NON-EXISTENCE RESULTS AND GROWTH PROPERTIES FOR HARMONIC MAPS AND FORMS

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1. Introduction and results

(1.1) In [L1] Lemaire showed that any harmonic map from a 2-disc with constant Dirichlet data must be constant. Non-existence results for harmonic maps from higher dimensional discs and other domains were first obtained in a preliminary report by the second author [W]. Improved non-existence results for the special case of rotationally symmetric harmonic maps from Euclidean space to a sphere have been given by Jäger and Kaul [J-K2]. Also related are L^2 vanishing theorems given by Sealey [S2]. The regularity results of Schoen and Uhlenbeck [S-U] depend on growth properties for harmonic maps which they derived for energy minimizing maps; such growth properties (which imply Liouville theorems) are also dealt with by Price and Simon [P-S]. The present paper developed from [W] which, in turn, built on [G-R-S-B] and [S1].

We obtain non-existence results which include those of [W] under geometrical assumptions on the domain which are weaker than those used by the above authors. Our proof depends on an identity (2.5) for vector bundle-valued harmonic p-forms which in the case of a rotationally symmetric harmonic map specializes to a first integral of the radial differential equation. This

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identity also implies growth properties (4.1) which give:

(a) Liouville theorems (4.1.1) for domains with narrow negative curvature bounds,

(b) an explicit estimate (4.1.2) of the energy $\int_{D_\varepsilon} |\omega|^2$ of a harmonic p-form ω over a small ball D_ε in terms of the energy over a fixed larger ball. Such inequalities restrict the singularities of ω see [H-W,S-U].

In § 5 we show that for "equivariant" harmonic maps between discs with certain rotationally symmetric metrics, the image of the map must lie in a disc of a certain radius; this gives <u>non</u>-existence results for equivariant harmonic maps with certain <u>non</u>-constant boundary values. In the case of rotationally symmetric maps from a Euclidean disc to a Euclidean sphere such results have been given by Jäger and Kaul [J-K2]. Our proof generalizes to some other cases of equivariant harmonic maps between manifolds but the results are somewhat technical and are therefore omitted. For some contrasting existence results see [H-K-W,J-K1].

Throughout the paper let $M = M^m$ denote a smooth m-dimensional Riemannian manifold (mostly $m > 2$); all data will be assumed smooth unless otherwise indicated.

(1.2) To describe our non-existence results more precisely we consider the following <u>geometric situation</u>

<u>Let</u> $M_1 \subset M^m$ <u>be a compact submanifold of</u> M^m <u>of codimension at least</u> 3 . <u>Let</u> T_r <u>denote the tube of points of</u> M^m <u>at a distance</u> r <u>from</u> M_1 . <u>Finally, let</u> R <u>be a positive number such that</u> <u>for all</u> $r \in (0,R)$ <u>the following condition holds</u>:

$(*_1)$ $\min_{T_r}(\text{sum of the principal curvatures of } T_r)$

$$> (\text{largest principal curvature of } T_r) .$$

(The sign convention is such that small spheres have positive principal curvatures.)

(1.3) <u>Examples</u>: (i) For any given $M_1 \subset M^m$ $(*_1)$ will hold for any sufficiently small R since codim $M_1 - 1$ principal curvatures behave as $1/r$ as $r \to 0$ (the other principal curvatures being bounded as $r \to 0$).

(ii) Let M_1 be a point so that the T_r are the distance spheres around it. Then:

If M^m is the standard sphere S^m , $(*_1)$ holds for any $R \leq \pi/2$;

If M^m is complex (resp. quaternionic) projective space with maximum sectional curvature $= 4$ then the principal curvatures are $2 \cot 2r$ with multiplicity 1 (resp. 3) and $\cot r$ with multiplicity $m-2$ (resp. $m-4$) [B-K]. Hence $(*_1)$ holds if $(m-2)\cot R > \operatorname{tg} R$ (resp. $> 3 \operatorname{tg} R$) ;

If M^m is a simply connected non-compact symmetric space of rank 1 then $(*_1)$ holds for any R since the principal curvatures are constant and > 0 over each distance sphere.

(iii) Let M^m be the unit sphere S^m and let M_1 be a great sphere S^{m_1} of M^m . The principal curvatures of the distance tubes T_r are $-\tan r$ with multiplicity m_1 and $\cotan r$ with multiplicity $m-1-m_1$. Hence $(*_1)$ holds for any R with $(\cot R)\cdot(m-2-m_1) > (\tan R)m_1$.

Our non-existence theorem for harmonic maps can now be stated:

(1.4) <u>THEOREM</u>: <u>In the geometric situation</u> (1.2) <u>let</u> D <u>be a domain with smooth boundary such that</u> (i) $M_1 \subset D \subset \bigcup_{r<R} T_r$ <u>and</u> (ii) <u>at all points</u> $x \in \partial D$ <u>the outer normals</u> $\hat{V}$ <u>of the tube</u> T_r <u>through</u> x <u>point outward from</u> D , i.e. $\langle \hat{V}, n \rangle \geq 0$ <u>where</u> n <u>denotes the outer normal to</u> ∂D <u>at</u> x .

<u>Then any smooth harmonic map</u> $\phi : D \to N$ <u>to an arbitrary smooth Riemannian manifold</u> N <u>with constant Dirichlet data</u> $\phi(\partial D) = y \in N$ <u>is constant</u>. <u>In particular no non-trivial homotopy class of maps</u> $(D, \partial D) \to (N, y)$ <u>has a harmonic representative</u>.

(1.5) <u>Remarks</u>: (i) Contrast the theorem with the existence results of Hamilton and Lemaire which assert for compact smooth Riemannian manifolds M,N with $\partial N = \phi$ if <u>either</u> N has non-positive sectional curvatures [H] or $\pi_2(N) = 0$ and dim M = 2 [L2] then every relative homotopy class of mappings $(M, \partial M) \to N$ has a harmonic representative.

