AMPLE DIVISORS ON NORMAL GORENSTEIN SURFACES

by

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In [So1] and [VdV] the classical adjunction process was improved to give a powerful structure theory for very ample divisors on projective surfaces. This theory has been used in many investigations in more and more general forms, e.g. [Io] , [Li] , [So2] , [So3] , [So4] , [So5] .

In [La+Pa] by using Mori theory (and also in [So4] as a biproduct of a more general adjunction process) the "stable" part of the theory was extended to arbitrary ample divisors on smooth surfaces. The definitive version of these results should cover arbitrary ample divisors on normal Gorenstein surfaces S, i.e. normal surfaces for which the dualizing sheaf ω_S is invertible. This is the natural generality because:

a) it includes the canonical models of general type surfaces and

b) in greater generality the dualizing sheaf is not invertible and new phenomena occur-in particular iteration of the process leads to non Cartier divisors and generally does not behave well.

In this paper heavily influenced by the work of F. Sakai ([Sa1], [Sa2], [Sa3]) I carry out this generalization to normal Gorenstein surfaces. Though the methods are more intricate than earlier methods the final results are very simple; ample divisors on Gorenstein surfaces have the same structure as on smooth surfaces.

I need some terminology. A pair (S,L) consisting of an ample line bundle L on a normal Gorenstein surface S is called a *quadric*, if S is biholomorphic to a possibly singular quadric $\mathcal{Q} \subseteq \mathbb{P}^3$ and L is isomorphic to the restriction of $\mathcal{O}_{\mathbb{P}^3}(1)$ to $\mathcal{Q}$. A pair (S,L) is called a *geometrically ruled surface*, if S is a holomorphic $\mathbb{P}^1$ bundle, $p : S \to R$, over a non-singular curve R and the restriction L_f of L to a fibre f of p is $\mathcal{O}_f(1)$.

Theorem. Let L be an ample line bundle on a normal Gorenstein surface S. The following are equivalent:

a) $\omega_S \otimes L$ *is numerically effective, i.e. degree* $(\omega_S \otimes L)_C \geq 0$ *for all irreducible curves C on S,*

b) *there is a $k > 0$ such that* $(\omega_S \otimes L)^k$ *is spanned by global sections,*

c) (S,L) *is neither* $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ *nor* $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ *nor a quadric nor a geometrically ruled surface*

d) <u>there is a</u> $k > 0$ <u>such that</u> $h^0((\omega_S \otimes L)^k) > 0$.

The next theorem analyzes the structure of the map associated to $\Gamma((\omega_S \otimes L)^k)$. The following terminology is convenient. A pair (S,L) consisting of a normal Gorenstein surface and an ample line bundle $L \approx \omega_S^{-1}$ is called a <u>Gorenstein del Pezzo surface</u> (see [Br] for the classification of these surfaces). A pair (S,L) is called a <u>conic bundle</u> if there is a holomorphic surjection $p : S \to R$, with connected fibre onto a smooth curve R with the property that for some $k > 0$ and some very ample line bundle $\mathcal{L}$ on R

$$(\omega_S \otimes L)^k \approx p^*\mathcal{L} .$$

If L is spanned by global sections then it is shown (1.4) that there is another equivalence, i.e. $h^1(\mathcal{O}_S) \neq h^1(\mathcal{O}_C)$ for smooth $C \in |L|$.

Further it is shown (3.1) that $\omega_S \otimes L$ for ample L always has enough global sections to seperate non-rational singularities. If L is very ample and $h^1(\mathcal{O}_S) = 0$ it is shown (3.3) that $\omega_S \otimes L$ numerically effective implies that $\omega_S \otimes L$ is spanned by global sections.

The "stable" structure of the map (i.e. the structure for $N >> o$) associated to $\Gamma((\omega_S \otimes L)^N)$ is also as nice as in the smooth case!

Theorem. *Let L be an ample line bundle on a normal irreducible Gorenstein surface S. Assume that $\omega_S \otimes L$ is numerically effective. There is an $N > 0$ such that $(\omega_S \otimes L)^N$ is spanned and the map $\phi: S \to \mathbb{P}_{\mathbb{C}}$ associated to $\Gamma((\omega_S \otimes L)^N)$ has connected fibres and a normal image.*

a) *if* $\dim \phi(S) = 0$ *then*, $\omega_S \approx L^{-1}$, *these Gorenstein Del Pezzo surfaces are classified in* [Br],

b) *if* $\dim \phi(S) = 1$ *then* $\phi: S \to \phi(S)$ *has general fibre* f *biholomorphic* $\mathbb{P}^1$ *and* $L_f \approx \mathcal{O}_f(2)$,

c) *if* $\dim \phi(S) = 2$ *then the set of positive dimensional fibres of* ϕ *are mapped onto a set* F *of smooth points of* $\phi(S)$; *in particular* $\phi(S)$ *is Gorenstein. Further* $L_{S-\phi^{-1}(F)}$ *extends over the smooth punctures* F *to give an ample line bundle* L' *on* $\phi(S)$ *such that* $\omega_{\phi(S)} \otimes L'$ *is ample and satisfies* $\omega_S \otimes L = \phi^*(\omega_{\phi(S)} \otimes L')$. *Further given* $x \in F$, $f = \phi^{-1}(x)$ *is a smooth* $\mathbb{P}^1$ *with* $L \cdot f = 1$, $\omega_S \cdot f = -1$.

The proofs are given in stages. In § 0 background material is collected.
In § 1 $\omega_S \otimes L^2$ is studied first culminating in theorem (1.2) which says precisely when it is ample. This leads to theorem (1.3) which shows a number of the equivalences to $\omega_S \otimes L$ being numerically effective. The section ends with Theorem (1.4) which shows that for spanned ample L, $h^1(\mathcal{O}_S) \neq h^1(\mathcal{O}_C)$ for smooth $C \in |L|$ is equivalent to numerical effectivity.

§ 2 is devoted to showing theorem (2.1) which relates numerical effectivity of $\omega_S \otimes L$ to the spannedness of $(\omega_S \otimes L)^N$ for

$N > 0$. The structure of the map $\phi: S \to \mathbb{P}_{\mathbb{C}}$ associated to $\Gamma((\omega_S \otimes L)^N)$ is a biproduct of the proof.

In § 3 the spannedness properties of $\omega_S \otimes L$ itself are studied.

In a sequel we will carry out the generalization (by inductions in the manner of [So3]) to n dimensional irreducible normal Gorenstein varieties.

I have been greatly influenced by the work of F. Sakai as communicated by both conversations and preprints and I would like to express my thanks to him.

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§ 0 BACKGROUND MATERIAL

In this section I collect background material that will be needed. Most of it is well known but scattered about. I follow tha notation of [So1] and [So2] . I have found F. Sakai's papers [Sa1],[Sa2],[Sa3] very helpful.

I follow the convention of not distinguishing between a vector bundle and its locally free sheaf of germs of holomorphic sections.

Throughout this section S will denote a complex analytic normal projective surface with at most Gorenstein singularities. Recall that the Gorenstein condition is equivalent for a surface to the dualizing sheaf, ω_S , being invertible.

