SPECIAL MANIFOLDS

(generalizations of Einstein metrics)

a chapter for the Arthur Besse book
on Einstein manifolds

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Contents

A . Introduction .

In this chapter we discuss some generalizations of Einstein metrics, that is, a few classes of Riemannian manifolds characterized by tensorial conditions, which are consequences of the Einstein metric equation. Among such generalizations, we restrict our consideration to those which have been studied in the differential geometric literature and can be illustrated by interesting examples.

Since the Einstein condition is an algebraic linear equation on r, it is to some extent natural that we first consider linear differential equations generalizing it. The simplest one is $Dr = 0$, providing a local characterization of products of Einstein manifolds. Its most immediate consequences consist in turn in imposing on Dr the natural linear conditions, which correspond to vanishing of certain irreducible components of Dr under the action of the orthogonal group. The bundle where Dr takes values splits into 3 irreducible invariant subbundles, giving rise, besides $Dr = 0$, to 6 conditions of this type, presented (in dimensions $n \geq 3$) by the following table :

condition	known examples of compact manifolds of this type, other than Einstein or locally product ones
$dr = 0$, i.e., r is a Codazzi tensor. Dr takes values in the invariant subbundle denoted by S. Equivalent conditions : 1) $\delta R = 0$ (harmonic curvature) 2) If $n \geq 4$: $\delta W = 0$ and constant scalar curvature	1) Compact conformally flat manifolds with constant scalar curvature 2) Compact quotients of $(\mathbb{R} \times \overline{M},\ dt^2 + f^{4/n}(t)\cdot\overline{g})$, where $(\overline{M},\overline{g})$ is Einstein with scalar curvature $\overline{u} > 0$ and f is a positive solution of $d^2f/dt^2 - \frac{1}{4}n(n-1)^{-1}\cdot\overline{u}f^{1-4/n} = cf$ with a constant $c < 0$
$2(n-1)(n+2)D[r - (2n-2)^{-1}ug] = (n-2)du \odot g$. Dr is a section of the invariant subbundle denoted by Q.	Compact quotients of $(\mathbb{R} \times \overline{M},\ dt^2 + f^{-2}(t)\cdot\overline{g})$, where $(\overline{M},\overline{g})$ is Einstein with scalar curvature $\overline{u} < 0$ and f is a positive solution of

(This implies $\delta W = 0$)	$d^2f/dt^2 - 2(n-1)^{-1}(n-2)^{-1}\cdot\bar{u}f^3 = cf$ with a constant $c > 0$
$(D_X r)(X,X) = 0$ for all vectors X. Dr is a section of the invariant subbundle denoted by A. (This implies constant scalar curvature)	1) Compact quotients of naturally reductive homogeneous Riemannian manifolds 2) Nilmanifolds covered by the generalized Heisenberg groups of A. Kaplan
$d[r - (2n-2)^{-1}ug] = 0$, i.e., $r - (2n-2)^{-1}ug$ is a Codazzi tensor. $Dr \in C^\infty(Q \oplus S)$. Equivalent condition (if $n \geq 4$): $\delta W = 0$ (harmonic Weyl tensor)	1) Compact conformally flat manifolds 2) Compact manifolds locally isometric to $(M_1 \times M_2 , f\cdot(g_1 \times g_2))$, where $\dim M_i = n_i \geq 1$, (M_i, g_i) has scalar curvature u_i and either a) (M_1, g_1) of constant curvature, $(M_2, g_2$ Einstein with $n_1(n_1-1)u_2 + n_2(n_2-1)u_1 =$ (e.g., $n_1 = 1$), and f is an arbitrary positive function on M_1 , or b) $n_1 = 2$, (M_2, g_2) two-dimensional or of constant curvature, $f = \lvert 2u_2 + n_2(n_2-1)u_1 \rvert^{2/(3-n)} > 0$
$\delta r = 0$ (constant scalar curvature) $Dr \in C^\infty(S \oplus A)$.	Every compact manifold admits such a metric. For details, see Chapter "F" .
$(D_X[r - 2(n+2)^{-1}ug])(X,X) = 0$ for each vector X . $Dr \in C^\infty(Q \oplus A)$	See the examples for $Dr \in C^\infty(Q)$ and $Dr \in C^\infty(A)$.

In Sections D through G we are concerned with four of these conditions, except for $Dr \in C^\infty(S \oplus A)$, dealt with by Chapter "F", and $Dr \in C^\infty(Q \oplus A)$, since very little is known about it.

Section C is devoted to Codazzi tensors, discussed separately for reasons explained in IV.6. Finally, in Section H we study oriented Riemannia four-manifolds satisfying $\delta W^+ = 0$, which is a natural linear condition on Dr relative to the special orthogonal group.

B . Natural linear conditions on Dr .

XVI. 1 . In order to discuss the irreducible components of Dr under the orthogonal group action, let us first recall that, for any Riemannian manifold (M,g) , Dr is a 3-tensor field having two additional algebraic properties, the one coming from the symmetry of r and the latter from the Bianchi identity $\delta r = -\frac{1}{2}\,du$ (see I.50). Thus, Dr is a section of the vector bundle $H = H(M,g) \subset T^*M \otimes S^2M \subset \otimes^3 T^*M$ the fibre of which, at any point $x \in M$, consists of all 3-linear maps ξ of T_xM into $\mathbb{R}$ such that $\xi(X,Y,Z) = \xi(X,Z,Y)$ and $\sum_{i=1}^{n}\left[\xi(X_i,X_i,X) - \frac{1}{2}\,\xi(X,X_i,X_i)\right] = 0$ for any $X,Y,Z \in T_xM$ and any orthonormal basis $X_1,\ldots,X_n$ of T_xM , $n = \dim M$.

A discussion of the irreducible components of Dr can also be found in A. Gray's article [Gr3].

XVI. 2 . Given a Riemannian manifold (M,g) , $\dim M = n \geq 3$, one has the following natural vector bundle homomorphisms associated with $\otimes^3 T^*M$: the contraction $\gamma : \otimes^3 T^*M \to T^*M$, the partial alternation $\alpha : \otimes^3 T^*M \to \Lambda^2 M \otimes T^*M$, the partial symmetrization $\sigma : \otimes^3 T^*M \to \otimes^3 T^*M$ and the mapping $\phi : T^*M \to H(M,g)$, given by

$$(\gamma(\xi))X = \sum_{i=1}^{n} \xi(X_i,X_i,X) ,$$

$$(\alpha(\xi))(X,Y,Z) = \tfrac{1}{2}\left[\xi(X,Y,Z) - \xi(Y,X,Z)\right] ,$$

$$(\sigma(\xi))(X,Y,Z) = \tfrac{1}{3}\left[\xi(X,Y,Z) + \xi(Y,Z,X) + \xi(Z,X,Y)\right] .$$

$$(\phi(\zeta))(X,Y,Z) = (X,Y)\cdot\zeta(Z) + (X,Z)\cdot\zeta(Y) + 2n(n-2)^{-1}(Y,Z)\cdot\zeta(X) ,$$

for $\xi \in \otimes^3 T_x^*M$, $X,Y,Z \in T_xM$, $\zeta \in T_x^*M$ and any orthonormal basis $X_1,\ldots,X_n$ of T_xM , $x \in M$. Since $(n-2)\gamma \circ \phi = (n-1)(n+2)\cdot \mathrm{Id}_{T^*M}$ and $(n-2)(\phi(\zeta),\xi) = (7n-6)(\gamma(\xi),\zeta)$ for any $\zeta \in T^*M$ and $\xi \in H(M,g)$, it follows that the n-dimensional invariant subbundle $Q = Q(M,g) = \mathrm{Im}\,\phi$ of $H = H(M,g)$ coincides with the orthogonal complement of $H \cap \mathrm{Ker}\,\gamma$ in H . The subbundles $S = S(M,g) = H \cap \mathrm{Ker}\,\alpha \subset \mathrm{Ker}\,\gamma$ and $A = A(M,g) = H \cap \mathrm{Ker}\,\sigma \subset \mathrm{Ker}\,\gamma$ of H are mutually orthogonal. It is now easy to verify that

$$H = Q \oplus S \oplus A$$

is an orthogonal decomposition of H into a direct sum of invariant (i.e., naturally defined) subbundles (explicitly, any $\xi \in H$ has the components $\xi_Q = (n-2)(n-1)^{-1}(n+2)^{-1}\phi(\gamma(\xi))$, $\xi_S = \sigma(\xi-\xi_Q)$). Using standard arguments of i theory one can prove (cf. [Gr3], I. 72) that this is the unique irreducit orthogonal decomposition of H . Moreover, the pairwise direct sums of the subbundles Q, S, A are easily seen to admit the following characterizations : for any $\xi \in H$,

(i) $\xi \in S \oplus A$ if and only if $\gamma(\xi) = 0$.

(ii) $\xi \in Q \oplus S$ if and only if $\alpha\left[\xi - (n-1)^{-1}\gamma(\xi) \otimes g\right] = 0$.

(iii) $\xi \in Q \oplus A$ if and only if $\sigma\left[\xi - 4(n+2)^{-1}\gamma(\xi) \otimes g\right] = 0$.

XVI. 3 . Let (M,g) be an n-dimensional Riemannian manifold, $n \geq 3$. For a tensor field $T \in C^\infty(\otimes^{k+1}T^*M)$, we define its divergence $\delta T \in C^\infty(\otimes^k T^*M)$

$$(\delta T)(X_1,\ldots,X_k) = -\operatorname{tr}_g\left[(Y,Z) \mapsto (D_Y T)(Z,X_1,\ldots,X_k)\right] .$$

Thus, the divergences δR and δW of R and W are sections of $T^*M \otimes \Lambda^2$ and, using an obvious switch of the arguments, we may view them as sections of $\Lambda^2 M \otimes T^*M$. Under this identification, the second Bianchi identity gives

$$\delta R = -\,dr \quad , \qquad \delta W = -\frac{n-3}{n-2}\, d\left[r - (2n-2)^{-1}ug\right] ,$$

where, for any symmetric 2-tensor field b , db denotes the exterior deriv tive of b (viewed as a T^*M-valued 1-form), i.e., $db = 2\alpha(Db)$. We shall say that (M,g) has harmonic curvature (resp., harmonic Weyl tensor) if $\delta R = 0$ (resp., if $\delta W = 0$). A symmetric 2-tensor field b on (M,g) will be called a Codazzi tensor if $db = 0$, i.e., if b satisfies the Codazzi equation $(D_X b)(Y,Z) = (D_Y b)(X,Z)$ for arbitrary tangent vectors X,Y,Z .

XVI. 4 . For a Riemannian manifold (M,g) , $\dim M = n \geq 3$, the natural linear conditions that can be imposed on Dr can be characterized, in view of XVI.2 and XVI.3 , as follows :

(i) $Dr \in C^\infty(Q)$ if and only if $D\left[r - (2n-2)^{-1}ug\right] = \frac{1}{2}(n-2)(n+2)^{-1}(n-1)^{-1}d$

(ii) $Dr \in C^\infty(S)$ is equivalent to each of the following conditions :

a) $\alpha(Dr) = 0$;

b) $dr = 0$, i.e., r is a Codazzi tensor ;

c) $\delta R = 0$, i.e., (M,g) has harmonic curvature ;

Y,Z , so that it is zero.

XVI. 9 . Theorem (M. Berger, cf. [BE], [Ry], [Si1], [W], [Bo2], [Gr3]). Every Codazzi tensor b with constant trace on a compact Riemannian manifold (M,g) with non-negative sectional curvature K is parallel. If, moreover, $K > 0$ at some point, then b is a constant multiple of g .

