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TRIANGULINE GALOIS REPRESENTATIONS AND SCHUR FUNCTORS

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Abstract. Given a B-pair W and a Schur functor S, we show under some general assumptions that W is trianguline if and only if $S(W)$ is. This is an extension of earlier work of Di Matteo. We derive some consequences on the behavior of local Galois representations under morphisms of Langlands dual groups. We attach to a Schur functor a map between the trianguline deformation spaces defined by Hellmann, and we study congruence loci on the Hecke-Taylor-Wiles varieties constructed by Breuil, Hellmann and Schraen for unitary groups.

Contents

1. Introduction

The original motivation for this paper comes from the intention of generalizing a result of [Cont16b] that we recall here very approximately. Fix a prime p larger than 3. Consider an abstract Hecke algebra $\mathcal{H}$ for GSp_4, spherical outside Np for a fixed integer N and of Iwahoric level at p. Let $\mathscr{E}$ be the GSp_4-eigenvariety constructed by Andreatta, Iovita and Pilloni in [AIP15]. There exists a homomorphism θ from $\mathcal{H}$ to the ring of rigid analytic functions $\mathcal{O}(\mathscr{E})$ over $\mathscr{E}$ interpolating the systems of Hecke eigenvalues of classical Siegel modular forms of tame level N and Iwahoric level at p. We denote by G_K the absolute Galois group of a local or global field K. In [Cont16b, Section 10.3], two loci on $\mathscr{E}$ are defined:

* the Galois symmetric cube locus, as the locus of points x of $\mathscr{E}$ such that the associated Galois representation $\rho_x\colon G_{\mathbb{Q}} \to \mathrm{GSp}_4(\overline{\mathbb{Q}}_p)$ takes values in the image of the symmetric cube representation $\mathrm{Sym}^3\colon \mathrm{GL}_2(\overline{\mathbb{Q}}_p) \to \mathrm{GSp}_4(\overline{\mathbb{Q}}_p)$;
* the automorphic symmetric cube locus, as the locus of points x of $\mathscr{E}$ such that the associated system of Hecke eigenvalues θ_x derives from that of a p-adic overconvergent

modular form for GL_2 via a suitable morphism of Hecke algebras attached to the symmetric cube representation.

Theorem 1.1. *[Cont16b, Theorem 10.10] The Galois and automorphic symmetric cube loci on $\mathscr{E}$ coincide.*

Thanks to the wide range of reductive algebraic groups for which eigenvarieties are available, one may wish to generalize Theorem 1.1 to other representations than the symmetric cube. We summarize the proof by a chain of implications below, where the property "trianguline" is intended in the sense of (φ, Γ)-modules or B-pairs (see Definition 2.4). Consider the following statements:

(1) x is a point of the Galois symmetric cube locus on $\mathscr{E}$;
(2) the restriction of ρ_x to a decomposition group at p is trianguline;
(3) there exists a continuous representation $\rho_y\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ that satisfies $\rho_x \cong \mathrm{Sym}^3\rho_y$;
(4) the restriction of the representation ρ_y at point (3) to a decomposition group at p is trianguline;
(5) the representation ρ_y at point (3) is attached to a p-adic overconvergent automorphic form for GL_2;
(6) the point x belongs to the automorphic symmetric cube locus on $\mathscr{E}$.

Under some technical hypotheses, there are implications

$$\begin{tikzcd}
(1) \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & (2) \arrow[dr, Rightarrow] & & & \\
& (3) \arrow[r, Rightarrow] & (4) \arrow[r, Rightarrow] & (5) \arrow[r, Rightarrow] & (6)
\end{tikzcd} \tag{1.1}$$

The implication (1) $\Longrightarrow$ (2) is an application of [KPX14, Corollary 6.3.13], a very general result that can be applied whenever the eigenvariety under consideration admits a Zariski-dense set of "refined crystalline" points in the sense of [KPX14, Definition 6.4.1] (see for instance the application in [BHS17, Section 2.2]). The arrow (5) $\Longrightarrow$ (6) follows immediately from the definition of the symmetric cube morphism of Hecke algebras. On the other hand, the implication (4) $\Longrightarrow$ (5) relies on the work of Emerton and Kisin on the overconvergent Fontaine-Mazur conjecture, that is very specific to the case of representations $G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$. We will need to assume an analogue of this conjecture in the more general cases we are interested in.

The goal of this paper is to fill the diagram (1.1) by generalizing the results (1) $\Longrightarrow$ (3) [Cont16a, Lemma 3.11.5] and (2) + (3) $\Longrightarrow$ (4) [Cont16a, Proposition 3.10.25] to an algebraic representation $S\colon \mathrm{GL}_n \to \mathrm{GL}_m$, for arbitrary integers n and m, defined by a Schur functor. This generalization is the content of Proposition 6.3, that relies essentially on the main results of the paper, Theorems 4.2 and 5.2. We summarize them in the following statement. Let S be as above, let K be a p-adic field and let $\rho\colon G_K \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ be a continuous representation.

Theorem 1.2.

* *If $S \circ \rho\colon G_K \to \mathrm{GL}_m(\overline{\mathbb{Q}}_p)$ is potentially trianguline (see Definition 2.4) then ρ is potentially trianguline (Theorem 4.2).*
* *If $S \circ \rho\colon G_K \to \mathrm{GL}_m(\overline{\mathbb{Q}}_p)$ is trianguline and it admits a triangulation whose parameters are derived from those of ρ (see Remark 4.8), then ρ is trianguline (Theorem 5.2).*

Our results are actually more general than Theorem 1.1, in the sense that we work in the category of B-pairs that contains as a strict subcategory that of p-adic representations of G_K. Accordingly, trianguline Galois representations are replaced by triangulable B-pairs. We also remark that when the property "trianguline" is replaced by "de Rham", the analogues of the previous results are proved by Di Matteo in [DiM13a, Theorem 2.4.2].

In the last section we describe an application of diagram (1.1) to the Hecke-Taylor-Wiles varieties for unitary groups that are studied by Breuil, Hellmann and Schraen in [BHS17]. We summarize our results here.

Let n be an integer such that $p > 2n+1$. Let F^+ be a totally real number field and let F be a totally imaginary quadratic extension of F^+. Let G be a unitary group over F^+ such that G is split over F, compact at infinity and isomorphic to GL_n at all places in Σ_p. Let k be a finite field of characteristic p and let $\overline{\rho}\colon G_F \to \mathrm{GL}_n(k)$ be a residual representation appearing in a space of automorphic forms for G. For a p-adic place v of F and a decomposition subgroup G_{F_v} of G_F, the couples $(\rho|_{G_{F_v}}, \delta)$ consisting of a trianguline characteristic 0 lift ρ of $\overline{\rho}_v \coloneqq \overline{\rho}|_{G_{F_v}}$ and an n-tuple of parameters of a triangulation of $\rho|_{G_{F_v}}$ live naturally on a "trianguline deformation space" $X^{\square}_{\mathrm{tri},v}$ [Hel12, Theorem 1.1]. Breuil, Hellmann and Schraen [BHS17] attached to G and to $\overline{\rho}$ a "Hecke-Taylor-Wiles variety" $X_p(\overline{\rho})$ that is strictly related both to the eigenvariety for G and $\overline{\rho}$ (see for instance [Che11, Definition 2.2]) and to the trianguline deformations spaces $X^{\square}_{\mathrm{tri},v}$. In particular there is a natural morphism $X_p(\overline{\rho}) \to \prod_{v|p} X^{\square}_{\mathrm{tri},\overline{\rho}_v}$.

As before, consider an algebraic representation $S\colon \mathrm{GL}_n \to \mathrm{GL}_m$. We study the morphisms induced by S between the trianguline deformation spaces and the Hecke-Taylor-Wiles varieties attached to two groups G_n and G_m on F^+ that are isomorphic to GL_n and GL_m, respectively, at p-adic places. We show the following.

Theorem 1.3.

* *For every p-adic place v of F, the representation S induces morphisms of trianguline deformation spaces*
$$S_{\mathrm{tri},v}\colon X^{\square}_{\mathrm{tri},\overline{\rho}_v} \to X^{\square}_{\mathrm{tri},S\,\circ\,\overline{\rho}_v}.$$
* *Assume part of the standard modularity conjectures for G_n and G_m, as reformulated in Conjecture 7.4. Then $\prod_{v\in V} S_{\mathrm{tri}}$ induces a morphism S_{TW} fitting in the following commutative diagram:*

$$\begin{array}{ccc} X_p(\overline{\rho}) & \longrightarrow & X^{\square}_{\mathrm{tri},\overline{\rho}_v} \\ \Big\downarrow{\scriptstyle S_{TW}} & & \Big\downarrow{\scriptstyle \prod_{v|p} S_{\mathrm{tri},v}} \\ X_p(S\circ\overline{\rho}) & \longrightarrow & X^{\square}_{\mathrm{tri},S\,\circ\,\overline{\rho}_v} \end{array}$$

We define an *S-congruence locus* on $X_p(S\circ\overline{\rho})$ as the locus of points whose associated local Galois representation are obtained via S from GL_n-valued representations. By combining Theorems 1.2 and 1.3, we give a characterization of the S-congruence locus that can be seen as a vast generalization of [Cont16b, Theorem 10.10].

Theorem 1.4.

∗ *Every point x of the S-congruence locus is the "twin" of a point x' in the image of the morphism S_{TW}, that is x and x' carry the same set of local Galois representation but possibly different triangulations. (Theorem 7.18).*

∗ *The image of S_{TW} consists of the irreducible components of maximal dimension of the S-congruence locus (Corollary 7.20).*

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Notation. In the following p is a fixed prime number. We denote by $\overline{\mathbb{Q}}_p$ an algebraic closure of the field $\mathbb{Q}_p$. We extend the p-adic valuation on $\overline{\mathbb{Q}}_p$ to a valuation on $\overline{\mathbb{Q}}_p$ and we let $\mathbb{C}_p$ be the completion of $\overline{\mathbb{Q}}_p$ with respect to this choice. By a "p-adic field" we will always mean a finite extension of $\mathbb{Q}_p$. Given a local or global field F we denote by G_F its absolute Galois group, equipped with the profinite topology. Throughout the whole text the letters K and E will denote two p-adic fields.

For every $n \geq 1$ we write $\mathbb{1}_n$ for the $n \times n$ unit matrix.

Let K be a finite extension of $\mathbb{Q}_p$ and let X be a rigid analytic space over K. We denote by $\mathcal{O}(X)$ the K-algebra of rigid analytic functions on X, and by $\mathcal{O}(X)^\circ$ the $\mathcal{O}_K$-subalgebra of functions of norm bounded by 1. We say that X is *wide open* if there exists an admissible covering $\{X_i\}_{i\in\mathbb{N}}$ of X by affinoid domains X_i such that, for every i, $X_i \subset X_{i+1}$ and the map $\mathcal{O}(X_{i+1}) \to \mathcal{O}(X_i)$ induced by the previous inclusion is compact.

Given a formal scheme $\mathrm{Spf}\,(A)$ over $\mathrm{Spf}\,(\mathcal{O}_K)$, we denote by $\mathrm{Spf}\,(A)^{\mathrm{rig}}$ the rigid analytic space over K attached to $\mathrm{Spf}\,(A)$ by Berthelot's construction, described for instance in [deJ95, Section 7].

2. Generalities on B-pairs

We refer to [Ber08] and [DiM13b] for the basic definitions concerning B-pairs. Let $\mathbf{B}$ be a topological ring equipped with a continuous action of G_K. We call semilinear $\mathbf{B}$-representation of G_K a free $\mathbf{B}$-module of finite rank M equipped with a semilinear action of G_K, meaning that $g(bm) = g(b)g(m)$ for every $b \in \mathbf{B}$, $m \in M$ and $g \in G_K$. We denote by $\mathrm{Sem}_{G_K}(\mathbf{B})$ the category whose objects are the semilinear $\mathbf{B}$-representations of G_K and whose morphisms are the G_K-equivariant morphisms of $\mathbf{B}$-modules. We call rank of an object of $\mathrm{Sem}_{G_K}(\mathbf{B})$ its rank as a $\mathbf{B}$-module.

