi))$ coincides with $H^i_{Dol}(X, \mathcal{H}om_X(E_*, F_*))$ the Dolbeault cohomology with coefficients in $\mathcal{H}om_X(E_*, F_*)$ that is defined by Simpson [23]. For a Higgs bundle (E, φ), by the isomorphism (4.3), we have

$$H^i_{Dol}(X, E) \simeq \operatorname{Ext}^i_X((\mathcal{O}_X, 0), (E, \varphi)).$$

5. Moduli spaces on curves

Let X be a smooth projective curve of genus $g \geq 2$ over an algebraically closed field k. Let $D = p_1 + \ldots + p_n$ be a reduced divisor. Let $\bar{M} = \bar{M}(d, \{\alpha_i^j, n_i^j\})$ (or, $\bar{M}_{\text{Higgs}} = \bar{M}_{\text{Higgs}}(d, \{\alpha_i^j, n_i^j\})$) be the moduli scheme of Seshadri equivalence classes of parabolically semi-stable bundles E_* (or, parabolically semi-stable Higgs bundles (E_*, φ), resp.) with $\deg E = d$, $\operatorname{rk}(E) = r$, flag structure of type $(n_i^1, \ldots, n_i^{l_i})$ at p_i and weights $\{\alpha_i^1, \ldots, \alpha_i^{l_i}\}$ at p_i (note that $r = n_i^1$). Let $F_i = \operatorname{Flag}(n_i^1, \ldots, n_i^{l_i})$ be a flag variety of type $(n_i^1, \ldots, n_i^{l_i})$.

Theorem 5.1 (Mehta and Seshadri[15]). *The moduli scheme $\bar{M}(d, \{\alpha_i^j, n_i^j\})$ is an irreducible normal projective variety of dimension $1 + r^2(g-1) + \sum_{i=1}^n \dim F_i$. It is smooth at points corresponding to parabolically stable bundles.*

Let us verify this theorem by calculating $\operatorname{Ext}^i_X(E_*, E_*)$ (for irreducibility, see the proof of Theorem 4.1 of [15]). Though those are calculated by usual cohomology theory using isomorphism (3.3), we shall show it by a method which can be applicable to non-locally free sheaves in higher dimensional cases. By our duality theorem, $\operatorname{Ext}^2_X(E_*, E_*) = 0$. So, the moduli scheme $\bar{M}$ is normal and it is non-singular at points corresponding to parabolically stable bundles. E_* has a parabolic sub-sheaf E (with special structure). Set $G_* = E_*/E$. The natural short exact sequence

$$0 \longrightarrow E \longrightarrow E_* \longrightarrow G_* \longrightarrow 0,$$

yields a long exact sequence

$$0 \longrightarrow \operatorname{Hom}_X(E_*, E_*) \xrightarrow{f} \operatorname{Hom}_X(E, E_*)$$
$$\xrightarrow{\delta} \operatorname{Ext}^1_X(G_*, E_*) \xrightarrow{g} \operatorname{Ext}^1_X(E_*, E_*) \xrightarrow{h} \operatorname{Ext}^1_X(E, E_*) \longrightarrow 0.$$

Let us assume that E_* is parabolically stable, then it is parabolically simple i.e. $\operatorname{Hom}_X(E_*, E_*) = k \cdot id_{E_*}$. By Corollary 1.3, $\operatorname{Ext}^i_X(E, E_*) \simeq \operatorname{Ext}^i_X(E, E)$. Since

$\hat{G}_* \otimes_X \Omega^1_X(D) \simeq \hat{G}_*$, by the duality theorem, $\mathrm{Ext}^1_X(G_*, E_*) \simeq \mathrm{Hom}_X(E_*, \hat{G}_*)^\vee$. The parabolic sub-sheaf $\hat{E}$ of E_* is annihilated by all parabolic homomorphisms $\phi : E_* \to \hat{G}_*$. So, if $E_{p_i} = V_i^1 \supset \cdots \supset V_i^{l_i} \supset V_i^{l_i+1} = 0$ is the flag structure which determines the parabolic structure of E_* at p_i, then

$$\begin{aligned}\mathrm{Hom}_X(E_*, \hat{G}_*) &\simeq \mathrm{Hom}_X(E_*/\hat{E}, \hat{G}_*)\\ &= \oplus_{i=1}^n \{f \in \mathrm{End}_k(V_i^1) \mid f(V_i^j) \subseteq V_i^{j+1} (j = 1, \ldots, l_i)\}\end{aligned}$$

Thus, we conclude that the dimension of the tangent space at E_* is

$$\dim_k \mathrm{Ext}^1(E_*, E_*) = 1 + r^2(g-1) + \sum_{i=1}^n \dim F_i. \tag{5.1}$$

For the moduli scheme of parabolic Higgs bundles, we have the following.

