HILBERT SCHEMES OF ABELIAN GROUP ORBITS

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To the memory of Professor Nobuo Sasakura

Abstract. We construct a crepant resolution of the quotient variety $\mathbf{A}^3/G$ in the canonical manner when G is an abelian subgroup of $\mathrm{SL}(3,k)$.

0. Introduction

Let k be any algebraically closed field and G a finite subgroup of $\mathrm{SL}(3,k)$ of order prime to the characteristic of k. The group G acts on the affine 3-space $\mathbf{A}^3$ and the quotient space $\mathbf{A}^3/G$ is a normal Gorenstein variety with trivial canonical sheaf.

No natural choice of a crepant smooth resolution of $\mathbf{A}^3/G$ seems to be known in the general theories such as the theory of minimal models in birational geometry and the theory of torus embeddings. In this article we give a crepant smooth resolution of $\mathbf{A}^3/G$ canonical in some sense when G is abelian. The G-Hilbert scheme (or The G-orbit Hilbert scheme) $\mathrm{Hilb}^G := \mathrm{Hilb}^G(\mathbf{A}^3)$ introduced in [IN96] is in fact such a resolution. The G-Hilbert scheme is roughly speaking a largest connected subset of the set consisting of all G-invariant zero-dimensional subschemes of $\mathbf{A}^3$ of length $n := |G|$, the order of G. The zero-dimensional subschemes of $\mathbf{A}^3$ of length n are something which substitute for n-points in $\mathbf{A}^3$, therefore those G-invariant zero-dimensional subschemes of length n are substitutes for G-orbits in $\mathbf{A}^3$ of length n. Hence Hilb^G will be a substitute of $\mathbf{A}^3/G$, the space of G-orbits. We will prove that for G abelian Hilb^G is a smooth torus embedding associated to a certain fan in $\mathbf{R}^3$ with apices junior elements of G (Theorem 4.2). As a corollary to it, we will see

Theorem 0.1. *For any abelian subgroup G of $\mathrm{SL}(3,k)$ of order prime to the characteristic of k, $\mathrm{Hilb}^G(\mathbf{A}^3)$ is a crepant smooth resolution of $\mathbf{A}^3/G$.*

Theorem 0.1 might sound a little unexpected because $\mathrm{Hilb}^n(\mathbf{A}^3)$ is known to be very singular. We conjecture that the same is true for any finite

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subgroup G of $\mathrm{SL}(3,k)$ if the order of G is prime to the characteristic of k. The first half of the present article is devoted to describing Hilb^G as a toric variety in arbitrary dimension. In the second half of it we will prove that if G is an abelian subgroup of $\mathrm{SL}(3,k)$, the fan associated with Hilb^G is nonsingular, that is, Hilb^G is smooth. Some examples will be given in Section 5 and Section 6. See also [Reid97] and Ito-Nakajima [INkjm98].

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1. Lattices and G-graphs

1.1. **Abelian subgroups.** Let G be a finite abelian subgroup of $\mathrm{GL}(r,k)$ of order prime to the characteristic of k and n the order of G. Without loss of generality we may assume that any element of G is a diagonal matrix.

Let μ_n be the group of n-th roots of unity and $\mathbf{G}_m$ the group of units in k. We choose and fix a primitive n-th root ζ of unity. Given a set F we denote its cardinality by $|F|$. We denote by $\mathbf{Z}_0$ (resp. $\mathbf{R}_0$) the set of all nonnegative integers (resp. nonnegative real numbers).

Definition 1.2. Let $N_0 := \oplus_{i=1}^r \mathbf{Z}e_i$ be a free $\mathbf{Z}$ module with e_i a $\mathbf{Z}$-basis, $M_0 := \mathrm{Hom}_{\mathbf{Z}}(N_0, \mathbf{Z}) = \oplus_{i=1}^r \mathbf{Z}f_i$ and $\Delta := \sum_{i=1}^r \mathbf{R}_0 e_i$ where $f_i(e_j) = \delta_{ij}$. In what follows we assume that any element of the group G is a diagonal $r \times r$-matrix, therefore we denote it by an r-vector consisting of r diagonal coefficients. Given a nontrivial element g of G, we write $g = (\zeta^{a_1}, \cdots, \zeta^{a_r})$ for some $0 < a_i < n$ and identify it with an r-vector $(a_1/n)e_1 + \cdots + (a_r/n)e_r \in N_0 \otimes_{\mathbf{Z}} \mathbf{Q}$. We define $N := N_0 + \sum_{g \in G} \mathbf{Z}g$ and $M = \mathrm{Hom}_{\mathbf{Z}}(N, \mathbf{Z})$. We note $N/N_0 \simeq G$ and $M_0/M \simeq G^\vee$. In fact, we see that $e^{2\alpha(g)\pi i} \in \mu_n$ is well-defined for $\alpha \in M_0/M$ and $g \in G$, which gives a nondegenerate pairing $N/N_0 \times M_0/M \to \mu_n$. This pairing enables us to identify $G^\vee$ with M_0/M. We call this isomorphism *weight* and denote it by $\mathrm{wt} : M_0/M \to G^\vee$. We denote $a \equiv b$ for $a, b \in M_0$ if $\mathrm{wt}(a) = \mathrm{wt}(b)$, as is the same, $a - b \in M$.

1.3. **The semigroup $S(I)$.** Let M_0^0 (resp. M_0^+) be the semigroup generated by 0 and f_i (resp. f_i) $(1 \le i \le r)$.

Since by the semigroup homomorphism $f_i \mapsto x_i$ there is an obvious bijection between the set of subsemigroups of M_0^0 and the set of monomial subsemigroups in the polynomial ring $k[\mathbf{A}^r] = k[x_1, \cdots, x_r]$, we will often identify them if no confusion is possible.

Suppose that an ideal I of $k[\mathbf{A}^r]$ is generated by monomials. The monomial generators of I, in an obvious manner, give rise to a subsemigroup $S(I)$ of $k[\mathbf{A}^r]$. Let $S(I)^\perp$ be the set of monomials in x_i which do not belong to $S(I)$. Then we easily see that $1 \notin S(I)$, $fS(I) \subset S(I)$ for any monomial f.

From a certain reason we will explain in Subsection 2.1, we are particularly interested in those I satisfying the condition (ii) in Definition 1.4. Content-Length: 68561 X-Lines: 2130 Status: RO

Definition 1.4. For a subsemigroup S of M_0^+ we define a subset $S^\perp$ of M_0^0 by $S^\perp := M_0^0 \setminus S$. The subset $S^\perp$ of M_0^0 is called a G-*graph* if the following conditions are true:

(i) $S + M_0^+ \subset S$, or as is equivalent, if $a+b \in S^\perp$ for some $a, b \in M_0^0$, then $a \in S^\perp$ and $b \in S^\perp$,
(ii) $\mathrm{wt} : S^\perp \to G^\vee$ is a bijection

where $\mathrm{wt}(\alpha)(g) = \alpha(g)$ $(\alpha \in S(I)^\perp, g \in G)$.

We denote $S^\perp$ by $\Gamma(S^\perp)$ or simply by Γ. We note $0 \in \Gamma(S^\perp)$ because $S \subset M_0^+$. If $G \subset SL(r,k)$, then $f_1 + f_2 + \cdots + f_r \not\in \Gamma(S^\perp)$. Therefore $\Gamma(S^\perp)$ consists of at most r $(r-1)$ -dimensional graphs $\Gamma_k := \Gamma(S^\perp) \cap (\sum_{i\neq k} \mathbf{R} f_i)$.

Definition 1.5. Let Γ be a G-graph. We define the weight map $\mathrm{wt}_\Gamma : M_0^0 \to \Gamma$ as follows. For any $u \in M_0^0$ there exists a unique element $v \in \Gamma$ such that $\mathrm{wt}(v) \equiv \mathrm{wt}(u)$. We denote v by $\mathrm{wt}_\Gamma(u)$. Then we define a cone $\sigma(\Gamma)$, a subsemigroup $S(\Gamma)$ of M and a $\mathbf{Z}$-submodule $M(\Gamma)$ of M by

$$\sigma(\Gamma) := \{\alpha \in N_0 \otimes_{\mathbf{Z}} \mathbf{R}; \alpha(v - \mathrm{wt}_\Gamma(v))) \geq 0, \forall v \in M_0^0\},$$
$$S(\Gamma) := \sum_{v \in M_0^0} \mathbf{Z}_0 (v - \mathrm{wt}_\Gamma(v)) \ (\subset M),$$
$$M(\Gamma) := \sum_{v \in M_0^0} \mathbf{Z} (v - \mathrm{wt}_\Gamma(v)) \ (\subset M).$$

Lemma 1.6. *For any finite abelian subgroup G of* $\mathrm{GL}(r,k)$*, there exists a G-graph.*

Proof. If G is cyclic, then let $g = (\cdots, 1/n)$ be a generator of G. Let Γ be a G-graph with $\Gamma = \{x f_r ; x \in [0, n-1]\}$. Then it is clear that Γ is a G-graph. If G is not cyclic, then we prove Lemma by the induction on r. We may assume there exists a $g \in G$ such that $g = (\cdots, 1/\ell)$ is of the maximal order ℓ in G. Then there exists a subgroup H of $G \cap GL(r-1, k)$ such that G/H $(\simeq \mathbf{Z}/\ell\mathbf{Z})$ is generated by the image of g. By the induction hypothesis there exists an H-graph $\Gamma' \subset \oplus_{i=1}^{r-1} \mathbf{Z} f_i$. Let $\Gamma := \Gamma' \times ([0, \ell-1] f_r)$. Then it is clear that Γ is a G-graph. □

Lemma 1.7. *Let Γ be a G-graph. Then $S(\Gamma)$ is a finitely generated semigroup and $M(\Gamma) = M$.*

Proof. Though it seems clear, we give a proof. First we note that $M(\Gamma) \subset M$ and $M \cap M_0^0 \subset M(\Gamma)$ because $\mathrm{wt}_\Gamma(v) = 0$ for $v \in M \cap M_0^0$. Since $M \cap M_0^0$ generates M as a group, $M \subset M(\Gamma)$. It follows $M(\Gamma) = M$. There exists a finite subset A of M_0^0 such that M_0^0 is generated by A and $M \cap M_0^0$ as a semigroup. Any $w \in M_0^0$ is written as $w = v + a$ $(\exists a \in A, v \in M \cap M_0^0)$.