Let $\pi : \bar{S} \to S$ denote the projective desingularization of S , minimal in the sense that the fibres of π contain no smooth rational curves E satisfying $E \cdot E = -1$. A fundamental fact is that:

(0.1) $$\pi^*\omega_S \approx \omega_{\bar{S}} + \Delta$$

where Δ is an effective divisor [cf. Sa1] . I will often do computations on $\bar{S}$. If C is an irreducible curve on S , then $\bar{C}$ will denote the proper transform of C . Clearly

(0.2) $$\omega_S \cdot C = (\omega_{\bar{S}} + \Delta) \cdot \bar{C} \geq \omega_{\bar{S}} \cdot \bar{C} .$$

If L is a line bundle on S then $\bar{L}$ will often denote π^*L (except when $L = \omega_S$ since this would lead to a confusion of bars). This is very convenient when calculating but the reader should be careful since given a $C \in |L|$, it is not necessarily true that $\bar{C} \in |\bar{L}|$. Note that for any such L on S there is by (0.1) a natural inclusion:

$$(0.3) \qquad \Gamma(\omega_{\bar{S}} \otimes \bar{L}) \to \Gamma(\pi^*(\omega_S \otimes L)) \approx \Gamma(\omega_S \otimes L)$$

Since S is Cohen-Macaulay and ω_S is invertible, Serre duality for a locally free sheaf E on S takes the usual form:

$$(0.4) \qquad h^i(S,E) = h^{2-i}(S,\omega_S \otimes E^*) \quad \text{for} \quad i = 0,1,2 .$$

The Kodaira-Ramanujam vanishing theorem holds in its usual forms (see [Sh+So], [Sa3] for more details).

(0.5) Theorem. Let L be a line bundle on S such that $L \cdot L > 0$. Assume that L is numerically effective, i.e. $L \cdot C \geq 0$ for each irreducible curve C on S. Then:

$$h^i(\omega_S \otimes L) = 0 = h^{2-i}(L^{-1}) \quad \text{for} \quad i \geq 1 .$$

Proof. There is the exact sequence:

$$0 \to \pi_*\omega_{\bar{S}} \to \omega_S \to \mathcal{S} \to 0$$

where $\mathcal{S}$ is a skyscraper sheaf supported on the singular points of S.

Tensoring the sequence with L and using the long exact cohomology sequence, the theorem reduces to showing that

$h^i((\pi_* \omega_{\bar{S}}) \otimes L) = 0$ for $i \geq 1$. Notes that $\pi_*(\omega_{\bar{S}} \otimes \pi^* L) \approx \pi_*(\omega_{\bar{S}}) \otimes L$ and that by the Grauert-Riemenschneider vanishing theorem [Gra + Ri], $\pi_{(i)}(\omega_{\bar{S}} \otimes \pi^* L) = 0$ for $i \geq 1$. Using this and the Leray spectral sequence for π and $\omega_{\bar{S}} \otimes \pi^* L$ the theorem reduces to showing that $h^i(\omega_{\bar{S}} \otimes \pi^* L) = 0$ for $i \geq 1$. This is the usual Kodaira-Pamonujam vanishing theorem applied to $\pi^* L$.

□

(0.6) Let $\nu(S) = h^\circ(\pi_{(1)}(\mathcal{O}_{\bar{S}}))$. This number is a measure of the singularities on S. If it is 0 then all the singularities of S are rational; if it is 1 then all but one singularity are rational and that one is single elliptic [see [Sa1] for a short discussion and references]. By the Leray spectral sequence it is quickly checked that for any line bundle L on S:

(0.6.1) $\nu(S) = \chi(\mathcal{O}_S) - \chi(\mathcal{O}_{\bar{S}}) = \chi(L) - \chi(\bar{L})$.

From this comparison and the Leray spectral sequence it is easy to check that the Riemann-Roch formula takes its usual form.

(0.6.2) Theorem. Let L be a line bundle on S. Then $\chi(L) = (L \cdot L - \omega_S \cdot L)/2 + \chi(\mathcal{O}_S)$.

The following lemma is useful.

(0.6.3) Lemma. If $\nu(S) = 0$, then S is a local complete intersection. In particular if D is an effective ample divisor on S then

$$H_1(D,\mathbb{Z}) \to H_1(S,\mathbb{Z}) \to 0.$$

Proof. If $\nu(S) = 0$ then all singularities are rational Gorenstein. These are well known to be hypersurface singularities, e.g. [Ar]. On local complete intersections the usual first Lefschetz theorem holds, e.g. [F + L, theorem (9.8)]. □

In general the following holds.

(0.6.4) Lemma. Let D be an effective ample divisor on S. Then D is connected and the restriction map $\Gamma(\mathcal{O}_S) \to \Gamma(\mathcal{O}_D)$ is injective.

Proof. Use the Kodaira vanishing theorem and the exact sequence:

$$0 \to [D]^{-1} \to \mathcal{O}_S \to \mathcal{O}_D \to 0.$$

□

(0.7) By a Gorenstein Del Pezzo surface I mean a normal irreducible Gorenstein projective surface S such that ω_S^{-1} is ample. These are studied in [Br]. A good example to keep in mind is the cone in $\mathbb{P}^3$ on a smooth cubic curve in $\mathbb{P}^2$.

If S is a Gorenstein Del Pezzo surface then $h^1(\mathcal{O}_S) = 0$ or 1. In the former case S has only rational singularities and S is birational to $\mathbb{P}^2$. In the latter case S has exactly one elliptic singularity with the possibly empty remaining set of singularities rational and S is birational to a $\mathbb{P}^1$ bundle over an ellitpic curve. Further details are given in [Br].

The following result can be deduced from [Br] or proved exactly as in the smooth case [Ko+Oc] using (0.4), (0.5) and (0.6.2).

(0.7.1) Lemma. Let L be an ample line bundle on an irreducible normal Gorenstein projective surface S. If $\omega_S \approx L^{-3}$ then $(S,L) \approx (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$. If $\omega_S \approx L^{-2}$ then S is biholomorphic to an irreducible quadric Q in $\mathbb{P}^3$ and L is isomorphic to the restriction of $\mathcal{O}_{\mathbb{P}^3}(1)$ to Q.

(0.8) Conic bundles (S,L), were defined in the introduction. The reader should note that given a conic bundle, the general fibre f of the map $p:S \to R$ in the definition of conic bundle is a smooth rational curve f with $L \cdot f = 2$. A good example to keep in mind is the following.

(0.8.1) Example. Let $X = R \times \mathbb{P}^1$ and let $p':X \to R$ be the product projection on the first factor where R is some smooth Riemann surface. Let H_1, H_2 be two disjoint sections of p' that get projected to points by the projection q of X onto the second factor. Pick a point $x \in \mathbb{P}^1$ and let $E = p'^{-1}(x)$. Blow up the two points $E \cap H_1 = a$ and $E \cap H_2 = b$ to get a new surface X'. The proper transform E' of E has self intersection -2. Blow E' down to get a normal compact complex surface S with an isolated Gorenstein rational singularity. The map p' drops to a map $p:S \to R$. Let H'_1 and H'_2 denote the image in S of the proper transforms in X' of H_1 and H_2. Let y be a general point of R and let $f = p^{-1}(y)$. Then

$L = [H'_1] \otimes [H'_2] \otimes [f]^2$ is easily checked to be ample by the Nakai criterion. (S,L) is a conic bundle with $L \cdot L = 6$, and $(\omega_S + L) \cdot L = 4g$ where g is the genus of R. Note also that $\omega_S \approx p^*\omega_R \otimes [H'_1]^{-1} \otimes [H'_2]^{-1}$ and that $\omega_S \otimes L^2 \approx p^*(\omega_R \otimes [y]^2)$.

§ 1 Numerical Effectivity of $\omega_S \otimes L$

(1.0) In this section S always denotes a normal irreducible Gorenstein projective surface and L always denotes an ample line bundle on S. The notation of § 0 associated with the minimal desingularization $\pi:\bar{S} \to S$ is kept and used without comment.

The plan of the section is as follows. First I show that $\omega_S \otimes L^2$ and $\omega_{\bar{S}} \otimes \bar{L}^2$ are numerically effective with only one exception. Using this a key criterion (1.2) to recognize geometrically ruled surfaces (S,L) and quadrics is given. Using this the precise ampleness properties of $\omega_S \otimes L$ are worked out in (1.2.1). This result allows us to characterize the numerical effectivity of $\omega_S \otimes L$ in (1.3).

