# TWISTED BURNSIDE THEOREM

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Abstract. It is proved for a wide class of groups including polycyclic and finitely generated polynomial growth groups that the Reidemeister number of an automorphism $\phi$ is equal to the number of finite-dimensional fixed points of $\widehat{\phi}$ on the unitary dual, if one of these numbers is finite. This theorem is a natural generalization to infinite groups of the classical Burnside theorem. On the other hand it has important consequences in Topological Dynamics. In some sense our theorem is a reply to a remark of J.-P. Serre.

The main technical theorem proved in the paper gives a tool for a further progress.

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## 1. Introduction

**Definition 1.1.** Let $G$ be a countable discrete group and $\phi : G \to G$ an endomorphism. Two elements $x, x' \in G$ are said to be *$\phi$-conjugate* or *twisted conjugate,* iff there exists $g \in G$ with

$$x' = gx\phi(g^{-1}).$$

We shall write $\{x\}_\phi$ for the *$\phi$-conjugacy* or *twisted conjugacy* class of the element $x \in G$. The number of $\phi$-conjugacy classes is called the *Reidemeister number* of an endomorphism $\phi$ and is denoted by $R(\phi)$. If $\phi$ is the identity map then the $\phi$-conjugacy classes are the usual conjugacy classes in the group $G$.

If $G$ is a finite group, then the classical Burnside theorem (see e.g. [17, p. 140]) says that the number of classes of irreducible representations is equal to the number of conjugacy classes of elements of $G$. Let $\widehat{G}$ be the *unitary dual* of $G$, i.e. the set of equivalence classes of unitary irreducible representations of $G$.

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If $\phi : G \to G$ is an automorphism, it induces a map $\widehat{\phi} : \widehat{G} \to \widehat{G}$, $\widehat{\phi}(\rho) = \rho \circ \phi$. Therefore, by the Burnside's theorem, if $\phi$ is the identity automorphism of any finite group $G$, then we have $R(\phi) = \# \operatorname{Fix}(\widehat{\phi})$.

The attempts to generalize this theorem to the case of non-identical automorphism and of non-finite group were inspired by the dynamical questions and were the subject of a series of papers [4, 5, 3, 7, 8, 6].

In the present paper we introduce the following property RP for a countable discrete group $G$: $\phi$-class functions of any automorphism $\phi$ with $R(\phi) < \infty$ are periodic in a natural sense (Definition 3.6).

After some preliminary and technical considerations we prove **the main results** of the paper, namely

(1) **RP respects some extensions:** Suppose, there is an extension $H \to G \to G/H$, where the group $H$ is a characteristic RP-group; $G/H$ is finitely generated FC-group (i.e. a group with finite conjugacy classes). Then $G$ is an RP-group (Theorem 3.9).
(2) **Classes of RP groups:** Polycyclic groups and finitely generated groups of polynomial growth are RP-groups, as well as almost polycyclic groups (Theorems 4.2, 4.4, 4.6). Twisted Burnside theorem is valid for them (Theorem 4.5).
(3) **Twisted Burnside theorem:** Let $G$ be an RP group and $\phi$ its automorphism. Denote by $\widehat{G}_f$ the subset of the unitary dual $\widehat{G}$ related to finite-dimensional representations and by $S_f(\phi)$ the number of fixed points of $\widehat{\phi}_f$ on $\widehat{G}_f$. Then $R(\phi) = S_f(\phi)$ if one of this numbers is finite (Theorem 5.2).

In some sense our theory is a reply to a remark of J.-P. Serre [22, (d), p.34] that for compact infinite groups an analogue of Burnside theorem is not interesting: $\infty = \infty$. It turns out that for infinite discrete groups the situation differs significantly, and even in non-twisted situations the number of classes can be finite (for one of the first examples see another book of J.-P. Serre [23]). A number of examples of groups and automorphisms with finite Reidemeister numbers was obtained and studied in [3, 11, 10, 8, 6].

Using the same argument as in [7] one obtains from the twisted Burnside theorem the following dynamical and number-theoretical consequence which is very important for the study of the Reidemeister zeta-function.

Let $\mu(d)$, $d \in \mathbb{N}$, be the *Möbius function*, i.e.

$$\mu(d) = \begin{cases} 1 & \text{if } d = 1, \\ (-1)^k & \text{if } d \text{ is a product of } k \text{ distinct primes,} \\ 0 & \text{if } d \text{ is not square} - \text{free.} \end{cases}$$

Congruences for Reidemeister numbers: *Let $\phi : G \to G$ be an automorphism of a countable discrete* RP*-group $G$ such that all numbers $R(\phi^n)$ are finite. Then one has for all $n$,*

$$\sum_{d|n} \mu(d) \cdot R(\phi^{n/d}) \equiv 0 \mod n.$$

These theorems were proved previously in a number of special cases in [4, 5, 7, 8, 6].