Therefore

$$w - \mathrm{wt}_\Gamma(w) = v + a - \mathrm{wt}_\Gamma(a)$$

because $\mathrm{wt}_\Gamma v = 0$. Since $M \cap M_0^0$ is finitely generated, $S(\Gamma)$ is finitely generated. □

Lemma 1.8. *Let Γ be a G-graph and A a finite subset of $M_0^0 \setminus \Gamma$ such that $M_0^0 \setminus \Gamma = A + M_0^0$. Then $S(\Gamma)$ is generated by $v - \mathrm{wt}_\Gamma(v)$ $(v \in A)$ as a semigroup.*

Proof. Let $\lambda(v) := v - \mathrm{wt}_\Gamma(v)$ and we denote by $S^*(\lambda(A))$ the semigroup generated by $\lambda(A)$. First we see $nf_i \in \overset{\vee}{\sigma}(\Gamma) \cap M$ where $n = |G|$ because $\mathrm{wt}_\Gamma(nf_i) = 0$. Hence $\sigma(\Gamma) \subset \Delta$. Next we notice that $\sigma(\Gamma)$ is a g-dimensional cone, that is, there is a solution $\alpha \in \mathrm{Int}(\sigma(\Gamma))$ to the strict inequalities

$$\alpha(\lambda(v)) > 0 \quad (\forall v \in M_0^0 \setminus \Gamma). \tag{1}$$

We choose and fix such an $\alpha \in \mathrm{Int}(\sigma(\Gamma))$. Then since $\sigma(\Gamma) \subset \Delta$, we see that $\alpha(v) > 0$ for any nonzero $v \in M_0^0$ and the subset $\{v \in M_0^0; \alpha(v) < c\}$ is finite for any constant c.

Now we prove our Lemma by the induction on the value $\alpha(z)$ $(z \in M_0^0 \setminus \Gamma)$. If $z \in A$, then $\lambda(z) \in S^*(\lambda(A))$. Let $z \in M_0^0$. By the assumption there exist $v \in A$ and $b \in M_0^0$ such that $z = v + b$. We see

$$\lambda(z) = \lambda(v) + \lambda(\mathrm{wt}_\Gamma(v) + b). \tag{2}$$

We easily see $\alpha(z) = \alpha(v) + \alpha(b) > \alpha(\mathrm{wt}_\Gamma(v) + b)$. By the induction hypothesis $\lambda(\mathrm{wt}_\Gamma(v) + b) \in S^*(\lambda(A))$. This completes the proof. □

Definition 1.9. For any G-graph Γ we define

$$\begin{aligned}
U_{\sigma(\Gamma)} &:= \mathrm{Spec}\, k[\overset{\vee}{\sigma}(\Gamma) \cap M],\\
V(\Gamma) &:= \mathrm{Spec}\, k[S(\Gamma)],\\
I(\Gamma) &:= (w^v; v \notin \Gamma) \subset k[\mathbf{A}^r],\\
I^{\mathrm{vers}}(\Gamma) &:= (w^v - s^{v - \mathrm{wt}_\Gamma(v)} w^{\mathrm{wt}_\Gamma(v)}; v \in M_0^0)\\
&\qquad \subset k[S(\Gamma)] \otimes_k k[\mathbf{A}^r] \subset k[\overset{\vee}{\sigma}(\Gamma) \cap M] \otimes_k k[\mathbf{A}^r]
\end{aligned}$$

where s^v $(v \in M)$ is the torus coordinate of the toric varieties $U_{\sigma(\Gamma)}$ and $V(\Gamma)$. Let $Z(\Gamma)$ (resp. $Z^{\mathrm{vers}}(\Gamma)$) be a G-invariant subscheme of $\mathbf{A}^r$ (resp. a G-invariant subscheme of $\mathbf{A}^r_{V(\Gamma)}$) defined by the ideal $I(\Gamma)$ (resp. $I^{\mathrm{vers}}(\Gamma)$).

2. The G-orbit Hilbert scheme

2.1. **The G-orbit Hilbert scheme.** Let G be a finite subgroup of $\mathrm{GL}(r, k)$ and $n = |G|$. Let $\mathrm{Hilb}^n(\mathbf{A}^3)$ be the Hilbert scheme of n points in $\mathbf{A}^r$ and $S^n(\mathbf{A}^r)$ the n-th symmetric product of $\mathbf{A}^r$. We have a natural morphism π_n from $\mathrm{Hilb}^n(\mathbf{A}^r)$ onto $S^n(\mathbf{A}^r)$. Since π_n is G-equivariant, we have a natural morphism between their G-fixed point sets. We see easily that the quotient variety $\mathbf{A}^r/G$ is one of the irreducible components of the G-fixed point set $S^n(\mathbf{A}^r)^{G-\mathrm{inv}}$ with reduced structure. Then *the G-orbit Hilbert scheme* $\mathrm{Hilb}^G := \mathrm{Hilb}^G(\mathbf{A}^r)$ is by definition the unique irreducible component, endowed with reduced structure, of the G-fixed point set of $\mathrm{Hilb}^n(\mathbf{A}^r)$ which dominates $\mathbf{A}^r/G$ by the map π_n. Let π be the natural morphism from $\mathrm{Hilb}^G(\mathbf{A}^r)$ onto $\mathbf{A}^r/G$. We call a zero-dimensional subscheme (resp. a G-invariant subscheme) Z of $\mathbf{A}^r$ of length n *a cluster* (resp. *a G-cluster*). A G-cluster Z is smoothable if and only if $Z \in \mathrm{Hilb}^G(\mathbf{A}^r)$. Let I be the ideal defining the (G-invariant) smoothable cluster. Then we write $I \in \mathrm{Hilb}^n(\mathbf{A}^r)$ resp. $I \in \mathrm{Hilb}^G(\mathbf{A}^r)$ since there seems to be no fear of confusion.

For instance, if $r = 3$ and G is generated by $g = (1/6, 1/2, 3/6)$, then (x^6, y, z), (x^2, y^3, z), and (x^2, xy, y^2, z^2) are examples of ideals generated by monomials in $\mathrm{Hilb}^G(\mathbf{A}^3)$. See Table 2.

By [IN98, Lemma 9.4] if $I \in \mathrm{Hilb}^G(\mathbf{A}^r)$, then the quotient module $k[\mathbf{A}^r]/I$ is isomorphic to the group algebra $k[G]$ (the regular representation of G) as a G-module. In other words, $\dim(k[\mathbf{A}^r]/I)^\rho = 1$ $(\forall \rho \in G^\vee)$ where $(k[\mathbf{A}^r]/I)^\rho$ is the ρ-eigensubspace of $k[\mathbf{A}^r]/I$ for any character $\rho \in G^\vee$. This is just the condition (ii) in Definition 1.4.

Definition 2.2. Let $\mathrm{Hilb}^G_{\mathrm{norm}}(\mathbf{A}^r)$ be the normalization of $\mathrm{Hilb}^G(\mathbf{A}^r)$. Since $\mathrm{Hilb}^G_{\mathrm{norm}}(\mathbf{A}^r)$ is a normal torus embedding, it is covered with finitely many toric charts. To each chart there corresponds a cone of $N \otimes \mathbf{Q}$. Since $\mathrm{Hilb}^{|G|}(\mathbf{A}^r)$ is projective over $S^{|G|}(\mathbf{A}^r)$, so is $\mathrm{Hilb}^G(\mathbf{A}^r)$ over $\mathbf{A}^r/G$. Therefore there exists a fan $\{\sigma\}$ with support Δ and a covering of $\mathrm{Hilb}^G(\mathbf{A}^r)$ consisting of toric charts U_σ with $\dim \sigma = r$, each of which has a unique zero-dimensional toric stratum corresponding to a G-cluster $Z_\sigma \in \mathrm{Hilb}^G(\mathbf{A}^r)$ defined by a monomial G-ideal I_σ of $k[\mathbf{A}^r]$. There exists a G-graph Γ such that $Z_\sigma = Z(\Gamma)$, $I_\sigma = I(\Gamma)$, equivalently $\sigma = \sigma(\Gamma)$.

Lemma 2.3. *Assume that Γ is a G-graph. Let $Z^{\mathrm{vers}} := Z^{\mathrm{vers}}(\Gamma)$ and $V := V(\Gamma)$ in Definition 1.9. Let Z^{vers}_s be a closed fibre of Z^{vers} over $s \in V(k)$. Then*

(i) *Z^{vers} is V-flat and $Z^{\mathrm{vers}}_s \in \mathrm{Hilb}^G$ for any $s \in V(k)$. In particular, $Z(\Gamma) \in \mathrm{Hilb}^G$.*
(ii) *Γ is a k-basis of $k[\mathbf{A}^r]/I^{\mathrm{vers}}(\Gamma) \otimes k(s)$ for any $s \in V(k)$.*
(iii) *At any point $s \in V(k)$ Z^{vers} is versal for flat G-equivariant deformations of Z^{vers}_s with generic support, that is, whose support is not contained in $\mathbf{A}^r \setminus \mathbf{G}^r_m$.*

Proof. The scheme Z as well as V admits a torus action $w^v \mapsto \lambda^v w^v$, $s^{v-\mathrm{wt}_\Gamma(v)} \mapsto \lambda^{v-\mathrm{wt}_\Gamma(v)} s^{v-\mathrm{wt}_\Gamma(v)}$ $(\forall v \in M^0_0, \lambda \in \mathbf{G}^r_m)$. For $s \in \mathbf{G}^r_m(k)$, the number of solutions of the system of the equations

$$w^v = s^{v-\mathrm{wt}_\Gamma(v)} w^{\mathrm{wt}_\Gamma(v)} \quad (\forall v \in M^0_0) \tag{3}$$

is equal to $|\Gamma| = |G|$ by Lemma 1.7. Hence Z is flat over $\mathbf{G}^r_m$ $(\subset V)$. Let B be the subscheme of V consisting of all $s \in V$ such that $\dim_k k[\mathbf{A}^r]/I^{\mathrm{vers}}(\Gamma) \otimes k(s) \geq |G| + 1$. The subset B is invariant under $\lambda \in \mathbf{G}^r_m$, whence if $B \neq \emptyset$, then B contains $s=0$, the unique 0-stratum of V because B is closed. However $\dim_k k[\mathbf{A}^r]/I(\Gamma)=|G|$ because Γ is a G-graph. It follows that $B=\emptyset$, Z is V-flat. In particular, Z_s is smooth for $s\in\mathbf{G}_m^r$ and $Z_s\in\mathrm{Hilb}^G$ for any $s\in V(k)$. This proves (i).