(1.1) Lemma. Let L be an ample line bundle on a normal Gorenstein surface S. Assume that (S,L) is not $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$. Then $\omega_S \otimes L^2$ and $\omega_{\bar{S}} \otimes \bar{L}^2$ are numerically effective. Further $h^0(\omega_S \otimes L^2) \neq 0$.

Proof. Let C be an irreducible curve on $\bar{S}$. If $(\omega_{\bar{S}} + 2\bar{L})\cdot C < 0$ then $\bar{L}\cdot C = 0$ or $\bar{L}\cdot C > 0$. If $\bar{L}\cdot C = 0$ then by the Hodge index theorem $C\cdot C < 0$ and by the last sentence $\omega_{\bar{S}}\cdot C < 0$. Thus C is a smooth rational curve contained in a fibre of $\pi:\bar{S} \to S$ and satisfying $C\cdot C = -1$. This is not possible by our minimality assumption on $\bar{S}$.

If $\bar{L}\cdot C > 0$ then from $(\omega_{\bar{S}} + 2\bar{L})\cdot C<0$ we conclude that $\omega_{\bar{S}}\cdot C \le -3$. Therefore by the adjunction formula it follows that $C\cdot C > 0$. From this it follows that $\dim |nC| > 0$ for some $n > 0$ and therefore $h^0((\omega_{\bar{S}} \otimes \bar{L}^2)^N) = 0$ for all $N > 0$. Since $\bar{L}^k$ has sections for the $k >> 0$ it follows that $h^0(\omega_{\bar{S}} \otimes \bar{L}) = 0$. From this and the Kodaira vanishing theorem we conclude that $\chi(\omega_{\overline{S}} \otimes \overline{L}) - \chi(\omega_{\overline{S}} \otimes \overline{L}^2) = h^0(\omega_{\overline{S}} \otimes \overline{L}) - h^0(\omega_{\overline{S}} \otimes \overline{L}^2) = 0$. Therefore by the Riemann-Roch theorem

$$(\omega_{\overline{S}} + 3\overline{L}) \cdot \overline{L} = 0.$$

Since for $n > 0$ $h^0(\omega_{\overline{S}} \otimes \overline{L}^n) = \chi(\omega_{\overline{S}} \otimes \overline{L}^n) = p(n)$ is a not identically 0 degree 2 polynomial in n and since we have shown that $p(1) = p(2) = 0$, it follows that $h^0(\omega_{\overline{S}} \otimes \overline{L}^3) \neq 0$. By (0.3) $h^0(\omega_S \otimes L^3) \geq h^0(\omega_{\overline{S}} \otimes \overline{L}^3) \neq 0$. Since $(\omega_{\overline{S}} + 3\overline{L}) \cdot \overline{L} = (\omega_S + 3L) \cdot L$ that there is a section of $\omega_S \otimes L^3$ that is nowhere 0. Thus $\omega_S \approx L^{-3}$ and we can use lemma (0.7.1) to finish the proof that either $\omega_{\overline{S}} \otimes \overline{L}^3$ is numerically effective or $(S,L) \approx (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$.

If $\omega_S \otimes L^2$ is not numerically effective then there is an irreducible curve C on S such that $(\omega_S + 2L) \cdot C < 0$. From (0.2) we conclude that $(\omega_{\overline{S}} + 2\overline{L}) \cdot \overline{C} < 0$ and we can apply the first half of the proof.

Finally let me show that $h^0(\omega_S \otimes L^2) \neq 0$.

It is was, then by the Kodaira vanishing theorem (0.5) and the Riemann-Roch theorem (0.6.2):

$$0 = h^0(\omega_S \otimes L^2) = \chi(\omega_S \otimes L^2) = 2L \cdot L + \omega_S \cdot L + \chi(\mathcal{O}_S).$$

Since $\frac{L \cdot L}{2} + \frac{\omega_S \cdot L}{2} + \chi(\mathcal{O}_S) = h^0(\omega_S \otimes L) \geq 0$ we conclude that

$$\frac{L \cdot L}{2} + L \cdot (\omega_S + 2L)/2 \leq 0$$

which is absurd since L is ample and $\omega_S + 2L$ is numerically effective.

□

(1.2) Theorem. Let L be an ample line bundle on an irreducible normal Gorenstein surface S. Assume that there is an irreducible curve C on S such that $(\omega_S + 2L)\cdot C = 0$. Then either (S,L) is a quadric or geometrically ruled. In the latter case C is a smooth fibre of the map $p:S \to R$ giving the ruling and $\omega_S \otimes L^2 = p^*\mathcal{L}$ for some ample line bundle $\mathcal{L}$ on R.

Proof. $\pi^*(\omega_S + 2L)\cdot\overline{C} = 0$ and therefore by (0.2), $(\omega_{\overline{S}} + 2\overline{L})\cdot\overline{C} \le 0$. By (1.1) it follows that:

$$(\omega_{\overline{S}} + 2\overline{L})\cdot\overline{C} = 0.$$

Therefore since $\overline{L}\cdot\overline{C} = L\cdot C \ge 1$ it follows that either $\omega_{\overline{S}}\cdot\overline{C} \le -2$. Therefore either:

a) $\overline{C}\cdot\overline{C} > 0$

or

b) $\overline{C}\cdot\overline{C} = 0$, $\omega_{\overline{S}}\cdot\overline{C} = -2$, and $\overline{L}\cdot\overline{C} = 1$.

In case a) first note by (1.1) that $h^0(\omega_S \otimes L^2) > 0$. Since $\overline{C}\cdot\overline{C} > 0$ and $\pi^*(\omega_S + 2L)\cdot\overline{C} = 0$, it follows by the Hodge index theorem and the numerical effectivity of $\pi^*(\omega_S \otimes L^2)$ that the zeroes of the pullback under π of any not identically zero section of $\omega_S \otimes L^2$ must lie on the fiber of π. From this it follows that $\omega_S \approx L^{-2}$ and by (0.7.1), (S,L) is a quadric.

Case b) remains. Thus it can be assumed without loss of generality that $\omega_{\overline{S}}\cdot\overline{C} = -2$, $\overline{C}\cdot\overline{C} = 0$, $\overline{L}\cdot\overline{C} = 1$ and thus since $L\cdot C = \overline{L}\cdot\overline{C}$ it follows that $L\cdot C = 1$ and $\omega_S\cdot C = -2$.

From $\overline{C}\cdot\overline{C} = 0$ it follows that $\overline{C}$ is a smooth $\mathbb{P}^1$. By a standard argument it follows that there is a holomorphic map $\overline{p}:\overline{S} \to R$ with connected fibres that maps $\overline{S}$ onto a smooth curve R and such that $\overline{C}$ is a fibre of $\overline{p}$ in the neighborhood of which $\overline{p}$ is of maximal rank. Choose the fraction k such that:

*) $$(\pi^*(\omega_S + 2L) - k\overline{C})\cdot\overline{L} = 0.$$

Since $L\cdot C$ 0 either $\pi^*(\omega_S + 2L)\cdot\overline{L} = 0$ or $k > 0$. If $\pi^*(\omega_S + 2L)\cdot\overline{L} = 0$ then since $h^0(\omega_S \otimes L^2) \neq 0$ by lemma (1.1) it follows that $\omega_S \approx L^{-2}$. Therefore by (0.7.1), (S,L) is a quadric. Therefore it can be assumed that $k > 0$. In this case it follows from *) by the Hodge index theorem on $\overline{S}$ that