We would like to emphasize the following important remarks.

(1) As it follows from a key example, which we have studied with A. Vershik in [8], an RP-group can have infinite dimensional "supplementary" fixed representations. More precisely we discuss in that paper the case of a semi-direct product of the action of $\mathbb{Z}$ on $\mathbb{Z} \oplus \mathbb{Z}$ by a hyperbolic automorphism with finite Reidemeister number (four to be precise) and the number of fixed points of $\widehat{\phi}$ on $\widehat{G}$ equal or greater than five, while the number of fixed points on $\widehat{G}_f$ is four.

(2) The origin of this phenomenon lies in bad separation properties of $\widehat{G}$ for general discrete groups. A more deep study leads to the following general theorem.

Weak twisted Burnside theorem [25]: *The number $R_*(\phi)$ of Reidemeister classes related to twisted invariant functions on $G$ from the Fourier-Stieltjes algebra $B(G)$ is equal to the number $S_*(\phi)$ of generalized fixed points of $\widehat{\phi}$ on the Glimm spectrum of $G$, i. e. on the complete regularization of $\widehat{G}$, if one of $R_*(\phi)$ and $S_*(\phi)$ is finite.*

The argument is based on a non-commutative version of the well-known Riesz(-Markov-Kakutani) theorem, which identifies the space of linear functionals on algebra $A = C(X)$ and the space of regular measures on $X$. To prove the weak twisted Burnside theorem we first obtain a generalization of this theorem to the case of non-commutative $C^*$-algebra $A$ via Dauns-Hofmann sectional representation theorem (in the same paper [25]). The corresponding measures on Glimm spectrum are functional-valued. In extreme situation this theorem is tautological, but for group $C^*$-algebras of discrete groups in many cases one obtains some new tool for counting twisted conjugacy classes. This is a basis for an approach to the problem which is an alternative one to the approach developed in the present paper.

(3) In fact the main Theorem 3.9 allows to verify periodicity of $\phi$-class functions in a number of cases which are not in the classes described in Section 4. Keeping in mind that for hyperbolic groups $R(\phi)$ is always infinite [9] while in the "opposite" case the twisted Burnside theorem is proved, we can hope that various use of Theorem 3.9 can lead to a complete resolution of the problem. A support to this gives the fact that in all known examples twisted invariant functionals are coefficients of finite-dimensional representations.

The interest in twisted conjugacy relations has its origins, in particular, in the Nielsen-Reidemeister fixed point theory (see, e.g. [16, 3]), in Selberg theory (see, eg. [24, 1]), and Algebraic Geometry (see, e.g. [14]).

Concerning some topological applications of our main results, they are already described in [7]. The congruences give some necessary conditions for the realization problem for Reidemeister numbers in topological dynamics. The relations with Selberg theory will be presented in a forthcoming paper.

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## 2. Preliminary considerations

The following fact will be useful.

**Theorem 2.1** ([20, Theorem 1.41])**.** *If $G$ is a finitely generated group and $H$ is a subgroup with finite index in $G$, then $H$ is finitely generated.*

**Lemma 2.2.** *Let $G$ be finitely generated, and $H' \subset G$ its subgroup of finite index. Then there is a characteristic subgroup $H \subset G$ of finite index, $H \subset H'$.*

*Proof.* Since $G$ is finitely generated, there is only finitely many subgroups of the same index as $H'$ (see [15], [18, § 38]). Let $H$ be their intersection. Then $H$ is characteristic, in particular normal, and of finite index. □

**Lemma 2.3.** *Let $G$ be abelian. The twisted conjugacy class $H$ of $e$ is a subgroup. The other ones are cosets $gH$.*

*Proof.* The first statement follows from the equalities

$$h\phi(h^{-1})g\phi(g^{-1}) = gh\phi((gh)^{-1}, \quad (h\phi(h^{-1}))^{-1} = \phi(h)h^{-1} = h^{-1}\phi(h).$$

For the second statement suppose $a \sim b$, i.e. $b = ha\phi(h^{-1})$. Then

$$gb = gha\phi(h^{-1}) = h(ga)\phi(h^{-1}), \qquad gb \sim ga.$$

□

Let us denote by $\tau_g : G \to G$ the automorphism $\tau_g(\widetilde{g}) = g\widetilde{g}\phi(g^{-1})$ for $g \in G$. Its restriction on a normal subgroup we will denote by $\tau_g$ as well.