Let $B'=\{s\in V;\Gamma$ is not a $k(s)$-basis of $k[\mathbf{A}^r]/I^{\mathrm{vers}}(\Gamma)\otimes k(s)\}$. Then B' is a closed $\mathbf{G}_m^r$-invariant subscheme of V. It is easy to see that $0\not\in B'$. This shows $B'=\emptyset$, which proves (ii).

Let R be a complete local domain with $R/m\simeq k$ and K the quotient field, m the maximal ideal of R. Choose and fix $s_0\in V(k)$. Let X be an R-flat (embedded) G-equivariant deformation of Z_{s_0} with generic support, in other words, the support of X_K is not contained in $\mathbf{A}_K^r\setminus\mathbf{G}_{m,K}^r$. Let J be a G-invariant ideal of $\Gamma(O_{\mathbf{A}_R^r})$ defining X. Since X is flat and $X_0=Z_{s_0}$, Γ is a free R-basis of $\Gamma(O_X)\simeq R\otimes k[\mathbf{A}^r]/J$. Therefore for any w^v there exist a unique $\gamma\in\Gamma$ and $c\in R$ such that $w^v=c\cdot\gamma \mod J$. In other words, $w^v-cw^{\mathrm{wt}_\Gamma(v)}\in J$. Moreover J is generated by the elements of this form because $\Gamma(O_X)\simeq R\otimes k[\mathbf{A}^r]/J$. Hence we see that

$$J=\left(w^v-q_vw^{\mathrm{wt}_\Gamma(v)}\ (\forall v\in M_0^0\setminus\Gamma)\right) \tag{4}$$

for some $q_v\in R$. Suppose that we are given $v_j,u_i\in M_0^0$ such that

$$\sum_j(v_j-\mathrm{wt}_\Gamma(v_j))=\sum_i(u_i-\mathrm{wt}_\Gamma(u_i)). \tag{5}$$

Then we prove

$$\prod_j q_{v_j}=\prod_i q_{u_i}. \tag{6}$$

In fact, let $a:=\sum_j\mathrm{wt}_\Gamma(v_j)$ and $b:=\sum_i\mathrm{wt}_\Gamma(u_i)$. Then

$$w^{a+b}\prod_j q_{v_j}=w^b\prod_j w^{v_j}=w^a\prod_i w^{u_j}=w^{a+b}\prod_i q_{u_j}\mod J.$$

Since X is a generic-support-deformation, the relation (6) follows. This proves (iii). □

Corollary 2.4. *$Z(\Gamma)$ is smoothable for any G-graph Γ. The map $I\mapsto\Gamma(S(I)^\perp)$ is a bijection between the set of $I\in\mathrm{Hilb}^G(\mathbf{A}^r)$ generated by monomials and the set of G-graphs.*

See also [INkjm98] for the case where $G\subset\mathrm{SL}(3,k)$.

Lemma 2.5. *Let Γ be a G-graph and $\sigma(\Gamma)$ the same as before. Let τ be a one-codimensional face of $\sigma(\Gamma)$. Then there exists $v^*:=v_--v_+\in M$ indivisible in M by any positive integer ≥ 2 such that $v_-\in M_0^0$, $v_+=\mathrm{wt}_\Gamma(v_-)\in\Gamma$ and $v_\pm$ has no common factors in M_0^0* [1]*, $\tau=\sigma(\Gamma)\cap(v^*)^\perp$ and $\sigma(\Gamma)\subset\{v_-\geq v_+\}$, v^* is a generator of the lattice $\tau^\perp\cap M$.*

[1]This means that there is no nonzero $u\in M_0^0$ such that $v_\pm-u\in M_0^0$

Proof. Since τ is one codimensional in $\sigma(\Gamma)$, there exists $v^* := v_- - v_+ \in M$ such that $v_+ = \mathrm{wt}_\Gamma(v_-)$ and $\tau = \sigma(\Gamma) \cap \{v^*\}^\perp$, $\sigma(\Gamma) \subset \{v_- \geq v_+\}$ for some $v_\pm \in M_0$. We note that if $v^* := v_- - v_+ \in M$ such that $v_- \in M_0^0$, $v_+ = \mathrm{wt}_\Gamma(v_-) \in \Gamma$ and if $v_\pm$ has a common factor u, then $v_+ - u \in \Gamma$ because Γ is a G-graph. Similarly $v_- - u \in M_0^0$ and $\mathrm{wt}_\Gamma(v_- - u) = v_+ - u$. Hence without loss of generality we may assume that $v_\pm$ has no factors in common. Moreover we may assume that $v_- - v_+$ is primitive in M, that is, $v_- - v_+$ is divisible in M by no positive integer ≥ 2. In fact, suppose that there exist $e_\pm \in M_0^0$ with no common factors and a positive integer ℓ such that $v_- - v_+ = \ell(e_- - e_+)$ and $e_- - e_+ \in M$. It follows that $v_- + \ell e_+ = v_+ + \ell e_-$ and $v_\pm = \ell e_\pm$. Since Γ is a G-graph, $e_+ \in \Gamma$ whence $e_- \notin \Gamma$ and $\mathrm{wt}_\Gamma(e_-) = e_+$ because $e_- - e_+ \in M$. Moreover $\tau = \sigma \cap \{e_- = e_+\}$, $\sigma \subset \{e_- \geq e_+\}$. Thus we may assume $v_- - v_+$ is divisible in M by no positive integer ≥ 2. This also shows that v^* is a generator of the lattice $\tau^\perp \cap M$. This completes the proof of Lemma. □

Definition 2.6. Let Γ be a G-graph, v^*, v_+ and v_- as in Lemma 2.5. Suppose $v_\pm \neq 0$. A G-igsaw transform Γ' of Γ is then defined to be

$$G_{\mathrm{ig}}(v_+, v_-)(v) = v + c^+_{\max}(v)(v_- - v_+) \in \Gamma', \tag{7}$$

$$\Gamma' := G_{\mathrm{ig}}(v_+, v_-)(\Gamma) \tag{8}$$

where $c^+_{\max}(v) := \max\{c \in \mathbf{Z}; v - cv_+ \in M_0^0\}$.

We will see in Lemma 2.8 that Γ' is also a G-graph adjacent to Γ in the sense that $\sigma(\Gamma)$ and $\sigma(\Gamma')$ are adjacent, *i.e.* have a one-codimensional face in common.

2.7. **One parameter family of clusters.** Let $\mathfrak{m}$ and $\mathfrak{n}$ be respectively the maximal ideal of $k[\mathbf{A}^r]$ at the origin, and the ideal of $k[\mathbf{A}^r]$ generated by all G-invariant monomials vanishing at the origin.

Let $I \in \mathrm{Hilb}^G(\mathbf{A}^r)$ and $\{I(q); q \in \mathbf{A}^1\}$ a $k[q]$-flat G-equivariant deformation of I. Suppose $I(q) \subset \mathfrak{m}$. Then $\mathfrak{n} \subset I(q)$ because $k[\mathbf{A}^r]/I(q) \simeq k[G]$, hence the G-invariant part of $k[\mathbf{A}^r]/I(q)$ is spanned by constant functions. Then $\dim_k k[\mathbf{A}^q]/I(q)$ is constant so that it determines a natural morphism $\phi : \mathbf{A}^1 \to \mathrm{Gr}$ into the Grassmann variety $\mathrm{Gr} := \mathrm{Gr}(V, n)$ of n-codimensional subspaces of V where $V := k[\mathbf{A}^r]/\mathfrak{n}$, $n = |G|$. Let $B := \phi(\mathbf{A}^1)$, $\bar{B}$ the closure of B and $\phi(\infty) := \bar{B} \setminus B$. Since $\phi(\infty) \in \mathrm{Gr}$, it determines a unique ideal $I_{\phi(\infty)}$ of $k[\mathbf{A}^r]$ such that $\phi(\infty) = I_{\phi(\infty)}/\mathfrak{n} \in \mathrm{Gr}$.

Lemma 2.8. *Let Γ be a G-graph, τ and $v^* := v_- - v_+$ the same as in Lemma 2.5. Suppose $v_\pm \neq 0$. Define a deformation $\{I(\Gamma, v^*)(q)\}_{q \in \mathbf{A}^1}$ of $I(\Gamma)$ in the v^*-direction by*

$$I(\Gamma, v^*)(q) = (w^{v_-} - qw^{v_+}, w^v \ (v \in M_0^0 \setminus \Gamma, v - \mathrm{wt}_\Gamma(v) \notin \tau^\perp)). \tag{9}$$

Then $I(\Gamma, v^*)(\infty) := (\lim_{q \to \infty} I(\Gamma, v^*)(q)/\mathfrak{n}) + \mathfrak{n} = I(G_{\mathrm{ig}}(v_+, v_-)(\Gamma))$. In particular, $G_{\mathrm{ig}}(v_+, v_-)(\Gamma)$ is a G-graph.

Proof. First we prove $I(\Gamma, v^*)(0) = I(\Gamma)$. Let $\sigma = \sigma(\Gamma)$ and $J(\Gamma) := I(\Gamma, v^*)(0)$. Let s^v $(v \in M)$ be the torus coordinate of the torus embedding $U_\sigma := \operatorname{Spec} k[\overset{\vee}{\sigma} \cap M]$. Let $q = s^{v_- - v_+}$. It is clear that $J(\Gamma) \subset I(\Gamma)$. Conversely let $v \in M_0^0 \setminus \Gamma$. If $v - \mathrm{wt}_\Gamma(v) \not\in \tau^\perp$, then $w^v \in J(\Gamma)$. Assume $v - \mathrm{wt}_\Gamma(v) \in \tau^\perp$. It follows that there exists $n \geq 1$ such that $v - \mathrm{wt}_\Gamma(v) = n(v_- - v_+)$. Hence $v - nv_- \in M_0^0$ and $\mathrm{wt}_\Gamma(v) - nv_+ \in M_0^0$. Hence $w^v = w^{nv_-} w^{v - nv_-} \in w^{nv_-} k[\mathbf{A}^r] \subset J(\Gamma)$. Therefore $I(\Gamma) = J(\Gamma)$.