Proof. For any Codazzi tensor b , the Weitzenböck formula $I.$ can be rewritten as

$$\text{(XVI.9)} \qquad \delta Db + Dd(\mathrm{tr}_g b) = \overset{\circ}{R}(b) - b \circ r \ .$$

For any $x \in M$ and some orthonormal basis $X_1,\dots,X_n$ of T_xM $(n = \dim M)$, we have $b_x(X_i,X_j) = \lambda_i\delta_{ij}$ and, at x , $(b, \overset{\circ}{R}(b) - b \circ r) = -\sum_{i<j} R(X_i,X_j,X_i,X_j)(\lambda_i-\lambda_j)^2 \leq 0$. On the other hand, $\int_M (b,\delta Db)v_g = \int_M |Db|^2 v_g \geq 0$, so that our assertion follows from formula (XVI.9).

XVI. 10 . Let b be any symmetric 2-tensor field on a Riemannian manifold (M,g) . Given $x \in M$ and an eigenvalue λ of b_x , we shall denote by $V_\lambda(x) \subset T_xM$ the corresponding eigenspace. In every connected component of the open dense subset M_b of M , consisting of points at which the number of distinct eigenvalues of b is locally constant, the eigenvalues of b form mutually distinct smooth eigenvalue functions and, for such a function λ , the assignment $x \to V_{\lambda(x)}(x)$ defines a smooth eigenspace distribution V_λ of b . If λ and μ are such eigenvalue functions, then, for any vector fields X,Y,Z with $X \in C^\infty(V_\lambda)$, $Y \in C^\infty(V_\mu)$, the Leibniz rule yields

$$\text{(XVI.10)} \qquad (D_Zb)(X,Y) = (X,Y)\cdot Z\lambda + (\lambda-\mu)(D_ZX,Y) \ .$$

XVI. 11 . Proposition (cf. [De1], [HR]). Given a Codazzi tensor b on a Riemannian manifold (M,g) and an eigenvalue function λ of b , defined in a component of M_b , we have

(i) The eigenspace distribution V_λ is integrable.

(ii) Each integral manifold N of V_λ is umbilical in (M,g) . More precisely, for any eigenvalue function $\mu \neq \lambda$ and any sections X,Z of V_λ , Y of V_μ , the Y-component h^Y of the second fundamental form h of N is given by $h^Y(Z,X) = -(D_ZX,Y) = (\mu-\lambda)^{-1}(X,Z)\cdot Y\lambda$.

(iii) <u>If</u> $\dim V_\lambda > 1$, <u>then</u> λ <u>is constant along</u> V_λ .

<u>Proof</u>. For μ, X, Y, Z as in (ii), $(X,Y) = (Z,Y) = 0$ and hence (XVI.10) implies $(\lambda-\mu)([Z,X],Y) = (D_Z b)(X,Y) - (D_X b)(Z,Y)$ and $(\lambda-\mu)(D_Z X,Y) - (X,Z)\cdot Y\lambda = (D_Z b)(X,Y) - (D_Y b)(X,Z)$, so that (i) and (ii) follow from the Codazzi equation. If $\dim V_\lambda > 1$ and $X \in C^\infty(V_\lambda)$, we can find, locally, a non-zero $Y \in C^\infty(V_\lambda)$ with $(X,Y) = 0$. Applying (XVI.10) with $\lambda = \mu$, we obtain $|Y|^2 \cdot X\lambda = (D_X b)(Y,Y) = (D_Y b)(X,Y) = 0$, which completes the proof.

XVI. 12 . As an application of XVI.11, we shall now derive a <u>local classification of Codazzi tensors</u> b <u>having exactly two distinct eigenvalue functions</u> λ, μ (with $\dim V_\lambda \leq \dim V_\mu$). For simplicity, we assume in addition that $\dim V_\lambda > 1$ or $tr_g b$ is constant (cf. [De1] for the latter case) ; a similar argument works without this hypothesis. Let $\dim M = n \geq 3$.

By XVI.11, V_λ (resp., V_μ) is integrable and has umbilical leaves with mean curvature vector $H_\lambda = (\mu-\lambda)^{-1}(D\lambda)_\mu$ (resp., $H_\mu = (\lambda-\mu)^{-1}(D\mu)_\lambda$), the subscript convention being that $X = X_\lambda + X_\mu \in V_\lambda \oplus V_\mu = TM$.

(i) If $\dim V_\lambda > 1$ (more generally, if λ is constant along V_λ , cf. XVI.11.iii)) , we have $(D\lambda)_\lambda = (D\mu)_\mu = 0$ and so $H_\lambda = -(D\log|\lambda-\mu|)_\mu$, $H_\mu = -(D\log|\lambda-\mu|)_\lambda$. For the conformally related metric $\bar{g} = (\lambda-\mu)^2 g$, it is now easy to verify that V_λ and V_μ are totally geodesic in $(M,\bar{g})$, i.e. they are $\bar{g}$-parallel along each other and along themselves, and so the splitting $TM = V_\lambda \oplus V_\mu$ comes from a local Riemannian product decomposition of $(M,\bar{g})$. Therefore, we have locally

$$M = M_1 \times M_2 , \quad g = (\lambda-\mu)^{-2}(\bar{g}_1 \times \bar{g}_2), \quad b = (\lambda-\mu)^{-2}(\lambda\bar{g}_1 + \mu\bar{g}_2) ,$$

where $\lambda : M_2 \to \mathbb{R}$, $\mu : M_1 \to \mathbb{R}$ have disjoint ranges. Conversely, for Riemannian manifolds $(M_i, \bar{g}_i)$, i=1,2 , and functions λ, μ with these properties, the above formula defines a Riemannian manifold (M,g) with a Codazzi tensor b satisfying our conditions.

(ii) Let b have constant trace and assume that b is not parallel. By (i), $\dim V_\lambda = 1$ (since, in (i), λ and μ depend on separate variables). In view of XVI.11.iii),ii), μ and λ are constant along V_μ and the integral curve of V_λ are geodesics, i.e., for a fixed local unit section X of V_λ , $D_X X$

$= 0$. Each leaf of V_μ has mean curvature $\eta = (X,H_\mu) = (\lambda-\mu)^{-1}X\mu$, which is constant along the leaf. In fact, for any $Y \in C^\infty(V_\mu)$, $Y\eta = (\lambda-\mu)^{-1}YX\mu$ and $YX\mu = [Y,X]\mu$, while $([Y,X],X) = -(D_XY,X) = (Y,D_XX) = 0$, so that $[Y,X] \in C^\infty(V_\mu)$ and $YX\mu = 0 = Y\eta$. For $x \in M$ we can find (cf. XVI.11.i)) local coordinates $t, y_1, \ldots, y_{n-1}$ at x with $\partial_t = \partial/\partial t \in V_\lambda$, $\partial_i = \partial/\partial y_i \in$ $\in V_\mu$, $1 \le i < n$. Since $[\partial_i, \partial_t] = 0$, XVI.11.ii) yields $\partial_i|\partial_t|^2 =$ $= 2(\mu-\lambda)^{-1}|\partial_t|^2\partial_i\lambda = 0$, i.e., making a substitution in t, we may assume that $\partial_t = X$. Similarly, $\partial_t(\partial_i,\partial_j) = -2(D_{\partial_i}\partial_j,\partial_t) = 2\eta\cdot(\partial_i,\partial_j)$. Since $\partial_i\eta =$ $= 0$, we obtain $(\partial_i,\partial_j) = e^{2\Psi}\cdot\bar{g}_{ij}$ with $\partial_i\Psi = 0$ and $d\Psi/dt = \eta = (\lambda-\mu)^{-1}\partial_t\mu$, and $\partial_t\bar{g}_{ij} = 0$. This, together with the fact that $\mathrm{tr}_g b = C_o$ is constant, gives, locally,

$$M = I \times \bar{M}, \quad g = dt^2 + e^{2\Psi(t)}\cdot\bar{g}, \quad b = \lambda dt^2 + \mu e^{2\Psi(t)}\cdot\bar{g},$$
(XVI.13)
$$\lambda = C_o/n + (1-n)Ce^{-n\Psi(t)}, \quad \mu = C_o/n + Ce^{-n\Psi(t)},$$

where I is an interval, $(\bar{M},\bar{g})$ an $(n-1)$-dimensional Riemannian manifold, and C a real constant. Conversely, for any such data, and for an arbitrary function Ψ on I, (XVI.13) defines a Riemannian manifold (M,g) with a Codazzi tensor b of the type discussed above.

XVI.14. Theorem (A. Derdziński and C.-L. Shen, [DS]). _Let B be a Codazzi tensor of type (1,1) on a Riemannian manifold (M,g), x a point of M, λ and μ eigenvalues of B_x. Then the subspace $V_\lambda(x) \wedge V_\mu(x) \subset \Lambda^2 T_xM$, spanned by all exterior products of elements of $V_\lambda(x)$ and $V_\mu(x)$, is invariant under the curvature operator $R_x \in \mathrm{End}\,\Lambda^2T_xM$._

XVI.15. _Proof._ Adding a constant multiple of Id to B, we may assume that B is non-degenerate in a neighborhood M' of x. The automorphism B of TM' transforms g and D into the metric $G = B^*g$ and the connection $\bar\nabla =$ $= B^*D$ on M' (so that $G(X,Y) = g(BX,BY)$, $B(\bar\nabla_XY) = D_X(BY)$). Clearly, $\bar\nabla G = 0$ and the curvature tensor $\bar{R}$ of $\bar\nabla$ satisfies $\bar{R} = B^*R$, i.e., $G(\bar{R}(X,Y)Z,U) = g(R(X,Y)BZ,BU)$. As observed by N. Hicks [Hi], the Codazzi equation for B means that $\bar\nabla$ is torsion-free. Thus, the Riemannian connection D_G of G and its curvature 4-tensor R^G are given by $D_G = \bar\nabla$ and

$$R^G(X,Y,Z,U) = R(X,Y,BZ,BU). \tag{XVI.16}$$

Let $X \in V_\lambda(x)$, $Y \in V_\mu(x)$, $Z \in V_\nu(x)$, $U \in V_\xi(x)$. Using abbreviated notations like $R_{XYZU} = R(X,Y,Z,U)$, we have, by (XVI.16), $\nu\xi R_{XYZU} = R^G_{XYZU} = R^G_{ZUXY} = \lambda\mu R_{XYZU}$ and, similarly, $(\mu\xi-\lambda\nu)R_{XZUY} = (\mu\nu-\lambda\xi)R_{XUYZ} = 0$, while the Bianchi identity for R^G yields $0 = \nu\xi R_{XYZU} + \mu\xi R_{XZUY} + \mu\nu R_{XUYZ}$. Combining these equalities, we obtain the matrix equation

$$\begin{bmatrix} \lambda & \xi & \nu \\ \xi & \lambda & \mu \\ \nu & \mu & \lambda \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} R(X,Y,Z,U) \\ R(X,Z,U,Y) \\ R(X,U,Y,Z) \end{bmatrix} = 0 .$$

If $R_{XYZU} \neq 0$, the coefficient matrix satisfies the cofactor relations $(\lambda-\xi)(\lambda+\xi-\mu-\nu) = (\nu-\lambda)(\nu+\lambda-\mu-\xi) = (\lambda-\mu)(\lambda+\mu-\nu-\xi) = 0$, which easily imply that λ is equal to one of μ,ν,ξ. Therefore, evaluating R_x on four eigenvectors of B_x yields zero if more than two eigenspaces are involved. Hence $R_{XY}Z = 0$ if λ,μ,ν are mutually distinct. On the other hand, if $\lambda = \mu \neq \nu$, (XVI.16) gives $0 = B(R^G_{XY}Z + R^G_{YZ}X + R^G_{ZX}Y) = R_{XY}(BZ) + R_{YZ}(BX) + R_{ZX}(BY) = (\nu-\lambda)R_{XY}Z$, which completes the proof.