**Lemma 2.4.** $\{g\}_\phi k = \{g\,k\}_{\tau_{k^{-1}} \circ \phi}$.

*Proof.* Let $g' = f\,g\,\phi(f^{-1})$ be $\phi$-conjugate to $g$. Then

$$g'\,k = f\,g\,\phi(f^{-1})\,k = f\,g\,k\,k^{-1}\,\phi(f^{-1})\,k = f\,(g\,k)\,(\tau_{k^{-1}} \circ \phi)(f^{-1}).$$

Conversely, if $g'$ is $\tau_{k^{-1}} \circ \phi$-conjugate to $g$, then

$$g'\,k^{-1} = f\,g\,(\tau_{k^{-1}} \circ \phi)(f^{-1})k^{-1} = f\,g\,k^{-1}\,\phi(f^{-1}).$$

Hence a shift maps $\phi$-conjugacy classes onto classes related to another automorphism. □

**Corollary 2.5.** $R(\phi) = R(\tau_g \circ \phi)$.

**Theorem 2.6** (see [16])**.** *Let $A$ be a finitely generated Abelian group, $\psi : A \to A$ its automorphism. Then $R(\psi) = \#\,\mathrm{Coker}(\psi - \mathrm{Id})$, i.e. to the index of subgroup generated by elements of the form $x^{-1}\psi(x)$.*

*Proof.* By Lemma 2.3, $R(\psi)$ is equal to the index of the subgroup $H = \{e\}_\psi$. This group consists by definition of elements of the form $x^{-1}\psi(x)$. □

## 3. Extensions and Reidemeister classes

Consider a group extension respecting homomorphism $\phi$:

$$\tag{1}\begin{tikzcd} 0 \arrow[r] & H \arrow[r, "i"] \arrow[d, "\phi'"'] & G \arrow[r, "p"] \arrow[d, "\phi"] & G/H \arrow[r] \arrow[d, "\overline{\phi}"] & 0 \\ 0 \arrow[r] & H \arrow[r, "i"] & G \arrow[r, "p"] & G/H \arrow[r] & 0, \end{tikzcd}$$

where $H$ is a normal subgroup of $G$. The argument below, especially related the role of periodic points, has a partial intersection with [11, 12].

First of all let us notice that the Reidemeister classes of $\phi$ in $G$ are mapped epimorphically on classes of $\overline{\phi}$ in $G/H$. Indeed,

$$\tag{2} p(\widetilde{g})p(g)\overline{\phi}(p(\widetilde{g}^{-1})) = p(\widetilde{g}g\phi(\widetilde{g}^{-1})).$$

Suppose, $R(\phi) < \infty$. Then the previous remark implies $R(\overline{\phi}) < \infty$. Consider a class $D = \{h\}_{\tau_g\phi'}$, where $\tau_g(h) := ghg^{-1}$, $g \in G$, $h \in H$. The corresponding equivalence relation is

$$\tag{3} h \sim \widetilde{h}hg\phi'(\widetilde{h}^{-1})g^{-1}.$$

Since $H$ is normal, the automorphism $\tau_g : H \to H$ is well defined. We will denote by $D$ the image $iK$ as well. By (3) the shift $Dg$ is a subset of $Hg$ is characterized by

$$\tag{4} hg \sim \widetilde{h}(hg)\phi'(\widetilde{h}^{-1}).$$

Hence it is a subset of $\{hg\}_\phi \cap Hg$ and the partition $Hg = \cup(\{h\}_{\tau_g\phi'})g$ is a subpartition of $Hg = \cup(Hg \cap \{hg\}_\phi)$.

We need the following statements.

**Lemma 3.1.** *Suppose, the extension* (1) *satisfies the following conditions:*

1. $\#\operatorname{Fix}\overline{\phi} = k < \infty$,
2. $R(\phi) < \infty$.

