Holomorphic extension of representations

We report on the paper [1] whose introduction we attach below. We make a special emphasis for the group $G = \mathrm{Sl}(2,\mathbb{R})$ and refer to the overview article [2].

Let us consider a unitary irreducible representation $(\pi, \mathcal{H})$ of a simple, non-compact and connected Lie group G. Let us denote by K a maximal compact subgroup of G. According to Harish-Chandra, the Lie algebra submodule $\mathcal{H}_K$ of K-finite vectors of π consists of analytic vectors of the representation. We determine, and in full generality, their natural domain of definition as holomorphic functions:

Theorem 0.1. *Let $(\pi, \mathcal{H})$ be a unitary irreducible representation of G. Let $v \in \mathcal{H}$ be a non-zero K-finite vector and*

$$f_v : G \to \mathcal{H}, \quad g \mapsto \pi(g)v$$

the corresponding orbit map. Then there exists a maximal $G \times K_\mathbb{C}$-invariant domain $D_\pi \subseteq G_\mathbb{C}$, independent of v, to which f_v extends holomorphically. Explicitely:

(i) *$D_\pi = G_\mathbb{C}$ if π is the trivial representation.*
(ii) *$D_\pi = \Xi^+ K_\mathbb{C}$ if G is Hermitian and π is a non-trivial highest weight representation.*
(iii) *$D_\pi = \Xi^- K_\mathbb{C}$ if G is Hermitian and π is a non-trivial lowest weight representation.*
(iv) *$D_\pi = \Xi K_\mathbb{C}$ in all other cases.*

Let us explain the objects Ξ, Ξ^+ and Ξ^- in the statement. We form $X = G/K$, the associated Riemann symmetric space, and view X as a totally real submanifold of its affine complexification $X_\mathbb{C} = G_\mathbb{C}/K_\mathbb{C}$. The natural G-invariant complexification of X, the crown domain, is denoted by Ξ ($\subseteq X_\mathbb{C}$). For a domain $D \subseteq X_\mathbb{C}$ we denote by $DK_\mathbb{C}$ its preimage in $G_\mathbb{C}$.

In [3] we observed that a $G \times K_\mathbb{C}$-invariant domain of definition of f_v, say $D_v \subseteq G_\mathbb{C}$, must be such that G acts properly on $D_v/K_\mathbb{C} \subseteq X_\mathbb{C}$. By our work with Robert J. Stanton we know that we can choose D_v such that $D_v \supseteq \Xi K_\mathbb{C}$ (see [4], [5]). Therefore it is useful to classify all G-domains $\Xi \subseteq D \subseteq X_\mathbb{C}$ with proper action. As it turns out, they allow a simple description. We extract from theorems below:

Theorem 0.2. *Let $\Xi \subseteq D \subseteq X_\mathbb{C}$ be a G-invariant domain on which G acts properly. Then:*

(i) *If G is not of Hermitian type, then $D = \Xi$.*

(ii) *If G is of Hermitian type, then either $D \subseteq \Xi^+$ or $D \subseteq \Xi^-$ with Ξ^+ and Ξ^- two explicite maximal domains for proper G-action.*

Finally, let us emphasize that proofs in this paper are modelled after $G = \mathrm{Sl}(2, \mathbb{R})$ which was dealt with earlier in [3].

References

[1] B. Krötz, *Domains of holomorphy for irreducible unitary representations of simple Lie groups*, arXiv:math/0608284, submitted

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[3] B. Krötz and E.M. Opdam, *Analysis on the crown domain*, math.RT/0606213, to appear in GAFA

[4] B. Krötz and R. Stanton, *Holomorphic extensions of representations: (I) automorphic functions*, Ann. Math. **159** (2004), 641–724.

[5] —, *Holomorphic extensions of representations: (II) geometry and harmonic analysis*, GAFA **15** (2005), 190–245.

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