XVI. 17. Corollary (J. P. Bourguignon, [Bo3]). Let b be a Codazzi tensor on a Riemannian manifold (M,g). Then

(i) b commutes with r, $\overset{o}{R}(b)$ and with $Dd(\mathrm{tr}_g b) + \delta Db$.

(ii) The endomorphisms R, $g \owedge r$ and W of $\Lambda^2 TM$ commute with $g \owedge b$.

Proof. By XVI.14, $r(X,Y) = \overset{o}{R}(b)(X,Y) = 0$ for eigenvectors X,Y of b corresponding to distinct eigenvalues, so that r and $\overset{o}{R}(b)$ commute with b. Hence (i) follows from formula (XVI.9) (which also directly implies that r and b commute). Since $\Lambda^2 T_xM$ is spanned by the subspaces $V_\lambda(x) \wedge V_\mu(x)$ and $(g \owedge b)_x$ restricted to such a subspace is $\lambda+\mu$ times the identity, (ii) is immediate from XVI.14.

XVI. 18. An orthonormal basis $X_1,\dots,X_n$ of a Euclidean space E is said to diagonalize an algebraic curvature tensor $R \in \mathcal{C}(E)$ if all exterior products $X_i \wedge X_j$, $i < j$, are eigenvectors of R (viewed as an endomorphism of $\Lambda^2 E$), i.e., if $R(X_i,X_j)X_k = 0$ whenever i,j,k are mutually distinct. Following H. Maillot ([Ma1], [Ma2]) we shall call $R \in \mathcal{C}(E)$ pure if it is diagonalized by some orthonormal basis of E. A Riemannian manifold (M,g)

will be said to have pure curvature operator (resp., pure Weyl tensor) if for any $x \in M$, R_x (resp., W_x) is pure. If $n = \dim E = 2$, every orthonormal basis of E diagonalizes any $R \in \mathcal{C}(E)$, while, for $n \geq 3$, such a basis diagonalizes R if and only if it diagonalizes $W(R)$ and $c(R)$ (cf. I.74). Thus, each of the following conditions implies that the Riemannian manifold (M,g) has pure curvature operator :

(i) $\dim M = 2$, or $W = 0$ (e.g., $\dim M = 3$, cf. I.78).

(ii) (M,g) is a hypersurface in a space of constant curvature (cf. the Gauss equation).

(iii) (M,g) is a Riemannian product of manifolds with pure curvature operator.

Moreover, it is obvious that

(iv) The property of having pure Weyl tensor is conformally invariant.

XVI. 19 . If $R \in \mathcal{C}(E)$ has pure Weyl component $W = W(R)$ (e.g., if R is pure), then all the Pontryagin forms $P_i(R) \in \Lambda^{4i}E^*$, $i \geq 1$, are zero (see [Ma1]). In fact, it is easy to see (cf. [Gre]) that $P_i(R) = P_i(W)$ and the subalgebra $P(R) = P(W)$ of ΛE^* generated by the $P_i(W)$ has another system of generators $\Omega_i \in \Lambda^{4i}E^*$, Ω_i being obtained by alternating the map $(Y_1,\ldots,Y_{4i}) \to \operatorname{tr}\left[W(Y_1 \wedge Y_2) \circ \ldots \circ W(Y_{4i-1} \wedge Y_{4i})\right]$. However, if an orthonormal basis $X_1,\ldots,X_n$ diagonalizes W, then $W(X_i \wedge X_j) \circ W(X_k \wedge X_l) = 0$ for mutually distinct i,j,k,l, which implies $P(W) = \Lambda^0 E^* = \mathbb{R}$.

XVI. 20 . Lemma . Let E be an oriented 4-dimensional Euclidean space and let $W \in \mathcal{W}(E)$ be pure. Then W^+ and W^-, viewed as endomorphisms of Λ^+E and Λ^-E, respectively, have equal spectra, i.e., $|W^+| = |W^-|$ and $\det W^+ = \det W^-$.

Proof. Since W is pure, we have $W(X_i \wedge X_j) = \lambda_{ij} X_i \wedge X_j$ for some orthonormal basis $X_1,\ldots,X_4$ of E and real numbers λ_{ij}. If $\{i,j,k,l\} = \{1,2,3,4\}$, then $*(X_i \wedge X_j) = \pm X_k \wedge X_l$, and so $\lambda_{ij} = \lambda_{kl}$, since W and $*$ commute by I.89. Elements of the form $X_i \wedge X_j \pm X_k \wedge X_l$ give now rise to bases of Λ^+E and Λ^-E, realizing equal spectra for W^+ and W^-.

XVI. 21 . Theorem (cf. [DS], [Bo3]). Let b be a Codazzi tensor on a Riemannian manifold (M,g) , $\dim M = n$.

(i) If b has n distinct eigenvalues at all points of a dense subset o then, for any $x \in M$, R_x is diagonalized by some orthonormal basis of diagonalizing b_x . The Pontryagin forms of (M,g) and the real Pontryagi classes of M are all zero.

(ii) If $n = 4$ and $b - (\mathrm{tr}_g b)g/4 \neq 0$ in a dense subset of M , then, a each $x \in M$, W_x is diagonalized by some orthonormal basis of T_xM , dia nalizing b_x . The Pontryagin form P_1 of (M,g) vanishes identically an $p_1(M,\mathbb{R}) = 0$. At each $x \in M$, W_x^+ and W_x^- have equal spectra.

XVI. 22 . Proof. (i) is obvious from XVI.14 and XVI.19 together with a co nuity argument. By XVI.19 and XVI.20, (ii) will follow if we show that W_x is diagonalized by an orthonormal eigenbasis of b_x , for any $x \in M_b$. In a neighbourhood M' of x , b has m distinct eigenvalue functions $\lambda($ $\ldots,\lambda(m)$, $2 \leq m \leq 4$, their multiplicities being $k_1,\ldots,k_m$ with $k_1 \leq$ $\ldots \leq k_m$. Four cases are possible : I. $m = 2$, $k_1 = k_2 = 2$; II. $m = 2$, $k_1 = 1$, $k_2 = 3$; III. $m = 3$, $k_1 = k_2 = 1$, $k_3 = 2$; IV. $m = 4$. In case XVI.12.i) implies that near x , g is conformal to a product of surface metrics, compatibly with the b-eigenspace decomposition of TM', and our sertion on W_x follows from XVI.18.i),iii),iv). Assume now case II (resp. case III). Since $V_{\lambda(1)} \wedge V_{\lambda(2)}$ (resp., $V_{\lambda(1)} \wedge V_{\lambda(2)}$ and $V_{\lambda(1)} \wedge V_{\lambda(3}$ are invariant, by XVI.14 and XVI.17.i), under the self-adjoint endomorphi W of $\Lambda^2 TM'$, we can choose an orthonormal b_x-eigenvector basis $X_1,\ldots,X$ of T_xM with $W(X_1 \wedge X_i) = \mu_i X_1 \wedge X_i$ for some μ_i , $2 \leq i \leq 4$. The fac that W commutes with $*$ for any orientation (cf. I.89) implies that all $X_i \wedge X_j$ are eigenvectors of W_x , as required. Finally, in case IV our assertion is immediate from (i) and XVI.18, which completes the proof.

D . The case $Dr \in C^\infty(Q \oplus S)$: Riemannian manifolds with harmonic Weyl tensor .

XVI. 23 . The n-dimensional Riemannian manifolds (M,g) for which Dr is a section of $Q \oplus S$, i.e., $r - (2n-2)^{-1}ug$ is a Codazzi tensor, can also be characterized by the condition $\delta W = 0$ (when $n \geq 4$), or by conformal flatness (when $n = 3$; see XVI.4.v)). In this section, we discuss these manifolds, always assuming that $n \geq 4$. They are then said to have harmonic Weyl tensor. This terminology is justified by the fact that $\delta W = 0$ implies the "Bianchi identity" $dW = 0$, so that W , viewed as a Λ^2M-valued 2-form, is both closed and co-closed (see XVI.42 for details).

After describing various examples of manifolds with harmonic Weyl tensor, we give some general theorems on the structure of their curvature operator (XVI. 28, XVI. 31) and a local classification result (XVI .32).

XVI. 24 . The simplest examples of manifolds with harmonic Weyl tensor :

(i) Manifolds with $Dr = 0$, locally isometric to products of Einstein manifolds.

(ii) Conformally flat manifolds $(W = 0)$. This class contains many <u>compact</u> examples, apart from spaces of constant curvature : the conformal inversion map $x \to |x|^{-2}x$ in the model space $\mathbb{R}^n$ immediately gives rise to a connected sum operation for such manifolds ([Ku]). Note that a Riemannian product is conformally flat if and only if both factors have constant sectional curvatures and either one of them is 1-dimensional, or the sum of their curvatures is zero.

(iii) The condition $DW = 0$ gives no new examples of manifolds with $\delta W = 0$: by a result of W. Roter (cf. [DR]), it implies that $DR = 0$ or $W = 0$ (see also [Mi] and, for $n = 4$, XVI.75.iii)). Moreover, the class of manifolds with $\delta W = 0$ is not closed under taking Riemannian products, unless both factors have constant scalar curvatures.

XVI. 25 . Other examples of manifolds with $\delta W = 0$ can be obtained by conformal deformations. Under a conformal change $g' = e^{2f}g$ of metric in dimension n, δW transforms like

(XVI.25) $$\delta_{g'}W_{g'} = \delta W - (n-3)W(Df,\cdot,\cdot,\cdot)$$

(cf. I. a), c)). Thus, we can proceed by taking a Riemannian manifold with $Dr = 0$ and finding on it a function f with $W(Df,\cdot,\cdot,\cdot) = 0$. An easy computation gives the following

Lemma. *Let (M,g) be a Riemannian product of two Einstein manifolds (M_i,g_i). For a non-zero vector X tangent to M we have $W(X,\cdot,\cdot,\cdot) = 0$ if and only if $n_2(n_2-1)u_1 + n_1(n_1-1)u_2 = 0$ and $W_i(X_i,\cdot,\cdot,\cdot) = 0$, $i = 1,2$, where $n_i = \dim M_i \geq 1$, u_i is the scalar curvature of g_i, X_i denotes the M_i-component of X and W_i is the Weyl tensor of g_i (defined to be zero if $n_i \leq 3$).*

XVI.26. The following constructions of examples are immediate from XVI.25:

(i) For an Einstein manifold $(\overline{M},\overline{g})$, a 1-dimensional manifold (I,dt^2) and a positive function F on I, the metric $g = dt^2 + F^2(t)\cdot\overline{g} = e^{2\log F(t)}\left[F^{-2}(t)dt^2 \times \overline{g}\right]$ on the product manifold $M = I \times \overline{M}$ satisfies $\delta W = 0$. Suppose now that $\overline{M}$ is compact, $I = \mathbb{R}$, F is periodic, with period T, and let Φ be an isometry of $(\overline{M},\overline{g})$. Clearly, the mapping $(t,y) \to (t + T, \Phi(y))$ of $\mathbb{R} \times \overline{M}$ is an isometry of (M,g), generating a properly discontinuous $\mathbb{Z}$ action on M. The quotient manifold $(M,g)/\mathbb{Z}$ (cf. [KN], p. 44 and § 10), with the "*twisted*" *warped product* metric determined by g, is then an example of a *compact Riemannian manifold with harmonic Weyl tensor*, diffeomorphic to a bundle with fibre $\overline{M}$ over the circle. In general, it is neither conformally flat (unless $\overline{g}$ is of constant curvature, cf. XVI.24.ii)), nor does it have parallel Ricci tensor (non-constant functions F for which this happens cannot be periodic and positive everywhere on $\mathbb{R}$).