(ii) There is an analagous theorem for harmonic p-forms with values in a Riemannian-connected vector bundle. This is formulated and proved in § 3 under the stronger geometric assumption:

$$(*_p) \qquad \min_{T_r}(\text{sum of the principal curvatures of } T_r)$$

$$> (2p-1) \quad (\text{largest principal curvature of } T_r) .$$

This curvature assumption is similar to Sealey's [S2] who has 2p instead of 2p-1 in his L^2-vanishing theorem.

(iii) It will also be clear from the proof (§ 2) that D , instead of being required to contain M_1 , may also have an "inner boundary" where the outer normals of the distance tubes T_r point into D ; then zero tangential Dirichlet data on the outer and zero normal data on the inner boundary force the harmonic form to be trivial.

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2. The basic identity

There will be no geometric assumptions in this section; these will only be needed (in § 3) to draw conclusions from the basic identity.

(2.1) Let D be a domain in a smooth Riemannian manifold. We shall assume that D has smooth boundary ∂D ; however the basic identity (2.5) immediately generalizes to a domain $\tilde{D}$ with non-smooth boundary provided $\tilde{D}$ can be approximated by domains with smooth boundary in such a way that the boundary integrals converge. For example $\tilde{D}$ may have piecewise smooth boundary. We denote the outer unit normal along ∂D by n .

For details on the following concepts see [E-L1, E-L2]. Let E be a Riemannian-connected vectorbundle over M , i.e. a real vector bundle equipped with metric $\langle\ ,\ \rangle$ and metric connection ∇ . These have natural extensions to E-valued forms (and other tensors):

$$\langle\omega,\tilde{\omega}\rangle = \frac{1}{p!}\sum_{(i_j)=1}^{m} \langle\omega(e_{i_1},\ldots,e_{i_p}),\tilde{\omega}(e_{i_1},\ldots,e_{i_p})\rangle$$

with $\{e_1,\ldots,e_m\}$ an orthonormal basis of T_xM ; and for vectorfields $X_1,\ldots$

$$(\nabla\omega)(X_1,\ldots,X_p) = \nabla(\omega(X_1,\ldots,X_p)) - \sum_j \omega(X_1,\ldots\nabla X_j,\ldots X_p) .$$

Let D and δ denote the corresponding exterior differential and codifferential operators:

$$d\omega(X_0,\ldots,X_p) = \sum_{j=0}^{p} (-1)^j (\nabla_{X_j}\omega)(X_0,\ldots\hat{X}_j,\ldots X_p) ,$$

$$\delta\omega(X_2,\ldots,X_p) = -\sum_{i=1}^{m} (\nabla_{e_i}\omega)(e_i,X_2,\ldots,X_p) .$$

An E-valued p-form ω is called harmonic iff it is closed $(d\omega=0)$ and coclosed $(\delta\omega=0)$.

(2.2) <u>Examples</u>: (i) The differential $d\Phi$ of a smooth map $\Phi : D \to N$ is a 1-form with values in the pull-back bundle $\Phi^{-1}TN$. Pull back the metric and Levi-Civita connection from TN . Then: Φ is a harmonic map iff $d\Phi$ is a harmonic 1-form.

(ii) The shape operator $S = -\nabla n$ of a hypersurface H in M (unit normal field n) is a TH-valued 1-form. $dS = 0$ are the constant curvature Codazzi equations and $\delta S = 0$ if H has constant mean curvature.

(iii) The curvature tensor R of an Einstein manifold is a 2-form with values in the bundle of skew-symmetric endomorphisms of TM ; $dR = 0$ is the second Bianchi-identity and $\delta R = 0$ since $\nabla\,Ric = 0$.

(2.3) We use the following notation: Given an E-valued p-form ω , a vector field X and an endomorphism field L , define E-valued forms ω_X , ω^L by

(2.3.1) $$\omega_X(V_1,\dots,V_{p-1}) = \omega(X,V_1,\dots,V_{p-1})$$

$$\omega^L(V_1,\dots,V_p) = \omega(LV_1,V_2,\dots,V_p) + \dots + \omega(V_1,\dots,LV_p) .$$

In particular, along ∂D we have the tangential and normal projections $P,Q : TM \to TM$

$$PX = X - \langle X,n\rangle n \ , \quad Q(X) = (1-P)X \ ,$$

which are used to define

$\omega^{tan}(X_1$

Then, for E-valu

The "integration by parts" formula now reads as follows

(2.4) LEMMA: For any harmonic p-form ω and C^1-vector field V defined on a domain D with smooth boundary we have

$$\int_D \{\langle\nabla_V\omega,\omega\rangle + \langle\omega^{\nabla V},\omega\rangle\} = \int_{\partial D} \langle\omega_V,\omega_n\rangle \ .$$

Proof: Let Z be the vector field representing the 1-form $X \to \langle\omega_V,\omega_X\rangle$, i.e.

$$\langle Z,X\rangle = \langle\omega_V,\omega_X\rangle \quad \text{for arbitrary } X \ .$$

To compute $\operatorname{div} Z = \sum \langle\nabla_{e_i} Z,e_i\rangle$ at $x \in D$ we may assume $\nabla_{e_i} e_i = 0$ at x to drop terms which cancel anyway. Then

$$\begin{aligned}\operatorname{div} Z &= \sum_i \nabla_{e_i}\langle\omega_V,\omega_{e_i}\rangle \\ &= \sum_i \{\langle(\nabla_{e_i}\omega)_V,\omega_{e_i}\rangle + \langle\omega_{\nabla_{e_i}V},\omega_{e_i}\rangle + \langle\omega_V,(\nabla_{e_i}\omega)_{e_i}\rangle\} \ .\end{aligned}$$

Now use $\langle(d\omega)_V,\tilde\omega\rangle = \langle\nabla_V\omega,\tilde\omega\rangle - \sum_i \langle(\nabla_{e_i}\omega)_V,\tilde\omega_{e_i}\rangle$ (note the normalization by $\frac{1}{(p-1)!}$ in the case of (p-1)-forms) to obtain

$$\operatorname{div} Z = \langle\nabla_V\omega,\omega\rangle + \langle\omega^{\nabla V},\omega\rangle - \langle(d\omega)_V,\omega\rangle - \langle\omega_V,\delta\omega\rangle \ ,$$

hence the lemma.