*Then*

$$\tag{5} R(\phi') \le k \cdot (R(\phi) - R(\overline{\phi}) + 1).$$

*If $G/H$ is abelian, let $g_i$ be some elements with $p(g_i)$ being representatives of all different $\overline{\phi}$-conjugacy classes, $i = 1, \dots, R(\overline{\phi})$. Then*

$$\tag{6} \sum_{i=1}^{R(\overline{\phi})} R(\tau_{g_i}\phi') \le k \cdot R(\phi).$$

*Proof.* Consider classes $\{z\}_{\phi'}$, $z \in G$, i.e. the classes of relation $z \sim hz\phi'(h^{-1})$, $h \in H$. The group $G$ acts on them by $z \mapsto gz\phi(g^{-1})$. Indeed,

$$\begin{aligned} g[\widetilde{h}h\phi(\widetilde{h}^{-1})]\phi(g^{-1}) &= (g\widetilde{h}g^{-1})(gh\phi(g^{-1}))(\phi(g)\phi(\widetilde{h}^{-1})\phi(g^{-1})) \\ &= (g\widetilde{h}g^{-1})(gh\phi(g^{-1}))\phi(g\widetilde{h}g^{-1}) \in \{gh\phi(g^{-1})\}_{\phi'}, \end{aligned}$$

because $H$ is normal and $g\widetilde{h}g^{-1} \in H$. Due to invertibility, this action of $G$ transposes classes $\{z\}_{\phi'}$ inside one class $\{g\}_\phi$. Hence, the number $d$ of classes $\{h\}_{\phi'}$ inside $\{h\}_\phi \cap H$

does not exceed the number of $g \in G$ such that $p(g)\overline{\phi}(p(g^{-1})) = \overline{e}$. Since two elements $g$ and $gh$ in one $H$-coset induce the same permutation of classes $\{h\}_{\phi'}$, the mentioned number $d$ does not exceed the number of $z \in G/H$ such that $z\overline{\phi}(z^{-1}) = \overline{e}$, i.e. $d \leq k$. This implies (5).

Now we discuss $\phi$-classes over $\overline{\phi}$-classes other than $\{\overline{e}\}_{\overline{\phi}}$ for an abelian $G/H$. An estimation analogous to the above one leads to the number of $z \in G/H$ such that $zz_0\overline{\phi}(z^{-1}) = z_0$ for some fixed $z_0$. But for an Abelian $G/H$ they form the same group $\mathrm{Fix}(\overline{\phi})$. This together with the description (4) of shifts of $D$ at the beginning of the present Section implies (6). □

**Lemma 3.2.** *Suppose, in the extension* (1) *the group $H$ is abelian. Then* $\#\,\mathrm{Fix}(\phi) \leq \#\,\mathrm{Fix}(\phi') \cdot \#\,\mathrm{Fix}(\overline{\phi})$.

*Proof.* Let $s : G/H \to G$ be a section of $p$. If $s(z)h$ is a fixed point of $\phi$ then

$$(s(z))^{-1}\phi(s(z)) = h\phi'(h^{-1}). \tag{7}$$

Hence, $z \in \mathrm{Fix}(\overline{\phi})$ and left hand side takes $k := \#\,\mathrm{Fix}(\overline{\phi})$ values $h_1, \dots, h_k$. Let us estimate the number of $s(z)h$ for a fixed $z$ such that $(s(z))^{-1}\phi(s(z)) = h_i$. These $h$ have to satisfy (7). Since $H$ is abelian, if one has

$$h_i = h\phi'(h^{-1}) = \widetilde{h}\phi'(\widetilde{h}^{-1}),$$

then $h^{-1}\widetilde{h} \in \mathrm{Fix}(\phi')$ and we are done. □

**Theorem 3.3.** *Let $A$ be a finitely generated Abelian group, $\psi : A \to A$ its automorphism with $R(\psi) < \infty$. Then $\#\,\mathrm{Fix}(\psi) < \infty$.*

*Moreover, $R(\psi) \geq \#Fix(\psi)$.*

*Proof.* Let $T$ be the torsion subgroup. It is finite and characteristic. We obtain the extension $T \to A \to A/T$ respecting $\phi$. Since $A/T \cong \mathbb{Z}^k$, $\mathrm{Fix}(\overline{\psi} : A/T \to A/T) = \overline{e}$, by [16],[3, Sect. 2.3.1]. Hence, by Lemma 3.2, $\#\,\mathrm{Fix}(\psi) \leq \#\,\mathrm{Fix}(\psi')$, $\psi' : T \to T$. For any finite abelian group $T$ one clearly has $\#\,\mathrm{Fix}(\psi') = R(\psi')$ by Theorem 2.6 (cf. [3, p. 7]). Finally, $R(\psi') \leq R(\psi)$ by (6). □

Let us remind the following definitions of a class of groups.

**Definition 3.4.** A group with finite conjugacy classes is called FC-*group.*

In a FC-group the elements of finite order form a characteristic subgroup with locally infinite abelian factor group; a finitely generated FC-group contains in its center a free abelian group of finite index in the whole group [19].