We denote by $\mathbf{B}_{\mathrm{cris}}$, $\mathbf{B}_{\mathrm{st}}$, $\mathbf{B}_{\mathrm{dR}}^+$ and $\mathbf{B}_{\mathrm{dR}}$ the rings of periods defined by Fontaine in [Fon94]. Each of these objects is a topological $\mathbb{Q}_p$-algebra carrying a continuous action of G_K. We add an index E denote the extension of scalars from $\mathbb{Q}_p$ to E. We denote by t Fontaine's chosen generator of the maximal ideal of the complete discrete valuation ring $\mathbf{B}_{\mathrm{dR}}^+$. We denote by φ the Frobenius endomorphism of $\mathbf{B}_{\mathrm{cris}}$. We set $\mathbf{B}_e = \mathbf{B}_{\mathrm{cris}}^{\varphi=1}$. Let $\mathbf{B}_{e,E} = \mathbf{B}_e \otimes_{\mathbb{Q}_p} E$. We let G_K act on $\mathbf{B}_{e,E}$ via its action on the first factor, that is we set $g(b \otimes e) = g(b) \otimes e$ for every $b \in \mathbf{B}_E$, $e \in E$ and $g \in G_K$.

Definition 2.1. *A* B_K^E-pair *is a pair* (W_e, W_{dR}^+) *where:*

* W_e *is an object of* $\mathrm{Sem}_{G_K}(\mathbf{B}_{e,E})$*;*
* W_{dR}^+ *is a* G_K*-stable* $\mathbf{B}_{\mathrm{dR}}^+$*-lattice of* $\mathbf{B}_{\mathrm{dR},E} \otimes_{\mathbf{B}_{e,E}} W_e$*.*

We define the rank *of* (W_e, W_{dR}^+) *as the common rank of* W_e *and* W_{dR}^+*. Given two* B_K^E*-pairs* (W_e, W_{dR}^+) *and* $(W_e', W_{\mathrm{dR}}^{+,\prime})$*, a* morphism of B_K^E-pairs $(W_e, W_{\mathrm{dR}}^+) \to (W_e', W_{\mathrm{dR}}^{+,\prime})$ *is a pair* (f_e, f_{dR}^+) *where:*

* $f_e\colon W_e \to W_e'$ *is a morphism in* $\mathrm{Sem}_{G_K}(\mathbf{B}_{e,E})$*,*
* f_{dR}^+ *is a morphism in* $\mathrm{Sem}_{G_K}(\mathbf{B}_{\mathrm{dR}}^+)$*,*
* *with the obvious notations, the* $\mathbf{B}_{\mathrm{dR},E}$*-linear morphisms* $f_e[1/t]$*,* $f_{\mathrm{dR}}[1/t]\colon W_{\mathrm{dR}} \to W_{\mathrm{dR}}'$ *coincide.*

The objects and morphisms described above define the category of B_K^E*-pairs.*

For a B_K^E-pair (W_e, W_{dR}^+) we also set $W_{\mathrm{dR}} = \mathbf{B}_{\mathrm{dR},E} \otimes_{\mathbf{B}_{e,E}} W_e$, that is the same as $\mathbf{B}_{\mathrm{dR},E} \otimes_{\mathbf{B}_{e,E}} W_e$; it is an object of $\mathrm{Sem}_{G_K}(\mathbf{B}_{\mathrm{dR}})$. Given two B_K^E-pairs $X = (X_e, X_{\mathrm{dR}}^+)$ and $W = (W_e, W_{\mathrm{dR}}^+)$ such that $X_e \subset W_e$ and $W_{\mathrm{dR}}^+ \subset W_{\mathrm{dR}}^+$, we say that X is a *saturated* sub-B_K^E-pair of W if the lattice W_{dR}^+ is saturated in W_{dR}^+. The quotient W/X admits a natural structure of B_K^E-pair if and only if X is a saturated sub-B_K^E-pair of W.

Given a B_K^E pair W and finite extensions L/K and F/E, we can define a B_L^F-pair as $(F \otimes_E W)|_{G_L}$, with the obvious notations. Given a property ?, we say that W has ? potentially if there is a finite extension L/K such that $W|_{G_L}$ has ?.

Definition 2.2. *Let* $? \in \{\mathrm{cris}, \mathrm{st}, \mathrm{dR}\}$*. We say that a* B_K^E*-pair* (W_e, W_{dR}^+) *is (potentially)* $\mathbf{B}_{?,E}$*-admissible if* $\mathbf{B}_{?,E} \otimes_{\mathbf{B}_{e,E}} W_e$ *is (potentially) trivial in* $\mathrm{Sem}_{G_K}(\mathbf{B}?, E)$*. We call* (W_e, W_{dR}^+) *(potentially) crystalline, semi-stable, de Rham if it is (potentially) admissible for the corresponding ring.*

Recall that the properties of being de Rham and potentially semi-stable are equivalent for a B_K^E-pair by a result of Berger [Ber08, Théorème 2.3.5].

Let $\mathrm{Rep}_{G_K}(E)$ be the category of continuous, E-linear, finite-dimensional representation V of G_K. For an object V of $\mathrm{Rep}_{G_K}(E)$ we denote by $W(V)$ the B_K^E-pair $(\mathbf{B}_{e,E} \otimes_E V, \mathbf{B}_{\mathrm{dR}}^+ \otimes_E V)$. The rank of $W(V)$ is equal to the E-rank of V. Given two objects V, V' of $\mathrm{Rep}_{G_K}(E)$ and a morphism $f\colon V \to V'$, we define a morphism $W(f)\colon W(V) \to W(V')$ by $\mathbf{B}_{e,E}$-linearly extending f to the first element of $W(V)$ and $\mathbf{B}_{\mathrm{dR}}^+$ linearly to the second. The functor $W(\cdot)$ defined this way is fully faithful and identifies $\mathrm{Rep}_{G_K}(E)$ with a full tensor subcategory of the category of B_K^E-pairs. This is [Ber08, Théorème 3.2.3] when $E = \mathbb{Q}_p$ and an immediate consequence of it for general E.

For $? \in \{\mathrm{cris}, \mathrm{st}, \mathrm{dR}\}$, an object V of $\mathrm{Rep}_{G_K}(E)$ is (potentially) $\mathbf{B}_?$-admissible in the sense of Fontaine if and only if the B_K^E-pair $W(V)$ is (potentially) $\mathbf{B}_?$-admissible in the sense of Definition 2.2. This means that V is (potentially) crystalline, semi-stable, de Rham if and only if $W(V)$ is.

Let R be a ring equipped with an action of G_K. A character $\eta\colon G_K \to R^\times$ is G_K-equivariant (for the action of G_K on itself by conjugation) if and only if it takes values in $(R^\times)^{G_K}$, where we use the standard notation for the elements fixed by a group action. For a character $\eta\colon G_K \to (R^\times)^{G_K}$, we denote by $R(\eta)$ the a free R-module of rank 1 with a distinguished generator e_η, equipped with the semilinear action $g_\eta(\cdot)$ of G_K defined by $g_\eta(x \cdot e_\eta) = \eta(g)g(x) \cdot e_\eta$ for every $x \in R$, $g \in G_K$ and the action $g(\cdot)$ of G_K on R. For an object M of $\mathrm{Sem}_{G_K}(R)$, we define a new object of this category as $M(\eta) = M \otimes_R R(\eta)$. For every η as before, we define a B_K^E-pair $\mathbf{B}(\eta)$ of rank one as $(\mathbf{B}_{e,E}(\eta), \mathbf{B}_{\mathrm{dR}}^+(\eta))$. For a B_K^E-pair W, we define a new B_K^E-pair as $W(\eta) = W \otimes \mathbf{B}(\eta)$.

Definition 2.3. *Let* $\mathbf{F}$ *be a field.*

* *An object* M *of* $\mathrm{Sem}_{G_K}(\mathbf{F})$ *is split triangulable if there exists a filtration* $0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$ *such that, for* $1 \le i \le n$, M_i *is a rank* i *subobject of* M *in* $\mathrm{Sem}_{G_K}(\mathbf{F})$. *If moreover* $M_i/M_{i-1} \cong \mathbf{F}(\eta_i)$ *for* $1 \le i \le n$ *and characters* $\eta_i\colon G_K \to (\mathbf{F}^\times)^{G_K}$, *then we say that* M *is split triangulable by characters* $\eta_1, \eta_2, \ldots, \eta_n$.
* *An object* M *of* $\mathrm{Sem}_{G_K}(\mathbf{F})$ *is triangulable if there is a finite extension* $\mathbf{F}'$ *of* $\mathbf{F}$ *such that* $\mathbf{F}' \otimes_{\mathbf{F}} M$ *is triangulable as an object of* $\mathrm{Sem}_{G_K}(\mathbf{F}')$.

Let M be an object of some rank n in $\mathrm{Sem}_{G_K}(\mathbf{F})$ and let $(e_1, e_2, \ldots, e_n)$ be a $\mathbf{F}$-basis of M. To every $g \in G_K$ we attach the only matrix $(a_{ij}^g)_{i,j} \in \mathrm{GL}_n(\mathbf{B})$ satisfying $g(e_i) = \sum_j a_{ij}^g e_j$ for every $i \in \{1, 2, \ldots, n\}$ (we leave implicit the dependence of $(a_{ij}^g)_{i,j}$ on the choice of a basis). Then:

* the object M is trivial if and only if it admits a basis with respect to which the matrix $(a_{ij}^g)_{i,j}$ is the identity for every $g \in G_K$;
* the object M is triangulable if and only if it admits a basis with respect to which the matrix $(a_{ij}^g)_{i,j}$ is upper triangular for every $g \in G_K$;

Definition 2.4.

* *A* B_K^E*-pair* W *is split triangulable if there exists a filtration* $0 = W_0 \subset W_1 \subset \ldots \subset W_n = W$ *such that, for* $1 \le i \le n$, W_i *is a rank* i *sub-*B_K^E*-pair of* W. *If moreover* $W_i/W_{i-1} \cong B_K^E(\eta_i)$ *for* $1 \le i \le n$ *and characters* $\eta_i\colon G_K \to (\mathbf{F}^\times)^{G_K}$, *then we say that* W *is split triangulable with ordered parameters* $\eta_1, \eta_2, \ldots, \eta_n$.
* *A* B_K^E*-pair* W *is triangulable if there is a finite extension* F *of* E *such that the* B_K^F *pair* $F \otimes_E W$ *is split triangulable.*
* *An object* V *of* $\mathrm{Rep}_{G_K}(E)$ *is (potentially, split) trianguline if* $W(V)$ *is (potentially, split) triangulable.*

The ring $\mathbf{B}_{e,E}$ is a principal ideal domain by [DiM13b, Proposition 1.2.2]. Let $\mathbf{F}_E$ be its fraction field, equipped with the action of G_K induced by that on $\mathbf{B}_{E,e}$. For every finite extension L of K, we have $\mathbf{B}_{\mathrm{dR},E}^{G_L} = \mathbf{F}_E^{G_L} = L$.

The following lemma gives equivalent conditions for a B_K^E-pair to be triangulable.

Lemma 2.5. *[DiM13b, Corollary 2.2.2] Let* $W = (W_e, W_{\mathrm{dR}}^+)$ *be a* B_K^E*-pair. The following conditions are equivalent:*

(1) the B_K^E*-pair* W *is split triangulable;*
(2) the semilinear $\mathbf{B}_{e,E}$*-representation* W_e *of* G_K *is split triangulable;*
(3) the semilinear $\mathbf{F}_E$*-representation* $\mathbf{F}_E \otimes_{\mathbf{B}_{e,E}} W_e$ *of* G_K *is split triangulable.*

In particular the same equivalence holds if we replace "split triangulable" by "triangulable".

Thanks to the lemma we will be able to study the triangulinity of a B_K^E-pair W by studying the triangulability of a $\mathbf{F}_E$-representation of G_K.

3. Generalities on Schur functors

Let q be a positive integer. Consider a decreasing integer partition $\underline{u}$ of q, that is a tuple $\underline{u} = (u_1, u_2, \ldots, u_r)$, for some $r \geq 1$, satisfying $u_1 \geq u_2 \geq \ldots \geq u_r \geq 1$ and $u_1 + u_2 + \ldots + u_r = q$. In the following we will refer to $\underline{u}$ simply as a partition, leaving the other properties implicit. If q and r are not specified, we will write $r(\underline{u})$ for the length of $\underline{u}$ and $q(\underline{u})$ for the sum $u_1 + u_2 + \ldots + u_{r(\underline{u})}$. Note that in [DiM13a, Section 1.4] the quantity $r(u)$ is defined in a slightly different way.