(ii) Let (M_1,g_1) be a space of constant sectional curvature K_1, (M_2,g_2) an Einstein manifold with scalar curvature $u_2 = -n_2(n_2-1)K_1$, where $n_2 = \dim M_2$. For an *arbitrary* positive function F on M_1, *the warped product metric* $g = F^2g_1 + F^2g_2 = e^{2\log F}(g_1 \times g_2)$ *on* $M_1 \times M_2$

has harmonic Weyl tensor. As in (i), g is, in general, not of type XVI.24.i) or ii). Choosing (M_1,g_1) to be a sphere and (M_2,g_2) a simply connected compact complex manifold having $c_1 < 0$, endowed with a Kähler-Einstein metric (cf. "KE"7, "KE"18), we obtain here examples of compact simply connected manifolds satisfying $\delta W = 0$ and neither $W = 0$, nor $Dr = 0$.

XVI. 27 . Further examples obtained by conformal deformations. Let a Riemannian manifold (M,g) have recurrent conformal curvature ([AM]) in the sense that $2|W|^2 \cdot DW = d(|W|^2) \otimes W$. By XVI.25, the metric $g' = |W|^{2/(3-n)} \cdot g$, defined wherever $W \neq 0$ $(n = \dim M \geq 4)$, satisfies $\delta_{g'} W_{g'} = 0$. As shown by W. Roter ([Ro]), locally, in dimensions $n \geq 5$, the only analytic manifolds with recurrent conformal curvature are those with $W = 0$, or $DR = 0$, or products of surfaces with spaces of constant curvature. In dimension 4 there are more such examples, e.g., all Riemannian products of surfaces. Thus, for $(M,g) = (M_1,g_1) \times (M_2,g_2)$, where $\dim M_1 = 2$ and g_2 is of constant curvature or $\dim M_2 = 2$, the metric g' defined above has harmonic Weyl tensor. Since, for such g, $|W|$ is proportional to $|K_1 + K_2|$, where K_i is the Gaussian (resp., constant sectional) curvature of g_i, this construction gives many examples of compact manifolds with harmonic Weyl tensor, including simply connected ones. In fact, if M_1 and M_2 are compact and $K_2 \neq 0$ everywhere, we may rescale g_1 to obtain $|W_{g_1 \times g_2}| > 0$, so that g' is defined everywhere on M.

XVI. 28 . As an immediate consequence of XVI.4.v), XVI.21 and XVI.18, we obtain the following

Theorem . Let (M,g) be a Riemannian manifold with harmonic Weyl tensor, $\dim M = n \geq 4$.

(i) If r has n distinct eigenvalues at all points of a dense subset of M, then (M,g) has pure curvature operator and $p_i(M,\mathcal{R}) = 0$ for $i \geq 1$.

(ii) If $n = 4$ and $r - ug/4 \neq 0$ in a dense subset of M, then (M,g) has pure curvature operator, $p_1(M,\mathcal{R}) = 0$ and, at each point $x \in M$,

W_x^+ and W_x^- have equal spectra.

XVI. 29 . Corollary (cf. [DS], [De3], [Bo3]). Let an oriented Riemannian four-manifold (M,g) satisfy $W^- = 0$ and have harmonic Weyl tensor. Then $W \otimes (r - ug/4) = 0$ everywhere in M, so that, if (M,g) is analytic, it must be conformally flat or Einstein.

In fact, by XVI.28.ii), $|W^+| = |W^-| = 0$ wherever $r - ug/4 \neq 0$.

XVI. 30 . Proposition (cf. [Mat], [Ta], [Gr3]). Let a Kähler manifold (M,J,g) of real dimension $n \geq 4$ have harmonic Weyl tensor. Then its Ricci tensor is parallel.

In fact, since r commutes with J (cf. "K." 44.), our assertion is immediate from XVI.4.v) and XVI.8. Note that in this case (M,J,g) is, locally, a product of Kähler-Einstein manifolds. Using XVI.24.ii), one can now easily conclude that a conformally flat Kähler manifold which is not flat must be 4-dimensional and locally isometric to a product of surfaces with mutually opposite constant curvatures (cf. [YM] and I.)).

XVI. 31 . Theorem (D. DeTurck and H. Goldschmidt, [DTG]). Let (M,g) be a Riemannian manifold with harmonic Weyl tensor, $\dim M = n \geq 4$. Suppose that, at some point $x \in M$, $W_x = 0$ and r_x has n distinct eigenvalues. Then $W = 0$ in a neighborhood of x. Consequently, if (M,g) is analytic, it must be conformally flat.

Proof. In a neighborhood of x, we can find an orthonormal frame field $X_1,\ldots,X_n$ diagonalizing r. By XVI.4.v), XVI.14 and XVI.17.i), it also diagonalizes W, so that the essential components of W are $w_{ij} = w_{ji} = W(X_i,X_j,X_i,X_j)$, $i \neq j$. Condition $\delta W = 0$ and its consequence $dW = 0$ (cf. XVI.23, XVI.42) now easily give

$$D_k w_{ik} = \sum_{j \neq i,k} (w_{ik} - w_{ij})\Gamma^k_{jj} \quad , \quad i \neq k ,$$

$$D_k w_{ij} = (w_{ij} - w_{kj})\Gamma^k_{ii} + (w_{ij} - w_{ki})\Gamma^k_{jj} \quad , \quad i \neq j \neq k \neq i ,$$

where D_i is the directional derivative along X_i and $\Gamma^k_{ij} = (D_{X_i}X_j, X_k)$. Thus, the w_{ij} satisfy a first order linear system of equations solved for the derivatives. Since $w_{ij}(x) = 0$ by hypothesis, our assertion follows.

XVI. 32 . In dimension four, we can state the following local classification result. For the proof, see XVI.70.

Proposition . *Let* (M,g) *be an oriented Riemannian four-manifold with harmonic Weyl tensor. Suppose that* W^{+} *, viewed as an endomorphism of* $\Lambda^{+}M$ *, has less than three distinct eigenvalues at any point. In a neighborhood of each point* $x \in M$ *at which* $W \neq 0$ *and* $r \neq ug/4$ *,* g *is obtained by a conformal deformation of a product of surface metrics as in* XVI.27. *In terms of* g *, that product metric equals* $|W|^{2/3} \cdot g$.

E . Condition $Dr \in C^\infty(S)$: Riemannian manifolds with harmonic curvature .

XVI. 33 . The study of Riemannian manifolds for which $Dr \in C^\infty(S)$ (i.e., r is a Codazzi tensor, which is also equivalent to $\delta R = 0$, cf. XVI.4.ii)) is additionally motivated by their relationship with the Yang-Mills connections (see XVI.35). In view of the second Bianchi identity $dR = 0$ (formula I.19.b)), it is natural to call them manifolds with harmonic curvature.

XVI.34. By a recent result of D. DeTurck and H. Goldschmidt [DTG], all manifolds with $\delta R = 0$ are analytic in suitable local coordinates. On the other hand, in dimensions $n \geq 4$, these manifolds are characterized by having harmonic Weyl tensor and constant scalar curvature. Thus, all arguments of Section D remain valid for manifolds with harmonic curvature. Therefore we have, for such manifolds, Bourguignon's theorems XVI.28.ii) and XVI.29 ([Bo3]), the Matsushima-Tanno theorem XVI.30, theorem XVI.31 due to DeTurck and Goldschmidt , as well as XVI.28.i) and XVI.32, with the obvious simplifications : in XVI.28 we may replace the "dense subset" by "a point", while, in XVI.29 and XVI.31, the analyticity hypotheses become superfluous.

In this section we describe the known examples of compact Riemannian manifolds with harmonic curvature and give two further results (XVI.37, XVI.39) on such manifolds, including a classification theorem. Finally, we state a few open problems and discuss the harmonicity of arbitrary algebraic curvature tensor fields.

XVI. 35 . Condition $\delta R = 0$ also appears in the following context. Given a vector bundle E over a compact Riemannian manifold (M,g) and a fibre metric g_E in E , one assigns to any connection ∇ in E , compatible with g_E , the Yang-Mills integral $YM(\nabla) = \int_M |R^\nabla|^2 \cdot v_g$, R^∇ being the curvature of ∇ . A connection ∇ is a critical point of this Yang-Mills functional if and only if $\delta^\nabla R^\nabla = 0$ (cf. [Pa] and "F"). For $E = TM$

and $g_E = g$, this is just condition $\delta R = 0$.

XVI. 36 . Examples . According to XVI.34, many (usually non-compact) manifolds with harmonic curvature can be constructed by imposing on the examples described in XVI.24, XVI.26 and XVI.27 the additional condition that their scalar curvature be constant. This gives the following compact manifolds with harmonic curvature :

(i) Compact manifolds with $Dr = 0$.

(ii) Compact conformally flat manifolds with constant scalar curvature. If the Yamabe conjecture is true in the conformally flat case (cf. "F"), it will, thus, yield a metric with $\delta R = 0$ on every compact conformally flat manifold. As observed by J. Lafontaine [Laf], the fact that every conformal class contains a metric whose scalar curvature is either non-positive and constant, or positive everywhere (see [Au]) immediately implies the existence of metrics with harmonic curvature on all compact conformally flat manifolds admitting no metric with $W = 0$ and $u > 0$ (e.g., on connected sums of conformally flat manifolds with tori, cf. XVI.24.ii) and "T").

(iii) Compact manifolds locally isometric to products of manifolds with $\delta R = 0$.

(iv) Given a compact Einstein manifold $(\bar{M},\bar{g})$ with scalar curvature $\bar{u} > 0$, $\dim \bar{M} = n - 1 \geq 2$, a non-constant positive periodic function F on $\mathbb{R}$ and an isometry Φ of $(\bar{M},\bar{g})$, it is easy to verify that the manifold $(M,g)/\mathbb{Z}$, defined as in XVI.26.i), has constant scalar curvature if and only if we have, for $f = F^{n/2}$ and a constant C_o ,

$$\text{(XVI.36.iv)} \qquad d^2f/dt^2 - \tfrac{1}{4}\, n(n-1)^{-1}\cdot\bar{u}\, f^{1-4/n} = \tfrac{n}{4}\, C_o f \ .$$

Choosing f to be a non-constant positive periodic solution of this equation (for its existence, see XVI.38), we obtain from our construction examples of compact Riemannian manifolds with harmonic curvature, namely, bundles with fibre $\bar{M}$ over S^1 endowed with twisted warped product metrics (cf. [De2]). These manifolds never satisfy $Dr = 0$ and are not conformally flat unless $\bar{g}$ has constant sectional curvature (cf. XVI.26.i)).

XVI . 37 . Theorem (M. Berger, cf. [BE], [Si1], [Bo2], [Gr3]). Every compact Riemannian manifold with harmonic curvature and non-negative sectional curvature K satisfies $Dr = 0$. If, moreover, $K > 0$ at some point, then the manifold is Einstein.

In fact, this is immediate from XVI.4.ii) and XVI.9.

XVI . 38 . Lemma . Given a C^2 function v on a closed interval I and f_1 , $f_2 \in I$, the following conditions are equivalent :

(i) The equation

(XVI.38.i) $\qquad d^2f/dt^2 = \frac{1}{2} v'(f)$

has a non-constant periodic C^2 solution f on $\mathbb{R}$ with range $[f_1, f_2]$.

(ii) $f_1 < f_2$, $v(f_1) = v(f_2)$, $v'(f_1) \cdot v'(f_2) \neq 0$, and $v(f_0) > v(f_1)$ for all $f_0 \in (f_1, f_2)$.