**Lemma 3.5.** *An automorphism $\phi$ of a finitely generated* FC-*group $G$ with $R(\phi) < \infty$ has a finite number of fixed points.*

*Proof.* Let $A$ be the center of $G$. As it was indicated, $A$ has a finite index in $G$ and hence, by Theorem 2.1, is f.g. Since $A$ is characteristic, one has an extension $A \to G \to G/A$ respecting $\phi$. One has $R(\phi') \leq R(\phi) \cdot |G/A|$ by Lemma 3.1 and $\#\,\mathrm{Fix}(\phi') \leq R(\phi) \cdot |G/A|$ by Theorem 3.3. Then $\#\,\mathrm{Fix}(\phi') \leq R(\phi) \cdot |G/A|^2$ by Lemma 3.2. □

**Definition 3.6.** We say that a group $G$ has the property RP if for any automorphism $\phi$ with $R(\phi) < \infty$ the characteristic functions $f$ of Reidemeister classes (hence all $\phi$-central functions) are periodic in the following sense.

There exists a finite group $K$, its automorphism $\phi_K$, and epimorphism $F : G \to K$ such that

(1) The diagram

$$\begin{tikzcd} G \arrow[r, "\phi"] \arrow[d, "F"'] & G \arrow[d, "F"] \\ K \arrow[r, "\phi_K"] & K \end{tikzcd}$$

commutes.

(2) $f = F^* f_K$, where $f_K$ is a characteristic function of a subset of $K$.

**Remark 3.7.** By (2) there is only one class $\{g\}_\phi$ which maps onto $\{F(g)\}_{\phi_K}$.

**Lemma 3.8.** *Suppose, $G$ is f.g. and $R(\phi) < \infty$. Then characteristic functions of $\phi$-conjugacy classes are periodic if their left shifts generate a finite dimensional space.*

*Proof.* From the supposition it follows that the stabilizer of each $\phi$-conjugacy class has finite index. Hence, the common stabilizer of all $\phi$-conjugacy classes under left shifts is an intersection of finitely many subgroups, each of finite index. Hence, its index is finite. By Lemma 2.2 there is some smaller subgroup $G_S$ of finite index which is normal and $\phi$-invariant. Then one can take $K = G/G_S$. Indeed, it is sufficient to verify that the projection $F$ is one to one on classes. In other words, that each coset of $G_s$ enters only one $\phi$-conjugacy class, or any two elements of coset are $\phi$-conjugated. Consider $g$ and $hg$, $g \in G$, $h \in G_s$. Since $h$ by definition preserves classes, $hg = xg\phi(x^{-1})$ for some $x \in G$, as desired. □

**Theorem 3.9.** *Suppose, the extension* (1) *satisfies the following conditions:*

(1) *$H$ has* RP*;*

(2) *$G/H$ is* FC *f.g.*

*Then $G$ has* RP*.*

*Proof.* We have $R(\overline{\phi}) < \infty$, hence $\# \operatorname{Fix}(\overline{\phi}) < \infty$ by Lemma 3.5. Then by Lemma 3.1 $R(\phi') < \infty$. Now we can apply the supposition that $H$ has RP and find the normal subgroup $H_K := \operatorname{Ker} F \subset H$ of finite index. Hence, we can take a quotient by $H_K$ of the extension (1):

$$\begin{tikzcd} H \arrow[r, hook] \arrow[d] & G \arrow[d, "F_1"] \arrow[dr, "p"] & \\ H/H_K \arrow[r, hook] \arrow[d, equal] & G/H_K \arrow[r, "p"] \arrow[d, equal] & G/H \arrow[d, equal] \\ K \arrow[r, hook] & G_1 \arrow[r, "p"] & \Gamma. \end{tikzcd}$$

The quotient map $F_1 : G \to G/H_K$ takes $\{g\}_\phi$ to $\{g\}_\phi$ and it is a unique class with this property (we conserve the notations $e, g, \phi$ for the quotient objects). Indeed, this follows from the commutativity of the upper square in the diagram above and from Remark 3.7.

By Lemma 3.8 (applied to $G_1$ and concrete automorphism $\phi$) for the purpose to find a map $F_2 : G_1 \to K_1$ with properties (1) and (2) of the Definition 3.6 it is sufficient to verify that shifts of the characteristic function of $\{h\}_\phi \subset G_1$ form a finite dimensional space, i.e. the shifts of $\{h\}_\phi \subset G_1$ form a finite collection of subsets of $G_1$. After that one can take the composition

$$G \xrightarrow{F_1} G_1 \xrightarrow{F_2} K_1$$

to complete the proof of theorem.