We recall that to a partition $\underline{u} = (u_1, u_2, \ldots, u_r)$ we can attach a Young diagram whose columns have lengths $u_1, u_2, \ldots, u_r$. A Young tableau of shape $\underline{u}$ is a filling of the Young diagram attached to $\underline{u}$ with positive integers. We call a Young tableau semistandard if its entries are strictly increasing from top to bottom along each column, and weakly increasing along each row from left to right.

Let R be a commutative ring with unity. To a partition $\underline{u}$ we can attach in the standard way a Schur functor $\mathbb{S}^{\underline{u}}$ from the category of R-modules to itself; we leave the dependence on R implicit. We refer to [FH91, Lecture 6] for the definition of $\mathbb{S}^{\underline{u}}$. We recall that $\mathbb{S}^{\underline{u}} = (0)$ if $r(\underline{u}) > n$. If M is a free R-module of rank n, then the R-module $\mathbb{S}^{\underline{u}}(M)$ is also free and we denote by $r_{\underline{u}}(n)$ its rank.

Let $\underline{u}$ be a partition. We write $q = q(\underline{u})$ in this paragraph. Let $\mathrm{Std}_n(R)$ denote a free R-module of rank n equipped with an ordered basis $(e_i)_{1\leq i\leq n}$ and with the standard representation of $\mathrm{GL}_n(R)$ with respect to this basis. The elementary tensors $e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_q}$ form an R-basis of the tensor power $\mathrm{Std}_n(R)^{\otimes q}$. We make this basis into an ordered basis via the lexico-graphic order; we denote it by $(\otimes^q e_i)$. We let $g \in \mathrm{GL}_n(R)$ act on an elementary tensor $e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_q} \in \mathrm{Std}_n(R)^{\otimes q}$ by $g(e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_q}) = g(e_{i_1}) \otimes g(e_{i_2}) \otimes \ldots \otimes g(e_{i_q})$, and we extend this action R-linearly to $\mathrm{Std}_n(R)^{\otimes q}$. The R-module $\mathbb{S}^{\underline{u}}\mathrm{Std}_n(R)$ is a free R-module of rank $r_{\underline{u}}(n)$ and a $\mathrm{GL}_n(R)$-stable direct summand of the R-module $\mathrm{Std}_n(R)^{\otimes q}$. More precisely, it is a direct summand of $\bigotimes_{i=1}^r \Lambda^{u_i}\mathrm{Std}_n(R)$. We use the ordered basis $(e_i)_{1\leq i\leq n}$ to define an ordered basis of $\mathbb{S}^{\underline{u}}\mathrm{Std}_n(R)$ in the following way. Let $\mathbf{T}$ be the set of semistandard Young tableaux of shape $\underline{u}$ taking all their entries in the set $\{1, 2, \ldots, n\}$. Let $T \in \mathbf{T}$. Denote by $i_{a,b}^T$ the integer at the intersection of the a-th column from the top and the b-th row from the left. Let e_T be the element of $\bigotimes_{i=1}^r \Lambda^{u_i}\mathrm{Std}_n(R)$ given by

$$(e_{i_{1,1}^T} \wedge e_{i_{1,2}^T} \wedge \ldots \wedge e_{i_{1,u_1}^T}) \otimes (e_{i_{2,1}^T} \wedge e_{i_{2,2}^T} \wedge \ldots \wedge e_{i_{2,u_2}^T}) \otimes \ldots \otimes (e_{i_{r,1}^T} \wedge e_{i_{r,2}^T} \wedge \ldots \wedge e_{i_{r,u_r}^T}).$$

Let $[e_T]$ be the projection of e_T to the quotient $\mathbb{S}^{\underline{u}}\mathrm{Std}_n(R)$ of $\bigotimes_{i=1}^r \Lambda^{u_i}\mathrm{Std}_n(R)$. We give the set $\mathbf{T}$ the following total ordering: given $T, T' \in \mathbf{T}$ we say that $T < T'$ if, proceeding in the lexico-graphic order, the first value of (a, b) such that $i_{a,b}^T \neq i_{a,b}^{T'}$ satisfies $i_{a,b}^T < i_{a,b}^{T'}$. When T varies over the ordered set $\mathbf{T}$, the string $(e_T)_{T\in\mathbf{T}}$ forms an ordered R-basis of $\mathbb{S}^{\underline{u}}\mathrm{Std}_n(R)$ [FH91, Problem 6.15]. In the following, when we speak of the basis $(e_T)_{T\in\mathbf{T}}$ we always leave implicit the fact that it depends on the choice of an ordered basis $(e_i)_{i\in\{1,\ldots,n}$ of V.

We denote by $\mathbb{S}^{\underline{u}}_n(R)$ the representation $\mathrm{GL}_n(R) \to \mathrm{GL}_{r_{\underline{u}}(n)}$ defining the action of $\mathrm{GL}_n(R)$ on $\mathbb{S}^{\underline{u}}\mathrm{Std}_n(R)$ with respect to the ordered R-basis $(e_T)_{T\in\mathbf{T}}$.

Our choice of ordering on $\mathbf{T}$ is explained by the following remark.

Remark 3.1. *For every positive integer d, let B_d denote the Borel subgroup of $\mathrm{GL}_d(R)$ consisting of upper triangular matrices and let T_d denote the torus of diagonal matrices in $\mathrm{GL}_d(R)$. An easy check shows that:*

* *the images of T_n and B_n via the representation $\mathbb{S}^{\underline{u}}_n$ are contained in $T_{r_{\underline{u}}(n)}$ and $B_{r_{\underline{u}}(n)}$, respectively;*
* *the preimages of $T_{r_{\underline{u}}(n)}$ and $B_{r_{\underline{u}}}(n)$ via the representation $\mathbb{S}^{\underline{u}}_n$ are contained in T_n and B_n, respectively.*

The properties on Borel subgroups depend on our chosen ordering of the R-basis of $\mathbb{S}^{\underline{u}}\mathrm{Std}_n(R)$. Note that such a choice may be unimportant when working with the linear representation $\mathbb{S}^{\underline{u}}_n$, but it becomes significant when we move to semilinear representations.

Remark 3.2. *Let $\underline{u} = (u_1, u_2, \dots, u_d)$ be any partition. For a positive integer k, let $\underline{u}+k$ denote the partition $(u_1 + k, u_2 + k, \dots, u_d + k)$. Then $\mathbb{S}^{\underline{u}+k}_n = \mathbb{S}^{\underline{u}}_n \otimes \det^k$.*

When the characteristic of R is zero, the representation $\mathbb{S}^{\underline{u}}_n(R)$ is irreducible. When $R = \mathbb{C}$ the morphism $\mathbb{S}^{\underline{u}}_n$ defines the $\mathbb{C}$-representation of $\mathrm{GL}_n(\mathbb{C})$ of highest weight $\underline{u}$, that is unique up to isomorphism. For arbitrary R, if $r(\underline{u}) = n$ and $u_1 = u_2 = \ldots = u_n = u$ for a positive integer u then the representation $\mathbb{S}^{\underline{u}}_n(R)$ is $\det^u$.

Now suppose that R carries an action of G_K and that M is an object of rank n in $\mathrm{Sem}_{G_K}(R)$. We choose an ordered R-basis $(e_i)_{1\le i\le n}$. We let $g \in G_K$ act on $M^{\otimes q}$ by setting $g(e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_q}) = g(e_{i_1} \otimes g(e_{i_2}) \otimes \ldots \otimes g(e_{i_q}))$ for every elementary tensor $e_{i_1}\otimes e_{i_2}\otimes\ldots\otimes e_{i_q}$, and extending semilinearly. This action leaves the R-submodule $\mathbb{S}^{\underline{u}}(M)$ stable, making it into an object of $\mathrm{Sem}_{G_K}(R)$ that we still denote by $\mathbb{S}^{\underline{u}}(M)$. Note that the action of G_K on $M^{\otimes q}$, hence on $\mathbb{S}^{\underline{u}}(M)$, depends on our choice of basis for M and for $M^{\otimes q}$.

Remark 3.3. *With notations as before, we identify M with $\mathrm{Std}_n(R)$ via the chosen ordered basis $(e_i)_{1\le i\le n}$ and we give $\mathbb{S}^{\underline{u}}(M)$ the ordered basis $(e_T)_{T\in\mathbf{T}}$ constructed in this section. For $g \in G_K$, let A^g be the matrix in $\mathrm{GL}_n(R)$ describing the action of g on the basis $(e_i)_{1\le i\le n}$ of M. Then g acts on the basis $(e_T)_{T\in\mathbf{T}}$ of $\mathbb{S}^{\underline{u}}(M)$ via the matrix $\mathbb{S}^{\underline{u}}_n(A^g)$.*

We recall a standard result.

Lemma 3.4. *[Hun86, Theorem 1] Let n be a positive integer. Let $\underline{u}$ be a partition that satisfies $(*)$. Then the morphism $\mathrm{GL}_n \to \mathrm{GL}_n/\ker\mathbb{S}^{\underline{u}}_n$ is an isogeny. The kernel of $\mathbb{S}^{\underline{u}}_n$ is the algebraic group μ_q of q-th roots of unity, embedded in GL_n via $\mu \mapsto \mu\mathbb{1}_n$.*

When choosing a partition $\underline{u}$ we will often make the following assumption:

$$(*)\qquad\qquad \text{neither (a) } r(\underline{u}) > n \text{ nor (b) } u_1 = u_2 = \ldots = u_{r(\underline{u})}.$$

When (a) holds the representation $\mathbb{S}^{\underline{u}}_n$ is trivial, and when (b) holds it is a power of the determinant. We bear no interest in these two cases: for these representations the conclusions of Theorems 4.2 and 5.2 are trivally false, if $n > 1$, or trivially true, if $n = 1$. Note that $(*)$ coincides with the assumption on the partition in [DiM13a, Sections 2.4, 3.3], even if the hypothesis there is phrased differently.

Let $\underline{u}$ be a partition and let $W = (W_e, W_{\mathrm{dR}}^+)$ be a B_K^E-pair. As in [DiM13b, Section 1.4], we define a B_K^E-pair $\mathbb{S}^{\underline{u}}(W)$ as $(\mathbb{S}^{\underline{u}}(W_e), \mathbb{S}^{\underline{u}}(W_{\mathrm{dR}}^+))$. If V is an object of $\mathrm{Rep}_{G_K}(E)$, then $\mathbb{S}^{\underline{u}}(W(V)) \cong W(\mathbb{S}^{\underline{u}}(V))$ in the category of B_K^E-pairs.

4. Schur functors and potentially trianguline B-pairs

We recall a result of Di Matteo.

Theorem 4.1. *Let $\underline{u}$ be a partition satisfying $(*)$. Let W be a B_K^E-pair.*

(1) If W is de Rham then $\mathbb{S}^{\underline{u}}(W)$ is de Rham.

(2) If $\mathbb{S}^{\underline{u}}(W)$ is de Rham, then there exists a finite extension F/E and a character $\mu\colon G_K \to F^\times$ such that $W(\mu)$ is de Rham.

(3) Statements (1) and (2) hold if we replace "de Rham" with "crystalline".

Part (1) follows by a straightforward semilinear algebra computation. Parts (2) and (3) are Theorems 2.4.2 and 3.3.2 of [DiM13a], respectively.

The goal of this section is to prove an analogue of Theorem 4.1(1,2), with "de Rham" replaced by "potentially triangulable". Our result is the following.

Theorem 4.2. *Let $\underline{u}$ be a partition satisfying $(*)$. Let W be a B_K^E-pair. Then:*

(1) if W is split triangulable of parameter $\underline{\delta}$ then $\mathbb{S}^{\underline{u}}(W)$ is split triangulable of parameter $\mathbb{S}^{\underline{u}} \circ \underline{\delta}$;

(2) if $\mathbb{S}^{\underline{u}}(W)$ is split triangulable, then W is potentially split triangulable.