Theorem 5.2. *The moduli scheme $\bar{M}_{Higgs}(d, \{\alpha^i_j, n^i_j\})$ is an irreducible normal quasi-projective variety of dimension*

$$2(1 + r^2(g-1) + \sum_i \dim F_i) + \sum_{i,j} (n_i^j - n_i^{j+1})^2 - 1.$$

Moreover, $\bar{M}_{Higgs}(d, \{\alpha^i_j, n^i_j\})$ is smooth at points corresponding to parabolically stable Higgs bundles.

Proof. For normality and smoothness, it is enough to show that for all parabolically semi-stable Higgs bundles (E_*, φ), $\mathrm{Ext}^2_X((E_*, \varphi), (E_*, \varphi)) = 0$. By the spectral sequence (4.2), we get an exact sequence

$$\begin{aligned}0 &\longrightarrow \mathrm{Hom}_X((E_*, \varphi), (E_*, \varphi)) \xrightarrow{f} \mathrm{Hom}_X(E_*, E_*) \longrightarrow \mathrm{Hom}_X(E_*, E_* \otimes_X \Omega^1_X(D))\\ &\xrightarrow{g} \mathrm{Ext}^1_X((E_*, \varphi), (E_*, \varphi)) \xrightarrow{h} \mathrm{Ext}^1_X(E_*, E_*) \xrightarrow{\delta} \mathrm{Ext}^1_X(E_*, E_* \otimes_X \Omega^1_X(D))\\ &\xrightarrow{g} \mathrm{Ext}^2_X((E_*, \varphi), (E_*, \varphi)) \longrightarrow 0.\end{aligned}$$

By the Serre duality theorem for our case, we have isomorphisms,

$$\begin{aligned}\mathrm{Ext}^i_X(E_*, E_*) &\simeq \mathrm{Ext}^{1-i}_X(E_*, \hat{E}_* \otimes_X \Omega^1_X(D))^\vee\\ \mathrm{Ext}^i_X(E_*, E_* \otimes_X \Omega^1_X(D)) &\simeq \mathrm{Ext}^{1-i}_X(E_*, \hat{E}_*)^\vee\end{aligned}$$

and $\delta^\vee$ is given by $f \mapsto f\varphi - \hat{\varphi} f$. Hence,

$$\mathrm{Ext}^2_X((E_*, \varphi), (E_*, \varphi)) \simeq \mathrm{Hom}_X((E_*, \varphi), (\hat{E}_*, \hat{\varphi}))^\vee.$$

Let $f : (E_*, \varphi) \to (\hat{E}_*, \hat{\varphi})$ be a parabolic homomorphism. $\mathcal{E} = (E_*, \varphi)$ has a Jordan-Hölder filtration $\mathcal{E} = \mathcal{E}^0 \supset \mathcal{E}^1 \supset \cdots \supset \mathcal{E}^m \supset \mathcal{E}^{m+1} = 0$. Set $\mathcal{G}^j = \mathcal{E}^j/\mathcal{E}^{j+1}$. If f is not zero, it induces a non-zero map $\mathcal{G}^i \to \hat{\mathcal{G}}^j \hookrightarrow \mathcal{G}^j$ for some i, j. Since $\mathcal{G}^i$ and $\mathcal{G}^j$ are stable and have same slope, it must be an isomorphism. But $\hat{\mathcal{G}}^j \neq \mathcal{G}^j$. This contradiction shows that $\mathrm{Ext}^2_X((E_*, \varphi), (E_*, \varphi)) = 0$.