Let $\Gamma' := G_{\mathrm{ig}}(v_+, v_-)(\Gamma)$. Then we see

$$M_0^0 \setminus \Gamma' = \{v \not\in \Gamma, v - \mathrm{wt}_\Gamma(v) \not\in \tau^\perp\} \cup \{v \in \Gamma; v - v_+ \in M_0^0\}. \tag{10}$$

We can easily prove that $I(\Gamma, v^*)(q)$ is coflat over $\mathbf{A}^1_{v^*} \simeq \operatorname{Spec} k[q]$ and that Γ is a k-basis of $k[\mathbf{A}^3]/I(\Gamma, v^*)(q)$. Since $I(\Gamma, v^*)(q) \subset \mathfrak{m}$, this implies that $\mathfrak{n} \subset I(\Gamma, v^*)(q)$. Therefore we can define $I(\Gamma, v^*)(\infty) := \lim_{q \to \infty}(\Gamma, v^*)(q)$ as in Subsection 2.7. Hence $\mathfrak{n} \subset I(\Gamma, v^*)(\infty)$ Since $I(\Gamma, v^*)(\infty)$ is $\mathbf{G}_m^r$-invariant, there is a G-graph Γ'' such that $I(\Gamma, v^*)(\infty) = I(\Gamma'')$. We prove $\Gamma' = \Gamma''$. Since $|\Gamma'| = |\Gamma''| = |G|$, it suffices to prove $\Gamma'' \subset \Gamma'$. If $v \not\in \Gamma$ and $v - \mathrm{wt}_\Gamma(v) \not\in \tau^\perp$, then $w^v \in I(\Gamma, v^*)(q)$ for any q, hence $w^v \in I(\Gamma, v^*)(\infty)$. If $v \in \Gamma$ and $v - v_+ \in M_0^0$, then $w^v = w^{v - v_+} w^{v_+} \in I(\Gamma, v^*)(\infty)$. Hence by (10) $(M_0^0 \setminus \Gamma') \subset (M_0^0 \setminus \Gamma'')$. Hence $\Gamma'' \subset \Gamma'$. It follows that $I(\Gamma, v^*)(\infty) = I(G_{\mathrm{ig}}(v_+, v_-)(\Gamma))$. This completes the proof. □

Corollary 2.9. *Let Γ be a G-graph and $\Gamma' := G_{\mathrm{ig}}(v_+, v_-)(\Gamma)$. Let $v^* := v_- - v_+$, $v_+ \in \Gamma$, $v_- \in \Gamma'$ and $h := s^{v_- - v_+}$ the same as before. Then $k[S(\Gamma)][1/h] \simeq k[S(\Gamma')][h]$.*

Proof. For any $v \in M_0^0$, there exists a unique integer $n \geq 0$ such that $\mathrm{wt}_{\Gamma'}(v) = \mathrm{wt}_\Gamma(v) + n(v_- - v_+)$, that is, $v - \mathrm{wt}_\Gamma(v) = v - \mathrm{wt}_{\Gamma'}(v) + n(v_- - v_+)$. In particular, $v_+ - \mathrm{wt}_{\Gamma'}(v_+) = v_+ - v_-$. Hence we have

$$w^{v - \mathrm{wt}_\Gamma(v)} = w^{v - \mathrm{wt}_{\Gamma'}(v)} w^{n(v_- - v_+)} \tag{11}$$

$$w^{v_- - v_+} = (w^{(v_+ - v_-)})^{-1}. \tag{12}$$

This proves the isomorphism. □

Remark 2.10. Thus by Corollary 2.9 we can glue together all $V(\Gamma)$ $(\Gamma \in \mathrm{Graph}(G))$ to obtain an irreducible variety $W(G) := T_M(\mathrm{Graph}(G))$. We see also that there is a zero-dimensional (universal) subscheme Z^{vers} over $W(G)$ such that $Z^{\mathrm{vers}} \times_{W(G)} V(\Gamma) \simeq Z^{\mathrm{vers}}(\Gamma)$. The torus embedding $T_M(\mathrm{Fan}(G))$ is the normalization of $W(G)$. It is not known whether $V(\Gamma)$ as well as Hilb^G is normal.

Theorem 2.11. *Let G be a finite abelian subgroup of $\mathrm{GL}(r, k)$ whose order is prime to the characteristic of k. Let $\mathrm{Graph}(G)$ be the set of all G-graphs and $\mathrm{Fan}(G)$ the set of all $\sigma(\Gamma)$ with $\Gamma \in \mathrm{Graph}(G)$. Then*

(i) $\mathrm{Fan}(G)$ *is a finite fan with its support* Δ,

(ii) $\mathrm{Fan}(G)$ *is obtained from a G-graph by G-igsaw tranformations,*

(iii) $T_M(\mathrm{Graph}(G)) \simeq \mathrm{Hilb}^G(\mathbf{A}^r)$ *and it is projective over* $\mathbf{A}^r/G$,

(iv) $T_M(\mathrm{Fan}(G))$ *is the normalization of* $\mathrm{Hilb}^G(\mathbf{A}^r)$, *projective over* $\mathbf{A}^r/G$.

Proof. First we notice that $\mathbf{A}^r/G \simeq \mathrm{Spec}\ k[\overset{\vee}{\Delta} \cap M]$, the latter of which we denote by $T_M(\Delta)$ where $\overset{\vee}{\Delta}$ is the dual cone of Δ in $M_{\mathbf{R}} := M \otimes \mathbf{R}$.

Now we prove (iii) and (iv). In fact, let $W(G) = T_M(\mathrm{Graph}(G))$ and X the normalization of it. By the universality of $\mathrm{Hilb}(\mathbf{A}^r)$ the subscheme Z^{vers} gives rise to a morphism $h : W(G) \to \mathrm{Hilb}^G := \mathrm{Hilb}^G(\mathbf{A}^r)$. We prove that h is an isomorphism. Let H be the normalization of Hilb^G. Since Hilb^G hence H admits a T_M-action, H is a normal torus embedding. Hence we have a sequence of morphisms of normal torus embeddings

$$X \xrightarrow{\tilde{h}} H \to \mathbf{A}^r/G$$

where $\tilde{h}$ is the morphism induced from $h : X \to \mathrm{Hilb}^G$. If $\tilde{h}$ is not an isomorphism, some toric 1-strata will be contracted by $\tilde{h}$, hence by h because X is finite over $W(G)$. This contradicts Lemma 2.3 (ii). Hence $X \simeq H$, whence $W(G)$ is finite and generically bijective over Hilb^G. Since $W(G)$ is locally isomorphic to Hilb^G by Lemma 2.3 (ii), $W(G) \simeq \mathrm{Hilb}^G$. Hence $W(G)$ is projective over $\mathbf{A}^r/G$. The normalization $T_M(\mathrm{Fan}(G))$ of $W(G)$ is also projective over $\mathbf{A}^r/G$ because it is finite over $W(G)$.

Next we prove (i) and (ii). Since $T_M(\mathrm{Fan}(G)) \simeq \mathrm{Hilb}^G(\mathbf{A}^r)$ is projective over $\mathbf{A}^r/G \simeq T_M(\Delta)$, the support of $\mathrm{Fan}(G)$ is exactly Δ. Let $B := nM_0^+ + M_0^0$. Since $w^v \in I(\Gamma)$ $(\forall v \in B)$ for any G-graph Γ, Γ is contained in the fixed finite subset $M_0^0 \setminus B$ of M_0^0. Therefore there are only finitely many G-graphs. Hence $\mathrm{Fan}(G)$ is a finite fan with its support Δ. For any G-graph Γ, the subscheme $Z(\Gamma)$ is smoothable by Lemma 2.3. Hence by Lemmas 2.5 and 2.8 and the connectivity of Δ any G-graph is obtained from a given G-graph by G-igsaw transformations. □

3. G-GRAPHS IN DIMENSION THREE

In what follows we study $\mathrm{Hilb}^G(\mathbf{A}^3)$ when $G \subset \mathrm{SL}(3, k)$.

3.1. **G-admissible fans.** Let $n = |G|$ and $g = (a/n, b/n, c/n) \in G$ such that $1 \leq a, b, c \leq n-1$. Then the sum $(a+b+c)/n$ is equal to either 1 or 2. If it is equal to 1 (resp. 2), then we call g *junior* (resp. *senior*). Let $\mathrm{Jun}(G)$ be the set of all junior elements in G, and $\mathrm{Jun}^*(G) := \{e_1, e_2, e_3\} \cup \mathrm{Jun}(G)$.

We notice $\mathbf{A}^3/G \simeq T_M(\Delta)$, a torus embedding. We subdivide Δ into simplicies with apices in $\mathrm{Jun}^*(G)$ in an arbitrary manner to get a fan $\mathrm{F_{AN}}$. $\mathrm{F_{AN}}$ is called *G-admissible* if any 3-dimensional cone $\tau \in \mathrm{F_{AN}}$ is generated by three vertices $v_1, v_2, v_3 \in \mathrm{Jun}^*(G)$ such that $\tau \cap \Delta \cap N = \{v_1, v_2, v_3\}$. By [Roan94], the torus embedding $T_M(\mathrm{F_{AN}})$ is a crepant smooth resolution of $\mathbf{A}^3/G$ if $\mathrm{F_{AN}}$ is G-admissible. Any crepant smooth resolution of $\mathbf{A}^3/G$ is obtained this way.

The purpose of this section is to prove that given an abelian subgroup G of $\mathrm{SL}(3,k)$ the fan $\mathrm{Fan}(G)$ is G-admissible in the above sense and that $\mathrm{Hilb}^G \simeq T_M(\mathrm{Fan}(G))$.

Lemma 3.2. *Assume $G \subset \mathrm{SL}(3,k)$. Let Γ be a G-graph and $p_k = \max\{u \in \mathbf{Z}; uf_k \in \Gamma\}$ $(k \in \mathbf{Z}/3\mathbf{Z})$. Then there exist unique nonnegative integers a_i, b_i such that*

$$(p_i+1)f_i \equiv a_{i+1}f_{i+1} + b_{i-1}f_{i-1} \in \Gamma.$$

Proof. We first prove that $(p_1+1)f_1 \equiv jf_2 + kf_3 \in \Gamma$ for some j, k. In fact, by the definition of p_1 we have $(p_1+1)f_1 \notin \Gamma$. Then by Definition 1.4 there exists a unique $if_1 + jf_2 + kf_3 \in \Gamma$ such that $(p_1+1)f_1 \equiv if_1 + jf_2 + kf_3 \in \Gamma$. In particular, $(p_1+1)f_1 \neq if_1 + jf_2 + kf_3$. If $i \geq 1$, then $(i-1)f_1 + jf_2 + kf_3 \in \Gamma$ by Definition 1.4, which contradicts Definition 1.4 because $p_1 f_1 \equiv (i-1)f_1 + jf_2 + kf_3 \in \Gamma$ and $p_1 f_1 \in \Gamma$. Therefore $i = 0$. □

Lemma 3.3. (Unique Valley Lemma) *Suppose $r \geq 0$, $s \geq 0$. Assume*

$$rf_{k+1} + sf_{k-1} \in \Gamma, \quad (r+1)f_{k+1} + sf_{k-1} \in \Gamma,$$
$$rf_{k+1} + (s+1)f_{k-1} \in \Gamma, \quad (r+1)f_{k+1} + (s+1)f_{k-1} \notin \Gamma.$$

Then $r = a_{k+1}$, $s = b_{k-1}$ and $(p_k+1)f_k \equiv rf_{k+1} + sf_{k-1}$.