**) $$(\pi^*(\omega_S + 2L) - k\overline{C})\cdot(\pi^*(\omega_S + 2L) - k\overline{C}) \leq 0$$

with equality only if $\pi^*(\omega_S + 2L) - k\overline{C}$ is algebraically equivalet to 0. Using the numerical effectivity of $\omega_S + 2L$ we see that the left hand expression in **) is 0. Since $\overline{S}$ is birationally ruled by $\overline{p}$ it follows that there exist an $M > 0$ such that $M(\pi^*(\omega_S + 2L) - K\overline{C}) = p^*E$

for a line bundle E on R with $c_1(E) = 0$ in $H^2(R,\mathbb{Z})$. In particular there is some $N > 0$ such that $\pi^*(\omega_S \otimes L^2)^N = \overline{p}^*L$ where L is ample on R. Thus choosing an N' so that $L^{N'}$ is very ample it follows that the map $\phi: \overline{S} \to \mathbb{P}_{\mathbb{C}}$, associated to $\Gamma(\pi^*(\omega_S \otimes L^2)^{NN'}) \approx \Gamma(L^{N'})$ factors $\phi = r \circ \overline{p}$ where r, the map associated to $\Gamma(L^{N'})$ is an embedding. Thus $\overline{p}$ descends to a map $p: S \to R$ and $(\omega_S \otimes L^2)^N = p^*L$. From the fact that $\overline{p}$ was of maximal rank near $\overline{C}$ and that $\overline{p}$ descends to p it is easy to check that p must give a biholomorphism near $\overline{C}$. Thus p is a $\mathbb{P}^1$ bundle near C with C a smooth fibre. Since $L \cdot C = 1$ it follows that $L \cdot f = 1$ for any other fibre f of p. Thus f is irreducible since L is ample and since $N(\omega_S + 2L) \cdot f = p^*L \cdot f = 0$ the whole argument can be repeated to show that p is of maximal rank in a neighborhood of f. Knowing that p is a fibre bundle it follows that N can be chosen = 1. There the case when $\overline{C} \cdot \overline{C} = 0$ is done.

□

(1.2.1) Corollary. Let L be an ample line bundle on an irreducible normal Gorenstein surface S. Then either:

a) $\omega_S \otimes L^2$ is ample,

or

b) (S,L) is $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ or a quadric,

or

c) (S,L) is geometrically ruled and $\omega_S \otimes L^2 = p^*L$ for some ample line bundle L on R where $p: S \to R$ gives the ruling.

Proof. If it is assumed that (S,L) is not $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$, from (1.1) then it follows that $\omega_S \otimes L^2$ is numerically effective. If $\omega_S \otimes L^2$ is not ample, then it follows by the Nakai criterion that there is an irreducible curve $C \subseteq S$ with $(\omega_S + 2L)\cdot C = 0$. Use theorem (1.2). □

(1.2.2) Remark. In light of the above result it would be natural to call only those (S,L) that occur in c) above geometrically ruled. This would exclude only the quadric which is reasonable since for the smooth quadric (S,L), there is no unique ruling determined by L.

(1.3) Theorem. Let L be an ample line bundle on a normal irreducible Gorenstein surface S. The following are equivalent:

a) (S,L) is neither geometrically ruled nor a quadric nor equal to $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(e))$ for $e = 1$ or 2,

b) $\omega_S \otimes L$ is numerically effective,

c) $h^0((\omega_S \otimes L)^N) \neq 0$ for some $N > 0$.

Proof. First note that a) ⇒ b). This will follow from the following claim.

Claim. a) implies that $\omega_S^{t-1} \otimes L^t$ is ample for
$t \in \{2^j \mid j = 1,2,3 \ldots \}$.

<u>Proof of Claim.</u> I use induction on j. For $j = 1$ the fact that $\omega_S^{2^j-1} \otimes L^{2^j}$ is ample follows from (1.2.1). Assume now that there is a $K > 1$ such that $\omega^{2^{j-1}} \otimes L^{2^j}$ is ample for $j < K$. Thus with $j = K-1$:

$$\mathcal{L} = \omega_S{}^{2^{K-1}-1} \otimes L^{2^{K-1}} \qquad \text{is ample.}$$

By (1.2.1)

$$\omega_S \otimes \mathcal{L}^2 = \omega_S{}^{2^K-1} \otimes L^{2^K} \quad \text{is ample unless} \quad (S,\mathcal{L})$$

is geometrically ruled or a quadric or $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$. If $(S,\mathcal{L})$ is geometrically ruled then choosing a generic fibre f of the ruling it follows that $\omega_S \cdot f = -2$ and $\mathcal{L} \cdot f = 1$. This implies that:

$$-2(2^{K-1}-1) + 2^{K-1} = 1$$

which is absurd for $K > 1$. If $(S,\mathcal{L})$ is a quadric then $\omega_S \approx \mathcal{L}^{-2}$ or $\omega_S \approx L^{-2}$ which is ruled out by a) and (0.7.1). If $(S,\mathcal{L})$ is $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ then $L = \mathcal{O}_{\mathbb{P}^2}(m)$ and

$$-3(2^{K-1}-1) + 2^{K-1}m = 1$$

or

$$(m - 3)\ 2^{K-1} = -2\ .$$

The only solution for $K > 1$ is $m = 2,\ K = 2$ which is ruled out by a). By the claim a) implies that for any irreducible curve C on S

$$(2^j - 1)\omega_s \cdot C + 2^j L \cdot C > 0 \quad \text{for} \quad j = 1,2,3 \ldots$$

Dividing by 2^j and letting $j \to \infty$ gives

$$(\omega_s + L) \cdot C \quad 0 \quad \text{proving b).}$$

Next note that b) $\Rightarrow$ c). To see this assume that $h^0((\omega_s \otimes L)^n) = 0$ for all $n > 0$. By b) $\omega_s \otimes L$ is numerically effective and therefore $\omega_s^{n-1} \otimes L^n$ is ample. By the Kodaira vanishing theorem (0.9)

$$\chi((\omega_s \otimes L)^n) = 0 \quad \text{for all} \quad n > 0.$$

By the Riemann-Roch theorem (0.6.2)

$$\text{*)} \qquad 0 = (\omega_s + L) \cdot (\omega_s + L) = \omega_s(\omega_s + L) = \chi(\mathcal{O}_s)\ .$$

Hence $L \cdot (\omega_s + L) = 0$. By the Hodge index theorem (pass to $\bar{S}$ if you wish) and the numerical effectivity of $\omega_S \otimes L$ I conclude that $\omega_S + L$ is numerically equivalent to zero. Therefore ω_S^{-1} is ample and by the Kodaira vanishing theorem (0.5) $\chi(\mathcal{O}_S) = 1$ contradicting *) and proving c).

Finally an easy direct check shows that c) $\Rightarrow$ a.

□

(1.3.1) Corollary. *Let L be an ample line bundle on a normal irreducible Gorenstein projective surface* S. *If* $h^0((\omega_S \otimes L)^n) \neq 0$ *for some* $n > 0$ *then*

$$h^0((\omega_S \otimes L)^n) = \frac{(\omega_S+L)(\omega_S+L)}{2} n^2 - n(\omega_S+L)\omega_S + \chi(\mathcal{O}_S).$$

and

$$(\omega_S + L)\cdot(\omega_S + L) \geq 0.$$

Proof. Obvious.

□

The following is a useful alternate characterization of the numerical effectivity of $\omega_S \otimes L$.

(1.4) Theorem. *Assume that* L *is ample line bundle spanned by global sections on an irreducible normal Gorenstein projective surface* S. *Then* $\omega_S \otimes L$ *is numerically effective if and only if* $g(L) > h^1(\mathcal{O}_S)$ *where* $g(L)$ *denotes the genus of a smooth* $C \in |L|$.