Let us observe, that we can apply Lemma 3.8 because the group $G_1$ is finitely generated: we can take as generators all elements of $K$ and some pre-images $s(z_i) \in G_1$ under $p$ of a finite system of generators $z_i$ for $\Gamma$. Indeed, for any $x \in G_1$ one can find some product of $z_i$ to be equal to $p(x)$. Then the same product of $s(z_i)$ differs from $x$ by an element of $K$.

Let us prove that the mentioned space of shifts is finite-dimensional. By Lemma 2.4 these shifts of $\{h\}_\phi \subset G_1$ form a subcollection of

$$\{x\}_{\tau_y \circ \phi}, \quad x, y \in G_1.$$

Hence, by Corollary 2.5 it is sufficient to verify that the number of different automorphisms $\tau_y : G_1 \to G_1$ is finite.

Let $x_1, \dots, x_n$ be some generators of $G_1$. Then the number of different $\tau_y$ does not exceed

$$\prod_{j=1}^{n} \#\{\tau_y(x_j) \mid y \in G_1\} \le \prod_{j=1}^{n} |K| \cdot \#\{\tau_z(p(x_j)) \mid z \in \Gamma\},$$

where the last numbers are finite by the definition of FC for $\Gamma$. $\square$

## 4. Polycyclic and polynomial growth groups

Now we describe using Theorem 3.9 some classes of RP groups. Of course these classes are only a small part of possible corollaries of this theorem.

Let $G' = [G, G]$ be the *commutator subgroup* or *derived group* of $G$, i.e. the subgroup generated by commutators. $G'$ is invariant under any homomorphism, in particular it is normal. It is the smallest normal subgroup of $G$ with an abelian factor group. Denoting $G^{(0)} := G$, $G^{(1)} := G'$, $G^{(n)} := (G^{(n-1)})'$, $n \ge 2$, one obtains *derived series* of $G$:

$$G = G^{(0)} \supset G' \supset G^{(2)} \supset \cdots \supset G^{(n)} \supset \dots \tag{8}$$

If $G^{(n)} = e$ for some $n$, i.e. the series (8) stabilizes by trivial group, the group $G$ is *solvable*;

**Definition 4.1.** A solvable group with derived series with cyclic factors is called *polycyclic group*.

**Theorem 4.2.** *Any polycyclic group is* RP.

*Proof.* By Lemma 2.3 any commutative group is RP. Any extension with $H$ being the commutator subgroup $G'$ of $G$ respects any automorphism $\phi$ of $G$, because $G'$ is evidently characteristic. The factor group is abelian, in particular FC.

Since any polycyclic group is a result of finitely many such extensions with finitely generated (cyclic) factor groups, starting from Abelian group, applying inductively Theorem 3.9 we obtain the result. $\square$

**Theorem 4.3.** *Any finitely generated nilpotent group is* RP*.*

*Proof.* These groups are supersolvable, hence, polycyclic [21, 5.4.6, 5.4.12]. □

**Theorem 4.4.** *Any finitely generated group of polynomial growth is* RP*.*

*Proof.* By [13] a finitely generated group of polynomial growth is just as finite extension of a f.g. nilpotent group $H$. The subgroup $H$ can be supposed to be characteristic, i.e. $\phi(H) = H$ for any automorphism $\phi : G \to G$. Indeed, let $H' \subset G$ be a nilpotent subgroup of index $j$. Let $H$ be the subgroup from Lemma 2.2. By Theorem 2.1 it is finitely generated. Also, it is nilpotent as a subgroup of nilpotent group (see [18, § 26]).

Since a finite group is a particular case of FC group and f.g. nilpotent group has RP by Theorem 4.3, we can apply Theorem 3.9 to complete the proof. □

The proof of the following twisted Burnside theorem can be extracted from Theorem 5.2, but we formulate it separately due to importance of the classes under consideration.

**Theorem 4.5.** *The Reidemeister number of any automorphism $\phi$ of a f.g. group of polynomial growth or polycyclic group is equal to the number of finite-dimensional fixed points of $\widehat{\phi}$ on the unitary dual of this group.*

**Theorem 4.6.** *Any almost polycyclic group, i.e. an extension of a polycyclic group with a finite factor group, is* RP*.*

*Proof.* The proof repeats almost literally the proof of Theorem 4.4. One has to use the fact that a subgroup of a polycyclic group is polycyclic [21, p. 147]. □

## 5. Twisted Burnside theorem for RP groups

**Definition 5.1.** Denote by $\widehat{G}_f$ the subset of the unitary dual $\widehat{G}$ related to finite-dimensional representations.