Proof. Let $k = 1$ for simplicity. By Definition 1.4 there exists $af_1 + bf_2 + cf_3 \in \Gamma$ such that $(r+1)f_2 + (s+1)f_3 \equiv af_1 + bf_2 + cf_3$. Since $(r+1)f_2 + (s+1)f_3 \notin \Gamma$, we have $(r+1)f_2 + (s+1)f_3 \neq af_1 + bf_2 + cf_3$. If $b \geq 1$, then $rf_2 + (s+1)f_3 \equiv af_1 + (b-1)f_2 + cf_3 \in \Gamma$, which contradicts Definition 1.4 because $rf_2 + (s+1)f_3 \in \Gamma$. Hence $b = 0$. Similarly $c = 0$. It follows that $(r+1)f_2 + (s+1)f_3 \equiv af_1 \in \Gamma$. Since $f_1 + f_2 + f_3 \equiv 0$, we have $rf_2 + sf_3 \equiv (a+1)f_1$, where $rf_2 + sf_3 \in \Gamma$. It follows from Definition 1.4 that $(a+1)f_1 \notin \Gamma$. Therefore $a = p_1$. □

Definition 3.4. We call $\mathfrak{v}_k := a_{k+1}f_{k+1} + b_{k-1}f_{k-1}$ $(k \in \mathbf{Z}/3\mathbf{Z})$ *a valley* of Γ if the condition of Lemma 3.2 is true. We also call $\mathfrak{s}_k := p_k f_k$ *a summit* of Γ. The vector $rf_{k+1} + sf_{k-1}$ of Γ is called *a geometric valley* of Γ if the conditions in Lemma 3.3 are true.

Lemma 3.3 is true even if $r = 0$ or if $s = 0$. By Lemma 3.3 Γ has at most a unique geometric valley on $\Gamma_k := \Gamma \cap (\mathbf{R}f_i + \mathbf{R}f_j)$ $(i, j \neq k)$. It gives a geometric way of computing a_k and b_k.

3.5. **Classification of G-graphs.** By Lemma 3.3 any G-graph Γ is one of the following:

(i) a linear graph,
(ii) a planar graph with no geometric valleys,
(iii) a planar graph with a geometric valley,
(iv) three planar graphs with no geometric valleys,
(v) a planar graph with a geometric valley and two planar graph with no geometric valleys,

(vi) two planar graphs, each with a geometric valley and a planar graph with no geometric valley,
(vii) three planar graphs, each with a geometric valley.

Definition 3.6. Let Γ be a G-graph. Let $\mathfrak{v}_i := a_{i+1}f_{i+1} + b_{i-1}f_{i-1}$ be the valleys of Γ ($i \in \mathbf{Z}/3\mathbf{Z}$). Then we define

$$\begin{aligned} v_i &:= (p_i+1)f_i - \mathfrak{v}_i, \\ w_i &:= f_1 + f_2 + f_3 - v_i, \\ v_c = w_c &:= f_1 + f_2 + f_3. \end{aligned}$$

A G-graph Γ is called *v-crepant* (resp. *w-crepant*) if $v_1 + v_2 + v_3 = f_1 + f_2 + f_3$ (resp. $w_1 + w_2 + w_3 = f_1 + f_2 + f_3$).

We call Γ *crepant* if Γ is either v-crepant or w-crepant. We remark that no G-graph Γ is v-crepant and w-crepant.

Lemma 3.7. *Let Γ be a G-graph. Then $S(\Gamma) = \overset{\vee}{\sigma}(\Gamma) \cap M$. If Γ is v-crepant (resp. w-crepant), then $S(\Gamma) = \mathbf{Z}_0 v_1 + \mathbf{Z}_0 v_2 + \mathbf{Z}_0 v_3$ (resp. $S(\Gamma) = \mathbf{Z}_0 w_1 + \mathbf{Z}_0 w_2 + \mathbf{Z}_0 w_3$). In particular, $S(\Gamma) = \overset{\vee}{\sigma}(\Gamma) \cap M$.*

Proof. By Lemma 1.8 and Subsection 3.5, $S(\Gamma)$ is generated by v_i, w_i and v_c ($i = 1, 2, 3$). If Γ is v-crepant, then $v_c = v_1 + v_2 + v_3$ and $w_i = v_c - v_i$. Hence $S(\Gamma) = \mathbf{Z}_0 v_1 + \mathbf{Z}_0 v_2 + \mathbf{Z}_0 v_3$. Since $M = M(\Gamma)$, we have $S(\Gamma) = \overset{\vee}{\sigma}(\Gamma) \cap M$. The rest is similar. □

Notation 3.8. Let Γ be a G-graph and v_i the valleys of Γ as in Lemma 3.2. For $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$ we denote Γ by $\Gamma = \Gamma(a, b)$ (resp. $\Gamma^*(a, b)$) if Γ is v-crepant (resp. w-crepant). For the same a and b we also define

$$\begin{aligned} \|a\| = a_1 + a_2 + a_3, &\quad a * b = a_1 b_2 + a_2 b_3 + a_3 b_1, \\ d(a, b) = 1 + \|a\| &+ \|b\| + a * a + a * b + b * b. \end{aligned}$$

Lemma 3.9. *Let Γ be a G-graph.*

(i) *If $\Gamma = \Gamma(a, b)$, then $w_i = v_{i-1} + v_{i+1}$, $p_i = a_i + b_i$ and $|\Gamma| = d(a, b)$,*
(ii) *If $\Gamma = \Gamma^*(a, b)$, then $v_i = w_{i-1} + w_{i+1}$, $p_i = a_i + b_i + 1$ and $|\Gamma| = 3 + \|a\| + \|b\| + d(a, b)$.*

Proof. Clear. □

3.10. **Crepant G-graphs.** Any v-crepant Γ in the first six classes is viewed as one of the special cases of crepant graphs in (vii). In fact, $\Gamma_k := \Gamma \cap \{x_k = 0\}$ has no geometric valley if and only if $a_{k-1} b_{k+1} = 0$. Therefore if at least one of a_i or b_j equals zero, (vii) is reduced to the cases (i)-(vi). For instance, the case (i) is obtained by setting $a = (a_1, 0, 0)$ and $b = (b_1, 0, 0)$ with $a_1 > 0$, $b_1 > 0$, in which case $\Gamma = \Gamma(a, b) = \{i f_1; i \in [0, a_1 + b_1]\}$. It is v-crepant but cannot be w-crepant. Since various proofs and computations are carried out more or less in the same manner as (vii), we mainly discuss the case (vii) in what follows. In the case (vii) Γ is given by

$$\Gamma = \begin{Bmatrix} if_k + jf_{k+1}; & (i,j) \in [0, p_k] \times [0, b_{k+1}] \text{ or} \\ (k \in \mathbf{Z}/3\mathbf{Z}) & (i,j) \in [0, a_k] \times [b_{k+1}+1, p_{k+1}] \end{Bmatrix}$$

where $p_k = a_k + b_k$.

Let $\Gamma = \Gamma^*(a, b)$ be a w-crepant G-graph. Then $p_i = a_i + b_i + 1 \geq 1$ by Lemma 3.9. Therefore (i)-(iii) is impossible.

Let us consider the case (iv). Then by Lemma 3.9 we have

$$|\Gamma| = 1 + \|p\| + p * p = 4 + 2\|p\| + a * a + a * b + b * a + b * b.$$

Hence $3 + \|a\| + \|b\| + b * a = 0$, which is impossible. Similarly we derive a contradiction in the cases (v) and (vi). Thus there are no w-crepant G-graphs in (iv)-(vi). However there do exist infinitely many w-crepant G-graphs in (vii). If Γ is w-crepant with $a_k = b_k = 0$, then $\Gamma = \{0, f_1, f_2, f_3\}$, and G is an abelian group $\simeq \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$ of order 4 consisting of diagonal matrices with entries ± 1 and determinant one. See also Corollary 2.4.

The following Lemmas follow from Definition 2.6 readily.

Lemma 3.11. *Let $\Gamma = \Gamma(a, b)$ be a v-crepant G-graph, and $v_1 = v_- - v_+$, $v_- = (p_1 + 1)f_1$ and $v_+ = a_2 f_2 + b_3 f_3$. Then the G-igsaw transformation $G_{\mathrm{ig}}(v_-, v_+)(\Gamma)$ of Γ is a crepant G-graph adjacent to Γ and it is given by*

$$G_{\mathrm{ig}}(v_+, v_-)(\Gamma) = \begin{cases} \Gamma^*(a - f_2, b - f_3) & \text{if } a_2 \geq 1, b_3 \geq 1 \\ \Gamma(a', b') & \text{if } a_2 \geq 1, b_3 = 0 \\ \Gamma(a'', b'') & \text{if } a_2 = 0, b_3 \geq 1 \end{cases} \tag{13}$$

where

$$\begin{aligned} a' &= (([p_2/a_2] - 1)(p_1 + 1) + a_1, (1 + [p_2/a_2])a_2 - p_2 - 1, 0), \\ b' &= (p_1 + 1, p_2 - [p_2/a_2]a_2, a_3), \\ a'' &= (p_1 + 1, b_2, p_3 - [p_3/b_3]b_3), \\ b'' &= (([p_3/b_3] - 1)(p_1 + 1) + b_1, 0, (1 + [p_3/b_3])a_3 - p_3 - 1). \end{aligned}$$

Proof. Let v_i be as before and $\Gamma' = G_{\mathrm{ig}}(v_+, v_-)(\Gamma)$. Then by Lemma 2.8, Γ' is a G-graph. Therefore we have $v'_i := v_i(\Gamma')$ and $w'_i := w_i(\Gamma')$ by Definition 3.6. The G-graph Γ' is given as above. In fact, suppose $a_2 \geq 1, b_3 \geq 1$. Then we see that $p_1(\Gamma') = p_1(\Gamma) + 1$, $a_2(\Gamma') = a_2(\Gamma) - 1$ and $b_3(\Gamma') = b_3(\Gamma) - 1$ and for the other invariants $a_i(\Gamma') = a_i(\Gamma)$ $(i = 1, 3)$, $b_j(\Gamma') = b_j(\Gamma)$ $(j = 1, 2)$ and $p_k(\Gamma') = p_k(\Gamma)$ $(k = 2, 3)$. Therefore $p_2(\Gamma') = a_2(\Gamma') + b_2(\Gamma')$, $\Gamma' = \Gamma^*(a - f_2, b - f_3)$ and $w_i(\Gamma') = (1 - \delta_{i1})(f_1 + f_2 + f_3) - v_i$. In particular, $w'_1 = -v_1$, $w'_1 + w'_2 + w'_3 = f_1 + f_2 + f_3$. Hence Γ' is w-crepant.