<u>Proof.</u> If $\omega_S \otimes L$ is not numerically effective then using (1.3) it is an immediate check that $h^1(\mathcal{O}_S) = g(L)$.

Therefore it can be assumed that $h'(\mathcal{O}_S) = g(L)$. By (1.3) I must only deduce that (S,L) is $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$, $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ on a geometrically ruled surface, on a quadric. Choose a smooth $C \in |L|$. If $g(L) = 0$ the result is folklore [e.g. [So2], lemma (0.6.1)]. Therefore it can be assumed that $g(L) > 0$. Let $\pi:\bar{S} \to S$ be the minimal desingularization of S. Since C is smooth it doesn't meet the singular set of S and therefore $\pi^{-1}(C)$ is biholomorphic to C. Consider the diagram:

$$\begin{array}{ccccccccc} 0 & \to & \pi^*L^{-1} & \to & \mathcal{O}_{\bar{S}} & \to & \mathcal{O}_{\pi^{-1}C} & \to & 0 \\ & & & & \uparrow \pi^* & & \wr\wr & & \\ 0 & \to & L^{-1} & \to & \mathcal{O}_S & \to & \mathcal{O}_C & \to & 0 \end{array} .$$

Since $h^1(L^{-1}) = h^1(\pi^*L^{-1}) = 0$ by (0.5) it follows that:

$$\begin{array}{ccccc} 0 & \to & H^1(\mathcal{O}_{\bar{S}}) & \to & H^1(\mathcal{O}_{\pi^{-1}(C)}) \\ & & \uparrow \pi^* & & \wr\wr \\ 0 & \to & H^1(\mathcal{O}_S) & \to & H'(\mathcal{O}_C) \end{array} .$$

Since $h^1(\mathcal{O}_S) = h^1(\mathcal{O}_C)$ by hypothesis it follows that the restriction:

$$*)\qquad \begin{cases} H^1(\mathcal{O}_{\bar{S}}) \to H^1(\mathcal{O}_{\pi^{-1}(C)}) \\ \text{is an isomorphism.} \end{cases}$$

Consider the adjunction sequence:

$$0 \to \omega_S \to \omega_S \otimes L \to \omega_C \to 0.$$

By (0.5) $h^1(\omega_S \otimes L) = 0$. Therefore it follows from our hypotheses and Serre duality that the connecting homomorphism from $H^0(\omega_C)$ to $H^1(\omega_S)$ is an isomorphism and thus that the map $H^0(\omega_S \otimes L)$ to $H^0(\omega_C)$ is the zero map. This implies that $h^0(\omega_S) = 0$. If $h^0(\omega_S)$ was not zero, then choosing a smooth $C \in |L|$ not contained in the zero set of a not identically zero section of ω_S, it would follow that the map $H^0(\omega_S \otimes L) \to H^0(\omega_C)$ was not the zero map. Therefore I include by Serre duality

$$**) \qquad h^2(\mathcal{O}_S) = 0.$$

Using *), **), and the Leray spectral sequence for π and $\mathcal{O}_{\overline{S}}$ it follows that:

$$***) \qquad h^2(\mathcal{O}_{\overline{S}}) = 0 \quad \text{and} \quad \pi_{(1)}(\mathcal{O}_{\overline{S}}) = 0.$$

Since $h^1(\mathcal{O}_{\overline{S}}) = g(L)$ and $h^2(\mathcal{O}_{\overline{S}}) = 0$ it follows that the Albanese map gives a holomorphic mapping $\overline{r}:\overline{S} \to R$ of $\overline{S}$ into a genus $g(L)$ curve R. Since R is a curve and since $\overline{r}$ arises from the Albanese mapping it follows that $\overline{r}$ has connected fiber and R is smooth. Since by ***) $\pi_1(\mathcal{O}_{\overline{S}}) = 0$ the fibres of π over singularities are composed of rational curves. Therefore $\overline{r}$ descends to a holomorphic surjection:

$$r:S \to R.$$

The map

$$r_C:C \to R$$

is a biholomorphism. This is obvious from Hurwitz's formula if $g(L) > 1$ since r_C is onto C being ample must meet all fiber of r) and both curves have the same genus of $g(L) = 1$ I must rule out the case that r_C is a non-trivial unramified cover. To see this it sufficies to show that:

$$r_{C*}:H_1(C,\mathbb{Z}) \to H_1(R,\mathbb{Z})$$

is onto. Since r has connected fibres it follows that $r_*:H_1(S,\mathbb{Z}) \to H_1(R,\mathbb{Z})$ has connected fibres. Therefore I must only show that

$$H_1(C,\mathbb{Z}) \to H_1(S,\mathbb{Z}) \to 0.$$

By ***) $\pi_{(1)}(\mathcal{O}_{\bar{S}}) = 0$ and therefore this follows from (0.6.3).

Since $r_C:C \to R$ is a biholomorphism it follows that $L \cdot f = 1$ for a general fiber of r and that a general fibre of r is $\mathbb{P}^1$. From this it follows that $\omega_S \otimes L^2$ is trivial on a general fibre of r and cannot be ample in S. The theorem follows immediately from Corollary (1.2.1).

□

(1.4.1) <u>Remark.</u> The higher dimensional analogue of this follows from (1.4) by induction and the results of this paper exactly as for the smooth case in [So3]. I will go over this in the sequel of this paper on general Gorenstein varieties.

§ 2 Ampleness Properties of $\omega_S \otimes L$ for ample L

(2.0) Throughout this section S is a normal irreducible projective Gorenstein surface and L is an ample line bundle on S. Let ln(S,L) denote the logarithmic Kodaira dimension of the pair. This concept due to Iitaka (for $L = [D]$ with D 2 reduced effective divisor, is defined as follows. Let A denote the set of positive integer n such that $h^0((\omega_S \otimes L)^n) > 0$ and for each such n let $\phi_n : S \to \mathbb{P}_{\mathbb{C}}$ denote the meromorphic map associated to $\Gamma((\omega_S \otimes L)^n)$. If A is empty set $\ln(S,L) = -\infty$. Otherwise

$$\ln(S,L) = \max_{n \in A} \dim \phi_n(S).$$

The class of (S,L) with $\ln(S,L) = -\infty$ was classified in theorem (1.3).

(2.1) Main Theorem. Let L be an ample line bundle on a normal irreducible Gorenstein projective surface. Assume that $\ln(S,L) \neq -\infty$. Then there is an $N > 0$ such that $(\omega_S \otimes L)^N$ is spanned by global sections and such that the map $\phi : S \to \mathbb{P}_{\mathbb{C}}$ associated to $\Gamma((\omega_S \otimes L)^N)$ has connected fibrea and a normal image.

(2.1.0) If $\dim \phi(S) = 0$ then (S,L) is a Gorenstein Del Pezzo surface.

(2.1.1) If $\dim \phi(S) = 1$ then (S,L) is a conic bundle and $\phi : S \to \phi(S)$ is the projection in the definition of the conic bundle.

(2.1.2) If dim $\phi(S) = 2$ then the image F of the positive dimensional fibres of ϕ under ϕ are smooth points of $S'=\phi(S)$. Each positive dimensional fibre is an irreducible curve C biholomorphic to $\mathbb{P}^1$ such that $L\cdot C = 1$. $L_{S-\phi^{-1}(F)}$ extends to an ample line bundle L' on $\phi(S)$ such that $\phi^*(\omega_{S'} \otimes L') \approx \omega_S \otimes L$ and $\omega_{S'} \otimes L'$ is ample on S'.