By specializing the theorem to the case where $W = W(V)$ for an E-linear representation V of G_K, we obtain the following corollary.

Corollary 4.3. *Let $\underline{u}$ be a partition satisfying $(*)$ and let V be an object of $\mathrm{Rep}_{G_K}(E)$. Then:*

(1) if V is split trianguline of parameter $\underline{\delta}$, then $\mathbb{S}^{\underline{u}}(V)$ is split trianguline of parameter $\mathbb{S}^{\underline{u}} \circ \underline{\delta}$;

(2) if $\mathbb{S}^{\underline{u}}(V)$ is split trianguline, then V is potentially trianguline.

Proof. (of Theorem 4.2) Let $\underline{u}$ and W be as in the statement of the theorem. Note that $\mathbf{F}_E \otimes_{\mathbf{B}_{e,E}} \mathbb{S}^{\underline{u}} W_e = \mathbb{S}^{\underline{u}}(\mathbf{F}_E \otimes_{\mathbf{B}_{e,E}} W_e)$ in $\mathrm{Sem}_{G_K}(\mathbf{F}_E)$. By the implication (1) $\implies$ (3) in Lemma 2.5, the object $\mathbb{S}^{\underline{u}}(\mathbf{F}_E \otimes_{\mathbf{B}_{e,E}} W_e)$ is triangulable in $\mathrm{Sem}_{G_K}(\mathbf{F}_E)$. On the other hand, the implication (3) $\implies$ (1) gives that if $\mathbf{F}_E \otimes_{\mathbf{B}_{e,E}} W_e$ is triangulable then the B_K^E-pair W is triangulable. We reduce Theorem 4.2 to a statement about semilinear representations of G_K. More precisely, the theorem follows by specializing the next lemma to the representation $V = \mathbf{F}_E \otimes_{\mathbf{B}_{e,E}} W_e$.

Let V be an object of rank n in $\mathrm{Sem}_{G_K}(\mathbf{F}_E)$.

Lemma 4.4.

(1) If the action of G_K on V is triangulable, then the action of G_K on $\mathbb{S}^{\underline{u}}(V)$ is triangulable.

(2) If the action of G_K on $\mathbb{S}^{\underline{u}}(V)$ is triangulable, then the action of G on $\mathbb{S}^{\underline{u}}(V)$ is potentially triangulable.

We prove part (1). Let $(e_i)_{i\in\{1,\dots,n\}}$ be a $\mathbf{F}_E$-basis of V in which G_K acts via upper triangular matrices. For $g\in G_K$ let $\sigma_g\in \mathrm{GL}_n(\mathbf{F}_E)$ be the matrix describing the action of g on V in the basis $(e_i)_{i\in\{1,\dots,n\}}$. Then the matrix describing the action of g on $\mathbb{S}^{\underline{u}}(V)$ in the basis $(e_T)_{T\in\mathbf{T}}$ is $\mathbb{S}^{\underline{u}}_n\sigma_g$. Since the matrix σ_g is upper triangular, the matrix $\mathbb{S}^{\underline{u}}_n\sigma_g$ is also upper triangular by Remark 3.1, as desired.

We prove part (2). We will need to apply a few results that we restate here.

Lemma 4.5. *[DiM13b, Lemma 3.1.1] Let $0\to X'\to X\to X''\to 0$ be an exact sequence in $\mathrm{Sem}_{G_K}(\mathbf{F}_E)$. Then X is triangulable if and only if X' and X'' are.*

Lemma 4.6. *[DiM13b, Theorem 3.2.2] If X and Y are two objects in $\mathrm{Sem}_{G_K}(\mathbf{F}_E)$ such that $X\otimes Y$ is triangulable, then X and Y are potentially triangulable.*

Lemma 4.7. *[Del82, Proposition 3.1] Let $\mathbf{F}$ be a field of characteristic zero, G a reductive algebraic group over $\mathbf{F}$ and let $S\colon G\to \mathrm{GL}(V)$ be a finite-dimensional faithful representation of G over $\mathbf{F}$. Then every irreducible, finite-dimensional $\mathbf{F}$-linear representation of G appears as a direct factor of $S^{\otimes k}$ for some positive integer k.*

Let $\underline{u}$ be a partition of an integer q such that $\underline{u}$ satisfies $(*)$ and assume that $\mathbb{S}^{\underline{u}}(V)$ is triangulable. Let $\underline{v}$ be another partition of the same integer, also satisfying $(*)$. We show that $\mathbb{S}^{\underline{v}}(V)$ is also triangulable. To this purpose we apply Lemma 4.7 to the reductive algebraic group G over $\mathbf{F}_E$ defined as $\mathrm{GL}_n/\ker\mathbb{S}^{\underline{u}}_n$. We make this precise. The $\mathbf{F}_E$-module $\mathbb{S}^{\underline{u}}(V)$, equipped with the action of G given by $\mathbb{S}^{\underline{u}}_n\colon G\to\mathrm{GL}(\mathbb{S}^{\underline{u}}(V))$, is a faithful representation of G. By Lemma 3.4 the kernels of $\mathbb{S}^{\underline{u}}_n$ and $\mathbb{S}^{\underline{v}}_n$ coincide the group μ_q embedded diagonally in GL_n. In particular $G=\mathrm{GL}_n/\ker\mathbb{S}^{\underline{v}}_n$, so that $\mathbb{S}^{\underline{v}}(V)$, equipped with the action of G given by $\mathbb{S}^{\underline{v}}_n\colon G\to\mathrm{GL}(\mathbb{S}^{\underline{v}}(V))$, is a faithful representation of G. By Lemma 4.7 the irreducible representation $\mathbb{S}^{\underline{v}}(V)$ of G appears as a subquotient of $(\mathbb{S}^{\underline{u}}(V))^{\otimes k}$ for some positive integer k. Since $\mathbb{S}^{\underline{u}}(V)$ is triangulable by assumption, its tensor power $(\mathbb{S}^{\underline{u}}(V))^{\otimes k}$ is also triangulable. In particular its direct factor $\mathbb{S}^{\underline{v}}(V)$ is trangulable by Lemma 4.5.

Now consider the object $\mathrm{Sym}^{q-1}V\otimes V$ in $\mathrm{Sem}_{G_K}(\mathbf{F}_E)$. By the Littlewood-Richardson rule (see for instance [FH91, Appendix 8]) the $\mathbf{F}_E$-linear representation $\mathrm{Sym}^{q-1}V\otimes V$ of GL_n can be decomposed in a direct sum of irreducible representations of the form $\mathbb{S}^{\underline{v}}(V)$, where $\underline{v}$ is a partition of q. By the result of the previous paragraph, every direct factor $\mathbb{S}^{\underline{v}}(V)$ is triangulable as an object of $\mathrm{Sem}_{G_K}(\mathbf{F}_E)$. We deduce that $\mathrm{Sym}^{q-1}V\otimes V$ is triangulable by Lemma 4.5. Now Lemma 4.6 gives that V is potentially triangulable, as desired. This concludes the proof of Lemma 4.4(2), hence of Theorem 4.2. $\square$

Remark 4.8.

1. *Suppose that W is triangulable with respect to ordered parameters $\eta_i\colon G_K\to E^\times$, $1\le i\le n$. Let $\underline{\eta}\colon G_K\to\mathrm{GL}_n(E)$ be the representation sending $g\in G_K$ to the diagonal matrix $(\eta_1(g),\dots,\eta_n(g))$. By Remark 3.1, the representation $\mathbb{S}^{\underline{u}}_n$ maps the diagonal torus of $\mathrm{GL}_n(E)$ to the diagonal torus of $\mathrm{GL}_{r_{\underline{u}}(n)}(E)$. In particular the representation $\mathbb{S}^{\underline{u}}_n\circ\underline{\eta}\colon G_K\to\mathrm{GL}_{r_{\underline{u}}(n)}(E)$ maps an element $g\in G_K$ to a diagonal matrix that we write as $(\eta_T(g))_{T\in\mathbf{T}}$ for the usual index set $\mathbf{T}$ of an E-basis of $\mathbb{S}^{\underline{u}}\mathrm{Std}_n(E)$ and characters $\eta_T\colon G_K\to E^\times$. Then it is immediate to see that the B_K^E-pair $\mathbb{S}^{\underline{u}}(W)$ is triangulable with respect to the ordered parameters $(\eta_T)_{T\in\mathbf{T}}$.*
2. *Each diagonal entry of $\mathbb{S}^{\underline{u}}_n\circ\underline{\eta}$ is expressed by a monomial of degree q in the variables $\eta_1,\dots,\eta_n$.*

5. Schur functors and trianguline B-pairs

We adapt the notion of *strict triangulinity* from [KPX14, Definition 6.3.1] to the context of B-pairs.

Definition 5.1. *Let W be a B_K^E-pair of rank n admitting a triangulation $\{W_i\}_{0\leq i\leq n}$ with ordered parameters $\eta_i\colon G_K\to K^\times$, $1\leq i\leq n$. We say that W is* strictly triangulable *with respect to the given ordered parameters if, for $1\leq i\leq n$, W_i is the only saturated, rank i sub-B_K^E-pair of W that contains W_{i-1} and satisfies $W_i/W_{i-1}\cong \mathbf{B}(\eta)$. We call an E-linear representation V of G_K strictly trianguline if $W(V)$ is strictly triangulable.*

We refer to Remark 7.2 for a condition on the parameter $(\eta_i)_{1\leq i\leq n}$ that guarantees that a B_K^E pair admitting a triangulation of this parameter is strictly trianguline.

Note that the property of being strictly trianguline with respect to some parameters depends on the choice of an ordering of the parameters. If L is a finite extension of K and W is a B_K^E-pair that is strictly trianguline with respect to ordered parameters $(\eta_i)_{1\leq i\leq n}$, then $W|_{G_L}$ is strictly trianguline with respect to the ordered parameters $(\eta_i|_{G_L})_{1\leq i\leq n}$.

Under a strict triangulinity assumption, we can improve Theorem 4.2 in the following way. Let $\underline{u}$ be a partition satisfying $(*)$. Let $W=(W_e,W_{\mathrm{dR}}^+)$ be a B_K^E-pair such that $\mathbb{S}^{\underline{u}}(W)$ is triangulable. Let L be a finite extension of K such that $W|_{G_L}$ is triangulable. It exists by Theorem 4.2. Let $(\eta_i)_{1\leq i\leq n}$ be the ordered set of parameters of the triangulation. Let $(\eta_T)_{T\in\mathbf{T}}$ be the ordered set of parameters defined in Remark 4.8.

Theorem 5.2. *If $\mathbb{S}^{\underline{u}}(W)$ is strictly triangulable with respect to the ordered parameters $(\eta_T|_{G_L})_{T\in\mathbf{T}}$ then W is split triangulable.*

Proof. As usual we can restrict ourselves to the split triangulable case. Let $\{W_i\}_{1\leq i\leq n}$ be a triangulation of $W|_{G_L}$ and write $W_i=(W_{i,e},W_{i,\mathrm{dR}}^+)$. Also write $W_i/W_{i-1}=((W_i/W_{i-1})_e,(W_i/W_{i-1})_{\mathrm{dR}}^+)$. For every $i\in\{1,\dots,n\}$ let $\overline{e}_{i,e}$ and $\overline{e}_{i,\mathrm{dR}}^+$ be generators of $(W_i/W_{i-1})_e$ over $\mathbf{B}_{e,E}$ and of $(W_i/W_{i-1})_{\mathrm{dR}}^+$ over $\mathbf{B}_{\mathrm{dR},E}^+$, respectively. Lift them to elements $e_{i,e}$ and $e_{i,\mathrm{dR}}^+$ of $W_{i,e}$ and $W_{i,\mathrm{dR}}^+$, so that:

* $(e_{i,e})_{1\leq i\leq n}$ is an ordered $\mathbf{B}_{e,E}$-basis of W_e such that the set $(e_{j,e})_{1\leq j\leq i}$ generates $W_{i,e}$ for $1\leq i\leq n$,
* $(e_{i,\mathrm{dR}}^+)_{1\leq i\leq n}$ is an ordered $\mathbf{B}_{\mathrm{dR},E}^+$-basis of W_{dR}^+ such that the set $(e_{j,\mathrm{dR}}^+)_{1\leq j\leq i}$ generates $W_{i,\mathrm{dR}}^+$ for $1\leq i\leq n$.