Proof. For non-constant f , equation (XVI.38.i) is equivalent to $(df/dt)^2 = v(f) - C$ with $C \in \mathbb{R}$. Given such an f , weakly monotone in an interval $[t_0, t]$, we have $v(f_0) \geq C$ for all f_0 in the closed interval I_0 joining $f(t_0)$ and $f(t)$, and

$$t - t_0 = \varepsilon \int_{f(t_0)}^{f(t)} (v(f) - C)^{-1/2} df \, , \quad \varepsilon = \pm 1 \, .$$

Since v is C^2 , finiteness of this integral gives $v'(f_0) \neq 0$ whenever $f_0 \in I_0$ and $v(f_0) = C$; thus, $v \geq C$ in I_0 implies $v > C$ in Int I_0 . Moreover, f is a Morse function (otherwise, by the uniqueness of solutions of (XVI.38.i), it would be constant) and, since $|df/dt|$ is a function of f , all critical points of f are absolute maxima or minima. Therefore (i) implies (ii). Conversely, assuming (ii), we can define the assignment $t \to f(t)$ implicitly by the above integral formula with $\varepsilon = 1$, $t_0 = 0$, $f(t_0) = f_1$ and for $t \in [0, T/2]$ with T such that $f(T/2) = f_2$. The extension of f to $\mathbb{R}$, characterized by $f(T+t) = f(T-t) = f(t)$, is C^1 and satisfies (XVI.38.i) outside a discrete set, so that it is C^2 . This gives (i), and completes the proof.

XVI. 39 . Theorem (A. Derdziński, [De2]) . Let (M,g) be a compact Riemannian manifold with harmonic curvature, $\dim M = n \geq 3$. If its Ricci tensor is not parallel and has, at each point, less than three distinct eigenvalues, then (M,g) is isometrically covered by one of the compact manifolds constructed in XVI.36.iv). Conversely, each of those manifolds has the stated properties.

XVI. 40 . Proof . Fix $x \in M$ with $r_x \neq u(x)g_x/n$ and $(Dr)_x \neq 0$. By XVI.4.ii) and XVI.12.ii), near x , g is given by (XVI.13) with some $(n-1)$-dimensional Riemannian manifold $(\bar{M},\bar{g})$, a function Ψ on an interval I and with $b = r$. Computing, in (XVI.13), r from g and comparing it with b , we see that $(\bar{M},\bar{g})$ is Einstein and $f = e^{n\Psi/2}$ satisfies equation (XVI.36.iv), where $\bar{u}$ is the scalar curvature of $\bar{g}$. The elementary symmetric functions of the eigenvalues of r are analytic on M (cf. XVI.34) and, by (XVI.13), one of them is non-constant, since $(Dr)_x \neq$ $\neq 0$. A suitable regular level of such a function gives an extension of $(\bar{M},\bar{g})$ to a compact Einstein manifold, whose universal covering space we denote by $(\hat{M},\hat{g})$.

In terms of (XVI.13), one easily verifies that the curves $I \ni t \to$ $\to c(t) = (t,y) \in I \times \bar{M} \subset M$ are geodesics and satisfy

$$\text{(XVI.40)} \qquad d^2 f^{2/n}/dt^2 = (1-n)^{-1} r(\dot{c},\dot{c}) f^{2/n} .$$

Since (M,g) is complete and analytic (cf. XVI.34), this linear differential equation implies that $f^{2/n}$ has an analytic extension to $\mathbb{R}$. This extension is non-zero everywhere. In fact, if it vanished at $t_o \in \mathbb{R}$, equation (XVI.36.iv), rewritten for $f^{2/n}$, would imply that $\bar{u} > 0$ (since $df^{2/n}/dt \neq 0$ at $t = t_o$ by the uniqueness of solutions of (XVI.40)) and would determine, up to a sign, the power series expansion of $f^{2/n}$ at t_o Hence $f^{2/n}$ would be one of the "obvious" solutions (linear, trigonometric or hyperbolic) of the rewritten equation, which would give $Dr = 0$ near x , contradicting our hypothesis. Therefore f can be extended to a positive analytic function on $\mathbb{R}$, again denoted by f .

The warped product manifold $(M',g') = (\mathbb{R} \times \hat{M}, dt^2 + f^{4/n}(t)\cdot\hat{g})$ is analytic, complete (see [BON], p. 23) and has an open subset isometric to a subset of (M,g). The universal covering space of (M,g) is therefore isometric to (M',g') (cf. [KN], p. 252) and so $(M,g) = (M',g')/\Gamma$, Γ being a discrete group of isometries. Clearly, Γ preserves the product foliations of $M' = \mathbb{R} \times \hat{M}$ (tangent to the eigenspace distributions of r) and, passing to a finite covering space, we may assume that Γ preserves the orientation in the $\mathbb{R}$-direction. Using equation (XVI.36.iv) with $(Dr)_x \neq 0$, one easily concludes that all elements of Γ operate on $\mathbb{R} \times \hat{M}$ as product maps of a translation of $\mathbb{R}$, keeping f invariant, with an isometry of $(\hat{M},\hat{g})$. For some element $T \times \Phi$ of Γ, the translation $T \in \mathbb{R}$ must be non-zero, for otherwise the projection $M' \to \mathbb{R}$ would define an unbounded function on M. Thus, f is periodic, and, by XVI.38, $\bar{u} > 0$ and $C_0 < 0$. The $\mathbb{Z}$ action on M' generated by $T \times \Phi$ defines a finite covering space $(M',g')/\mathbb{Z}$ of (M,g), which is of the type described in XVI.36.iv). This completes the proof.

XVI. 41 . Some open problems .

(i) Does there exist a compact simply connected Riemannian manifold with harmonic curvature and non-parallel Ricci tensor ? (cf. [Bo4]).

(ii) Does there exist a compact 4-dimensional Riemannian manifold with harmonic curvature which satisfies neither $Dr = 0$ nor $W = 0$ and is not locally isometric to a Riemannian product ? (cf. [Bo3], Remarque 7.2.i) ; also, note that XVI.27 and XVI.4.ii)d) easily yield non-compact examples of this sort).

(iii) Are there new compact manifolds with harmonic curvature (that is, with constant scalar curvature) among the examples described in XVI.26.ii) and XVI.27 ? *(See the footnote)*

(iv) Do there exist locally homogeneous Riemannian manifolds with $\delta R = 0$ and $Dr \neq 0$? (note that, for locally homogeneous manifolds, conformal flatness implies $Dr = 0$; see [AK]).

Note. The construction of XVI.26.ii) actually gives some new examples of compact manifolds with harmonic curvature. See A. Derdziński, "An easy construction of new compact Riemannian manifolds with harmonic curvature" (preliminary report).

XVI. 42 . Let $\overline{R}$ be an algebraic curvature tensor field on a Riemannian manifold (M,g) , $\dim M = n \geq 4$. We say that $\overline{R}$ is <u>closed</u> (as a $\Lambda^2 M$-valued 2-form) if it satisfies the "second Bianchi identity" $d\overline{R} = 0$, where $(d\overline{R})(X,Y,Z,U,V) = (D_X\overline{R})(Y,Z,U,V) + (D_Y\overline{R})(Z,X,U,V) + (D_Z\overline{R})(X,Y,U,V)$, and that $\overline{R}$ is <u>co-closed</u> if $\delta\overline{R} = 0$. Let $\overline{R} = \overline{U} + \overline{Z} + \overline{W}$ be the decomposition of $\overline{R}$ in the sense of I.72 , and set $\overline{r} = c(\overline{R})$ and $\overline{u} = \mathrm{tr}_g \overline{r}$.

<u>Proposition</u> (K. Nomizu [N], J. P. Bourguignon [Bo1]). <u>In the above notations</u>,

(i) <u>Conditions</u> $d\overline{U} = 0$, $\delta\overline{U} = 0$ <u>and</u> $d\overline{u} = 0$ <u>are mutually equivalent</u>.

(ii) <u>Conditions</u> $d\overline{Z} = 0$ <u>and</u> $\delta\overline{Z} = 0$ <u>are mutually equivalent</u>. <u>They are satisfied if and only if</u> $d\overline{r} = 0$, <u>i.e.</u>, <u>if</u> $\overline{r}$ <u>is a Codazzi tensor</u>.

(iii) <u>If, moreover,</u> $d\overline{R} = 0$, <u>then</u> $d\overline{W} = 0$ <u>if and only if</u> $\delta\overline{W} = 0$, <u>which is in turn equivalent to</u> $d[\overline{r} - (2n-2)^{-1}\overline{u}g] = 0$.

(iv) <u>If</u> $d\overline{R} = 0$, <u>then</u> $\overline{R}$ <u>is co-closed if and only if so are</u> $\overline{U}$, $\overline{Z}$, $\overline{W}$.

<u>Proof</u>. (i) and (ii) are obvious from (I.74) together with the easily verified fact that, for any symmetric 2-tensor field a , conditions $\delta(a \circledwedge g) = 0$ and $da = 0$ are equivalent provided $\mathrm{tr}_g a$ is constant, and $d(a \circledwedge g) = 0$ if and only if $da = 0$. The latter equivalence, together with $\overline{R} = (n-2)^{-1} g \circledwedge [\overline{r} - (2n-2)^{-1}\overline{u}g] + \overline{W}$ (cf. (I.74)) and the formula $\delta\overline{W} = -(n-3)(n-2)^{-1} d[\overline{r} - (2n-2)^{-1}\overline{u}g]$ (obtained from $d\overline{R} = 0$ by contraction, cf. XVI.3), yields (iii). Since $\delta\overline{R} = -d\overline{r}$ and $2\delta\overline{r} = -d\overline{u}$ whenever $d\overline{R} = 0$ (cf. XVI.3 and I.50), condition $d\overline{R} = \delta\overline{R} = 0$ implies that $\overline{u}$ is constant. Hence (iv) is immediate from (i) and (ii), which completes the proof.

F . The case $Dr \in C^\infty(Q)$.

XVI. 43 . The class of Riemannian manifolds with $Dr \in C^\infty(Q)$ (see XVI.4.i for an equivalent condition) has already been discussed in the literature ([Gr3], [Sv], [Sva]). In dimensions $n \geq 4$, all these manifolds have harmonic Weyl tensor (cf. XVI.4) and so all results of Section D remain valid for them. In this section we construct examples of compact manifolds with $Dr \in C^\infty(Q)$ and $Dr \neq 0$, and prove a pinching theorem. We also discuss some questions related to the local classification problem for such manifolds.

XVI. 44 . Examples . By XVI.43, an obvious construction procedure is to impose condition $Dr \in C^\infty(Q)$ on the manifolds described in Section D, which immediately gives rise to the following examples of compact manifolds with $Dr \in C^\infty(Q)$:

(i) Compact manifolds with $Dr = 0$.

(ii) Bundles with fibre $\bar{M}$ over S^1 with twisted warped product metrics constructed as in XVI.26.i), where the compact Einstein manifold $(\bar{M},\bar{g})$ has scalar curvature $\bar{u} < 0$ and $f = F^{-1}$ is a non-constant positive periodic solution of $d^2f/dt^2 = 2(n-1)^{-1}(n-2)^{-1}\cdot\bar{u}f^3 + Cf$ with a constant $C > 0$ (such an f exists by XVI.38). These examples never satisfy $Dr = 0$. Some of them (those for which $\bar{g}$ is of constant curvature) are conformally flat.

XVI. 45 . Proposition . Let (M,g) be a compact Riemannian manifold with $Dr \in C^\infty(Q)$ and

(XVI.45) $\qquad (n-1)(n+6)\, r \leq 4u\ , \quad n = \dim M \geq 3$

(the latter condition holds, e.g., if r is sufficiently C^0 close to ϕg for a function $\phi < 0$). Then $Dr = 0$.

Corollary . Let g_0 be an Einstein metric with negative scalar curvature on a compact manifold M . If a metric g on M satisfies $Dr \in C^\infty(Q)$ and is sufficiently C^2 close to g_0 , then g has parallel Ricci tensor.