(2.5) PROPOSITION (The basic identity): Let ω be a harmonic p-form with values in a Riemannian-connected vector bundle defined on a domain D in M with smooth boundary. Let V be an arbitrary C^1 vector field on D; let V^{tan} denote its tangential component on ∂D. Then

$$\int_{\partial D}\langle V,n\rangle|\omega^{norm}|^2 - \int_{\partial D}\langle V,n\rangle|\omega^{tan}|^2 + 2\int_{\partial D}\langle\omega_{V^{tan}},\omega_n\rangle + \int_D|\omega|^2 \operatorname{div} V - 2\int_D\langle\omega^{\nabla V},\omega\rangle = 0 \ .$$

(2.6) COROLLARY: With hypotheses as for the above Proposition, suppose that V can be chosen such that

(2.6.1) $\quad |\omega|^2 \operatorname{div} V - 2 \langle \omega^{\nabla V}, \omega \rangle \geq 0$

with equality if and only if $\omega \equiv 0$. Further suppose that $\omega^{norm} = 0$ on the "inner boundary" i.e. the part of ∂D with $\langle V,n \rangle < 0$ and $\omega^{tan} = 0$ where $\langle V,n \rangle \geq 0$.

Then $\quad \omega \equiv 0$.

The growth property will follow from the basic identity by estimating the volume integral over balls D_r , see § 4 .

(2.7) Proof of Proposition (2.5): Let D' be a domain with $\overline{D'} \subset D$. Let $\psi_t : D' \to M$ be the flow of V , i.e. the solution to the initial value problem $\frac{d}{dt}\psi_t(x) = V(\psi_t(x))$, $\psi_o(x) = x$. For sufficiently small t , $\psi_t(D') \subset D$. We define forms ω_t on D' by transporting ω by the flow, viz.

(2.7.1) $\quad \omega_t(X_1,\ldots,X_p) = (\omega\circ\psi_t)(d\psi_t(X_1),\ldots,d\psi_t(X_p))$.

The proof consists of computing $\frac{d}{dt}\int_{D'} |\omega_t|^2\big|_{t=0}$ in two different ways, (a) directly using integration by parts, (b) after the change of variable $y = \psi_t(x)$. The difference of the two formulae thus obtained is the basic identity.

(a) First calculation

For any point $x \in M$ and any $X_1,\ldots,X_p \in T_xM$ set $\tilde{X}_j = d\psi_t(X_j)$. Note that $\nabla_V \tilde{X}_j = \frac{\nabla}{dt} d\psi_t(X_j)\big|_{t=0} = \nabla_{X_j} V$ because of the symmetry of the Levi-Civita connection.

From the definition of ω_t

$$\frac{\nabla}{dt}\{\omega_t(X_1,\dots,X_p)\}\Big|_{t=0} = (\nabla_V\omega)^t(X_1,\dots,X_p) + \omega(\nabla_V\tilde{X}_1,X_2,\dots,X_p) + \dots\Big|_{t=}$$

hence

$$\frac{\nabla}{dt}\omega_t\Big|_{t=0} = \nabla_V\omega + \omega^{\nabla V}$$

and

$$\frac{d}{dt}<\omega_t,\omega_t>\Big|_{t=0} = 2<\nabla_V\omega,\omega> + 2<\omega^{\nabla V},\omega> \quad .$$

Finally with Lemma (2.4):

$$\text{(2.7.2)} \qquad \frac{d}{dt}\int_{D'}|\omega_t|^2 = 2\int_{\partial D'}<\omega_V,\omega_n>$$

$$= 2\int_{\partial D'}<V,n>|\omega^{norm}|^2 + 2\int_{\partial D'}<\omega_{V^{tan}},\omega_n> \quad .$$

(b) <u>Second calculation</u>

$$\int_{D'}|\omega_t|^2 = \int_{D'}|\omega(\psi_t(x))(d\psi_t,\dots,d\psi_t)_x|^2 dx$$

$$= \int_{\psi_t(D')}|\omega(y)(d\psi_t,\dots,d\psi_t)_{\psi_t^{-1}(y)}|^2 \det((d\psi_t)_{\psi^{-1}(y)})^{-1}dy$$

Now, for $y \in \psi_t(D')$ let $e_i(t)$ be orthonormal parallel vector fields along the curve $c(t) = \psi_t^{-1}(y)$ and abbreviate $X_i(t) = d\psi_t(e_i(t)) \in T_yM$. Then

$$\int_{D'}|\omega_t|^2 = \int_{\psi_t(D')}\frac{1}{p!}\sum_{(i_j)=1}^{m}|\omega(y)(X_{i_1},\dots,X_{i_p})|^2\det(X_1,\dots,X_m)^{-1}$$

As before we have $\frac{\nabla}{dt}X_i(t) = \nabla_{e_i}V$ and therefore

$$\frac{d}{dt}\det(X_1,\dots,X_m) = \det(X_1,\dots,X_m)\cdot \operatorname{trace}\nabla V$$

$$\frac{d}{dt}\frac{1}{p!}\sum|\omega(y)(X_{i_1},\dots,X_{i_p})|^2\Big|_{t=0} = 2<\omega^{\nabla V},\omega>(y) \quad .$$

This gives

$$\text{(2.7.3)} \qquad \frac{d}{dt} \int_{D'} |\omega_t|^2 \Big|_{t=0} = \int_{\partial D'} \langle V,n\rangle |\omega|^2 + 2 \int_{D'} \langle \omega^{\nabla V}, \omega\rangle - \int_{D'} |\omega|^2 \operatorname{div} V ,$$

and the difference between 2.7.2 and 2.7.3 is the basic identity for D' .