If we admit connectedness, the dimension of $\mathrm{Ext}^1_X((E_*,0),(E_*,0))$ with stable E_* is the dimension of $\bar{M}_{\mathrm{Higgs}}$. In this case,

$$\dim_k \mathrm{Ext}^1_X((E_*,0),(E_*,0)) = \dim_k \mathrm{Ext}^1_X(E_*,E_*) + \dim_k \mathrm{Ext}^1_X(E_*,\hat{E}_*).$$

Let us set $G_* = E_*/\hat{E}_*$. Then we have

$$0 \longrightarrow \mathrm{Hom}_X(E_*,E_*) \longrightarrow \mathrm{Hom}_X(E_*,G_*) \longrightarrow \mathrm{Ext}^1_X(E_*,\hat{E}_*) \xrightarrow{a} \mathrm{Ext}^1_X(E_*,E_*) \longrightarrow \mathrm{Ext}^1_X(E_*,G_*) \longrightarrow 0.$$

The dual map of a is

$$\mathrm{Hom}_X(E_*,\hat{E}_* \otimes_X \Omega^1_X(D)) \longrightarrow \mathrm{Hom}_X(E_*,E_* \otimes_X \Omega^1_X(D)).$$

Hence, $\mathrm{Ext}^1_X(E_*,G_*) = 0$. Clearly, $\mathrm{Hom}_X(E_*,G_*) = \mathrm{Hom}_X(G_*,G_*)$ and its dimension is $\sum_{i,j}(n_i^j - n_i^{j+1})^2$. Thus, we obtain the desired dimension.

Connectedness is proved by the method of Simpson (cf. Corollary 11.10 of [20]). We have a $\mathbb{G}_m$-action on $\bar{M}_{\mathrm{Higgs}}$ by $(E_*,\varphi) \mapsto (E_*,t\varphi)$. By Corollary 5.12 of [25] for each parabolically semi-stable Higgs sheaf (E_*,φ), $\lim_{t\to 0}(E_*,t\varphi)$ always exists in $\bar{M}_{\mathrm{Higgs}}$. Let (E_*,φ) be a parabolically stable Higgs sheaf fixed by the $\mathbb{G}_m$-action. Suppose that $\varphi \neq 0$. By the proof of Corollary 11.10 of [20], it is enough to prove that there is a parabolic Higgs bundle (F_*,ψ) not isomorphic to (E_*,φ), such that $\lim_{t\to\infty}(F_*,t\psi) = (E_*,\varphi)$. By the proof of Theorem 8 of [22], there is a decomposition such that $E_* = \oplus_{p=0}^l E_*^p$ and $\varphi(E_*^p) \subseteq E_*^{p-1} \otimes_X \Omega^1_X(\log D)$. Since (E_*,φ) is parabolically stable, $\text{par-}\mu(E_*^0) < \text{par-}\mu(E_*) < \text{par-}\mu(E_*^l)$. Then, $\text{par-deg}(\mathcal{H}om_X(E_*^l,E_*^0)_*) < 0$ and so, $\deg(\mathcal{H}om_X(E_*^l,E_*^0)) < 0$. By the Riemann-Roch theorem, $\mathrm{Ext}^1_X(E_*^l,E_*^0) \simeq H^1(X,\mathcal{H}om_X(E_*^l,E_*^0)) \neq 0$. Now by the same argument as the proof of Lemma 11.9 of [20], we can find a desired (F_*,ψ). □

Remark 5.1. In [10] Konno constructed a moduli space of stable parabolic Higgs bundles on a Riemann surface as a hyperkähler quotient by a gauge group. His definition of parabolic Higgs bundles is different from ours. He defined a parabolic Higgs bundle as a pair (E_*,φ) of parabolic bundle E_* and a parabolic homomorphism $\varphi : E_* \to \hat{E}_* \otimes_X \Omega^1_X(D)$.

Let P be the moduli functor for $\bar{M}_{\mathrm{Higgs}}$. Let NP be a sub-functor of P such that $(E_*,\varphi) \in NP(T)$ if and only if $\varphi \in \mathrm{Hom}_{X_T}(E_*,\hat{E}_* \otimes_X \Omega^1_X(D))$. Clearly, NP is a closed sub-functor of P. We constructed $\bar{M}_{\mathrm{Higgs}}$ as a geometric quotient by an algebraic group $G = \mathrm{PGL}(V)$ of some G-scheme R. On X_R, we have a universal family of parabolically semi-stable Higgs sheaves $(\tilde{E}_*,\tilde{\varphi})$ and surjections $V \otimes_k \mathcal{O}_{X_R} \twoheadrightarrow \tilde{E}$. Then there exists a unique closed sub-scheme NR of R such that a morphism $f : T \to R$ factors through NR if and only if $f^*(\tilde{\varphi}) \in \mathrm{Hom}_{X_T}(f^*(\tilde{E}_*), f^*(\tilde{E})\hat{}_* \otimes_X \Omega^1_X(D))$ (cf. [25] Corollary 2.3). Clearly, NR is G-invariant. Hence, we obtain the geometric quotient $\overline{NM}$ of NR by G. $\overline{NM}$ is a closed sub-scheme of $\bar{M}$. By the proof of Theorem 2.3, $\overline{NM}$ is the moduli scheme for the functor NP (modulo Seshadri-equivalence relations) and it contains the coarse moduli scheme NM of stable objects as an open sub-scheme. Fix a line bundle L. Let NM_L be the fibre of the canonical map $NM \to \mathrm{Pic}_X$ over L. Then NM_L is the moduli scheme which corresponds to the moduli space constructed by Konno. He proved that his moduli space is a hyperkähler manifold of dimension $2((g-1)(r^2-1)+\sum_{i=1}^{n}\dim F_i)$.