For $g\in G_K$, let A_e^g be the matrix in $\mathrm{GL}_n(\mathbf{B}_{e,E})$ defining the action of g on the basis $(e_i)_{1\leq i\leq n}$. If $g\in G_L$, the matrix A_e^g is upper triangular since $\{W_{i,e}\}_{1\leq i\leq n}$ is a triangulation of $W_e|_{G_L}$ in $\mathrm{Sem}_{G_L}(\mathbf{B}_{e,E})$. Now let $(e_{T,e})_{T\in\mathbf{T}}$ be the ordered $\mathbf{B}_{e,E}$-basis of $\mathbb{S}^{\underline{u}}(W_e|_{G_L})$ constructed from $(e_i)_{1\leq i\leq n}$ as in Section 3. For $T\in\mathbf{T}$, let $W_{U,e}$ be the subobject of $\mathbb{S}^{\underline{u}}(W_e)$ generated over $\mathbf{B}_{e,E}$ by the set $(e_U)_{U\in\mathbf{T},U\leq T}$. The set $\{W_{T,e}\}_{T\in\mathbf{T}}$ defines a triangulation of $\mathbb{S}^{\underline{u}}(W_e|_{G_L})$ by the characters $(\eta_T|_{G_L})_{T\in\mathbf{T}}$. By Remark 3.3, the matrix describing the action of $g\in G_K$ on the basis $(e_T)_{T\in\mathbf{T}}$ is $\mathbb{S}_n^{\underline{u}}(A_e^g)$.

By repeating the reasoning of the preceding paragraph starting with the basis $(e_{i,\mathrm{dR}}^+)_{1\leq i\leq n}$ of W_{dR}^+, we obtain an ordered basis $(e_{T,\mathrm{dR}}^+)_{T\in\mathbf{T}}$ of $\mathbb{S}^{\underline{u}}(W_{\mathrm{dR}}^+)$ and we define a triangulation $\{W_{T,\mathrm{dR}}^+\}_{T\in\mathbf{T}}$ of $\mathbb{S}^{\underline{u}}(W_{\mathrm{dR}}^+|_{G_L})$. We let $A_{\mathrm{dR}}^{g,+}$ be the element of $\mathrm{GL}_n(\mathbf{B}_{\mathrm{dR},E}^+)$ defining the action of $g\in G_K$ on the basis $(e_i)_{1\leq i\leq n}$, so that $\mathbb{S}_n^{\underline{u}}(A_{\mathrm{dR}}^{g,+})$ defines the action of g on the basis $(e_T)_{T\in\mathbf{T}}$. The matrix $A_{\mathrm{dR}}^{g,+}$ is upper triangular for $g\in G_L$. We deduce from the construction of our bases that the couple $(W_{T,e}, W^+_{T,\mathrm{dR}})$ defines a B^E_K-pair W_T for every $T \in \mathbf{T}$, and that the set $\{W_T\}_{T\in\mathbf{T}}$ is a triangulation of $\mathbb{S}^{\underline{u}}(W)$ with respect to the ordered parameters $(\eta_T|_{G_L})_{T\in\mathbf{T}}$.

Now let $(X_T)_{T\in\mathbf{T}}$ be a triangulation of $\mathbb{S}^{\underline{u}}(W)$ with respect to the ordered parameters $(\eta_T)_{T\in\mathbf{T}}$. The restriction $(X_T|_{G_L})_{T\in\mathbf{T}}$ is a triangulation of $\mathbb{S}^{\underline{u}}(W|_{G_L})$ with respect to the ordered parameters $(\eta_T|_{G_L})_{T\in\mathbf{T}}$. By assumption $\mathbb{S}^{\underline{u}}(W|_{G_L})$ is strictly split triangulable with respect to the aforementioned parameters, so the triangulations $X_T|_{G_L}$ and W_T must coincide. In particular the filtration $\{W_T\}_{T\in\mathbf{T}}$ is stable under every $g \in G_K$ (not only $g \in G_L$) since X_T is a triangulation of $\mathbb{S}^{\underline{u}}(W)$. This implies that the matrices $\mathbb{S}^{\underline{u}}_n(A^g_e)$ and $\mathbb{S}^{\underline{u}}_n(A^{g,+}_{\mathrm{dR}})$ are upper triangular. Then Remark 3.1 implies that A^g_e and $A^{g,+}_{\mathrm{dR}}$ are upper triangular for every $g \in G_K$, hence that $\{W_i\}_{1\le i\le n}$ is a triangulation of W. $\square$

Definition 5.3. *We say that an E-linear representation V of G_K is strictly trianguline with ordered parameters $(\delta_i)_{1\le i\le n}$ if the B-pair $W(V)$ is strictly triangulable with the same ordered parameters.*

If V is refined trianguline in the sense of [KPX14, Definition 6.4.1], then V is strictly trianguline by [KPX14, Lemma 6.4.2]. However the condition of refined triangulinity is too strong for our purposes since it implies potential semi-stability.

By combining Theorems 4.2 and 5.2 we obtain the following corollary.

Corollary 5.4. *Let $\underline{u}$ be a partition satisfying $(*)$. Let V be an E-representation of G_K. Then V is strictly trianguline with ordered parameters $(\delta_i)_{1\le i\le n}$ if and only if $\mathbb{S}^{\underline{u}}V$ is strictly trianguline with ordered parameters $(\delta_T)_{T\in\mathbf{T}}$ constructed from $(\delta_i)_{1\le i\le n}$ as in Remark 4.8.*

6. Lifting trianguline representations via isogenies

Let G be a reductive group over $\mathbb{Q}$. In light of Theorem 5.2, the following definition makes sense.

Definition 6.1. *We say that a continuous representation $\rho\colon G_{\mathbb{Q}_p} \to G(\overline{\mathbb{Q}}_p)$ is strictly trianguline if there exists a homomorphism $\mu\colon \mathbb{G}_m \to G$ over $\overline{\mathbb{Q}}_p$ satisfying the following property: for every algebraic representation $S\colon G \to \mathrm{GL}_n$ defined over $\overline{\mathbb{Q}}_p$, the representation $S\circ\rho\colon G_{\mathbb{Q}_p} \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ is strictly trianguline with respect to the n-tuple of parameters $S\circ\mu(\overline{\mathbb{Q}}_p)\colon \overline{\mathbb{Q}}_p^\times \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$. In this case we also say that ρ is trianguline with parameter μ.*

In this section L is a number field, H and H' be two algebraic groups and $\pi\colon H' \to H$ is an isogeny. Given a continuous representation $\rho\colon G_L \to H(\overline{\mathbb{Q}}_p)$ with some prescribed local properties, we can investigate whether there exists a representation $\rho'\colon G_L \to H'(\overline{\mathbb{Q}}_p)$, with the same local properties, satisfying $\pi\circ\rho' \cong \rho$. When the required local properties are:

(1) unramifiedness outside of a finite set of places containing the places above p;
(2) a p-adic Hodge theoretic property at p, taken from the set {Hodge Tate, de Rham, semi-stable, crystalline};

the lifting problem is studied and solved in [Win95]. The same question is studied in depth in the Ph.D. thesis of Hoang Duc [Hoa15], that includes a treatment of the problem of minimizing the set of ramification primes of the lift. In this section we study the analogue of the problem above when (2) is replaced by the property that ρ is strictly trianguline at p, in the special case where the isogeny π is of the form $H' \to \mathbb{S}^{\underline{u}} H'$ for a partition $\underline{u}$ satisfying $(*)$.

The following result guarantees that a lift always exists in our situation. It is a corollary of a theorem of Tate [Tat77, Theorem 4].

Proposition 6.2. *Suppose that* $\ker \pi$ *is a torus. Let* $\rho\colon G_L \to H(\overline{\mathbb{Q}}_p)$ *be a continuous representation. There exists a representation* $\rho'\colon G_L \to H'(\overline{\mathbb{Q}}_p)$ *such that* $\pi \circ \rho' \cong \rho$.

Proposition 6.3. *Keep the notations and hypoteses of Proposition 6.2 and suppose that:*

(1) ρ *is unramified outside of a finite set of places;*

(2) ρ *is strictly trianguline at the places of* L *above* p*;*

(3) the trianguline parameter $\mu\colon \mathbb{G}_m \to H$ *of* ρ *admits a lift* $\mu'\colon \mathbb{G}_m \to H'(\overline{\mathbb{Q}}_p)$.

Let $\rho'\colon G_{\mathbb{Q}_p} \to H'(\overline{\mathbb{Q}}_p)$ *be a representation lifting* ρ*. Then* ρ' *is unramified outside of a finite set of places and strictly trianguline at* p *with parameter* μ'.

Assumption (3) in the corollary is an analogue of the condition on the lifting of the Hodge-Tate parameter in [Win95, Théorème 1.1.3]. Note that the existence of ρ' is guaranteed by Proposition 6.2.

Proof. By a result of Conrad [Conr11, Lemma 5.2 and Proposition 5.3] the representation ρ' is unramified outside of a finite set of places. Let v be a place of L above p and let ρ_v and ρ'_v be the restrictions of ρ and ρ', respectively, to a decomposition group at v. Let q be the order of $\ker \pi$. Let $\underline{u}$ be a partition of q satisfying $(*)$. Let $S'\colon H' \to \mathrm{GL}_n$ be a representation. We need to show that $S' \circ \rho'_v$ is strictly trianguline of parameters $S' \circ \mu'$. By Lemma 3.4 the kernel of the representation $\mathbb{S}_n^{\underline{u}} \circ S'\colon H' \to \mathrm{GL}_{r_{\underline{u}}(n)}$ contains $(S')^{-1}(\mu_q \mathbb{1}_n)$, that in turn contains $\ker \pi$ since π is a central isogeny with kernel of order q. In particular $\mathbb{S}_n^{\underline{u}} \circ S'$ factors as $S \circ \pi$ for a representation $S\colon H \to \mathrm{GL}_{r_{\underline{u}}(n)}$. By composing with ρ'_v we obtain

$$\mathbb{S}_n^{\underline{u}} \circ S' \circ \rho'_v \cong S \circ \pi \circ \rho'_v \cong S \circ \rho_v, \tag{6.1}$$

where the last equivalence comes from the definition of ρ'. The representation $S \circ \rho_v$ is trianguline of parameters $S \circ \mu$ because ρ_v is strictly trianguline of trianguline parameter μ by assumption. From the equivalence 6.1 we deduce that $\mathbb{S}_n^{\underline{u}} \circ S' \circ \rho'_v$ is trianguline of parameter $S \circ \mu$. Now $\mu = \pi \circ \mu'$ for the parameter μ' provided by condition (3) in the statement, so the trianguline parameter of $\mathbb{S}_n^{\underline{u}} \circ S' \circ \rho'_v$ is $S \circ \pi \circ \mu'$, that is $\mathbb{S}_n^{\underline{u}} \circ S' \circ \mu'$ by definition of S. Thanks to Corollary 5.4, we conclude that the representation $S' \circ \rho'_v$ is strictly trianguline of parameters $S' \circ \mu'$ because the representation $\mathbb{S}_n^{\underline{u}} \circ S' \circ \rho'_v$ is strictly trianguline of parameters $\mathbb{S}_n^{\underline{u}} \circ S' \circ \mu'$. Since this is true for every choice of S', we obtain the thesis. $\square$

Let $\rho\colon G_L \to G(E)$ be a continuous representation. We denote by $\overline{\operatorname{Im} \rho}$ the Zariski-closure over E of the image of ρ. The following is a corollary of Propositions 6.2 and 6.3.

Corollary 6.4. *Let H be a reductive group over $\mathbb{Q}$ equipped with a central isogeny $\pi\colon H \to \overline{\operatorname{Im}\rho}$ over E. Suppose that:*

(1) ρ is unramified outside of a finite set of places;

(2) ρ is strictly trianguline at the places of L above p;

(3) the trianguline parameter $\mu\colon \mathbb{G}_m \to \overline{\operatorname{Im}\rho}$ of ρ admits a lift $\mu'\colon \mathbb{G}_m \to H(E)$.