Finally approximate D by $D' \subset D$.

3. Geometric conditions for non-existence results

(3.1) We proceed in two steps: In (3.2) we give conditions on the principal curvatures of tubes around submanifolds (announced in (1.2)) which ensure th a certain normal vector field V satisfies the positivity condition 2.6.1; then we give in (3.5) curvature assumptions for M which imply the conditio (3.2) for the principal curvatures of distance tubes. The resulting non-existence theorem (3.3,3.6) repeats corollary 2.6 but is a better result because of the much more explicit assumptions.

Step 1 (compare 1.2). Let $M_1 \subset M$ be a compact submanifold of codimension $\geq 2p+1$. Let T_r denote the distance tube around M_1 consisting of the points in M at distance r from M_1. Let $\hat{V} = \frac{\partial}{\partial r}$ be the outer unit normal vector field along T_r. Our method gives the best result if we choose the vector field

$$V = \lambda \cdot \hat{V} \qquad \text{with } \lambda : M \to \mathbb{R} \text{ constant on each } T_r \text{ and satisfying}$$
$$\frac{\lambda'}{\lambda}(r) = K(r) := \text{largest principal curvature of } T_r .$$

(3.2) First geometric condition. Let the principal curvatures of the tubes T_r satisfy:

$$(*_p) \qquad \min_{T_r}(\text{sum of principal curvatures}) > (2p-1)\cdot(\text{largest principal curvature of } T_r)$$

for all r, $0 < r < R$.

3.2.1 Example: Since at least $2p$ principal curvatures of T_r behave as $\frac{1}{r}$ as $r \to 0$ while the remaining ones stay bounded from below $(*_p)$ is always satisfied for R small enough.

(3.3) Non-existence theorem: Let $M_1 \subset M$ be a submanifold and let $R > 0$ be such that (3.2) is satisfied. Let D be a domain with smooth boundary such that (i) $M_1 \subset D \subset \bigcup_{r<R} T_r$ and (ii) at all point $x \in \partial D$ the outer normal $\hat{\nu}$ of the tube T_r through x points outward from D. Then any smooth harmonic p-form ω on D for which ω^{tan} vanishes on ∂D must vanish on D.

Proof: In view of (2.6) it suffices to show, that (3.2) implies (2.6.1). Since $\nabla\hat{\nu}\big|_{T_r} = S_r$ = shape operator of T_r (for the inner normal $-\hat{\nu}$) and since $\nabla V = \lambda\cdot\nabla\hat{\nu} + d\lambda\cdot\hat{\nu}$ we can compute $\langle\omega^{\nabla V},\omega\rangle$ by using an orthonormal basis of eigenvectors of ∇V ! Denoting the principal curvatures by k_i we have

$$\begin{aligned}
\text{radial eigenvalue of } \nabla V &: \ \lambda\cdot\lambda'/\lambda = \lambda\cdot K(r) \ , \\
\text{tangential eigenvalues of } \nabla V &: \ \lambda\cdot k_i \le \lambda\cdot K(r) \ , \\
\operatorname{div} \nabla V &: \ \lambda\cdot(\operatorname{trace} S_r + \lambda'/\lambda) \ , \\
\text{and} \qquad \langle\omega^{\nabla V},\omega\rangle &\le \ p\cdot\lambda\cdot K(r)\cdot\langle\omega,\omega\rangle \ .
\end{aligned}$$

With this (2.6.1) follows trivially from $(*_p)$.

(3.4) Curvature assumptions which imply (3.2).

(3.4.1) We start from the following basic fact: The shape operators S_r of a family of metrically parallel hypersurfaces T_r are controlled by the curvature tensor R of M via the differential equation:

$$-\frac{\nabla}{dr} S_r = R^m(\,.\,,\frac{\partial}{\partial r})\frac{\partial}{\partial r} + S_r\cdot S_r \ .$$

(Equivalently S_r can be calculated in terms of matrix solutions of the Jacobi equation, $S_r = J'\circ J^{-1}(r)$, see e.g. [K].)

(3.4.2) General case. For $0 < \dim M_1 < \dim M$ one does not have a geometrically satisfactory comparison theory in terms of upper and lower curvature bounds of M and in terms of the shape operators of M_1. Known are explicit estimates for Jacobi fields with linearly dependent initial conditions (i.e. eigendirections of S_r) which control these Jacobi fields, their derivatives and their rotation against parallel fields [K]. These estimates cannot be stated as comparison results with constant curvature situations and are therefore somewhat messy. They show however, that explicit lower bounds for a radius R can be given such that (3.2) is satisfied.

(3.4.3) The case of distance spheres, i.e. M_1 a point. For M a symmetric space, the Jacobi equation can be solved explicitly in terms of the curvature tensor leading to explicit values for R as in examples 1.3(ii). For arbitrary M with curvature bounds $\delta \le K^M \le \Delta$ one has comparison estimates for the principal curvatures of distance spheres. Denote the solution of the initial value problem: $f'' + \kappa \cdot f = 0$, $f(0) = 0$, $f'(0) = 1$ by $f = s_\kappa$. Then [K], as long as $s_\Delta(r) > 0$,

(3.4.4) $\frac{s'_\Delta}{s_\Delta}(r) \le$ (principal curv. of spheres of radius r in M) $\le \frac{s'_\delta}{s_\delta}(r)$.

