Universal abelian covers of normal surface singularities

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This talk described joint work with Jonathan Wahl on "splice-quotient singularities." We consider normal surface singularities (X, o) whose links are rational homology spheres ($\mathbb{Q}$HS for short). The $\mathbb{Q}$HS condition is equivalent to the condition that the resolution graph Γ of a minimal good resolution be a *rational tree*, i.e., Γ is a tree and all exceptional curves are genus zero. The resolution graph Γ determines and is determined by the topology of X (or its link). The graphs Γ that arise this way, and hence the possible topologies, are easy to describe, but explicit description of even a single analytic structure for a given topology was generally not possible before splice-quotients.

Splice-quotient singularities developed out of the observation that the *universal abelian cover* (branched only at the singular point) of (X, o) is surprisingly often a complete intersection singularity. This was observed first for (X, o) weighted homogeneous ([5]), next cusp-quotient ([7]), and greatly extended in the work described here. In the weighted homogeneous case the universal abelian cover is a Brieskorn-Hamm-Pham complete intersection, that is, it is defined by $n-2$ equations:

$$a_{i1}x_1^{m_1} + \cdots + a_{in}x_n^{m_n} = 0, \quad i = 1, \ldots, n-2\,,$$

with all maximal minors of the coefficient matrix (a_{ij}) non-singular; moreover, the exponents m_i and coefficient matrix can be easily read off from the singularity. Splice-quotients are a broad generalization of this situation.

Denote the link of the singularity by Σ. If the leaves of the resolution graph Γ are labelled $i = 1, \ldots, n$, the splice quotient construction associates a coordinate x_i of $\mathbb{C}^n$ with each leaf and specifies $n-2$ complete intersection equations describing a singularity with link the universal abelian cover of Σ, and also describes an explicit action of the covering transformation group.

More specifically, denote the intersection form of the resolution $A(\Gamma)$. Denote $d = \det(A(\Gamma))$ and let

$$L = (\ell_{vw})_{v,w\in\mathrm{vert}(\Gamma)} := \mathrm{adj}(-A(\Gamma))\,,$$

so $A(\Gamma)^{-1} = \frac{-1}{d}L$. The $n-2$ complete intersection equations are grouped into groups of $\delta_v - 2$ equations for each node v of Γ, where δ_v is valency of v. The equations associated to a node v are weighted homogeneous of total weight ℓ_{vv} with each variable x_i given weight ℓ_{vi}. These equations (one also allows addition of terms of higher weight) determine a normal surface singularity $(V, 0)$ whose topology is the universal abelian cover of the topology determined by Γ.

One has a natural isomorphism $H_1(\Sigma) \cong \mathbb{Z}^n/A(\Gamma)\mathbb{Z}^n$. Each standard basis element e_i of Z_n thus determines a generator of $D := H_1(\Sigma)$. If we let this generator act on $\mathbb{C}^n$ by the diagonal matrix $\operatorname{diag}(e^{2\pi i \ell_{i1}/d}, \dots, e^{2\pi i \ell_{in}/d})$, we get an action of D on $\mathbb{C}^n$. If one chooses the complete intersection equations defining $(V, 0)$ to be D–invariant then D acts freely on $V - \{0\}$ and the quotient $(V, 0)/D$ is a normal surface singularity (X, o) whose minimal good resolution graph is Γ. This is what we call a *splice-quotient singularity.* Splice-quotient singularities exist for most Γ, but there are (weak) conditions on Γ for the complete intersection equations to exist and be equivariant with respect to the group action. The above results are in [9].

Suppose $z_i \colon (X, 0) \to (C, 0)$ is an analytic function whose zero set lifted to the minimal good resolution of (X, o) is a smooth connected curve germ transversely intersecting the exceptional curve corresponding to the leaf i of Γ in one point. Then we call f an *end curve function* for the leaf i. In general z_i will vanish to some order $d_i \geq 1$ on the curve germ.

Theorem (End Curve Theorem). *An end curve function z_i exists for each leaf i of Γ if and only if (X, o) is a splice-quotient. A choice of root $x_i = z_i^{1/d_i}$ is well defined on the universal abelian cover of (X, o). These x_i are the coordinates for the splice quotient description.*

A corollary is the positive answer to the conjecture in [8] that if (X, o) is rational or minimally elliptic then it is a splice quotient; this was first proved by Okuma [10]. The end curve theorem has been used by Okuma and Nemethi [11, 3, 4] to compute the geometric genus of splice quotients and thus resolve positively for splice quotients the Casson Invariant Conjecture as well as the Nemethi Nicolaescu generalizations of it.

There is a natural arithmetic analog. A standard "dictionary" that developed out of proposals of Mazur and Manin pairs 3–manifolds with number fields, knots with primes, and so on. A natural analog of universal abelian covers of QHS links belonging to complete intersections would be that the ring of integers of a Hilbert class field be a complete intersection over the ground ring. This is in fact true and was proved by de Smit and Lenstra [1]. The analogy between splice singularities and Hilbert class fields is enticing, since it is an open problem to compute Hilbert class fields in general, while the explicit splice singularity description is easily computed from the resolution diagram.

References

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