Then there exists a continuous representation $\rho'\colon G_L \to H'(E)$ that lifts ρ via π and is:

* *unramified outside of a finite set of places;*
* *strictly trianguline at p with parameter μ'.*

7. Congruence loci on Hecke-Taylor-Wiles varieties

We apply the results of the previous section to the study of certain p-adic Langlands lifts as maps between the "Hecke-Taylor-Wiles varieties" studied by Breuil, Hellmann and Schraen [BHS17]. We begin by recalling some definitions from there.

In the following n is a positive integer and p a prime number larger than $2n+1$. Let F^+ be a totally real number field and let F be a totally imaginary quadratic extension of F^+. We denote by Σ_p the set of places of F^+ above p. Let G be a unitary group over F^+ such that G is split over F, compact at infinity and isomorphic to GL_n at all places in Σ_p. Let L be a p-adic field with ring of integers $\mathcal{O}_L$ and residue field k_L. We say that a representation $\rho\colon G_F \to \mathrm{GL}_n(\mathcal{O}_L)$ is of automorphic origin if it is attached to a maximal ideal of a Hecke algebra acting on the space of L-valued automorphic forms for G of some fixed tame level $U^p \subset \mathbb{A}_{F^+}^{p\infty}$. We denote by Σ the support of U^p, so that $\overline{\rho}$ will be unramified outside $\Sigma \cup \Sigma_p$. Fix a representation ρ of automorphic origin. Let $\overline{\rho}\colon G_F \to \mathrm{GL}_n(k_L)$ be the reduction of ρ modulo the maximal ideal of $\mathcal{O}_L$. Let ζ_p be a p-th root of unity; we assume that $\overline{\rho}|_{G_{F(\zeta_p)}}$ is absolutely irreducible. For every place v of F^+ above p, we denote by $\widetilde{v}$ a place of F above v. Up to enlarging L, we can assume that for every v the field $F_{\widetilde{v}}$ admits $[F_{\widetilde{v}}\colon \mathbb{Q}_p]$ homomorphisms to L. For every v let $G_{F_{\widetilde{v}}}$ be a decomposition group for F at $\widetilde{v}$ and let $\overline{\rho}_{\widetilde{v}}$ be the restriction of $\overline{\rho}$ to $G_{F_{\widetilde{v}}}$.

Let $\mathcal{C}_{k_L}$ be the category of local Artinian k_L-algebras with residue field k_L. Let $W(k_L)$ be the ring of Witt vectors of k_L and let L_0 be its field of fractions. Let $v \in \Sigma$. Let $D_{\overline{\rho}_{\widetilde{v}}}^{\square}\colon \mathcal{C}_{k_L} \to \mathrm{Sets}$ be the functor that associates with $A \in \mathcal{C}_{k_L}$ the set of continuous representations $\rho'\colon G_{F_{\widetilde{v}}} \to \mathrm{GL}_n(A)$ such that $\pi_A \circ \rho' = \overline{\rho}_{\widetilde{v}}$, where π_A denotes reduction of the coefficients modulo the maximal ideal of A. By Schlessinger's criterion the functor $D_{\overline{\rho}_{\widetilde{v}}}^{\square}$ is represented by a complete local Noetherian $W(k_L)$-algebra $R(\overline{\rho}_{\widetilde{v}})^{\square}$ with residue field k_L. We call *framed deformation space* for $\overline{\rho}_{\widetilde{v}}$ and denote by $\mathfrak{X}_{\overline{\rho}_{\widetilde{v}}}^{\square}$ the rigid analytic space $\mathrm{Spf}\,(R(\overline{\rho}_{\widetilde{v}})^{\square})^{\mathrm{rig}} \times_{L_0} L$ over L. We also set $\mathfrak{X}_{\overline{\rho}_p}^{\square} \cong \prod_{v\in\Sigma_p} \mathfrak{X}_{\overline{\rho}_{\widetilde{v}}}^{\square}$ and $\mathfrak{X}_{\overline{\rho}_p}^{\square} \cong \prod_{v\in\Sigma_p} \mathfrak{X}_{\overline{\rho}_{\widetilde{v}}}^{\square}$.

Let $R_{\overline{\rho}_{\widetilde{v}}}^{\overline{\square}}$ the maximal reduced and p-torsion free quotient of $R_{\overline{\rho}_{\widetilde{v}}}^{\square}$. Let $R^{\mathrm{loc}} = \widehat{\bigotimes}_{v\in\Sigma_p} R_{\overline{\rho}_{\widetilde{v}}}^{\overline{\square}}$. Let g be a positive integer. We define formal series rings $R_\infty = R^{\mathrm{loc}}[[x_1,\dots,x_g]]$ and $S_\infty = \mathcal{O}_L[[y_1,\dots,y_{g'}]]$, where $g' = g + [F^+\colon \mathbb{Q}]\frac{n(n-1)}{2} + |\Sigma_p|n^2$. We suppose from now on that g satisfies the conditons in [BHS17, Théorème 3.4] and we give R_∞ an S_∞-algebra structure via the morphism given by *loc. cit.*.

Fix an isomorphism of algebraic groups $G \times_{F^+} F \cong \mathrm{GL}_{n,F}$. We set $G_p = \prod_{v\in\Sigma_p} G(F_v^+)$. The choice of the place $\widetilde{v}$ of F determines an isomorphism $i_{\widetilde{v}}\colon G(F_v^+) \cong \mathrm{GL}_n(F_{\widetilde{v}})$. We denote by B_v, T_v and N_v the inverse images via $i_{\widetilde{v}}$ of the subgroups of upper triangular, diagonal and upper unipotent matrices in $\mathrm{GL}_n(F_{\widetilde{v}})$, respectively. We denote by K_v the inverse image of $\mathrm{GL}_n(\mathcal{O}_{F_{\widetilde{v}}})$ via $i_{\widetilde{v}}$; it is a compact open subgroup of $G(\mathbb{F}_v^+)$. We define $B_p = \prod_{v\in\Sigma_p} B_v$ and analogously T_p, N_p and K_p. We also set $B_p^0 = B_p \cap K_p$, $N_p^0 = N_p \cap K_p$ and $T_p^0 = T_p \cap K_p$. We denote by $\widehat{T}_p$ and $\widehat{T}_p^0$ the rigid analytic spaces over $\mathbb{Q}_p$ parametrizing the continuous characters of T_p and T_p^0, and we set $\widehat{T}_{p,L} = \widehat{T}_p \times_{\mathbb{Q}_p} L$, $\widehat{T}_{p,L}^0 = \widehat{T}_p^0 \times_{\mathbb{Q}_p} L$.

Using the Taylor-Wiles-Kisin systems constructed in [Car+16], Breuil, Hellmann and Schraen construct a Hecke-Taylor-Wiles variety $X_p(\overline{\rho})$ (see [BHS17, Section 3] and Définition 3.6 there). It is a reduced rigid analytic space over L, defined as a closed subspace of $\mathrm{Spf}\,(R_\infty)^{\mathrm{rig}} \times_L \widehat{T}_{p,L}$ by means of Emerton's Jacquet functor. We set $\mathcal{W}_\infty = \mathrm{Spf}\,(S_\infty)^{\mathrm{rig}} \times \widehat{T}_{p,L}^0$ and we call it the *weight space*. We define a *weight morphism* $\omega_X \colon X_p(\overline{\rho}) \to \mathcal{W}_\infty$ as the composition of the inclusion $X_p(\overline{\rho}) \hookrightarrow \mathrm{Spf}\,(R_\infty)^{\mathrm{rig}} \times \widehat{T}_{p,L}$ and the morphism $\mathrm{Spf}\,(S_\infty)^{\mathrm{rig}} \times \widehat{T}_{p,L} \to \mathrm{Spf}\,(S_\infty)^{\mathrm{rig}} \times \widehat{T}_{p,L}^0$ given by the S_∞-algebra structure of R_∞ and the restriction of characters $\widehat{T}_{p,L} \to \widehat{T}_{p,L}^0$. If $\mathscr{E}(U^p)_{\overline{\rho}}$ denotes the $\overline{\rho}$ part of the eigenvariety for G (as defined for instance in [Che11, Definition 2.2] when $n=3$), equipped with a weight morphism $\omega\colon \mathscr{E}(U^p)_{\overline{\rho}} \to \widehat{T}(\mathcal{O}_{F^+}\otimes_{\mathbb{Z}} \mathbb{Z}_p)$, then there is a commutative diagram of reduced rigid analytic spaces

$$\begin{array}{ccc} \mathscr{E}(U^p)_{\overline{\rho}} & \hookrightarrow & X_p(\overline{\rho}) \\ \downarrow{\scriptstyle\omega} & & \downarrow{\scriptstyle\omega_X} \\ \widehat{T}(\mathcal{O}_{F^+}\otimes_{\mathbb{Z}}\mathbb{Z}_p) & \longrightarrow & \widehat{T}(\mathcal{O}_{F^+}\otimes_{\mathbb{Z}}\mathbb{Z}_p)\times \mathfrak{X}_{S_\infty} \end{array}$$

where the lower horizontal arrow is defined by the inclusion of a closed point in $\mathfrak{X}_{S_\infty}$.

Remark 7.1. *We recall that:*

(1) the weight morphism ω_X is flat and finite locally on the domain [BHS17, Proposition 3.10];

(2) both $\mathcal{W}_\infty$ and $X_p(\overline{\rho})$ are equidimensional of dimension $g + [F^+\colon \mathbb{Q}]\frac{n(n-1)}{2} + |\Sigma_p| n^2$; this is clear for $\mathcal{W}_\infty$ and it is given by [BHS17, Corollary 3.11] for $X_p(\overline{\rho})$;

(3) the image of an irreducible component of $X_p(\overline{\rho})$ via ω_X is Zariski-open in $\mathcal{W}_\infty$.

We refer to [BHS17, Définition 3.14] for the definition of a classical point of $X_p(\overline{\rho})$.

We denote by $\mathscr{T}_v$ and $\mathcal{W}_v$ the rigid analytic spaces over $\mathbb{Q}_p$ parametrizing continuous characters $F_v^{+,\times} \to \mathbb{C}_p$ and $\mathcal{O}_{F_v^+}^\times \to \mathbb{C}_p$, respectively. Restriction of characters induces a morphism of rigid analytic spaces $\mathscr{T}_v \to \mathcal{W}_v$. Let $\mathscr{T}_{v,L} = \mathscr{T}_v \times_{\mathbb{Q}_p} L$ and $\mathcal{W}_{v,L} = \mathcal{W}_v \times_{\mathbb{Q}_p} L$. The space $\mathscr{T}_{v,L}$ is a product of unit balls over L of dimension $n[F_v^+\colon \mathbb{Q}_p] + n$, so $\mathcal{W}_{v,L}$ is a product of unit balls of dimension $n[F_v^+\colon \mathbb{Q}_p] + n - 1$. We denote by $\mathscr{T}_{v,\mathrm{reg}}$ the complement in $\mathscr{T}_{v,L}$ of the set of L-points

$$\{x \mapsto x^{-\mathbf{i}}\}_{\mathbf{i}\in \mathbb{Z}_{\geq 0}^{\mathrm{Hom}(F_v^+,L)}} \cup \{x \mapsto x^{\mathbb{1}+\mathbf{i}}|x|_{F_v^+}\}_{\mathbf{i}\in \mathbb{Z}_{\geq 0}^{\mathrm{Hom}(F_v^+,L)}}$$

and by $\mathscr{T}_{v,\mathrm{reg}}^n$ the Zariski-open subset of $\mathscr{T}_{v,L}^n$ consisting of the points $(\delta_1,\dots,\delta_n)$ satisfying $\delta_i\delta_j^{-1} \in \mathscr{T}_{v,\mathrm{reg}}$ for all i,j with $i\neq j$. We denote by $\mathcal{W}_{v,\mathrm{reg}}^n$ the image of $\mathscr{T}_{v,\mathrm{reg}}^n$ in $\mathcal{W}_v^n \times_{\mathbb{Q}_p} L$.

Let $U_{\mathrm{tri}}^\square(\overline{\rho}_{\widetilde{v}})^{\mathrm{reg}}$ be the set of couples $(r, \underline{\delta})$ where:

* $r\colon G_{F_v} \to \mathrm{GL}_n(\mathcal{O}_L)$ is a framed trianguline lift of $\overline{\rho}_{\widetilde{v}}$;
* $\underline{\delta} \in \mathscr{T}_{v,\mathrm{reg}}^n$ is an ordered set of parameters of a triangulation of r.