Now it is easy to state a curvature assumption which, in view of (3.4.4) trivially implies (3.2) in case M_1 is a point:

(3.5) Second geometric condition. Assume curvature bounds $\delta \le K^M \le \Delta$ for M and choose $R > 0$ so that for $0 < r < R$ we have

(3.5_p) $(m-1) \cdot \frac{s'_\Delta}{s_\Delta}(r) > (2p-1) \cdot \frac{s'_\delta}{s_\delta}(r)$, (in particular $s'_\Delta(r) > 0$) .

(3.6) Non-existence theorem: Assume that M satisfies (3.5). Let D be a domain which is contained in a ball of radius R and geodesically star-

shaped relative to the midpoint. Then any harmonic p-*form* ω *on* D *for which* ω^{tan} *vanishes on* ∂D *must vanish.*

This is a special case of (3.3) since (3.5) implies (3.2).

4. Growth Properties

(4.1) THEOREM: Assume (3.5) for M. Let ω be a harmonic p-form defined on a ball D_R. If $\delta \geq 0$ then

$$\frac{s_\delta(r)^{2p-1}}{s_\Delta(r)^{m-1}} \cdot \int_{D_r} |\omega|^2 \text{ is non-decreasing for } 0 < r < R ;$$

if δ is arbitrary then

$$\frac{s_\delta(r)^{2p-1}}{s_\Delta(r)^{m-1}} \cdot \int_{D_r} s_\delta' \cdot |\omega|^2 \text{ is non-decreasing for } 0 < r < R .$$

(4.1.1) COROLLARY (Liouville type result): Let M be simply connected and of negative curvature such that $(m-1)\cdot|\Delta|^{1/2} - 2p\cdot|\delta|^{1/2} > 0$. (In particular (3.5) then holds for all R.) If ω is a harmonic p-form for which $\int_{D_r} |\omega|^2$ grows slower than $\exp(((m-1)|\Delta|^{1/2} - 2p|\delta|^{1/2})r)$ then ω vanishes.

Example: Consider on the dual of complex projective space (curvature $\subset [-4,-1]$) the complex structure as a (parallel) vector valued 1-form. The Dirichlet integral grows as $\exp(m\cdot r)$. The corollary forces harmonic 1-forms to vanish if the Dirichlet integral grows slower than $\exp((m-5)r)$.

(4.1.2) COROLLARY (Energy growth on small balls): Let $\delta \leq K^M \leq \Delta$ be curvature bounds for M. (Then there is $R = R(\delta,\Delta,$ injectivity radius$)$ such that (3.5) holds.) Let ω be a harmonic p-form defined on a ball D_r $(r < R)$. Let D_ε $(\varepsilon < r)$ be concentric balls. Then

$$\int_{D_\varepsilon} |\omega|^2 \leq \left(\frac{\varepsilon}{r}\right)^{m-2p} \cdot e^{c_1\cdot(r^2-\varepsilon^2)} \cdot \int_{D_r} |\omega|^2 ,$$

where $c_1 = c_1(\Delta,\delta,m,p,R)$ is an explicit constant.

Remark: Such growth properties have, for $p = 1$, been used by Hildebrandt-Widman [H-W] and Schoen-Uhlenbeck [S-U] to prove regularity results for minimizing harmonic maps. Note that in (4.1.2) no minimizing property of ω is assumed.

(4.2) Proof of (4.1): Pick in the basic identity (2.5) $V = s_\delta(r) \cdot \frac{\partial}{\partial r}$. First, (2.5) gives a differential inequality

$$(4.2.1) \quad |V| \cdot \frac{d}{dr} \int_{D_r} |\omega|^2 \geq |V| \cdot \int_{\partial D_r} \{|\omega^{\tan}|^2 - |\omega^{\mathrm{norm}}|^2\}$$

$$= \int_{D_r} \{|\omega|^2 \operatorname{div} V - 2\langle \omega^{\nabla V}, \omega\rangle\} .$$

Next (3.4.4) and (3.5) imply ($\lambda(r) = s_\delta(r)$)

$$(4.2.2) \quad |\omega|^2 \cdot \operatorname{div} V - 2\langle \omega^{\nabla V}, \omega\rangle \geq s_\delta(r) \cdot \left\{(m-1) \frac{s'_\Delta}{s_\Delta}(r) - (2p-1) \frac{s'_\delta}{s_\delta}(r)\right\} |\omega|^2$$

$$= \left[s'_\delta(r) \cdot \left\{(m-1) \cdot \frac{s'_\Delta}{s_\Delta} \cdot \frac{s_\delta}{s'_\delta}(r) - (2p-1)\right\}\right] \cdot |\omega|^2 .$$

We insert this in (4.2.1). Note that for $\delta \geq 0$ the expression in [ ]-brackets is the product of two positive non-increasing functions. Therefore we decrease the right side of (4.2.1) further by taking these functions out of the integral and replacing them by their value at the boundary. This gives for $\delta \geq 0$

$$(4.2.3) \quad \frac{d}{dr} \int_{D_r} |\omega|^2 \geq \left\{(m-1) \frac{s'_\Delta}{s_\Delta}(r) - (2p-1) \frac{s'_\delta}{s_\delta}(r)\right\} \cdot \int_{D_r} |\omega|^2 ,$$

which is the first part of (4.1).

For δ arbitrary we leave the factor $s'_\delta(r)$ under the integral and this time get from (4.2.1/2)

$$(4.2.4) \quad \frac{d}{dr}\int_{D_r} s_\delta' \cdot |\omega|^2 = s_\delta'(r) \cdot \int_{\partial D_r} |\omega|^2 \geq$$

$$\geq \left\{(m-1)\,\frac{s_\Delta'}{s_\Delta}(r) - (2p-1)\,\frac{s_\delta'}{s_\delta}(r)\right\} \cdot \int_{D_r} s_\delta' \cdot |\omega|^2 ,$$

which is the second part of (4.1) ($s_\delta'(r) \geq s_\Delta'(r) > 0$ is a consequence of (3.5)).