If $a_2 \geq 1, b_3 = 0$, then $v'_1 = (k + 1)v_1 + v_2$, $v'_2 = -v_1$, $v'_3 = v_3 - (k - 1)v_1$ where $k = [p_2/a_2]$. Hence $v'_1 + v'_2 + v'_3 = f_1 + f_2 + f_3$. This proves that Γ' is v-crepant. The rest is similar. $\square$

Lemma 3.12. *Let* $\Gamma = \Gamma^*(a, b)$ *be a* w*-crepant* G*-graph and* $w_1 = w_- - w_+$, $w_- = (a_2 + 1)f_2 + (b_3 + 1)f_3$ *and* $w_+ = p_1 f_1$. *Then* $G_{\mathrm{ig}}(w_+, w_-)(\Gamma)) = \Gamma(a + f_2, b + f_3)$. *In particular,* $G_{\mathrm{ig}}(v_+, v_-)(\Gamma)$ *is crepant and adjacent to* Γ.

Proof. This case is the converse to the first case in Lemma 3.11. □

Lemma 3.13. *Any* G*-graph is crepant if* $G \subset \mathrm{SL}(3, k)$.

Proof. Suppose that there is at least a crepant G-graph Γ. Then any G-graph adjacent to Γ is obtained by a G-igsaw transformation from Γ by Theorem 2.11 and it is crepant by Lemmas 3.11 and 3.12. Thus we can find adjacent crepant G-graphs successively so that we can eventually cover the whole cone Δ with $\sigma(\Gamma)$ for crepant G-graphs Γ. It remains to find a crepant G-graph.

If G is cyclic, then let $g = (r/n, s/n, 1/n)$ be a generator with $1 \le r \le s \le n - 1$ and $1 + r + s = n$. Let Γ be a linear graph in Subsection 3.5 (i) with $\Gamma = \{xf_3; x \in [0, n-1]\}$. Then it is easy to check that Γ is a crepant G-graph. We see $\Gamma = \Gamma(a, b)$ where $a_i = b_i = 0$ $(i = 1, 2)$, $a_3 = s$ and $b_3 = r$.

Next consider an abelian group G with two generators. Let m be the maximal order of elements in G. Then without loss of generality we may assume that G is a group of order ℓm generated by $g := (r/m, s/m, 1/m)$ and $f := (-1/\ell, 1/\ell, 0)$ for some $2 \le \ell \le m$. Let $\Gamma = \{pf_2 + qf_3; p \in [0, \ell - 1], q \in [0, m-1]\}$, a planar G-graph in Subsection 3.5 (ii) with no valleys. Let $a_1 = b_1 = b_2 = 0$, $a_2 = \ell - 1$, a_3 the unique integer such that $0 \le a_3 \le m - 1$, $a_3 = \ell s \mod m$, and $b_3 := m - 1 - a_3$ (≥ 0). Let $a = (a_i)$, $b = (b_i)$. Then it turns out that $\Gamma = \Gamma(a, b)$ and Γ is v-crepant. This proves the existence of crepant G-graphs. This completes the proof of Lemma. □

TABLE 1. G-graphs

type	v/w	conditions	geom. valleys
(i)	v	$a_k = a_{k+1} = b_k = b_{k+1} = 0$	none
(ii)	v	$a_k = b_k = a_{k-1}b_{k+1} = 0$	none
(iii)	v	$a_k = b_k = 0$	$\mathfrak{v}_k$
(iv)	v	$a_2b_1 = a_3b_2 = a_1b_3 = 0$	none
(v)	v	$a_kb_{k-1} = a_{k+1}b_k = 0$	$\mathfrak{v}_k$
(vi)	v	$a_{k+1}b_k = 0$	$\mathfrak{v}_k, \mathfrak{v}_{k+1}$
(vii)	v	$a_k \ge 1$, $b_k \ge 1$	$\mathfrak{v}_k$ $(k = 1, 2, 3)$
(vii)	w	$a_k \ge 0$, $b_k \ge 0$	$\mathfrak{v}_k$ $(k = 1, 2, 3)$

3.14. **Generators of $\sigma(\Gamma)$.** By Definition 1.5 $\sigma(\Gamma)$ is given explicitly. Let

$$\mathfrak{a}(a, b) = \begin{pmatrix} F(a_2, a_3, b_2, b_3) \\ F(a_3, a_1, b_3, b_1) \\ F(a_1, a_2, b_1, b_2) \end{pmatrix} \tag{14}$$

where $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$ and $F(x, y, u, v) = (xy + xv + uv)/|\Gamma|$.

It turns out that $\sigma(\Gamma)$ is generated by three junior elements of G via the isomorphism $G \simeq N/N_0$. Let $\Gamma = \Gamma(a,b)$ and

$$A = A(\Gamma) = \begin{pmatrix} a_1+b_1+1 & -a_2 & -b_3 \\ -b_1 & a_2+b_2+1 & -a_3 \\ -a_1 & -b_2 & a_3+b_3+1 \end{pmatrix}. \tag{15}$$

Then we have

$$\begin{aligned} A^{-1} &:= (\mathfrak{a}_1(\Gamma), \mathfrak{a}_2(\Gamma), \mathfrak{a}_3(\Gamma)) \\ &:= (\mathfrak{a}(a+f_2, b+f_3), \mathfrak{a}(a+f_3, b+f_1), \mathfrak{a}(a+f_1, b+f_2)). \end{aligned} \tag{16}$$

The column vectors $\mathfrak{a}_k(\Gamma)$ are junior elements of G via the isomorphism $G \simeq N/N_0$ because $(1,1,1)A = (1,1,1)$ by the v-crepancy of Γ. Next take $\Gamma = \Gamma^*(a,b)$ and set

$$A^* = A^*(\Gamma) = \begin{pmatrix} -a_1-b_1-1 & a_2+1 & b_3+1 \\ b_1+1 & -a_2-b_2-1 & a_3+1 \\ a_1+1 & b_2+1 & -a_3-b_3-1 \end{pmatrix} \tag{17}$$

and $(A^*)^{-1} = (\mathfrak{a}_1^*(\Gamma), \mathfrak{a}_2^*(\Gamma), \mathfrak{a}_3^*(\Gamma))$. The vectors $\mathfrak{a}_k^*(\Gamma)$ are junior elements of G via the isomorphism $G \simeq N/N_0$ where

$$\begin{aligned} \mathfrak{a}_1^*(\Gamma) &= \mathfrak{a}(a+f_3+f_1, b+f_1+f_2), \\ \mathfrak{a}_2^*(\Gamma) &= \mathfrak{a}(a+f_1+f_2, b+f_2+f_3), \\ \mathfrak{a}_3^*(\Gamma) &= \mathfrak{a}(a+f_2+f_3, b+f_3+f_1). \end{aligned} \tag{18}$$

Thus $\sigma(\Gamma(a,b))$ is generated by $\mathfrak{a}_k(\Gamma(a,b))$ $(k \in \mathbf{Z}/3\mathbf{Z})$, while $\sigma(\Gamma^*(a,b))$ is generated by $\mathfrak{a}_k^*(\Gamma^*(a,b))$ $(k \in \mathbf{Z}/3\mathbf{Z})$.

4. A crepant smooth resolution

Lemma 4.1. *Let* $\mathrm{Graph}(G)$ *be the set of all* G*-graphs. Let* $\mathrm{Fan}(G)$ *be the set of all* $\sigma(\Gamma)$ *such that* $\Gamma \in \mathrm{Graph}(G)$*. Then* $\mathrm{Fan}(G)$ *is a finite fan with* Δ *its support, which consists of exactly* $|G|$ *3-dimensional* G*-cones and their faces. The associated torus embedding* $T_M(\mathrm{Fan}(G))$ *is a crepant smooth resolution of* $T_M(\Delta) \simeq \mathbf{A}^3/G$*.*

Proof. By Lemma 3.7 $T_M(\mathrm{Fan}(G))$ is smooth. It remains to prove crepancy. Let $\Gamma \in \mathrm{Graph}(G)$ and let u_i be a v-basis or a w-basis of M. Then $X := T_M(\mathrm{Fan}(G))$ has a globally well-defined rational 3-form

$$\omega_{\mathrm{rat}} := \prod_{i=1}^{3} dw^{u_i}/w^{u_i} = |G| \prod_{i=1}^{3} dw^{f_i}/w^{f_i}. \tag{19}$$

By the crepancy of $\Gamma \in \mathrm{Graph}(G)$, $w^{f_1+f_2+f_3}\omega_{\mathrm{rat}}$ is a nonvanishing regular 3-form on X because char. k is prime to $n = |\Gamma|$, whence the dualizing sheaf of X is trivial. This proves Lemma. □

Theorem 4.2. $T_M(\mathrm{Fan}(G)) \simeq \mathrm{Hilb}^G(\mathbf{A}^3)$.

Proof. By Theorem 2.11 and Lemma 3.7 we see $\mathrm{Hilb}^G \simeq T_M(\mathrm{Graph}(G)) \simeq T_M(\mathrm{Fan}(G))$. $\square$

Remark 4.3. The crepancy of G-graphs is interpreted almost as smoothness of versal generic-support-deformation spaces of $Z(\Gamma)$ when $G \subset \mathrm{SL}(3,k)$.

With the notation in Subsection 3.1 $\mathbf{A}^3 \simeq \mathrm{Spec}\ k[\overset{\vee}{\Delta} \cap M_0] \simeq \mathrm{Spec}\ k[x,y,z]$ where x,y and z are the torus coordinates corresponding to the characters of M_0 f_1, f_2 and f_3 respectively. We also denote $k[x,y,z]$ by $k[\mathbf{A}^3]$.