Remarks. The proof will take the rest of this section. The proof that $\ln(S,L) \neq -\infty$ implies that $(\omega_S \otimes L)^N$ is spanned by global sections for some $N > 0$ will not be completely shown until the end of the proof. Note that once $(\omega_S \otimes L)^N$ is known to be spanned then a standard argument shows that for a possibly larger N, the map $\phi: S \to \mathbb{P}_{\mathbb{C}}$ associated to $\Gamma((\omega_S \otimes L)^N)$ has connected fibres and a normal image.

The conclusion of (2.1.2) implies that $\phi(S)$ is Gorenstein but is not claiming that each fibre C of ϕ is Cartier!

Proof. $\omega_S \otimes L$ can be assumed to be numerically effective by (1.3). It can be assumed without loss of generality that:

(2.2) $(\omega_S + L)\cdot L > 0.$

To see this note that if $(\omega_S + L)\cdot L = 0$ then $\pi^*(\omega_S + L)\cdot \overline{L} = 0$ and since $(\omega_S + L)\cdot(\omega_S + L) \geq 0$ it follows by the Hodge index theorem that $\omega_S \otimes L$ is numerically equivalent to 0. Since some power of $\omega_S \otimes L$ has a section it follows that $(\omega_S \otimes L)^N = \mathcal{O}_S$. (using (1.3.1) it follows that $N = 1$ and (S,L) is a Gorenstein

Del Pezzo surface, i.e. (S,L) is in (2.1.0) above.

Similarly it can be assumed that if C is an irreducible curve such that $(\omega_S + L)\cdot C = 0$ then:

(2.3) $\quad \overline{C}\cdot\overline{C} \le 0.$

If not then since $\pi^*(\omega_S + L)\cdot\overline{C} = 0$, the Hodge index theorem can be used again to deduce that $\omega_S \otimes L$ is numerically equivalent to zero. The same argument as in the last paragraph leads to (2.1.0).

Either $(\omega_S + L)\cdot(\omega_S + L) = 0$ or >0.
Assume first that

#) $\quad (\omega_S + L)\cdot(\omega_S + L) = 0.$

By (2.2) and (1.3.1) it follows that for some $N > 0$:

$$\dim |N\pi^*(\omega_S + L)| \ge 1.$$

Therefore there is an irreducible curve C on S such that $\overline{C}$ is part of the moving part of $|N\pi^*(\omega_S + L)|$ on $\overline{S}$. Therefore $\overline{C}\cdot\overline{C} \ge 0$. Combinded with (2.3) it follows that:

##) $\quad \overline{C}\cdot\overline{C} = 0.$

Since $\overline{C}$ is part of a curve $C \in |N\pi^*(\omega_S + L)|$ it follows that:

###) $\pi^*(\omega_S + L)\cdot\overline{C} = 0.$

Indeed $\pi^*(\omega_S + L)\cdot C = 0$ by #). By numerical effectivity of $\omega_S + L$ this implies that $\pi^*(\omega_S + L)\cdot C'$ for each irreducible component of C.

Consider that ###) implies $(\omega_{\overline{S}} + \overline{L})\cdot\overline{C} \leq 0$ by (0.2). Since $\overline{C}$ moves in $\overline{S}, \overline{L}\cdot\overline{C} \geq 1$. Therefore $\omega_{\overline{S}} \cdot \overline{C} \leq -1$. From ##) it follows that $\overline{C}$ is a smooth rational $\mathbb{P}^1$ and that there is a holomorphic surjection $\overline{p}:\overline{S} \to R$ onto a smooth curve R. Here $\overline{p}$ is of maximal rank in a neighborhood of $\overline{C}$. Consider the fraction λ such that:

###) $(\pi^*(\omega_S + L) - \lambda\overline{C})\cdot\overline{L} = 0$.

Since $\overline{C}$ moves on $\overline{S}, \overline{L}\cdot\overline{C} \geq 1$, and therefore by (2.2) it follows that $\lambda > 0$. Using ##) and ###) it follows that:

$$(\pi^*(\omega_S + L) - \lambda\overline{C})\cdot(\pi^*(\omega_S + L) - \lambda\overline{C}) \geq 0.$$

This and ####) imply by the Hodge index theorem that $\pi^*(\omega_S + L) - \lambda\overline{C}$ is numerically equivalent to 0. By the same argument as used in (1.2) it follows that $\overline{p}$ descends to a map $p:S \to R$ and $(\omega_S \otimes L)^n$ for some $n > 0$ is the pullback of the ample line bundle on R. From this it follows that (S,L) is a conic bundle.

It only remains to consider the case:

(2.4) $(\omega_S + L)\cdot(\omega_S + L) > 0.$

Let U denote the set of all irreducible $C \subseteq S$ such that $(\omega_S + L)\cdot C = 0$. If U is empty then by the Nakai criterion $\omega_S \otimes L$ is ample and I am in case (2.1.2). Therefore it can be assumed without loss of generality that U is non-empty. The structure of U is not hard to work out and the relevant structure was discussed by F. Sakai [Sa3]. I work out what I need below.

First note that $L\cdot C \geq 1$ and $\omega_S \cdot C \leq -1$. Since $(\omega_{\overline{S}} + \overline{L})\cdot\overline{C} \leq 0$ for $C \in U$ and since $\overline{L}\cdot\overline{C} \geq 1$ it follows that $\omega_{\overline{S}}\cdot\overline{C} \leq -1$. Since $\pi^*(\omega_S + L)\cdot\overline{C} = 0$ and $\pi^*(\omega_S + L)\cdot\pi^*(\omega_S + L) > 0$ it follows that $\overline{C}\cdot\overline{C} < 0$ by the Hodge index theorem. From this it follows that:

(2.5) $\overline{C}\cdot\overline{C} = -1 = \omega_{\overline{S}}\cdot\overline{C} = \omega_S\cdot C$ and $L\cdot C = \overline{L}\cdot\overline{C} = 1.$

In particular $\Delta\cdot\overline{C} = 0$ and so

(2.6) each $C \in U$ meet atworst rational singularities.

(2.7) <u>Lemma.</u> <u>The set U is finite. Given two district curves</u> C_1 <u>and</u> C_2 <u>belonging to U then</u> $\overline{C}_1\cdot\overline{C}_2 = 0.$

<u>Proof.</u> Since $\pi^*(\omega_S + L)\cdot\overline{C}$ for all $C \in U$ it follows from (2.4) and the Hodge index theorem that the intersection form of any finite subset of $\{\overline{C} \mid C \in U\}$ is negative definite. Thus

$$0 > (\overline{C}_1 + \overline{C}_2)\cdot(\overline{C}_1 + \overline{C}_2) = -1 + 2\ \overline{C}_1\cdot\overline{C}_2 - 1.$$

So $\overline{C_1}\cdot\overline{C_2} = 0$. It is well known that disjoint curves with negative self intersections give rise to linearly independent elements of $H^2(\overline{S},\mathbb{Q})$. Since $H^2(\overline{S},\mathbb{Q})$ is finite dimensional over $\mathbb{Q}$ it follows that $\{\overline{C} | C \in \mathcal{U}\}$ and hence $\mathcal{U}$ is finite.

□

(2.8) Lemma. Let R denote the set of rational singularities of S. Let $\mathcal{V}$ denote the set whose elements are the irreducible components of the fibres $\pi^{-1}(R)$ and all $\overline{C}$ where $C \in \mathcal{U}$. Then the set $\mathcal{V}$ is finite and has a negative definite intersection pairing. In particular any finite set of course of $\mathcal{V}$ can be contracted.

Proof. Using (2.7) it is clear that $\mathcal{V}$ is finite. By the Hodge index theorem (2.4) and $0 = \pi^*(\omega_S + L)\cdot E$ for all $E \in \mathcal{V}$ it follows that $\mathcal{V}$ has a negative definite intersection matrix. Therefore by Grauert's Contraction theorem [Gr] any finite set of curves of $\mathcal{V}$ can be contracted.