Proof of the proposition. From XVI.4.i) and XVI.3, we obtain $2(n-1)(\delta R)(X,Y,Z) = (X,Y)du(Z) - (X,Z)du(Y)$. The Weitzenböck formula (XVI.9), applied to XVI.4.i) implies, after taking divergence and using the formula $r(Du) = d\Delta u - \delta Ddu$ (cf. I.) that $(n-2)\delta Ddu = (n+6)r(Du) - 4(n-1)^{-1}udu$, which easily yields (cf. [Sva])

$$(n-2)\Delta|Du|^2 = 2(n+6)r(Du,Du) - 8(n-1)^{-1}u|Du|^2 - 2(n-2)|Ddu|^2 .$$

Integrating this and using the inequality (XVI.45), we conclude that $Ddu = 0$, so that u is constant and, by XVI.4.i), $Dr = 0$, as required.

XVI. 46 . By XVI.4.i), condition $Dr \in C^\infty(Q)$ can be rewritten as

(XVI.47) $\quad Db = \zeta \odot g \ , \quad \zeta = (n+2)^{-1}d(tr_g b) \ , \quad n = \dim M$,

where $b = r - (2n-2)^{-1}ug$. To determine the local structure of all solutions (M,g,b) of (XVI.47), let us first note that for any such solution, any eigenvalue functions λ , ν of b with $\lambda \neq \nu$ and any $X \in C^\infty(V_\lambda)$, $Z \in C^\infty(V_\nu)$, we have

(i) $\quad X\lambda = 3\zeta(X)$, $Z\lambda = \zeta(Z)$.

(ii) If λ is constant along V_λ (e.g., if $\dim V_\lambda > 1$, cf. XVI.11.iii)), then all eigenvalues of b are constant along V_λ , ζ annihilates V_λ and $\zeta = d\lambda$.

In fact, setting in formula (XVI.10) first $\lambda = \mu = \nu$ and $Z = Y = X$ and then $\lambda = \mu \neq \nu$ and $Y = X$, we obtain (i) from (XVI.47). If λ is constant along V_λ , (i) gives $\zeta(X) = 0$, $X\nu = \zeta(X) = 0$ and $Z\lambda = \zeta(Z)$, which proves (ii).

XVI. 48 . Local classification of the solutions (M,g,b) of (XVI.47) which are of generic type in the sense that, for each $x \in M$, b_x has n distinct eigenvalues and the restriction of ζ_x to each eigenspace is non-zero : Set $x_i = (n+2)\lambda_i - tr_g b$, where λ_i are the eigenvalue functions of b , $1 \leq i \leq n$. By XVI.46.i), $Xx_i = 2X(tr_g b) = 2(n+2)\zeta(X) \neq 0$ and $Yx_i = 0$ whenever $X \in V_i$, $X \neq 0$ and $Y \in (V_i)^\perp$, V_i being the eigenspace distribution of λ_i. Hence the x_i form, locally, a coordinate system and

$$g = \sum_i \psi_i dx_i^2 , \quad b = \sum_i \lambda_i \psi_i dx_i^2 , \quad \lambda_i = (2n+4)^{-1}(2x_i + \sum_j x_j)$$

with $\psi_i = |\partial_i|^2$, $\partial_i = \partial/\partial x_i$. Since $[\partial_i,\partial_j] = 0$, XVI.11.ii) yields $\partial_j\psi_i = -2(D_{\partial_i}\partial_i,\partial_j) = 2\psi_i(\lambda_j-\lambda_i)^{-1}\partial_j\lambda_i$ for $j \neq i$, which implies

$$\psi_i = \exp(q_i(x_i)) \cdot \prod_{j\neq i} |x_j-x_i| ,$$

where q_i are functions of one variable. Conversely, one easily verifies that in any domain M of $\mathbb{R}^n$ where the Cartesian coordinates x_i are mutually distinct, the above formulae, with arbitrary functions $q_i(x_i)$, define a generic type solution (M,g,b) of (XVI.47).

XVI. 49. There exist solutions (M,g,b) of (XVI.47) which are <u>not of generic type</u>, other than those with $Db = 0$:

<u>Example</u>. Let (M_1,g_1,b_1) satisfy (XVI.47) with the 1-form ζ_1, $\dim M_1 = n_1 \geq 1$, and denote by D_1, etc., the Riemannian connection, etc., of g_1. Assume that M_1 is simply connected and that, for some $\mu_o \in \mathbb{R}$, $T = (b_1 - \mu g_1)^{-1}$ exists, where $\mu = (n_1+2)^{-1}\mathrm{tr}_{g_1} b_1 + \mu_o$ and b_1 is viewed as a (1,1) tensor ; locally, these hypotheses are always satisfied. For any Riemannian manifold (M_2,g_2), we shall construct a solution (M,g,b) of (XVI.47) which is an <u>extension of</u> (M_1,g_1,b_1) <u>by</u> (M_2,g_2), that is

$$\text{(XVI.49)} \quad M = M_1 \times M_2 , \quad g = g_1 + e^{\phi}g_2 , \quad b = b_1 + \mu e^{\phi}g_2$$

for some function ϕ on M_1. Condition (XVI.47) for this (M,g,b) is equivalent to $2T(d\mu) = d\phi$. To prove that such a ϕ exists, i.e., that $d(T(d\mu)) = 0$, note that $d\mu = \zeta_1$ by (XVI.47) and so $((D_1)_X T)(Y,Z) = -T(d\mu,Y)T(X,Z) - T(d\mu,Z)T(X,Y)$, while (XVI.47) gives $D_1 d(\mathrm{tr}_{g_1} b_1) + \delta_1 D_1 b_1 = n_1 D_1 d\mu + \Delta_1\mu\cdot g_1$. Thus, by XVI.17.i), $D_1 d\mu$ commutes with b_1 and hence also with T, which implies $d(T(d\mu)) = 0$, as required.

XVI. 50. <u>Local classification of arbitrary solutions</u> (M,g,b) <u>of</u> (XVI.47) (at points of general position). Assume that $M = M_b$ and ζ annihilates a fixed eigenspace distribution V_μ (cf. XVI.46.ii) and XVI.48). We claim that, <u>if</u> b <u>is not parallel</u>, (M,g,b) <u>is, locally, obtained from a gene-</u>

ric type solution by a finite number of extension procedures as described in XVI.49. In fact, for $X \in C^\infty(V_\lambda)$, $Y \in C^\infty(V_\mu)$, $Z \in C^\infty(V_\nu)$ with $\lambda \neq \mu \neq \nu$, (XVI.10) and (XVI.47) yield $(\lambda-\mu)(D_Z X, Y) = (X,Z)\cdot\zeta(Y) = 0$, i.e., $D_Z X \in C^\infty((V_\mu)^\perp)$. Thus, by XVI.11, $(V_\mu)^\perp$ and V_μ are integrable and the leaves of $(V_\mu)^\perp$ are totally geodesic, while those of V_μ are umbilical. Hence we have, locally, formula (XVI.49), (M_i, g_i) being Riemannian manifolds with $b_1 \in S^2M_1$ (b_1 is "constant along M_2" in view of (XVI.47)), and ϕ is a function on $M_1 \times M_2$, while (M_1, g_1, b_1) satisfies (XVI.47) with $\zeta_1 = \zeta$, and $b_1(d_1\phi) - \mu d_1\phi = 2d\mu$, d_i being the differential along M_i. Since, by XVI.46.i), $d_2\mu = 0$, this gives $(b_1 - \mu g_1)(d_2 d_1 \phi) = 0$, and hence (general position !) $\phi = \phi_1 + \phi_2$ for some functions ϕ_i on M_i. Replacing g_2 by $\exp(\phi_2)g_2$, we obtain relation (XVI.49) with $\phi : M_1 \to \mathbb{R}$ and so (M,g,b) is an extension as in XVI.49. Repeating this procedure, we can successively split off all eigenspace distributions of b annihilated by ζ, which eventually leads to a generic type situation.

XVI. 51 . There is the obvious problem of finding a local classification (at points of general position) of all Riemannian manifolds with $Dr \in C^\infty(Q)$. An obvious idea would be to take our classification XVI.48, XVI.50 of the solutions (M,g,b) of (XVI.47) and impose on them the condition $b = r - (2n-2)^{-1}ug$. This direct approach leads, however, to hopeless computations.

G . Condition $Dr \in C^\infty(A)$: Riemannian manifolds such that $(D_X r)(X,X) = 0$ for all tangent vectors X .

XVI. 52 . A Riemannian manifold (M,g) satisfies the condition $Dr \in C^\infty(A)$ if and only if $\sigma(Dr) = 0$ (see XVI.4.iii), which is clearly equivalent to $(D_X r)(X,X) = 0$ for all $X \in TM$. This condition occurs as a consequence of some geometrical hypotheses. Namely, it holds (see [DN1] and XVI.55.i)) for all D'Atri spaces [VW], i.e., Riemannian manifolds (M,g) such that, for each $x \in M$, the local geodesic symmetry at x (assigning $\exp_x(-X)$ to $\exp_x X$, for $X \in T_x M$ close to 0) preserves, up to a sign, the volume element v_g . Moreover, all Riemannian manifolds (M,g) for which the operator $f \to (r, Ddf)$, acting on functions, commutes with Δ , satisfy $\sigma(Dr) = 0$ (see [Su]).

In the present section we describe various examples of manifolds with $\sigma(Dr) = 0$ and state a theorem on them.

XVI. 53 . Examples of Riemannian manifolds with $\sigma(Dr) = 0$, i.e., with $(D_X r)(X,X) = 0$ for all vectors X .

(i) Manifolds with $Dr = 0$.

(ii) All D'Atri spaces (cf. XVI.55.i)). This class contains (see [DN2]) all homogeneous Riemannian manifolds which are naturally reductive (for some G and $\mathfrak{p}$, cf. "E".74), in particular, the normal homogeneous Riemannian manifolds. Thus, all such manifolds satisfy $\sigma(Dr) = 0$, which gives rise to a large variety of examples of compact locally homogeneous manifolds with $(D_X r)(X,X) = 0$ for all vectors X , including all metric spheres in rank one symmetric spaces (as shown directly by Chen and Vanhecke [CV], cf. also [Z]), all nearly Kählerian 3-symmetric spaces (by Gray's result [Gr3], cf. [Gr2], [TV2]) and many compact Lie groups with suitable left-invariant metrics constructed by D'Atri and Ziller [DAZ]. Most of these examples have $Dr \neq 0$. Moreover, there exist homogeneous D'Atri spaces which are not naturally reductive (for any G and $\mathfrak{p}$). Namely, A. Kaplan constructed ([Ka1] , [Ka2], [Ka3], cf. [TV2]) a class of left-invariant metrics on certain 2-step nilpotent Lie groups, all of which have the D'Atri space property without being, in general, naturally reductive. Many of Kaplan's examples admit compact quotient manifolds ([Ka2]).

(iii) Riemannian products of manifolds with $\sigma(Dr) = 0$.

XVI. 54. Theorem (U. Simon [Si2], cf. [Gr3]). Let (M,g) be a compact Riemannian manifold with $(D_X r)(X,X) = 0$ for all $X \in TM$ and with non-positive sectional curvature K. Then $Dr = 0$. If, moreover, $K < 0$ at some point, then (M,g) is Einstein.

Proof. For any symmetric 2-tensor field b on (M,g) such that $(D_X b)(X,X) = 0$ for all vectors X, the Weitzenböck formula I. can easily be rewritten as

$$\delta Db + Dd(\mathrm{tr}_g b) = b \circ r + r \circ b - 2\overset{\circ}{R}(b) .$$

If $b = r$, then $\mathrm{tr}_g b$ is constant (cf. XVI.4) and our assertion can be obtained by integration, exactly as in XVI.9.