To prove corollary (4.1.1) observe that by assumption (4.1.1) the non-decreasing non-negative function

$$\frac{s_\delta(r)^{2p-1}}{s_\Delta(r)^{m-1}} \cdot \int_{D_r} s_\delta' \cdot |\omega|^2 \text{ has limit } 0 \text{ for } r \to \infty$$

and therefore vanishes.

To prove corollary (4.1.2) we apply Taylor's theorem to the trigonometric and/or hyperbolic functions s_δ , s_Δ to simplify (4.2.2) before inserting into (4.2.1) to get

$$(4.2.5) \quad \frac{d}{dr}\int_{D_r} |\omega|^2 \geq \frac{m-2p}{r} \cdot (1-c_1 \cdot r^2) \cdot \int_{D_r} |\omega|^2 ,$$

which says that

$$\frac{e^{c_1 \varepsilon^2}}{\varepsilon^{m-2p}} \cdot \int_{D_\varepsilon} |\omega|^2 \text{ is a non-decreasing function of } \varepsilon ,$$

which is equivalent to (4.1.2).

5. Non-existence of certain equivariant harmonic maps

(5.1) Let M be a smooth Riemannian manifold and let r be a C^∞ function on M with values in an interval I. Following Baird [B], call r a *generalized isoparametric function* if, on $M^* = M \setminus$ zero set of grad r,

(1) the integral curves of the unit normal vector field $V = \text{grad } r/|\text{grad } r|$ are geodesics,

(2) the principal curvatures are constant over each level set $M_a = r^{-1}(a)$,

(3) the differential of the projection $M_a \to M_b$ along integral curves of V maps the principal spaces of M_a, i.e. the eigenspaces of the shape operator of M_a, into principal spaces of M_b. It follows [B] that $|\text{grad } r|$ and Δr are functions of r alone; it is sometimes convenient to reparametrize so that $|\text{grad } r| \equiv 1$.

(5.2) Let $r : M \to I$ and $R : N \to \tilde{I}$ be generalized isoparametric functions on smooth Riemannian manifolds M, N. Let $\phi : M \to N$ be a smooth map. We say that ϕ is *equivariant* with respect to r and R if (1) ϕ maps level sets of r to level sets of R, i.e. there exists a smooth function $\alpha : I \to \tilde{I}$ such that $\alpha \circ r = R \circ \phi$.
(2) $d\phi$ maps normals to normals. Then we have a reduction theorem (cf. [B,P]).

(5.3) LEMMA: Let $\phi : M \to N$ be a smooth map which is equivariant with respect to generalized isoparametric functions r, R as above. Then ϕ is harmonic if and only if

(i) each map $\phi_r = \phi|_{M_r} : M_r \to N_{\alpha(r)}$ between level sets is harmonic,

(ii) α satisfies

(5.3.1) $\alpha''(r) + (\text{trace } S_r)\alpha'(r) = \text{trace } \langle \tilde{S}_{\alpha(r)} \circ d\phi_r, d\phi_r \rangle$

where S_r and $\tilde{S}_R$ are the shape operators of M_r and N_R respectively. In particular, trace $\langle \tilde{S}_{\alpha(r)} \circ d\phi_r, d\phi_r \rangle$ depends only on r.

Remark: In the case that each level surface N_R is a Euclidean sphere trace $\langle \tilde{S}_{\alpha(r)} \circ d\phi_r, d\phi_r \rangle$ is a multiple of the energy density of $d\phi_r$. Thus in this case, if ϕ is harmonic, the energy density of $d\phi_r$ is constant over each level surface M_r.

Proof: Let $e_1, \ldots, e_{m-1}$, V be an orthonormal frame for TM. Then

$$\tau(\phi) = \sum_{i=1}^{m-1} \tilde{D}_{d\phi(e_i)} d\phi(e_i) - \sum_{i=1}^{m-1} d\phi(D_{e_i} e_i) + \tilde{D}_{d\phi(V)} d\phi(V) - d\phi(D_V V)$$

where D and $\tilde{D}$ are the Levi-Civita covariant differentiations on M and N respectively. But

$$\sum_{i=1}^{m-1} D_{e_i} e_i = \sum_{i=1}^{m-1} \tan(D_{e_i} e_i) + \sum_{i=1}^{m-1} \langle D_{e_i} e_i, V\rangle V = \sum_{i=1}^{m-1} D^{M_r}_{e_i} e_i - (\text{trace } S)$$

where D^{M_r} denotes the Levi-Civita connection on M_r induced from that on Similarly, denoting the induced inner product on $TM_r^* \otimes TN_{\phi(r)}$ by $\langle\ ,\ \rangle$

$$\sum_{i=1}^{m-1} \tilde{D}_{d\phi(e_i)} d\phi(e_i) = \sum_{i=1}^{m-1} \tilde{D}^{N_R}_{d\phi(e_i)} d\phi(e_i) - \langle \tilde{S} \circ d\phi_r, d\phi_r \rangle \tilde{V}$$

where D^{N_R} denotes the Levi-Civita connection on N_R induced from that on N and $\tilde{V}$ is the unit normal to N_R. Thus, noting that $D_V V = 0$,

$$\tau(\phi) = \tau(\phi_r) + (\text{trace } S) d\phi(V) - \langle \tilde{S} \circ d\phi_r, d\phi_r \rangle \tilde{V} + \tilde{D}_{d\phi(V)} d\phi(V).$$

The first term is tangential to $M_{\alpha(r)}$, the other terms are normal. Thus

$\tau(\phi) = 0 \iff \tau(\phi_r) = 0$ and $\tilde{D}_{d\phi(V)} d\phi(V) + (\text{trace } S) d\phi(V) = \langle \tilde{S} \circ d\phi_r, d\phi_r \rangle \tilde{V}$. Writing the last equation in terms of derivatives of α yields (5.3.1).