Let $X^{\square}_{\mathrm{tri}}(\overline{\rho}_{\widetilde{v}})$ be the smallest closed rigid analytic subspace of $\mathfrak{X}^{\square}_{\overline{\rho}_{\widetilde{v}}} \times \mathscr{T}^n_{v,L}$ containing the $U^{\square}_{\mathrm{tri}}(\overline{\rho}_{\widetilde{v}})^{\mathrm{reg}}$.

Remark 7.2. *By [HS16, Lemma 2.12], the points of $U^{\square}_{\mathrm{tri}}(\overline{\rho}_{\widetilde{v}})^{\mathrm{reg}}$ are strictly trianguline in the sense of Definition 5.1.*

By composing the inclusion $X^{\square}_{\mathrm{tri}}(\overline{\rho}_{\widetilde{v}}) \hookrightarrow \mathfrak{X}^{\square}_{\overline{\rho}_{\widetilde{v}}} \times \mathscr{T}^n_{v,L}$ with the second projection and then with the morphism $\mathscr{T}^n_{v,L} \to \mathcal{W}^n_{v,L}$ given by the restriction of characters we obtain a *weight morphism* $\omega_v \colon X^{\square}_{\mathrm{tri}}(\overline{\rho}_{\widetilde{v}}) \to \mathcal{W}^n_{v,L}$. We set $X^{\square}_{\mathrm{tri}}(\overline{\rho}_p) = \prod_{v\in\Sigma_p} X^{\square}_{\mathrm{tri}}(\overline{\rho}_{\widetilde{v}})$, that is a closed rigid analytic subvariety of $\mathfrak{X}^{\square}_{\overline{\rho}_p} \times \widehat{T}_{p,L}$. This space is equipped with a weight morphism $\omega_p \colon X^{\square}_{\mathrm{tri}}(\overline{\rho}_p) \to \prod_{v\in\Sigma_p} \mathcal{W}_{v,L}$ defined by $\prod_{v\in\Sigma_p} \omega_v$.

We denote by $\mathbb{U}$ the open rigid analytic disc of radius 1 over L, that is the rigid analytic space $\mathrm{Spf}\,(\mathcal{O}_L[[T]])^{\mathrm{rig}}$. For every $v \in \Sigma_p$, let ιF_v^+ be the automorphism of $\widehat{T}(F_v^+)$ defined by

$$\iota_{F_v^+}(\delta_1, \dots, \delta_n) = \delta_{B_v} \cdot (\delta_1, \dots, \delta_i \cdot (\varepsilon \circ \mathrm{rec}_K)^{i-1}, \dots, \delta_n \cdot (\epsilon \circ \mathrm{rec}_K)^{n-1}),$$

where δ_{B_v} is the modulus character of $B_v(F_v^+)$, that is $\delta_{B_v} = |\cdot|^{n-1}_{F_v^+} \otimes |\cdot|^{n-3}_{F_v^+} \otimes |\cdot|^{1-n}_{F_v^+}$. Let ι be the automorphism

$$\mathrm{Id}_{\mathfrak{X}^{\square}_{\overline{\rho}_p}} \times (\iota_{F_v^+})_{v\in\Sigma_p} \colon \mathfrak{X}^{\square}_{\overline{\rho}_p} \times \widehat{T}_{p,L} \to \mathfrak{X}^{\square}_{\overline{\rho}_p} \times \widehat{T}_{p,L}.$$

By [BHS17, Théorème 3.20], there is an embedding

$$X_p(\overline{\rho}) \hookrightarrow \mathfrak{X}^{\square}_{\overline{\rho}_p} \times \mathfrak{X}^{\square}_{\overline{\rho}^p} \times \mathbb{U}^g \times \widehat{T}_{p,L} \tag{7.1}$$

that identifies $X_p(\overline{\rho})$ with a union of irreducible components of $\iota(X^{\square}_{\mathrm{tri}}(\overline{\rho}_p)) \times \mathfrak{X}^{\square}_{\overline{\rho}^p} \times \mathbb{U}^g$. For every $v \in \Sigma_p$, we denote by res_v the morphism $X_p(\overline{\rho}) \to X^{\square}_{\mathrm{tri}}(\overline{\rho}_{\widetilde{v}})$ defined as the composition of the inclusion $X_p(\overline{\rho}) \hookrightarrow \iota(X^{\square}_{\mathrm{tri}}(\overline{\rho}_p)) \times \mathfrak{X}^{\square}_{\overline{\rho}^p} \times \mathbb{U}^g$ with the projection to $\iota(X^{\square}_{\mathrm{tri}}(\overline{\rho}_{\widetilde{v}}))$ and the automorphism ι^{-1}. We also set $\mathrm{res}_p = \prod_{v\in\Sigma_p} \mathrm{res}_v \colon X_p(\overline{\rho}) \to X^{\square}_{\mathrm{tri}}(\overline{\rho}_p)$. If $v \in \Sigma - \Sigma_p$, we denote by res_v the morphism $X_p(\overline{\rho}) \to \mathfrak{X}^{\square}_{\overline{\rho}_v}$ obtaine by composing the inclusion $X_p(\overline{\rho}) \hookrightarrow \iota(X^{\square}_{\mathrm{tri}}(\overline{\rho}_p)) \times \mathfrak{X}^{\square}_{\overline{\rho}^p} \times \mathbb{U}^g$ with the projection to $\mathfrak{X}^{\square}_{\overline{\rho}_v}$.

As in [BHS17, Définition 3.21], we call an irreducible component X of $\mathfrak{X}^{\square}_{\overline{\rho}^p}$ *automorphic* if there exists an irreducible component $\mathfrak{X}^p$ of $\mathfrak{X}^{\square}_{\overline{\rho}^p}$ such that $\iota(X) \times \mathfrak{X}^p \times \mathbb{U}^g$ is an irreducible component of $X_p(\overline{\rho})$.

Let $\varpi_{\widetilde{v}}$ be a uniformizer of $F_{\widetilde{v}}$. For $i \in \{1, \dots, n\}$ we denote by $\beta'_i \colon \mathbb{G}_{m,F_{\widetilde{v}}} \to \mathrm{GL}_{n,F_{\widetilde{v}}}$ the cocharacter embedding $\mathbb{G}_m$ into the i-th entry of $T_{F_{\widetilde{v}}}$. We set $\beta_i = \prod_{j\leq i} \beta'_i$ and $\gamma_{\widetilde{v},i} = \beta_i(\varpi_{\widetilde{v}})$. Recall that we denote by $\delta_{B_v} \colon B_v(F_v^+) \to \mathbb{Q}^\times$ the modulus character of B_v. For $i \in \{1, \dots, n\}$, the map $(x, \underline{\delta}) \mapsto \underline{\delta}\delta^{-1}_{B_v}(\beta_i(\gamma_{\widetilde{v},i}))$ defines an analytic function on $X^{\square}_{\mathrm{tri}}(\overline{\rho}_{n,\widetilde{v}})$.

Definition 7.3. *Given $i \in \{1, \dots, n\}$ and a point $(x, \underline{\delta})$ of $X^{\square}_{\mathrm{tri}}(\overline{\rho}_{n,\widetilde{v}})$, we set $\mathrm{sl}_i(x, \underline{\delta}) = v_{F_v^+}(\underline{\delta}\delta^{-1}_{B_v}(\gamma_{\widetilde{v},i}))$ and we call it the i-th slope of $(x, \underline{\delta})$.*

Since the functions $\delta\delta^{-1}_{B_v}(\gamma_{\widetilde{v},i})$ are analytic on $X^{\square}_{\mathrm{tri}}(\overline{\rho}_{n,\widetilde{v}})$, the slopes $\mathrm{sl}_i \colon X^{\square}_{\mathrm{tri}}(\overline{\rho}_{n,\widetilde{v}}) \to \mathbb{R}^\times$ are locally constant functions.

The following conjecture of Breuil, Hellmann and Schraen can be seen as an analogue of the overconvergent Fontaine-Mazur conjecture that Kisin and Emerton proved for the group $\mathrm{GL}_{2/\mathbb{Q}}$.

Conjecture 7.4. *[BHS17, Conjecture 3.22] An irreducible component of $\prod_{v|p} X^{\square}_{\mathrm{tri}}(\overline{\rho}_{\widetilde{v}})$ is automorphic if and only if its intersection with $\prod_{v|p} U^{\square}_{\mathrm{tri}}(\overline{\rho}_{\widetilde{v}})^{\mathrm{reg}}$ contains a crystalline point.*

Given a $\overline{\mathbb{Q}}_p$-point x of $X_p(\overline{\rho})$ and a p-adic place v of F, we denote by $\rho_x\colon G_{F_{\widetilde{v}}} \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ and $\underline{\delta}_{x,v} \in \widehat{T}_{v,L}$ the p-adic Galois representation and the trianguline parameter at v, respectively, associated with x by the immersion $X_p(\overline{\rho}) \hookrightarrow \mathfrak{X}^{\square}_{\overline{\rho}_p} \times \mathfrak{X}^{\square}_{\overline{\rho}^p} \times \mathbb{U}^g \times \widehat{T}_{p,L}$.

Definition 7.5. *We call* strictly trianguline locus *of $X_p(\overline{\rho})$ and denote by $X_p(\overline{\rho})^{\mathrm{str}}$ the locus of $\overline{\mathbb{Q}}_p$-points x of $X_p(\overline{\rho})$ such that $\rho_{x,v}$ is strictly trianguline for every p-adic place v of F.*

The strictly trianguline locus is Zariski-open in $X_p(\overline{\rho})$. In particular it admits a unique structure of rigid $\overline{\mathbb{Q}}_p$-analytic subspace of $X_p(\overline{\rho})$.

Lemma 7.6.

(1) There exists an n-dimensional pseudocharacter $T_{\overline{\rho}_{\widetilde{v}}}\colon G_{F_{\widetilde{v}}} \to \mathcal{O}^{\circ}(X^{\square}_{\mathrm{tri}}(\overline{\rho}_{\widetilde{v}}))$ with the property that, for every $\overline{\mathbb{Q}}_p$-point x of $X^{\square}_{\mathrm{tri}}(\overline{\rho}_{\widetilde{v}})$, the specialization of $T_{\overline{\rho}_{\widetilde{v}}}$ at x is the trace pseudocharacter attached to ρ_x.

(2) There exists an n-dimensional pseudocharacter $T_{\overline{\rho}_{\widetilde{v}}}\colon G_{F_{\widetilde{v}}} \to \mathcal{O}^{\circ}(X_p(\overline{\rho}))$ with the property that, for every $\overline{\mathbb{Q}}_p$-point x of $X_p(\overline{\rho})$, the specialization of $T_{\overline{\rho}_{\widetilde{v}}}$ at x is the trace pseudocharacter attached to $\rho_{x,v}$.

(3) Let U be a connected wide open subdomain of $X_p(\overline{\rho})$ such that $\omega_X|_U\colon U \to \omega_X(U)$ is a finite morphism. Then the pseudocharacter T can be lifted over U to a continuous representation $\rho_U\colon G_{F_{\widetilde{v}}} \to \mathrm{GL}_n(\mathcal{O}^{\circ}(U))$, and for every $\overline{\mathbb{Q}}_p$-point x of U the specialization of ρ_U at x is equivalent to the representation ρ_x.

Proof. Part (1) follows from the argument in [BC09, Proposition 7.5.4], using the fact that $X^{\square}_{\mathrm{tri}}(\overline{\rho}_{\widetilde{v}})$ is wide open (or *nested* in the terminology of *loc. cit.*). The pseudocharacter of part (2) is obtained by composing with the morphism $\mathcal{O}^{\circ}(\iota(X^{\square}_{\mathrm{tri}}(\overline{\rho}_p))) \to \mathcal{O}^{\circ}(X_p(\overline{\rho}))$ induced by $\mathrm{res}_p\colon X_p(\overline{\rho}) \to X^{\square}_{\mathrm{tri}}(\overline{\rho}_p)$.