Let Γ be a G-graph and $Z(\Gamma)$ the associated G-cluster.

The generic-support G-equivariant deformation theory of the cluster $Z(\Gamma)$ is more or less the same for any crepant G-graph Γ. The cases (i)-(vi) are obtained by specialization of the case (vii)$_v$. Now suppose that $\Gamma := \Gamma(a,b)$ is a v-crepant G-graph in the class (vii). The ideal $I(\Gamma)$ defining $Z(\Gamma)$ is given by

$$I(\Gamma) := \begin{pmatrix} x^{p_1+1}, y^{p_2+1}, z^{p_3+1}, xyz \\ y^{a_2+1}z^{b_3+1}, z^{a_3+1}x^{b_1+1}, x^{a_1+1}y^{b_2+1} \end{pmatrix}.$$

Let $A^v := k[s_1,s_2,s_3]$ and Z^v a G-invariant subscheme of $\mathbf{A}^3_{A^v}$ defined by the ideal

$$I^v(\Gamma)(s) := \begin{pmatrix} x^{p_1+1} - s_1y^{a_2}z^{b_3}, y^{p_2+1} - s_2z^{a_3}x^{b_1}, z^{p_3+1} - s_3x^{a_1}y^{b_2} \\ y^{a_2+1}z^{b_3+1} - s_2s_3x^{p_1}, z^{a_3+1}x^{b_1+1} - s_3s_1y^{p_2} \\ x^{a_1+1}y^{b_2+1} - s_1s_2z^{p_3}, xyz - s_1s_2s_3 \end{pmatrix}.$$

This is the versal deformation given in Lemma 2.3. Similarly for a w-crepant G-graph $\Gamma := \Gamma^*(a,b)$ the versal deformation of $Z^w := Z(\Gamma)$ is given by

$$I^w(\Gamma)(t) := \begin{pmatrix} x^{p_1+1} - t_2t_3y^{a_2}z^{b_3}, y^{p_2+1} - t_3t_1z^{a_3}x^{b_1}, z^{p_3+1} - t_1t_2x^{a_1}y^{b_2} \\ y^{a_2+1}z^{b_3+1} - t_1x^{p_1}, z^{a_3+1}x^{b_1+1} - t_2y^{p_2} \\ x^{a_1+1}y^{b_2+1} - t_3z^{p_3}, xyz - t_1t_2t_3 \end{pmatrix}.$$

Thus in each case the distinguched term $xyz - s_1s_2s_3$ or $xyz - t_1t_2t_3$ is present in the ideal defining a versal deformation, which implies crepancy of $\mathrm{Hilb}^G(\mathbf{A}^3)$.

By Lemma 3.7 $U_{\sigma(\Gamma)} := \mathrm{Spec}\ k[\overset{\vee}{\sigma}(\Gamma) \cap M]$ turns out to be

$$\begin{aligned} U_{\sigma(\Gamma(a,b))} &:= \mathrm{Spec}\ k[w^{v_1}, w^{v_2}, w^{v_3}] \simeq \mathrm{Spec}\ k[s_1,s_2,s_3], \\ U_{\sigma(\Gamma^*(a,b))} &:= \mathrm{Spec}\ k[w^{w_1}, w^{w_2}, w^{w_3}] \simeq \mathrm{Spec}\ k[t_1,t_2,t_3]. \end{aligned}$$

See Definition 3.6 for the notation.

4.4. **The exceptional set.** Let $\pi : \mathrm{Hilb}^G \to \mathbf{A}^3/G$ be the natural morphism and $E(G)$ the execeptional set of π. Any irreducible component of $E(G)$ is by Theorem 4.2 a smooth torus embedding corresponding to a junior element $g \in G$, which we denote by $E(g)$. Hence $E(G) = \cup_{g\in \mathrm{Jun}(G)} E(g)$. Let $\mathrm{Graph}(G,g)$ be the set of $\Gamma \in \mathrm{Graph}(G)$ with $g \in \sigma(\Gamma)$, and $U_g(\sigma(\Gamma)) := \mathrm{Spec}\ k[\overset{\vee}{\sigma}(\Gamma) \cap g^{\perp} \cap M]$. Then $E(g)$ is covered with $U_g(\sigma(\Gamma))$ ($\Gamma \in \mathrm{Graph}(G,g)$).

Definition 4.5. Let $g_i \in \mathrm{Jun}(G)$ $(i = 1, 2, 3)$ be distinct elements. Then $E(g_1) \cap E(g_2)$ is a smooth rational curve if and only if $\mathrm{Graph}(G, g_1) \cap \mathrm{Graph}(G, g_2) \neq \emptyset$ if and only if $\sigma(g_1, g_2) := \mathbf{R}_0\, g_1 + \mathbf{R}_0\, g_2 \in \mathrm{Sk}^2 \mathrm{Fan}(G)$. Similarly $\cap_{i=1,2,3} E(g_i)$ is a point if and only if $\cap_{i=1,2,3}\ \mathrm{Graph}(G, g_i) \neq \emptyset$ if and only if $\sigma(g_1, g_2, g_3) := \sum_{i=1,2,3} \mathbf{R}_0\, g_i \in \mathrm{Sk}^3 \mathrm{Fan}(G)$. In this case $|\cap_{i=1,2,3}\ \mathrm{Graph}(G, g_i)| = 1$. Thus $E(G)$ is stratified into 0,1,2-strata labeled by $\mathrm{Sk}^p \mathrm{Fan}(G)$ $(p = 3, 2, 1)$ respectively. Let $\tau \in \mathrm{Sk}^p \mathrm{Fan}(G)$ and $E^0(\tau)$ a $(3-p)$-stratum of $E(G)$ corresponding to τ and $E(\tau)$ the closure of $E^0(\tau)$. This means that $E^0(\sigma) \cap E^0(\tau) = \emptyset$ for $\sigma \neq \tau$ and $E(\sigma) \subset E(\tau)$ if $\tau \prec \sigma$ $(\tau, \sigma \in \mathrm{Fan}(G))$, where $E^0(\tau) = \mathrm{Spec}\, k[\tau^\perp \cap M] \simeq \mathbf{G}_m^{3-p}$.

Definition 4.6. For a subset A of M_0 and $\tau \in \mathrm{Fan}(G)$ we define

$$\mathrm{Graph}(G, \tau) := \cap_{g \in \tau \cap \mathrm{Jun}(G)} \mathrm{Graph}(G, g),$$
$$\Gamma(\tau) := \cap_{\Gamma \in \mathrm{Graph}(G,\tau)}\, \Gamma, \quad \Gamma(\tau, A) := \cap_{A \subset \Gamma \in \mathrm{Graph}(G,\tau)}\, \Gamma.$$

Now one easily sees

Lemma 4.7. *Let $\Gamma = \Gamma(a - f_2, b - f_3)$ with $a_i, b_j \geq 1$ and $g := \mathfrak{a}_1(\Gamma) = \mathfrak{a}(a, b)$ with the notation in Subsection 3.14. Then*

$$\mathrm{Graph}(G, g) = \begin{Bmatrix} \Gamma^*(a - f_i - f_{i+1}, b - f_{i+1} - f_{i-1}) \\ \Gamma(a - f_{i+1}, b - f_{i+2})\ (i \in \mathbf{Z}/3\mathbf{Z}) \end{Bmatrix}.$$

Corollary 4.8. *Let the G-graph Γ and the junior element g be as above. Then $E(g)$ is a regular hexagon – $\mathbf{P}^2$ blown up at three distinct points.*

Remark 4.9. As in [IN98] there is a certain (rather complicated) correspondence between the set of irreducible exceptional subvarieties and the set of minimal G-submodules generating $I \in \mathrm{Hilb}^G$. This would be a kind of phenomenon generalizing [McKay80] from the view point of representations of G. However it seems difficult to understand this correspondence only from irreducible decompositions of tensor products. We omit it. A generalization of McKay correspondence in dimension three analogous to [GSV83] is given in [INkjm98].

5. Examples in dimension 3

5.1. **The case $G = G_{1,2,3}$.** We will explain the theory in the previous sections by the examples in smaller values. Let G be a cyclic group of order 6, $g = (1, 2, 3)/6$ a generator of G. We denote $G = G_{1,2,3}$ when necessary. Figure 1 plots all the junior elements for $G = G_{1,2,3}$. Any $\Gamma \in \mathrm{Graph}(G)$ is one of the following table (Table 2). Def_Γ is the set of toric parameters for deforming the G-cluster $Z(\Gamma)$. Let us explain the theory in this case in detail.

First we start with the ideal $I(\Gamma_1) = (x^6, y, z)$. It is clear $\dim k[\mathbf{A}^3]/I(\Gamma_1) = 6$, and that $k[\mathbf{A}^3]/I(\Gamma_1)$ is spanned by Γ_1. We can deform $Z(\Gamma_1)$ into smooth

G-subschemes by $(x^6 - s, y, z)$, whence $Z(\Gamma_1) \in \mathrm{Hilb}^G$. The versal deformation of $Z(\Gamma_1)$ is given by the ideal

$$(x^6 - s, y - tx^2, z - ux^3, x^4y - st, x^3z - us, xyz - stu)$$

where x^2 and x^3 are the unique elements of Γ_1 such that $\mathrm{wt}(y) = \mathrm{wt}(x^2)$, $\mathrm{wt}(z) = \mathrm{wt}(x^3)$. Def_{Γ_1} is now understood as 3 free parameters s, t, u hence may be understood as x^6, yx^{-2} and zx^{-3}. In this case there are two directions yx^{-2} and zx^{-3} of deformations which give rise to monomial G-ideals (*i.e.* G-ideals generated by monomials). This is written in Figure 2 as $b - 2a$ and $c - 3a$. The graph Γ_2 is obtained from Γ_1 by replacing $2a$ in any vector of Γ_1 by b repeatedly. In general, in the process of taking the limit of clusters in the direction $v_- - v_+$ the second G-graph is obtained from the first G-graph by replacing v_+ by v_- repeatedly.

For instance $v_+ = a + b$ and $v_- = c$ in the process from Γ_2 to Γ_4. Hence $a + b \in \Gamma_2$ (resp. $a + 2b$) is transformed into c (resp. $b + c$) so that Γ_2 is changed into Γ_4 under $(G_{1,2,3})_{\mathrm{ig}}(a + b, c)$. Figure 2 lists the possible solutions to the jigsaw puzzle (the G-igsaw puzzle !) between G-graphs when $G = G_{1,2,3}$.