(5.4) Now let $f(r)$ be a smooth positive function on $[0,b]$ with $\frac{d}{dr} f(r) = (\text{trace } S_r) f(r)$. Change the variable r to a new variable $t = t(r) = -\int_r^b \frac{1}{f(r)} dr$. As r increases from 0 to b , t increases from $-\infty$ to 0 and the differential equation (5.3.1) reads, for $\tilde{\alpha}(t) = \alpha(r(t))$,

$$\text{(5.4.1)} \qquad \frac{d^2\tilde{\alpha}}{dt^2} = f(r(t))^2 \cdot \langle \tilde{S}_{\tilde{\alpha}(t)} \circ d\phi_{r(t)}, d\phi_{r(t)} \rangle .$$

Now, if in addition,

(iii) for each r , $d\phi_r$ maps each principal space of $T(M_r)$ to a principal space of $T(N_{\alpha(r)})$, then it is possible to evaluate the right-hand side in terms of principal curvatures of M_r and $N_{\alpha(r)}$ and in certain circumstances, by finding a first integral of (5.4.1), we can estimate the maximum distance from 0 reached by the solution $\tilde{\alpha}(t)$. As general results are too technical to state we illustrate this in the simplest situation:

(5.5) Let M be a disc $D^m(b) = \{x \in \mathbb{R}^m : |x| < b\}$ equipped with polar coordinates $(r,\theta) \in [0,b) \times S^{m-1}$ and a smooth rotationally symmetric metric $ds^2 = dr^2 + h(r)^2 d\theta^2$. Then $r : M \to [0,b)$ is a generalized isoparametric function.

Similarly, let N be a disc $D^n(B)$ equipped with polar coordinates $(R,\Theta) \in [0,B) \times S^{n-1}$ and a smooth rotationally symmetric metric $d\tilde{s}^2 = dR^2 + H(R)^2 d\Theta^2$. Then $R : N \to [0,B)$ is a generalized isoparametric function. A smooth map $\phi : D^m(b) \to D^n(B)$ is equivariant iff it is of the form $\phi(r,\theta) = (R(r), \Theta(\theta))$, where $R : [0,b) \to [0,B)$ satisfies $R(0) = 0$ and $\Theta : S^{m-1} \to S^{n-1}$. In the special case $m=n$, $\Theta(\theta) = \theta$, such a map is <u>rotation-</u>

ally symmetric, i.e. equivariant with respect to the action of $O(m)$ on $D^m(b)$ and $D^m(B)$. By Lemma (5.3) and the subsequent remark a non-constant smooth equivariant map $\phi : D^m(b) \to D^n(B)$, $\phi(r,\theta) = (R(r),\Theta(\theta))$ is harmonic if and only if

(i) $\Theta : S^{m-1} \to S^{n-1}$ is a harmonic polynomial map, i.e. the restriction of a map $\mathbb{R}^m \to \mathbb{R}^n$ each component of which is a harmonic homogeneous polynomial of degree $d \geq 1$, and

(ii) $R : [0,b) \to [0,B)$ satisfies the ordinary differential equation

$$(5.6.3) \quad \frac{d}{dr}\left(h(r)^{m-1}\frac{dR}{dr}\right) = \frac{d(d+m-2)}{2}\, h(r)^{m-3}\, \frac{d}{dR}\left(H^2(R)\right)\Big|_{R(r)} .$$

We shall call d the homogeneity degree of ϕ.

For part (i) we use the fact [B-G-M], that since its components are eigenfunctions on the domain sphere, a harmonic map of constant energy density between Euclidean spheres is a harmonic polynomial map.

(5.6) We now assume that $D^m(b)$ and $D^n(B)$ are qualitatively like a Euclidean disc and sphere respectively; specifically we assume:

(5.6.1) the principal curvatures of the distance spheres $S_r = \{x \in D^m(b) : |x| = r\}$ of the domain are positive, i.e. the principal curvature vectors of S_r point towards the origin for all $r \in (0,b)$;

(5.6.2) (a) the principal curvatures of the distance spheres $\tilde{S}_R = \{x \in D^n(B) : |x| = R\}$ are non-negative on $(0,R_e]$ and non-positive on $[R_e,B)$ for some $R_e \in (0,B)$;

(b) $$\int_{R_e}^{B} \frac{dR}{H(R)} = \infty .$$

We take R_e to be the least number satisfying the condition (5.6.2)(a).

Note that the principal curvatures of S_r and $\tilde{S}_R$ are given by $+\frac{d}{dr}\ln h(r)$ and $+\frac{d}{dR}\ln H(R)$ respectively so that condition (5.6.1) is equivalent to demanding that $h'(r) > 0$, and similarly for condition (5.6.2). The condition (5.6.2)(b) may be interpreted as follows: There is a rotationally symmetric conformal map c of $D^n(B)\setminus\{0\}$ to the cylinder $\mathbb{R}\times S^{n-1}$ given by $(R,\Theta) \to (L(R),\Theta)$ defined by setting $L(R) = \int_{R_e}^{R} \frac{dR}{H(R)}$. Note that $L(R) \to -\infty$ as $R \to 0$. The condition (5.6.2)(b) is equivalent to $L(R) \to +\infty$ as $R \to B$, i.e. <u>the conformal map</u> $c : D^n(B)\setminus\{0\} \to \mathbb{R}\times S^{n-1}$ <u>covers the whole cylinder</u>. Note also that the "radial" sectional curvatures are $-\frac{h''}{h}$ resp. $-\frac{H''}{H}$.