XVI. 55. Theorem (J. E. D'Atri and H. K. Nickerson, [DN1], [DN2], [DA]).

(i) Every D'Atri space (M,g) satisfies the condition $(D_X r)(X,X) = 0$ for all vectors X.

(ii) Every naturally reductive Riemannian homogeneous manifold is a D'Atri space.

Proof. (i) For any Riemannian manifold (M,g), $x \in M$ and $X \in T_x M$,

$$(D_X r)(X,X) = \frac{d^3}{dt^3} \log \theta(\exp_x tX)\Big|_{t=0} ,$$

$\theta = \theta_x$ being the normal volume function centered at x (see I.40).
This follows from a direct computation in normal coordinates at x, using the easy equality $\sum_{i,j} g^{ij}\partial_k g_{ij} = \partial_k \log \det g_{ij}$ and relations obtained by differentiating (I.34). In a D'Atri space, the function $t \to \theta_x(\exp_x tX)$ is even, which completes the proof.

(ii) See [DN2] or [DA].

XVI. 56 . Some open problems.

(i) Find examples of Riemannian manifolds with $(D_X r)(X,X) = 0$ for all vectors X , which are neither locally homogeneous, nor locally isometric to Riemannian products and have non-parallel Ricci tensor.

(ii) Do there exist D'Atri spaces which are not locally homogeneous ? (cf. [TV1]).

H . Oriented Riemannian four-manifolds with $\delta W^+ = 0$.

XVI. 57 . For oriented Riemannian four-manifolds, relation $\delta W^+ = 0$ is equivalent to a linear condition on Dr , namely, to $d(r - ug/6) \in$ $\in C^\infty(\Lambda^- M \otimes T^*M)$ (cf. XVI.5). In the present section we discuss the manifolds satisfying this condition and show that some of them are in a natural conformal relationship with Kähler manifolds, which allows us to construct many examples of compact manifolds of this type. We also give some results concerning the behavior of the function $\det W^+$ on such manifolds.

XVI. 58 . In our discussion of oriented Riemannian four-manifolds (M,g) , it is convenient to use local trivializations of $\Lambda^+ M$ by eigenvectors of W^+ . In the open dense subset M_{W^+} of M , consisting of points at which the number of distinct eigenvalues of $W^+ \in C^\infty(\mathrm{End}\, \Lambda^+ M)$ is locally constant, we can, locally, choose C^∞ functions μ_i and mutually orthogonal C^∞ sections ω_i of Λ^+ with $|\omega_i|^2 = 2$ and $W^+(\omega_i) = \mu_i \omega_i$, $1 \leq i \leq 3$. Note that, for $x \in M$, every oriented orthonormal basis $X_1, \ldots, X_4$ of $T_x M$ gives rise to orthonormal bases $2^{-1/2}(\mp X_1 \wedge X_2 - X_3 \wedge X_4)$, $2^{-1/2}(\mp X_1 \wedge X_3 - X_4 \wedge X_2)$, $2^{-1/2}(\mp X_1 \wedge X_4 - X_2 \wedge X_3)$ of $\Lambda^\pm_x M$. This construction can easily be verified to be equivariant relative to the two-fold covering homomorphism $SO(4) \to SO(3) \times SO(3)$, so that each pair of suitably oriented orthonormal bases of $\Lambda^+_x M$ and $\Lambda^-_x M$ is obtained in this way from an oriented orthonormal basis of $T_x M$. Hence, by changing the signs of some of our ω_i , we get (cf. I.88)

(XVI.59)
$$\omega_i \omega_j = \omega_k = - \omega_j \omega_i \quad \text{if} \quad \varepsilon_{ijk} = 1 , \quad \omega_i^2 = - \mathrm{Id} ,$$
$$W^+ = \tfrac{1}{2} \sum_i \mu_i \omega_i \otimes \omega_i , \quad \sum_i \mu_i = 0 ,$$

where ε_{ijk} is the Ricci symbol (skew-symmetric in i,j,k , with $\varepsilon_{123} = 1$). The invariance of $\Lambda^+ M$ under parallel displacements implies

(XVI.60)
$$D\omega_i = \zeta_k \otimes \omega_j - \zeta_j \otimes \omega_k , \quad \text{if} \quad \varepsilon_{ijk} = 1$$

for certain C^∞ 1-forms ζ_i defined (locally) in M_{W^+} . Calculating the curvature of the Riemannian connection in $\Lambda^+ M$ in terms of the connection forms ζ_i , i.e., applying to (XVI.60) the Ricci identity I. we obtain

(XVI.61) $\quad d\zeta_i + \zeta_j \wedge \zeta_k = (\mu_i - u/6)\cdot\omega_i + \frac{1}{2}(r\omega_i + \omega_i r) \quad$ if $\varepsilon_{ijk} = 1$.

XVI. 62 . Proposition (cf. [Gr1], [Po], [De3]). Let (M,J,g) be a Kähler manifold of real dimension four, oriented in the natural way (so that the Kähler form ω is a section of Λ^+M). Then $W^+(\omega) = u\cdot\omega/6$ and $W^+(\eta) = -u\cdot\eta/12$ for any 2-form $\eta \in \Lambda^+M$ orthogonal to ω .

Proof. The Weitzenböck formula for 2-forms η (cf. I.),

$$2W(\eta) = \delta D\eta - \Delta\eta + 2(n-1)^{-1}(n-2)^{-1}u\cdot\eta + (n-4)(n-2)^{-1}(r\eta + \eta r)$$

implies, in dimension $n = 4$ and for $\eta = \omega$ with $D\omega = 0$, that $W(\omega) = u\cdot\omega/6$. In M_{W^+} (notation of XVI.58), set $\omega_i = \omega$ for a fixed i , so that $\mu_i = u/6$. From (XVI.60) and (XVI.61) , $\zeta_s = 0$ and $r\omega_s + \omega_s r = (u/3 - 2\mu_s)\omega_s$, if $s \neq i$. Since $r\omega_i = \omega_i r$ (cf. "K."44), this gives, by (XVI.59), $(u/3 - 2\mu_j)\omega_k = \omega_i[(u/3 - 2\mu_j)\omega_j] = \omega_i(r\omega_j + \omega_j r) = r\omega_k + \omega_k r = (u/3 - 2\mu_k)\omega_k$ for j,k such that $\varepsilon_{ijk} = 1$. Hence (cf. (XVI.59)) $\mu_j = \mu_k = -u/12$, as required.

XVI. 63 . Remark . By XVI.62, every Kähler manifold (M,J,g) of real dimension four satisfies the relation

(XVI.64) $\quad \# \operatorname{Spec}_{(\Lambda^+)} W^+ \leq 2$

i.e., the endomorphism W^+ of Λ^+M has, at each point, less than three distinct eigenvalues. By the conformal invariance of W , (XVI.64) will also hold for any metric conformal to our Kähler metric g . On the other hand, XVI.62 implies for (M,J,g) the equality $W^+ = u\cdot T$ for some tensor field T , which is parallel, since it is naturally determined by ω and Λ^+M . Therefore, $u\cdot\delta W^+ + W^+(Du,\cdot,\cdot,\cdot) = 0$. Consequently (cf. (XVI.25) and XVI.5), for any Kähler four-manifold (M,J,g), the metric $g' = g/u^2 = g/(24|W^+|^2)$, defined wherever $W^+ \neq 0$, satisfies the conditions $\delta_{g'}W^+_{g'} = 0$ and $\# \operatorname{Spec}_{(\Lambda^+)} W^+_{g'} \leq 2$; clearly, $g = (24g'(W^+_{g'}, W^+_{g'}))^{1/3}\cdot g'$. For a converse statement, see XVI.67.

XVI. 65 . Examples of compact oriented Riemannian four-manifolds with $\delta W^+ = 0$:

(i) Compact oriented four-manifolds with $Dr = 0$ (cf. XVI.57).

(ii) Compact manifolds satisfying the conformally invariant condition $W^+ = 0$. The only known examples of this type are compact conformally flat 4-manifolds, and (suitably) oriented manifolds conformal to a quotient of a K 3 surface with a Ricci-flat Kähler metric, or to the standard CP^2 , or to compact quotients of its dual $(CP^2)^*$ (cf. [AHS] and "D.").

(iii) For any compact Kähler manifold (M,J,g) with $\dim_{\mathbb{R}} M = 4$ and $u \neq 0$ everywhere, the conformally related compact manifold $(M,g') = (M,g/u^2)$ satisfies $\delta_{g'} W_{g'} = 0$ (for the natural orientation ; cf. XVI.63). Using small Kählerian deformations of g (see "K."), we obtain a large variety of such examples (e g., with the underlying manifold CP^2).

XVI. 66 . In the notations of XVI.58, formulae (XVI.59) and (XVI.60) easily imply that condition $\delta W^+ = 0$ is equivalent to

(XVI.66) $$d\mu_i = (\mu_i - \mu_j)\omega_k(\zeta_k) + (\mu_i - \mu_k)\omega_j(\zeta_j) \quad \text{if} \quad \varepsilon_{ijk} = 1 ,$$

where we set $\omega(\zeta) = -\iota_\zeta \omega = \omega(\cdot,\zeta)$ for any 2-form ω and any 1-form ζ .

XVI. 67 . We have the following converse of the statement given in XVI.63 :

Theorem (A. Derdziński, [De3]) . Let an oriented Riemannian four-manifold (M,g) satisfy the conditions (XVI.64) and $\delta W^+ = 0$. Then the metric $g' = (24|W^+|^2)^{1/3}\cdot g$, defined wherever $W^+ \neq 0$, is (locally) Kählerian with respect to some complex structure, defined explicitly (up to a sign) at points where $W^+ \neq 0$ and compatible with the original orientation. Moreover, $u_{g'} = \phi^{1/3}$, where $\phi/6$ is the simple eigenvalue of W^+ , and $g = g'/u_{g'}^2$.

Proof. In the notations of XVI.58, let $6\mu_i = -12\mu_j = -12\mu_k = \phi$ for some fixed i,j,k with $\varepsilon_{ijk} = 1$, the function ϕ being smooth wherever $W^+ \neq 0$. Clearly, $\phi^2 = 24|W^+|^2$, $g' = \phi^{2/3} g$ and, by (XVI.66),

(XVI.68) $$d\phi = 3\phi\omega_k(\zeta_k) = 3\phi\omega_j(\zeta_j) .$$

We claim that

(XVI.69) $$3\phi\cdot D_X\omega_i + d\phi \wedge \omega_i(X) + \omega_i(d\phi) \wedge X = 0$$

for any tangent vector X. In fact, let η be the 2-form constituting the left-hand side of (XVI.69). Any $\xi \in \Lambda^- M$ is orthogonal to $D_X \omega_i$ and commutes with ω_i (cf. XVI.58, I.35), so that $(\eta,\xi) = 0$. On the other hand, (XVI.59), (XVI.60) and (XVI.68) yield $(\omega_j,\eta) = (X, 6\phi\cdot\zeta_k + 2\omega_k(d\phi)) = 0$ and, similarly, $(\omega_k,\eta) = (\omega_i,\eta) = 0$, which proves (XVI.69). It is now easy to verify (cf. I.) that (XVI.69) means nothing but $D_{g'}(\phi^{2/3}\omega_i) = 0$, so that g' is a Kähler metric with Kähler form determined by g up to a sign (since ω_i is a simple eigenvector of W^+). Our assertion is now immediate from the obvious relations between the spectra of W^+_g and $W^+_{g'}$, acting on $\Lambda^+ M$, together with XVI.62.