In the case of usual Higgs bundles, their moduli space contains the cotangent bundle of the moduli space of stable bundles as an open subset (see Hitchin [7]). In our case, M_{Higgs} also contains a vector bundle on M and NM contains the cotangent bundle of M where M is the locus of stable parabolic bundles of $\bar{M}$. Let $M^{\circ}_{\mathrm{Higgs}}$ be the open sub-scheme of $\bar{M}_{\mathrm{Higgs}}$ consisting of points which correspond to (E_*,φ) with stable E_*. We have a canonical map $\pi : M^{\circ}_{\mathrm{Higgs}} \to M$. There is an étale surjective map $\rho : M' \to M$ and a parabolic sheaf E_* on $X_{M'}$ flat over M' which determines the map ρ. Let

$$\begin{aligned}\mathcal{F} &=(f_{M'})_*(\mathcal{H}om_{X_{M'}}(E_*, E_*\otimes_X \Omega^1_X(D)))\\ \mathcal{G} &=(f_{M'})_*(\mathcal{H}om_{X_{M'}}(E_*, \hat{E}_*\otimes_X \Omega^1_X(D))).\end{aligned}$$

Since dimensions of $\mathrm{Hom}_{X_m}(E_*|_{X_m}, E_*|_{X_m}\otimes_X \Omega^1_X(D))$ (or, $\mathrm{Hom}_{X_m}(E_*|_{X_m}, \hat{E}_*|_{X_m}\otimes_X \Omega^1_X(D))$) are constant, $\mathcal{F}$ (or, $\mathcal{G}$, resp.) is locally free. Let V (or, W, resp.) be a vector bundle over M' whose sheaf of sections is $\mathcal{F}$ (or, $\mathcal{G}$, resp.). The canonical flat family of parabolic Higgs bundles on X_V induces a map $V \to M^{\circ}_{\mathrm{Higgs}}$. Clearly this map induces a bijective map $V \to M^{\circ}_{\mathrm{Higgs}} \times_M M'$. Since both of them are smooth, it is an isomorphism. Moreover, we have a map $W \to NM \cap M^{\circ}_{\mathrm{Higgs}}$. By Serre duality theorem, W is the pull-back of the cotangent bundle T^*M of M. Since T^*M is smooth, by the Zariski's main theorem, we obtain a map $T^*M \to NM \cap M^{\circ}_{\mathrm{Higgs}}$. It is easy to see that the Zariski tangent space of NM at (E_*,φ) is $\mathrm{Ker}(\mathrm{Ext}^1_X((E_*,\varphi),(E_*,\varphi)) \xrightarrow{d} \mathrm{Ext}^1_X((E_*,\varphi),(G_*,\bar{\varphi})))$ where $G_* = E_*/\hat{E}_*$. Since $\mathrm{Ext}^1_X(E_*,G_*)=0$ and the natural map $\mathrm{Hom}_X(E_*,G_*) \to \mathrm{Hom}_X(E_*,G_*\otimes_X \Omega^1_X(D))$ is a zero map ($\bar{\varphi}=0$), $\mathrm{Hom}_X(E_*,G_*\otimes_X\Omega^1_X(D)) \simeq \mathrm{Ext}^1_X((E_*,\varphi),(G_*,\bar{\varphi}))$. $\mathrm{Coker}\, d \simeq \mathrm{Ext}^2_X((E_*,\varphi),(\hat{E}_*,\hat{\varphi})) \simeq \mathrm{Hom}_X((E_*,\varphi),(E_*,\varphi))^\vee$. Hence,

$$\dim_k \mathrm{Ker}\, d = 2((g-1)(r^2-1)+\sum_{i=1}^{n}\dim F_i) = \dim T^*M.$$

Since the map $T^*M \to NM \cap M^{\circ}_{\mathrm{Higgs}}$ is bijective, we conclude that it is an isomorphism.

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