□

(2.9) Lemma. Let $C \in \mathcal{U}$. C meets at mostone rational singularity. Assume that C does meet a singularity $x \in S$ and let $\pi^{-1}(x)_{red} = \sum_{i=1}^{r} E_i$. Then after possibly renumbering the E_i it follows that:

a) $\overline{C}\cdot E_1 = 1$ and $C\cdot E_i = 0$ for $i > 1$,

b) $E_i\cdot E_j = 1$ for $|i-j| = 1$ and 0 otherwise

c) C <u>can be blown down to give a smooth point in a new surface S'.</u>

<u>Proof.</u> Assume first that C meets a rational singularity x of S and let $\sum_{i=1}^{r} E_i = \pi^{-1}(x)$. Since x is rational Gorenstein $E_i \cdot E_i = -2$. Since $\{\overline{C}, E_1, \ldots, E_r\}$ has a negative definite intersection matrix by (2.8) it follows that

$$0 > (\overline{C} + E_i)\cdot(\overline{C} + E_i) = -1 + 2\overline{C}\cdot E_i + 2$$

and therefore $\overline{C}\cdot E_i = 0$ or 1. Assume that two distinct curves E_i and E_j meet $\overline{C}$. Then

$$0 > (\overline{C} + E_i + E_j)\cdot(\overline{C} + E_i + E_j) =$$

$$\overline{C}\cdot\overline{C} + E_i\cdot E_i + E_j \,.\, E_j + 2\overline{C}\cdot E_i + 2\overline{C}\cdot E_j + 2E_i\cdot E_j$$

$$= -1 -1 -2 + 2 + 2 + 2\, E_i\cdot E_j$$

implies that $E_i \cdot E_j = 0$. Let S* denote the smooth surface obtained by blowing down $\overline{C}$. Let E_i' and E_j' denote the image of E_i and E_j. Note that by the above $E_i'\cdot E_j' = 1$ and $E_i'\cdot E_i' = -1 = E_j'\cdot E_j'$. Since $\overline{C} \cup E_i \cup E_j$ can be contracted it follows that $E_i' \cup E_j'$ can be contracted on S* and thus

$$0 \quad (E_i' + E_j')\cdot(E_i' + E_j') = -1 + 2E_i'\cdot E_j' + 1 = 0$$

which is absurd. Thus $\overline{C}\cdot E_i = 1$ for precisely one E_i and $= 0$ for all the other i. Similar reasoning shows b) above. If it is shown that C meets at most one singularity then c) will follow by systematically blowing down $\overline{C}$, then the image of E_1, then the image of E_2 etc.

Assume therefore that C met two singularities x and x' of S. Let $\sum_{i=1}^{r} E_i = \pi^{-1}(x)_{red}$ as above and let $\sum_{i=1}^{k} E_i' = \pi^{-1}(x')_{red}$ satisfy the properties a) and b) also. Then $\overline{C}\cdot E_1' = 1 = \overline{C}\cdot E_1$. Hence by (2.8)

$$0 > (C + E_1 + E_1')\cdot(C + E_1 + E_1') = -1-2-2+2+2+2E_1\cdot E_1'.$$

Thus $E_1\cdot E_1' \leq 0$ and an argument exactly as in the first half of the proof of this lines gives a contradiction.

□

(2.10) Lemma. No two elements of $\mathcal{U}$ meet.

Proof. By (2.7) it can be assumed that C,C' are distinct elements of $\mathcal{U}$, meeting a singular point x, and $\overline{C}\cdot\overline{C'} = 0$.

Let $\sum_{i=1}^{r} E_i = \pi^{-1}(x)_{red}$ be as in the last lemma ordered for C. Blowing down $\overline{C}$ it follows that $E_1'\cdot E_1' = -1$ where E_i' is the image of E_i in the surface S_0 obtained from $\overline{S}$ by blowing down $\overline{C}$. Arguing with $\overline{C}$ replaced by E_1' it follows that unless $r = 1, E_1'\cdot\overline{C_0'}=0$ where $\overline{C_0'}$ denotes the image of $\overline{C'}$ in S_0. Proceding until $\overline{C} \cup \pi^{-1}(x)$ is contracted on $\overline{S}$ to give a smooth S_r it will follow that $\overline{C_r'}\cdot\overline{C_r'} = 0$ where $\overline{C_r'}$, the image of $\overline{C'}$ on S_r is a smooth $\mathbb{P}^1$. Since $\overline{C} \cup \pi^{-1}(x) \cup \overline{C'}$ can be contracted so can $\overline{C_r'}$ and hence $\overline{C_r'} \cdot \overline{C_r'} < 0$ contradicting the last sentence.

□

After blowing down all the $C \in \mathcal{U}$ I get a new surface S' with $\pi': S \to S'$ mapping $\mathcal{U}$ onto smooth points. Since L is algebraic $L_{S-\mathcal{U}}$ extends over the smooth punctions on $S'-\pi'(\mathcal{U})$ to give a line bundle L' on S' that is ample by the Nakai criterion. Also $\pi'^*(\omega_{S'} \otimes L') = \omega_S \otimes L$ and therefore given any irreducible C on S', $(\omega_{S'} + L') \cdot C > 0$ since otherwise the proper transform of C under π' would $\in \mathcal{U}$. Therefore $\omega_{S'} \otimes L'$ is ample by the Nakai criterion and for some $N > 0$ $(\omega_{S'} \otimes L')^N$ is very ample and in particular $(\omega_S \otimes L)^N = \pi'^*(\omega_{S'} \otimes L')^N$ is spanned and gives the map ϕ of (2.1.2).

□

§3 Spannedness of $\omega_S \otimes L$

The first result is very general. It would be easy to give a direct and very short proof for surfaces using (0.1) but the following will be needed in the papers sequel.

(3.1) Theorem. Let L be a numerically effective line bundle on an irreducible normal projective Gorenstein n dimensional variety X. Assume that the set of non rational singularities of X is a finite set $F = \{x_1,\dots,x_r\}$ with distinct points. Assume that $c_1(L)^n > 0$. Then $\Gamma(\omega_S \otimes L)$ contains global sections $s_1,\dots,s_r$ such that $s_i(x_j)$ is non-zero if and only if $i = j$.

Proof. Let $\pi:\overline{X} \to X$ denote a projective desingularization of X. There is the sequence

$$0 \to \pi_*\omega_{\overline{X}} \to \omega_X \to \mathcal{S} \to 0 \ .$$

It is a well known theorem of the Kemph [Ke] that $\mathcal{S}$ is supported precisely on the set of non rational singularities. Thus in our case $\mathcal{S}$ is a skyscraper sheaf supported precisely on F, i.e. $\mathcal{S}_x$ is non trivial if and only if $x \in F$. Tensoring with L the result will follow from $H^1(\pi_*\omega_{\overline{X}} \otimes L) = 0$. Using the Leray spectral sequence for π and $\omega_{\overline{X}} \otimes \pi^*L$ and the Grauert-Riemenschneider vanishing theorem [Gra+Ri], it follows that it will suffice to show that $H^1(\omega_{\overline{X}} \otimes \pi^*L)$. This follows from the Kawamata-Viehweg vanishing theorem [Ka,Vi].

□

(3.1.1) Remark. The exact structure S is not known to me even if S is a skyscraper sheaf; more information could be quite usefule. The ideal sheaf $\pi_*\omega_{\overline{X}} \otimes \omega_X^{-1}$ is called the "conductor ideal" sheaf in the literature.