XVI. 70 . <u>Proof of Proposition</u> XVI.32 : By XVI.28.ii), relations $|W^+| = |W^-| > 0$ and $\# \operatorname{Spec}_{(\Lambda^-)} W^- \leq 2$ hold near x. Thus, XVI.67 implies that the metric $g' = |W|^{2/3} g$ is Kählerian for <u>two complex structures</u> J^+, J^-, corresponding to different orientations in a neighborhood of x. The corresponding Kähler forms $\omega^\pm$ are sections of $\Lambda^\pm$, and therefore J^+ and J^- must commute (since $\omega^\pm = \mp X_1 \wedge X_2 - X_3 \wedge X_4$ for some local g'-orthonormal frame $X_1,\ldots,X_4$, cf. XVI.58). Hence $a = J^+J^-$ is a g'-parallel self-adjoint (1,1) tensor field with $a^2 = \mathrm{Id}$, $\det a = 1$ and $a \neq \pm\mathrm{Id}$ (note that $(J^+)^{-1} = -J^+$). Consequently, the (± 1)-eigenspaces of a form g'-parallel plane fields near x. Together with the conformal transformation rule for $|W|$, this completes the proof.

XVI. 71 . Let (M,g) be an oriented Riemannian four-manifold with $\delta W^+ = 0$. In the notations of XVI.58, formulae (XVI.59) and (XVI.61) yield $(\omega_i, d\zeta_i) = -\omega_i(\zeta_j,\zeta_k) + 2\mu_i + u/6$, $(\omega_j, d\zeta_i) = -\omega_j(\zeta_j,\zeta_k)$, $(\omega_k, d\zeta_i) = -\omega_k(\zeta_j,\zeta_k)$ and hence (cf. (XVI.60)), $\delta(\omega_i(\zeta_i)) = \omega_j(\zeta_i,\zeta_k) - \omega_k(\zeta_i,\zeta_j) + \omega_i(\zeta_j,\zeta_k) - 2\mu_i - u/6$, whenever $\varepsilon_{ijk} = 1$. Using (XVI.66), we obtain the equality (cf. [De3])

$$\text{(XVI.71)} \qquad \Delta\mu_i = 2\mu_i^2 + 4\mu_j\mu_k - u\cdot\mu_i/2 + 2(\mu_j-\mu_i)|\zeta_k|^2 + 2(\mu_k-\mu_i)|\zeta_j|^2$$

whenever $\varepsilon_{ijk} = 1$.

XVI. 72 . Proposition . Let an oriented Riemannian 4-manifold (M,g) satisfy the conditions $\delta W^+ = 0$ and $\det_{(\Lambda^+)} W^+ = 0$. Then $W^+ = 0$.

Proof. Locally in M_{W^+} , we have $\mu_i = 0$ for a fixed i (notations of XVI.58). Suppose that $W^+ \neq 0$ near $x \in M_{W^+}$. By (XVI.59), $\mu_j = -\mu_k \neq$ $\neq 0$, j,k being such that $\varepsilon_{ijk} = 1$, and so, by (XVI.66), $\omega_j(\zeta_j) =$ $= \omega_k(\zeta_k)$, which gives $|\zeta_j| = |\zeta_k|$. Hence (XVI.71) implies $0 = \Delta\mu_i =$ $= 4\mu_j\mu_k$. This contradiction completes the proof.

XVI. 73 . Proposition . Every oriented Riemannian 4-manifold with $\delta W^+ = 0$ satisfies the relation

$$(XVI.73)\qquad \Delta|W^+|^2 = -u\cdot|W^+|^2 + 36\det_{(\Lambda^+)}W^+ - 2|DW^+|^2.$$

Proof. In the notations of XVI.58, set $Y_i = (\mu_j - \mu_k)\omega_i(\zeta_i)$, if $\varepsilon_{ijk} = 1$. By (XVI.59) and (XVI.60), $|DW^+|^2 = \sum_i (|d\mu_i|^2 + 2|Y_i|^2)$. Computing $\Delta|W^+|^2$ from (XVI.71), we now easily obtain (XVI.73).

XVI. 74 . Theorem . Let (M,g) be a compact oriented Riemannian 4-manifold with $\delta W^+ = 0$ and $u \geq 0$. Viewing W^+ as an endomorphism of $\Lambda^+ M$, we have

$$(XVI.74)\qquad \int_M \det W^+ \cdot v_g \geq 0 ,$$

the inequality being strict unless $W^+ = 0$ identically.

Proof. The weak inequality follows from (XVI.73) by integration. Equality occurs there if and only if $DW^+ = 0$ and $u\cdot|W^+| = 0$, which implies that $|W^+|$ is constant. Thus, by (XVI.73), equality in (XVI.74) yields $\det_{(\Lambda^+)} W^+ = 0$ and hence $W^+ = 0$ in virtue of XVI.72.

XVI. 75 . Remarks .

(i) It follows immediately from (XVI.59) that every oriented Riemannian 4-manifold (M,g) satisfies the conditions $\check{W} = |W|^2\cdot g$ and $\check{W}^{\pm} =$ $= |W^{\pm}|^2 \cdot g$, where, for an algebraic curvature tensor T , $\check{T}(X,Y) =$ $= (T(X,\cdot,\cdot,\cdot),T(Y,\cdot,\cdot,\cdot))$, cf. I.94.a) , I.96 and [Ba]. Consequently, the vector bundle homomorphism $TM \ni X \to W^+(X,\cdot,\cdot,\cdot) \in T^*M \otimes \Lambda^+ M$ is injective wherever $W^+ \neq 0$. The transformation rule $\delta_{g'} W^+_{g'} = \delta W^+ -$

$- W^+(Df,\cdot,\cdot,\cdot)$ of δW^+ for conformally related metrics $g' = e^{2f}g$ on an oriented four-dimensional manifold (see (XVI.25) and XVI.5) implies that in the open subset of such a manifold defined by $W^+ \neq 0$, the given conformal class cannot contain two essentially distinct (i.e., not proportional with a constant factor) metrics with $\delta W^+ = 0$.

(ii) An oriented Riemannian four-manifold (M,g) satisfies the condition $DW^+ = 0$ if and only if either $W^+ = 0$, or g is locally Kählerian (Kählerian up to a two-fold isometric covering) in a way compatible with the orientation and has non-zero constant scalar curvature. In fact, if $W^+ \in C^\infty(S^2_0\Lambda^+M)$ (cf. I. 88) is parallel and non-zero, it must have a simple eigenvalue and a local section ω of the corresponding line subbundle of Λ^+M , normed by $|\omega|^2 = 2$, is parallel, so that our assertion follows from XVI.63.

(iii) Roter's theorem saying that Riemannian manifolds (M,g) with $DW = 0$ must have $W = 0$ or $DR = 0$ (cf. XVI.24.iii)) can be proved in dimension four as follows. If $W \neq 0$, g is locally Kählerian by (ii), so that XVI.30 implies $Dr = 0$, which, together with $DW = 0$, gives $DR = 0$ (cf. I. 74).

[Si1] U. Simon: Compact conformally symmetric Riemannian spaces. Math. Z. 132(1973), 173-177

[Si2] U. Simon: On differential operators of second order on Riemannian manifolds with nonpositive curvature. Colloq. Math. 31(1974), 223-229

[Sv] N. S. Sinyukov: Geodesic mappings of Riemannian manifolds (in Russian). Nauka, Moscow 1979

[Sva] E. N. Sinyukova: Geodesic mappings of the spaces L_n (in Russian). Izvestiya Vyssh. Ucheb. Zaved. (Mat.) No. 3(238) , (1982), 57-61

[Su] T. Sumitomo: On the commutator of differential operators. Hokkaido Math. J. 1(1972), 30-42

[Ta] S. Tanno: Curvature tensors and covariant derivatives. Ann. Mat. Pura Appl. 96(1973), 233-241

[TV1] F. Tricerri and L. Vanhecke: Naturally reductive homogeneous spaces and generalized Heisenberg groups. (to appear in Compos. Math.)

[TV2] F. Tricerri and L. Vanhecke: Homogeneous Structures on Riemannian Manifolds. L.M.S. Lecture Notes Series, No. 83. Cambridge Univ. Press, Cambridge, 1983

[VW] L. Vanhecke and T. J. Willmore: Interaction of tubes and spheres. Math. Ann. 263(1983), 31-42

[W] B. Wegner: Codazzi-Tensoren und Kennzeichnungen sphärischer Immersionen. J. Differential Geometry 9(1974), 61-70

[YM] K. Yano and I. Mogi: On real representations of Kaehlerian manifolds. Ann. Math. 61(1955), 170-189

[Z] W. Ziller: The Jacobi equation on naturally reductive compact Riemannian homogeneous spaces. Comment. Math. Helvetici 52(1977), 573-590

Some notations

$c(\bar{R})$ the Ricci contraction of an algebraic curvature tensor $\bar{R}$

$C^\infty(E)$ the space of C^∞ sections of a vector bundle E

$\mathcal{C}(E)$ the space of algebraic curvature tensors in a Euclidean space E

d exterior derivative of real or vector valued differential forms on a Riemannian manifold (M,g)

D the Riemannian connection (covariant derivative) of (M,g)

$h \owedge b$ the Kulkarni-Nomizu product of symmetric 2-tensors : $(h \owedge b)_{ijkl} = h_{ik}b_{jl} + h_{jl}b_{ik} - h_{jk}b_{il} - h_{il}b_{jk}$

K the sectional curvature of (M,g) : $K(\mathrm{span}(X,Y)) = R(X,Y,X,Y)$ for orthonormal vectors X,Y

r the Ricci tensor of (M,g) : $r_{ij} = R_{ikj}{}^{k}$

R the curvature tensor of (M,g) : sign convention $R(X,Y) = [D_Y, D_X] + D_{[X,Y]}$

$\overset{o}{R}(b)$ action of R on symmetric 2-tensors : $[\overset{o}{R}(b)]_{ij} = R_{ipjq}b^{pq}$

$S^2_o E$ the bundle of traceless symmetric 2-forms in a Riemannian vector bundle E

$tr_g b$ the trace of a covariant 2-tensor b on (M,g) : $tr_g b = g^{ij}b_{ij}$

u the scalar curvature of (M,g) : $u = tr_g r$

v_g the volume element of (M,g)

W the Weyl conformal tensor of (M,g)

W^+, W^- the self-dual (resp., anti-self-dual) component of W on an oriented Riemannian four-manifold (M,g)

$\mathcal{W}(E)$ the space of algebraic Weyl tensors in a Euclidean space E

δ the divergence (see XVI. 3)

$\Delta = -g^{ij}D_iD_j$ the Laplace operator of (M,g) , acting on functions

$\theta = \theta_x$ the normal volume function of (M,g) at $x \in M$ (in normal coordinates at x , $\theta = \det g_{ij}$)

$\Lambda^k E$ the k-th exterior power of a vector space (vector bundle) E

$\Lambda^+ M$, $\Lambda^- M$ the bundle of self-dual(resp., anti-self-dual) 2-forms on an oriented Riemannian 4-manifold (M,g)

$\odot$ the symmetric product

M_b the set of points in (M,g) at which the number of distinct eigenvalues of the symmetric tensor b is locally constant (see XVI.10)

Conventions :

$|\omega|^2 = \frac{1}{2}\omega_{ij}\omega^{ij}$, $(d\omega)_{ijk} = D_i\omega_{jk} + D_j\omega_{ki} + D_k\omega_{ij}$ for a 2-form ω

$|\overline{R}|^2 = \frac{1}{4}\overline{R}_{hijk}\overline{R}^{hijk}$, $|D\overline{R}|^2 = \frac{1}{4}D_s\overline{R}_{hijk}\cdot D^s\overline{R}^{hijk}$, $(\overline{R}(\omega))_{ij} = \frac{1}{2}\overline{R}_{ijpq}\omega^{pq}$

for an algebraic curvature tensor $\overline{R}$ and a 2-form ω

$(X \wedge Y)_{ij} = X_iY_j - X_jY_i$, $(dX)_{ij} = D_iX_j - D_jX_i$ for 1-forms X , Y