The non-existence theorem may now be stated

(5.7) <u>THEOREM</u>: Let $(D^m(b),ds^2)$, $(D^n(B),d\tilde{s}^2)$ be discs with rotationally symmetric metrics satisfying (5.6.1) and (5.6.2).
Let $\phi : (D^m(b),ds^2) \to (D^n(B),d\tilde{s}^2)$ be an equivariant harmonic map of homogeneity degree $d \geq 1$. Then there is a number $R_c \in (R_e,B)$ depending only on n, d and the metrics ds^2, $d\tilde{s}^2$ such that the image of ϕ is contained in the disc $\{y \in D^n(B) : |y| < R_c\}$. In fact we may take $R_c \in (R_e,B)$ to be the unique solution to the equation

$$\int_{R_e}^{R_c} \frac{dR}{H(R)} = \frac{\sqrt{d(d+m-2)}}{m-2}\,\gamma(h) \ .$$

where

$$\gamma(h) = \sup_{(0,b)} h(r)^{m-2} \int_r^b \frac{m-2}{h(\rho)^{m-1}}\, d\rho \ .$$

(5.8) <u>Remarks</u>: (i) It can easily be seen that, for any smooth rotationally symmetric metric, $\gamma(h)$ is finite; the condition (5.6.2(b)) then ensures that R_c exists.

(ii) We see that R_c depends only on the metrics ds^2 and $d\tilde{s}^2$ as claimed.

(iii) If $(D^m(b),ds^2)$ is the Euclidean disc and $(D^n(B),d\tilde{s}^2)$ the Euclidea sphere (minus one point) then $\gamma(h) = 1$ so that $R_c < \frac{\pi}{2} + \sin^{-1}\tanh \frac{\sqrt{d(d+m\cdot}}{m-2}$ In fact, in the case of a rotationally symmetric map, $m=n$, $d=1$, Jäger and Kaul |J-K2| show that we can take $R_c = \frac{\pi}{2}$ if $m \geq 7$ and give explicit computer estimates for R_c for other values of m . See also [Ba,E-L3].

(5.9) Proof of the Theorem: Writing $\phi(r,\theta) = (R(r),\Theta(\theta))$, by harmonici $R(r)$ satisfies (5.6.3) . As in (5.4) we change the variable $r \to t$ where

$$t = -\int_r^b \frac{m-2}{h(\rho)^{m-1}}\, d\rho$$

so that as r varies over $(0,b)$, t varies over $(-\infty,0)$ and setting $\tilde{R}(t) = R(r(t))$, (5.6.3) is transformed into

$$\frac{d^2\tilde{R}}{dt^2} = \frac{c^2}{2} h(r(t))^{2m-4} \frac{d}{dR} H^2(R)\Big|_{\tilde{R}(t)}$$

where $c = \sqrt{d(d+m-2)}/(m-2)$. Note that as $t \to -\infty$, $\frac{d\tilde{R}}{dt} = h(r(t)) \cdot \frac{dR}{dt} \to 0$. No confusion will arise if we now write R for $\tilde{R}$. Multiplying by $2(dR/dt$ and integrating we have for any t_1 , t_2 the energy equation

$$\text{(5.9.1)}\qquad \left\{\frac{dR}{dt}(t_2)\right\}^2 - \left\{\frac{dR}{dt}(t_1)\right\}^2 = c^2 \int_{t_1}^{t_2} h(r(t))^{2m-4} \frac{dH^2}{dR}\frac{dR}{dt}\, dt .$$

If the solution $R(t)$ does not reach $R = R_e$ there is nothing to prove, otherwise let $t_e \in (-\infty,t_b)$ be the least value of t with $R(t) = R_e$ and let $t_m \in [t_e,0)$ be the least value of t at which dR/dt changes sign - if there is no such value, set $t_m = 0$. Using the facts that $h(r(t))$ is an increasing function of t and that by (5.6.2)(a) $(dH^2/dR)_{R(t)}$ is non-negative for $t \in (-\infty, t_e]$ and non-positive for $t \in [t_e, t_m]$, we have that for all $t \in (-\infty, t_m]$:

$$(\frac{dR}{dt})^2 < c^2 h(r(t_e))^{2m-4} \int_{-\infty}^{t} \frac{dH^2}{dR} \frac{dR}{dt} dt$$

$$= c^2 h(r(t_e))^{2m-4} H^2(R(t)) .$$

Dividing by $H^2(R(t))$, taking the square root and integrating from t_e to t_m we obtain

$$\int_{R(t_e)}^{R(t_m)} \frac{dR}{H(R)} < c\, h(r(t_e))^{m-2} \cdot (t_m - t_e) \leq c\, \gamma(h) .$$

To finish the proof we have to show that $R(t) \leq R(t_m)$ for all t, i.e. any subsequent maximum of $R(t)$ is of the same or lesser magnitude. But for a subsequent maximum t'_m, we have from the energy equation (5.9.1),

$$\int_{t_m}^{t'_m} h(r(t))^{2m-4} \frac{dH^2}{dR} \frac{dR}{dt} dt = 0 .$$

Since $h(r(t))$ is increasing, it can be shown that $R(t'_m) \leq R(t_m)$. Thus $R(t) \leq R(t_m)$ for all $t \in (-\infty, 0)$ and the proof is complete.

(5.10) <u>Remarks</u>: (1) The theorem applies under the same conditions to equivariant harmonic maps from an annulus $(D^m(b) \setminus D^m(b'), ds^2)$ $(0 < b' < b)$ under the additional hypothesis that ϕ has zero normal derivative on the inner boundary $\partial D^m(b')$ (cf. Remarks (1.5)).

(2) (Suggested by L. Lemaire) Let N be a rotation symmetric "dumbbell" made of two equal spheres of the same dimension with holes removed joined

together smoothly by a tube. Choose the holes to have radius $< \frac{\pi}{2} - \sin^{-1} \tanh (\sqrt{m-1})$. Then there exists no rotationally symmetric harmonic map from the Euclidean m-sphere to the dumbbell. For the restriction of such a map to one of the hemispheres would have image covering a disc of radius $\geq \frac{\pi}{2} + \sin^{-1} \tanh \sqrt{m-1}$ contradicting the Theorem.

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