**Theorem 5.2** (Twisted Burnside Theorem)**.** *Let $G$ be an* RP *group and $\phi$ its automorphism. Denote by $S_f(\phi)$ the number of fixed points of $\widehat{\phi}_f$ on $\widehat{G}_f$. Then*

$$R(\phi) = S_f(\phi)$$

*if one of this numbers is finite.*

*Proof.* Let us start from the following observation. Let $\Sigma$ be the universal compact group associated with $G$ and $\alpha : G \to \Sigma$ the canonical morphism (see, e.g. [2, Sect. 16.1]). Then $\widehat{G}_f = \widehat{\Sigma}$ [2, 16.1.3]. The coefficients of (finite-dimensional) non-equivalent irreducible representations of $\Sigma$ are linear independent by Peter-Weyl theorem as functions on $\Sigma$. Hence the coefficients of finite-dimensional non-equivalent irreducible representations of $G$ as functions on $G$ are linearly independent as well.

It is sufficient to verify the following three statements:

1) If $R(\phi) < \infty$, than each $\phi$-class function is a finite linear combination of twisted-invariant functionals being coefficients of points of $\operatorname{Fix}\widehat{\phi}_f$.

2) If $\rho \in \operatorname{Fix}\widehat{\phi}_f$, there exists one and only one (up to scaling) twisted invariant functional on $\rho(C^*(G))$ (this is a finite full matrix algebra).

3) For different $\rho$ the corresponding $\phi$-class functions are linearly independent. This follows from the remark at the beginning of the proof.

Let us remark that the property RP implies in particular that $\phi$-central functions (for $\phi$ with $R(\phi) < \infty$) are functionals on $C^*(G)$, not only $L^1(G)$, i.e. are in the Fourier-Stieltijes algebra $B(G)$.

The statement 1) follows from the RP property. Indeed, this $\phi$-class function $f$ is a linear combination of functionals coming from some finite collection $\{\rho_i\}$ of elements of $\widehat{G}_f$ (these representations $\rho_1, \dots, \rho_s$ are in fact representations of the form $\pi_i \circ F$, where $\pi_i$ are irreducible representations of the finite group $K$ and $F : G \to K$, as in the definition of RP). So,

$$f = \sum_{i=1}^{s} f_i \circ \rho_i, \quad \rho_i : G \to \mathrm{End}(V_i), \quad f_i : \mathrm{End}(V_i) \to \mathbb{C}, \quad \rho_i \neq \rho_j,\ (i \neq j).$$

For any $g, \widetilde{g} \in G$ one has

$$\sum_{i=1}^{s} f_i(\rho_i(\widetilde{g})) = f(\widetilde{g}) = f(g\widetilde{g}\phi(g^{-1})) = \sum_{i=1}^{s} f_i(\rho_i(g\widetilde{g}\phi(g^{-1}))).$$

By the observation at the beginning of the proof concerning linear independence,

$$f_i(\rho_i(\widetilde{g})) = f_i(\rho_i(g\widetilde{g}\phi(g^{-1}))). \qquad i = 1, \dots, s,$$

i.e. $f_i$ are twisted-invariant. For any $\rho \in \widehat{G}_f$, $\rho : G \to \mathrm{End}(V)$, any functional $\omega : \mathrm{End}(V) \to \mathbb{C}$ has the form $a \mapsto \mathrm{Tr}(ba)$ for some fixed $b \in \mathrm{End}(V)$. Twisted invariance implies twisted invariance of $b$ (evident details can be found in [7, Sect. 3]). Hence, $b$ is intertwining between $\rho$ and $\rho \circ \phi$ and $\rho \in \mathrm{Fix}(\widehat{\phi}_f)$. The uniqueness of intertwining operator (up to scaling) implies 2). □

**Remark 5.3.** We were able to prove only one statement of the theorem in the terms of $\Sigma$ because of difficulties with an extension of $\phi$ to $\Sigma$.