(3.2) Theorem. Let L be an ample line bundle on an irreducible normal projective Gorenstein surface S. Assume that $L \cdot L \geq 3$ and that L is spanned by global sections. Let $F = \{x_1, \ldots, x_r\}$ denote the possibly empty set of distinct non-rational singularities and let x_{r+1} denote a rational singulartiy. Then there exist global sections $s_1, \ldots, s_{r+1}$ of $\omega_S \otimes L$ such that $s_i(x_j) \neq 0$ if and only if $i = j$.

Proof. Let $\pi: S' \to S$ be a blowing of S such that:

a) π gives a biholomorphism between

$$S' - \pi^{-1}(F \cup \{x_{r+1}\}) \quad \text{and} \quad S - F \cup \{x_{r+1}\},$$

and,

b) π gives a minimal desingularization of $F \cup \{x_{r+1}\}$.

Note that by (0.1)

*) $\omega_{S'} \approx \pi^*\omega_S + \Delta$

where Δ has $\pi^{-1}(F)$ as its exact support. Let $Z = \pi^{-1}(x_{r+1})$ be the fundamental cycle of x_{r+1} ; this divisor was studied by [Ar]. Consider:

$$0 \to \pi^*(\omega_S \otimes L) \otimes [\Delta]^{-1} \otimes [Z]^{-1} \to \pi^*(\omega_S \otimes L) \to S \to 0$$

where $S \approx \mathcal{O}_Z \oplus \mathcal{O}_\Delta$. If I show that

$$H^1(\pi^*(\omega_S \otimes L) \otimes [\Delta]^{-1} \otimes [Z]^{-1}) = 0$$

then it is straightforward to check the assertion of the theorem. Noting by *) that:

$$H^1(\omega_{S'} \otimes \pi^*(L) \otimes [Z]^{-1}) = H^1(\pi^*(\omega_S \otimes L) \otimes [\Delta]^{-1} \otimes [Z]^{-1})$$

it sufficies by (0.5) to show that $\mathcal{L} = \pi^*(L) \otimes [Z]^{-1}$ is numerically effective and satisfies $\mathcal{L}\cdot\mathcal{L} > 0$. Since x_{r+1} is rational Gorenstein it is a rational double point, i.e. $Z\cdot Z = -2$. Thus:

$$**) \qquad \mathcal{L}\cdot\mathcal{L} = (\pi^*(L) - Z)\cdot(\pi^*(L) - Z) = L\cdot L - 2 > 0.$$

Therefore I must only check that $\mathcal{L}$ is numerically effective.

To see this assume otherwise and let C be an irreducible curve such that $\mathcal{L}\cdot C < 0$. If $\pi^*(L)\cdot C = 0$ then since $\Delta\cdot Z = 0$ C is a component of Z. But by a basic result of Artin [Ar] on the fundamental cycle Z of a rational singulartiy $-Z\cdot C \geq 0$.

This implies that $\pi^*(L)\cdot C > 0$. Since L is ample and spanned a divisor $D \in |L|$ can be chosen so that $x \in D$ and $\pi(C)$ is not a component of D. Therefore $\pi^{-1}(D) - Z \in |L|$ is effective and does't contain C. Thus $L\cdot C = (\pi^{-1}(D) - Z)\cdot C \geq 0$.

□

(3.2.1) Remark. If L were the tensor product of two ample spanned line bundles L_1 and L_2 and $L\cdot L \geq 5$ then the above argument would show that $\Gamma(\omega_S \otimes L)$ has enough sections to pairwise seperate singularities. Recall that $g(L)$ is defined by $2g(L) - 2 = L\cdot L + \omega_S\cdot L$.

(3.3) Theorem. Let L be a very ample line bundle on an irreducible normal projective surface S. Assume that S is regular, i.e. $h^1(\mathcal{O}_S) = 0$. Then if $\ln(S,L) \neq -\infty$ or equivalently $g(L) \neq 0$:

$$\omega_S \otimes L \text{ is spanned by global sections.}$$

Proof. I must in view of theorem (3.2) only investigate the spanning at the smooth points of S. Let $x \in S$ be smooth. Choose a general $C \in |L|$ that contains x. By Bertini's theorem C is smooth. By theorem (1.4) the condition $\ln(S,L) \neq -\infty$ is equivalent to $g(L) \neq 0 = h^1(\mathcal{O}_S)$. The latter implies that ω_C is smooth. Thus using $h^1(\mathcal{O}_S) = 0$ and Serre duality it follows from

$$0 \to \omega_S \to \omega_S \otimes L \to \omega_C \to 0$$

that $\Gamma(\omega_S \otimes L) \to \Gamma(\omega_C) \to 0$. Thus $\omega_S \otimes L$ is spanned at x.

□

(3.3.1) Remark. The fine structure of the adjunction map associated to $\Gamma(\omega_S \otimes L)$ can be studied as in [So1] in the case $h^1(\mathcal{O}_S) = 0$. This will be done in a sequel. The restriction on $h^1(\mathcal{O}_S)$ can be somewhat relaxed but there are severe technical problems in general. If S had no lines, i.e. no smooth C biholomorphic to $\mathbb{P}^1$ with $L \cdot C = 1$ then the methods of [So1] and [VdV] would show spanning of $\omega_S \otimes L$ in general. The trouble is when a line meets the singular set. I can show that if $\ln(S,L) \neq -\infty$ then $\omega_S \otimes L$ is spanned off a finite set. Hopefully these problems can be resolved.

The final result of this section will be needed to do an induction for n dimensional varieties in the sequel.

(3.4) Theorem. Let L_1 and L_2 be two ample line bundles on a irreducible normal projective Gorenstein surface S. Assume that L_1 and L_2 are spanned by global sections. Unless $S \approx \mathbb{P}^2$ and $L_1 \approx L_2 \approx \mathcal{O}_{\mathbb{P}^2}(1)$ it follows that $\omega_S \otimes L_1 \otimes L_2$ is spanned by sections.

Proof. Again it is only necessary to show spanning at smooth points x. Let $\pi: S' \to S$ denote S blown up at x and let $L = L_1 \otimes L_2$. Then considering

$$0 \to \pi^*(\omega_S \otimes L) \otimes [\pi^{-1}(x)]^{-1} \to \pi^*(\omega_S \otimes L) \to \mathcal{O}_{\pi^{-1}(x)} \to 0$$

it sufficies to show that:

$$H^1(\pi^*(\omega_S \otimes L) \otimes [\pi^{-1}(x)]^{-1}) = 0.$$

The latter group is isomorphic to

$$H^1(\omega_{S'} \otimes \pi^*(L) \otimes [\pi^{-1}(x)]^{-2}).$$

It sufficies by (0.5) to show that $\pi^*(L) \otimes [\pi^{-1}(x)]^{-2}$ is numerically effective and satisfies

$$(\pi^*(L) - 2\pi^{-1}(x)) \ (\pi^*(L) - 2\pi^{-1}(x)) > 0.$$

If the above inequality is not satisfied then $L_1 \cdot L_1 = L_1 \cdot L_2 = L_2 \cdot L_2 = 1$. It is easy to check that the map associated to $\Gamma(L_i)$ for either i most map S finite to one generically one to me onto $\mathbb{P}^2$. From this and Zariski's main theorem it would follow that $(S, L_i) \approx (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$. Therefore only the inequality:

$$(\pi^*(L) - 2\pi^{-1}(x) \cdot C \geq 0$$

for an arbitrary irreducible curve on S' must be checked. If $\pi^*(L) \cdot C = 0$ then $C = \pi^{-1}(x)$ and the inequality is obvious. If $\pi^*(L) \cdot C > 0$ then it is only necessary by the argumentat the end of theorem (2.2) to choose a $D \in |L|$ such that D is singular at x and $\pi(C)$ does not belong to D. Clearly $D_i \in |L_i|$ can be chosen so that D_i contains x and $\pi(C)$ does not belong to D_i. Let $D = D_1 + D_2$.

□

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