TABLE 2. $G_{1,2,3}$-graphs/cones

No.	Γ	$I(\Gamma)$
1	$\{1, x, x^2, x^3, x^4, x^5\}$	(x^6, y, z)
2	$\{1, x, y, xy, y^2, xy^2\}$	(x^2, y^3, z)
3	$\{1, y, z, yz, y^2, y^2z\}$	(x, y^3, z^2)
4	$\{1, x, y, z, yz, y^2\}$	$(xy, y^2z, zx, x^2, y^3, z^2)$
5	$\{1, x, y, z, yz, zx\}$	(x^2, xy, y^2, z^2)
6	$\{1, x, x^2, z, xz, x^2z\}$	(x^3, y, z^2)
No.	Def_Γ	$\sigma(\Gamma)$
1	$x^6, y/x^2, z/x^3$	(e_2, e_3, g)
2	$x^2/y, y^3, z/xy$	(e_3, g, g^3)
3	$x/y^2z, y^3, z^2$	(e_1, g^3, g^4)
4	$xy/z, y^2z/x, zx/y^2$	(g, g^3, g^4)
5	$x^2/y, y^2/zx, z^2$	(g, g^2, g^4)
6	$x^3/z, y/x^2, z^2$	(e_2, g, g^2)

5.2. **Jigsaw puzzles.** For a given G-graph the possible solutions to the jigsaw puzzle are clear at first sight if the graph is complicated enough, as we see in the case of $G_{1,5,31}$ below. See Figure 3. Let $\Gamma = \Gamma(g^{10}, g^{11}, g^{17})$. Then the G-graph Γ is v-crepant. There are exactly three jigsaw puzzle solutions only in the v_k-directions where v_k is among $4a - 2b - c$, $5c - 2a - b$ and $4b - a - 3c$. For instance, the adjacent G-graph of Γ in the $(5c - 2a - b)$-direction is obtained by moving the hook at $2a + b$ consisting of 4 small black dots, each surrounded by a large white circle, into the position $5c$ as in the lower-left of the Figure 3. On the other hand there are exactly three jigsaw puzzle solutions only in the w_k-directions for any w-crepant Γ, for instance, $\Gamma = \Gamma(g^4, g^{10}, g^{11})$. See also Tbale 3 for another example of G-graphs/cones.

TABLE 3. $G_{1,2,4}$-graphs/cones

No.	Γ	$I(\Gamma)$
0	$\{1, x, y, z, xy, yz, zx\}$	(x^2, y^2, z^2, xyz)
1	$\{1, y, z, z^2, z^3, yz, yz^2\}$	(x, y^2, z^4, yz^3)
2	$\{1, z, x, x^2, x^3, zx, zx^2\}$	(y, z^2, x^4, zx^3)
3	$\{1, x, y, y^2, y^3, xy, xy^2\}$	(z, x^2, y^4, zx^3)
4	$\{1, z, z^2, z^3, z^4, z^5, z^6\}$	(x, y, z^7)
5	$\{1, y, y^2, y^3, y^4, y^5, y^6\}$	(x, y^7, z)
6	$\{1, x, x^2, x^3, x^4, x^5, x^6\}$	(x^7, y, z)
No.	Def_Γ	$\sigma(\Gamma)$
0	$x^2/y, y^2/z, z^2/x$	(g, g^2, g^4)
1	$x/z^2, z^4/y, y^2/z$	(e_1, g^2, g^4)
2	$y/x^2, x^4/z, z^2/x$	(e_2, g, g^2)
3	$z/y^2, y^4/x, x^2/y$	(e_3, g, g^4)
4	$z^7, x/z^2, y/z^4$	(e_1, e_2, g^2)
5	$y^7, z/y^2, x/y^4$	(e_3, e_1, g^4)
6	$x^7, y/x^2, z/x^4$	(e_2, e_3, g)

6. MISCELLANEOUS REMARKS

6.1. **A singular example in dimension four.** There is an example of singular $\mathrm{Hilb}^G(\mathbf{A}^4)$. Let G be an abelian subgroup of $\mathrm{SL}(4, k)$ consisting of all daigonal matrices with entries ± 1, which we denote by G_{2^3}. We will prove that $\mathrm{Hilb}^G(\mathbf{A}^4)$ is singular. The group G has order 8 with 6 junior elements and 12 monomial G-ideals. Among them there are essentially three different types of G-ideals whose G-graphs are Γ_1, Γ_2 and Γ_3 in the Table 4. Let $U_k := \mathrm{Spec}\, k[\overset{\vee}{\sigma}(\Gamma_k) \cap M]$. We see that U_k $(k = 1, 2)$ is smooth, while U_3 is singular. Hence $\mathrm{Hilb}^G(\mathbf{A}^4)$ is singular. The versal generic-support-G-equivariant deformation of $Z(\Gamma_k)$ $(k = 1, 2, 3)$ is given by the ideal

$$I(\Gamma_1)(s) := \begin{pmatrix} x - s_1 yzw, y^2 - s_2, z^2 - s_3, w^2 - s_4 \\ xyzw - s_1 s_2 s_3 s_4 \end{pmatrix},$$

$$I(\Gamma_2)(t) := \begin{pmatrix} x^2 - t_1 t_2 t_3 t_4, y^2 - t_1 t_4, z^2 - t_2 t_4, w^2 - t_3 t_4 \\ xy - t_1 zw, xz - t_2 yw, xw - t_3 yz \\ yzw - t_4 x, xyzw - t_1 t_2 t_3 t_4^2 \end{pmatrix},$$

$$I(\Gamma_3)(u) := \begin{pmatrix} x^2 - u_1, y^2 - u_2, z^2 - u_3, w^2 - u_4 \\ yz - u_5 xw, yw - u_6 xz, zw - u_7 xy \\ xyzw - u_1 u_2 u_7 \end{pmatrix}$$

where

$$u_2u_7 = u_3u_6 = u_4u_5,\ u_2u_3 = u_1u_4u_5^2,$$
$$u_2u_4 = u_1u_3u_6^2,\ u_3u_4 = u_1u_2u_7^2.$$

TABLE 4. G_{23}-graphs/cones

No.	Γ	$I(\Gamma)$
1	$\{1, y, z, w, yz, yw, zw, yzw\}$	(x, y^2, z^2, w^2)
2	$\{1, x, y, z, w, yz, yw, zw\}$	$(xy, xz, xw, x^2, y^2, z^2, w^2, yzw)$
3	$\{1, x, y, z, w, xy, xz, xw\}$	$(yz, yw, zw, x^2, y^2, z^2, w^2)$

6.2. **The case of no crepant resolutions.** There is an example of a smooth $\mathrm{Hilb}^G(\mathbf{A}^4)$ which is not a crepant resolution of $\mathbf{A}^4/G$. For instance let G be the subgroup of $\mathrm{SL}(4,k)$ of order two generated by minus the identity. We easily see that $\mathrm{Hilb}^G(\mathbf{A}^4)$ is smooth, while it is well known that $\mathbf{A}^4/G$ has no crepant resolutions because it is terminal.

6.3. **Abelian subgroups in** GL**.** There is also an example of singular Hilb^G for a finite abelian subgroup G in $\mathrm{GL}(3,k)$ by [Reid97]. Very recently R. Kidoh [Kidoh98] gave a complete description of $\mathrm{Hilb}^G(\mathbf{A}^2)$ for an abelian finite subgroup G of $\mathrm{GL}(2,k)$ by using two kinds of continued fractions. Her work essentially gives the G-graphs in our formulation explicitly.

6.4. **Quot schemes.** There are infinitely many algebro-geometric relatives of Hilbert schemes, that is, Grothedieck's Quot schemes of coherent sheaves.

Conjecture 6.5. *Suppose $G \subset \mathrm{SL}(3,k)$. Then any projective crepant resolution of $\mathbf{A}^3/G$ is isomorphic to $\mathrm{Quot}^G(F)$ for some G-invariant coherent sheaf F on $\mathbf{A}^3$.*

A similar conjecture was communicated to us by Nakajima in the quiver variety formulation.

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FIGURE 1. FAN($G_{1,2,3}$)

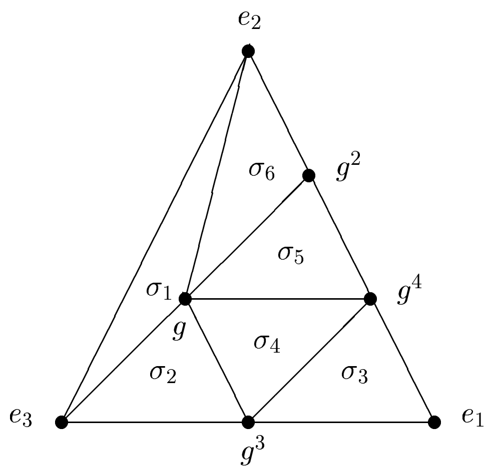

FIGURE 2. JIGSAW PUZZLE OF $G_{1,2,3}$-GRAPHS

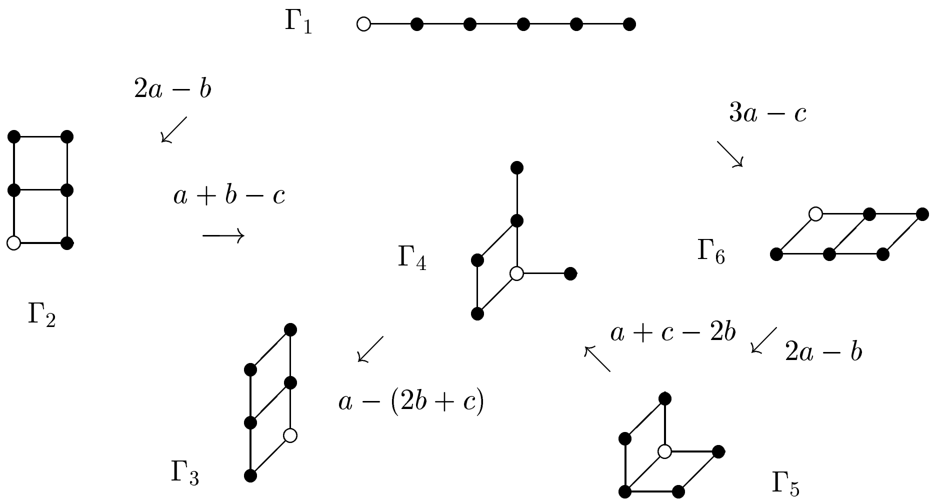

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Figure 3. JIGSAW PUZZLE OF $G_{1,5,31}$-GRAPHS