## References


1. J. Arthur and L. Clozel, *Simple algebras, base change, and the advanced theory of the trace formula*, Princeton University Press, Princeton, NJ, 1989. MR **90m:**22041
2. J. Dixmier, *C*-algebras*, North-Holland, Amsterdam, 1982.
3. A. Fel'shtyn, *Dynamical zeta functions, Nielsen theory and Reidemeister torsion*, Mem. Amer. Math. Soc. **147** (2000), no. 699, xii+146. MR **2001a:**37031
4. A. Fel'shtyn and R. Hill, *The Reidemeister zeta function with applications to Nielsen theory and a connection with Reidemeister torsion*, $K$-Theory **8** (1994), no. 4, 367–393. MR **95h:**57025
5. ______, *Dynamical zeta functions, congruences in Nielsen theory and Reidemeister torsion*, Nielsen theory and Reidemeister torsion (Warsaw, 1996), Polish Acad. Sci., Warsaw, 1999, pp. 77–116. MR **2001h:**37047
6. A. Fel'shtyn, F. Indukaev, and E. Troitsky, *Twisted Burnside theorem for two-step torsion-free nilpotent groups*, Preprint, Max-Planck-Institut für Mathematik, 2005.
7. A. Fel'shtyn and E. Troitsky, *A twisted Burnside theorem for countable groups and Reidemeister numbers*, Proc. Workshop Noncommutative Geometry and Number Theory (Bonn, 2003) (K. Consani, M. Marcolli, and Yu. Manin, eds.), Vieweg, Braunschweig, 2004, (Preprint MPIM2004-65), pp. 000–000.
8. A. Fel'shtyn, E. Troitsky, and A. Vershik, *Twisted Burnside theorem for type $II_1$ groups: an example*, Preprint 85, Max-Planck-Institut für Mathematik, 2004, (submitted to *Math. Res. Lett.*).
9. A. L. Fel'shtyn, *The Reidemeister number of any automorphism of a Gromov hyperbolic group is infinite*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **279** (2001), no. 6 (Geom. i Topol.), 229–240, 250. MR **2002e:**20081

10. Alexander Fel′shtyn, Richard Hill, and Peter Wong, *Reidemeister numbers of equivariant maps*, Topology Appl. **67** (1995), no. 2, 119–131. MR **MR1362078 (96j:**58139)
11. D. Gonçalves and P. Wong, *Twisted conjugacy classes in exponential growth groups*, Bull. London Math. Soc. **35** (2003), no. 2, 261–268. MR **2003j:**20054
12. Daciberg L. Gonçalves, *The coincidence Reidemeister classes of maps on nilmanifolds*, Topol. Methods Nonlinear Anal. **12** (1998), no. 2, 375–386. MR **MR1701269 (2000d:**55004)
13. Mikhael Gromov, *Groups of polynomial growth and expanding maps*, Inst. Hautes Études Sci. Publ. Math. (1981), no. 53, 53–73. MR **MR623534 (83b:**53041)
14. A. Grothendieck, *Formules de Nielsen-Wecken et de Lefschetz en géométrie algébrique*, Séminaire de Géométrie Algébrique du Bois-Marie 1965-66. SGA 5, Lecture Notes in Math., vol. 569, Springer-Verlag, Berlin, 1977, pp. 407–441.
15. Ph. Hall, *A characteristic property of soluble groups*, J. London Math. Soc. **12** (1937), 198–200.
16. B. Jiang, *Lectures on Nielsen fixed point theory*, Contemp. Math., vol. 14, Amer. Math. Soc., Providence, RI, 1983.
17. A. A. Kirillov, *Elements of the theory of representations*, Springer-Verlag, Berlin Heidelberg New York, 1976.
18. A. G. Kurosh, *The theory of groups*, Translated from the Russian and edited by K. A. Hirsch. 2nd English ed. 2 volumes, Chelsea Publishing Co., New York, 1960. MR MR0109842 (22 #727)
19. B. H. Neumann, *Groups with finite classes of conjugate elements*, Proc. London Math. Soc. (3) **1** (1951), 178–187. MR MR0043779 (13,316c)
20. Derek J. S. Robinson, *Finiteness conditions and generalized soluble groups. Part 1*, Springer-Verlag, New York, 1972, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 62. MR MR0332989 (48 #11314)
21. ———, *A course in the theory of groups*, second ed., Graduate Texts in Mathematics, vol. 80, Springer-Verlag, New York, 1996. MR **MR1357169 (96f:**20001)
22. Jean-Pierre Serre, *Linear representations of finite groups*, Springer-Verlag, New York, 1977, Translated from the second French edition by Leonard L. Scott, Graduate Texts in Mathematics, Vol. 42. MR MR0450380 (56 #8675)
23. ———, *Trees*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003, Translated from the French original by John Stillwell, Corrected 2nd printing of the 1980 English translation. MR **MR1954121 (2003m:**20032)
24. S. Shokranian, *The Selberg-Arthur trace formula*, Springer-Verlag, Berlin, 1992, Based on lectures by James Arthur. MR **93j:**11029
25. E. Troitsky, *Noncommutative Riesz theorem and weak twisted Burnside theorem*, Funct. Anal. Appl. **00** (2005), 00–00, (accepted). (Preprint 86, Max-Planck-Institut für Mathematik, 2004).



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