The ring of analytic functions of norm bounded by 1 on a wide open separated rigid analytic space is profinite by [BC09, Lemma 7.2.11(2)], hence the ring $\mathcal{O}^{\circ}(U)$ admits all of the previous properties; it is moreover local and Noetherian since U is connected and admits a finite morphism to $\omega_X(U)$. Then part (3) follows from a classical theorem of Nyssen and Rouquier [Nys96; Rou96, Corollary 5.2]. □

Note that by [BHS17, Proposition 3.10] $X_p(\overline{\rho}_m)$ admits an admissible covering by domains of the form required in part (2) of the above lemma.

Let $(x, \underline{\delta})$ be a point of $X^{\square}_{\mathrm{tri}}(\overline{\rho}_{n,\widetilde{v}})$ such that the representation ρ_x is crystalline and non-critical. Let $(\varphi_1, \ldots, \varphi_n)$ be the eigenvalues of the Frobenius automorphism of $D_{\mathrm{cris}}(\rho_x)$, with the ordering given by the triangulation of parameter δ, and let $(k_1, \ldots, k_n)$ be the Hodge-Tate weights of ρ_x, in decreasing order. Since $(x, \underline{\delta})$ is non-critical, [BHS17, Proposition 3.15] gives $\varphi_i = p^{k_i}\delta_i(\gamma_{\widetilde{v}})$ for every i. In particular $v_p(\delta_i(\gamma_{\widetilde{v}})) = v_p(\varphi_i) - k_i$, so

$$\mathrm{sl}_i(x, \underline{\delta}) = v_p(\varphi_i) - v_p(\delta_{B_{\widetilde{v}}}(\gamma_{\widetilde{v}})) - k_i. \tag{7.2}$$

Now let $(x, \underline{\delta}')$ be a point of $X_{\mathrm{tri}}^{\square}(\overline{\rho}_{n,\widetilde{v}})$ corresponding to a different triangulation of the representation ρ_x. Since we are working over the strict trianguline locus, the triangulation of parameter δ' induces an ordering $(\varphi_{\sigma(1)}, \dots, \varphi'_{\sigma(n)})$ of the eigenvalues of the crystalline Frobenius for some non-trivial permutation σ of $\{1, \dots, n\}$. For every i, Equation (7.2) gives

$$\mathrm{sl}_i(x, \underline{\delta}') = v_p(\varphi_{\sigma(i)}) - v_p(\delta_{B_{\widetilde{v}}}(\gamma_{\widetilde{v}})) - k_i = \mathrm{sl}_{\sigma(i)}(x, \underline{\delta}) + k_{\sigma(i)} - k_i. \tag{7.3}$$

Definition 7.7. *We call* family of representations of $G_{F_{\widetilde{v}}}$ *the datum* $\mathscr{F} = (A, \rho_{\mathscr{F}})$ *of:*

* *an affinoid L-algebra A that is a domain,*
* *a continuous representation $\rho_{\mathscr{F}}\colon G_{F_{\widetilde{v}}} \to \mathrm{GL}_n(A)$,*

such that the set of points x such that $\rho_{\mathscr{F}}$ specializes to a crystalline representation at x is Zariski-dense and accumulation in $\mathrm{Sp}(A)$. *We call* support *of the family the affinoid space* $\mathrm{Sp}(A)$ *and* dimension *of the family the L-dimension of* $\mathrm{Sp}(A)$.

Definition 7.8. *Let $\mathscr{F} = (\mathrm{Sp}(A), \rho_A)$ be a family of representations of $G_{F_{\widetilde{v}}}$. We say that $\mathscr{F}$ appears on $X_{\mathrm{tri}}^{\square}(\overline{\rho}_{n,\widetilde{v}})$ if there exists an embedding $\alpha\colon \mathrm{Sp}(A) \to X_{\mathrm{tri}}^{\square}(\overline{\rho}_{n,\widetilde{v}})$ such that the specialization of $\rho_{\mathscr{F}}$ at $x \in \mathrm{Sp}(A)$ is isomorphic to $\rho_{\alpha,x}$. We say that $\mathscr{F}$ appears j times on $X_{\mathrm{tri}}^{\square}(\overline{\rho}_{n,\widetilde{v}})$ if there are j distinct morphisms α satisfying the previous condition.*

Definition 7.9. *We say that the i-th weight of $\mathscr{F}$ is constant if there is an integer k such that the i-th Hodge-Tate weight of every crystalline specialization of $\rho_{\mathscr{F}}$ is k.*

We say that $\mathscr{F}$ has r constant weights if there exist r distinct indices i in the set $\{1, \dots, d\}$ with the property that the i-th weight of $\mathscr{F}$ is constant.

Remark 7.10. *Let $\mathscr{F}$ be a family with r constant weights. If $\mathscr{F}$ appears on $X_{\mathrm{tri}}^{\square}(\overline{\rho}_{n,\widetilde{v}})$, then its codimension in $X_{\mathrm{tri}}^{\square}(\overline{\rho}_{n,\widetilde{v}})$ is at least the L-codimension $r[F_{\widetilde{v}} \colon \mathbb{Q}_p]$ of $\dim_L \mathcal{W}_{v,L}^{n-r}$ in $\dim_L \mathcal{W}_{v,L}^{n}$, since only $n - r$ weights are allowed to vary.*

Proposition 7.11. *Assume that $\mathscr{F}$ has exactly r constant weights. Then $\mathscr{F}$ appears at most $r!$ times on $X_{\mathrm{tri}}^{\square}(\overline{\rho}_{\widetilde{v}})$.*

Proof. Let $I_{\mathscr{F}}^{\mathrm{cst}}$ be the set of indices of the constant weights of $\mathscr{F}$. Let ℓ be the number of appearances of the family $\mathscr{F}$ on $X_{\mathrm{tri}}^{\square}(\overline{\rho}_{m,\widetilde{v}})$. This means that there exist ℓ families $\mathscr{F}_1, \dots, \mathscr{F}_\ell$ that satisfy $\rho_{\mathscr{F}_i} \cong \rho_{\mathscr{F}}$ for $i = 1, \dots, \ell$. By the local constancy of the slopes, up to restricting the family $\mathscr{F}$ to a family of the same dimension we can suppose that all the slopes sl_i, $1 \leq i \leq n$, are constant on each of the families $\mathscr{F}_1, \dots, \mathscr{F}_\ell$. Given a point ρ_x of the family $\mathscr{F}$, we denote by $(\rho_x, \underline{\delta}_{x,1}), \dots, (\rho_x, \underline{\delta}_{x,\ell})$ the ℓ triangulations of ρ_x appearing in the families $\mathscr{F}_1, \dots, \mathscr{F}_\ell$. Consider two crystalline points ρ_x, ρ_y in the family $\mathscr{F}$ such that both x and y appear with ℓ distinct triangulations on the families $\mathscr{F}_1, \dots, \mathscr{F}_\ell$. This is possible since the rigid analytic subspaces of $X_{\mathrm{tri}}^{\square}(\overline{\rho}_{m,\widetilde{v}})$ supporting the families $\mathscr{F}_1, \dots, \mathscr{F}_\ell$ pairwise intersect in a Zariski-closed subspace, while the set of crystalline points is Zariski-dense in each of them since it is Zariski-dense in $\mathscr{F}$. We denote the triangulations of x and y on the family $\mathscr{F}_j$ by $(x, \underline{\delta}_{j,x})$ and $(y, \underline{\delta}_{j,y})$, respectively. Let $(k_{1,x}, \dots, k_{n,x})$ and $(k_{1,y}, \dots, k_{n,y})$ be the decreasing n-tuples of Hodge-Tate weights of ρ_x and ρ_y, respectively. Up to modifying our choice of the point y respecting the previous condition, we can suppose that

$$k_{i_1,x} - k_{i_2,x} = k_{i'_1,y} - k_{i'_2,y} \iff i_1, i_2, i'_1, i'_2 \in I_{\mathscr{F}}^{\mathrm{cst}}. \tag{7.4}$$

Indeed, the points y satisfying the first condition accumulate at x, so we can choose a point y as p-adically close to x as we want making the right hand side of Equation (7.4) arbitrarily large.

Let $i \in \{1, \dots, n\}$ and $j_1, j_2 \in \{1, \dots, \ell\}$ with $j_1 \neq j_2$. The triangulation of ρ_x of parameter $\underline{\delta}_{j_1}$ induces an ordering on the eigenvalues of the Frobenius automorphism of $D_{\mathrm{cris}}(\rho_x)$. The triangulation of parameter $\underline{\delta}_{j_2}$ gives a new ordering, obtained by acting on the indices of the previous one with a permutation $\sigma_{j_1,x}^{j_2}$ of $\{1, \dots, n\}$. In the same way we define a permutation $\sigma_{j_1,y}^{j_2}$ attached to the change of triangulations on ρ_y. By Equation (7.3) we have

$$\mathrm{sl}_i(x, \underline{\delta}_{j_2,x}) = \mathrm{sl}_{\sigma_{j_1,x}^{j_2}(i)}(x, \underline{\delta}_{j_1,x}) + k_{\sigma_{j_1,x}^{j_2}(i),x} - k_{i,x},$$
$$\mathrm{sl}_i(y, \underline{\delta}_{j_2,y}) = \mathrm{sl}_{\sigma_{j_1,y}^{j_2}(i)}(y, \underline{\delta}_{j_1,y}) + k_{\sigma_{j_1,y}^{j_2}(i),y} - k_{i,y}.$$

Now the slopes are constant on the families $\mathscr{F}_{j_1}$ and $\mathscr{F}_{j_2}$, so we deduce that

$$k_{\sigma_{j_1,x}^{j_2}(i),x} - k_{i,x} = k_{\sigma_{j_1,y}^{j_2}(i),y} - k_{i,y}.$$

Then condition (7.4) implies that either

$$i, \sigma_{j_1,x}^{j_2}(i) \in I_{\mathscr{F}}^{\mathrm{cst}}$$

or

$$i = \sigma_{j_1,x}^{j_2}(i) = \sigma_{j_1,y}^{j_2}(i).$$

Therefore $\sigma_{j_1,x}^{j_2}$ acts trivially on $\{1, \dots, n\} - I_{\mathscr{F}}^{\mathrm{cst}}$, which implies that there are at most $r!$ distinct choices for $\sigma_{j_1,x}^{j_2}$. Since distinct triangulations of ρ_x determine distinct values of $\sigma_{j_1,x}^{j_2}$, we conclude that $\ell \leq r!$. $\square$

7.1. **Schur functors on trianguline varieties.** Let $\underline{u}$ be a partition satisfying $(*)$, let $q = q(\underline{u})$ and let $m = r_{\underline{u}}(n)$. Suppose that $p > 2m + 1$. Let $S\colon \mathrm{GL}_n \to \mathrm{GL}_m$ be the representation of GL_n of highest weight $\underline{u}$. We omit the reference to $\underline{u}$ since the partition will be fixed throughout the rest of the text.

Lemma 7.12.
(1) For every $v \in \Sigma$, there exists a closed morphism of rigid analytic spaces

$$S_v\colon \mathfrak{X}_{\overline{\rho}_{\widetilde{v}}}^{\square} \to \mathfrak{X}_{S \circ \overline{\rho}_{\widetilde{v}}}^{\square}$$

that maps ρ_x to $S \circ \rho_x$.

(2) For every $v \in \Sigma_p$, there exists a closed morphism of rigid analytic spaces

$$S_{\mathrm{tri},v}\colon X_{\mathrm{tri}}^{\square}(\overline{\rho}_{\widetilde{v}}) \to X_{\mathrm{tri}}^{\square}(S_{\mathrm{tri}} \circ \overline{\rho}_{\widetilde{v}})$$

that maps $(\rho_x, \underline{\delta})$ to $(S \circ \rho_x, S \circ \underline{\delta})$.

Proof. FOr part (2), we show that the map

$$\mathfrak{X}_{\overline{\rho}_{\widetilde{v}}}^{\square} \times \mathscr{T}_v^n \to \mathfrak{X}_{S \circ \overline{\rho}_{\widetilde{v}}}^{\square} \times \mathscr{T}_v^m$$
$$(r, (\delta_i)_{1 \leq i \leq n}) \mapsto (S \circ r, S \circ \underline{\delta}).$$

inducess a morphism of rigid analytic spaces. It is sufficient to show that this is true separately for the maps $\mathfrak{X}_{\overline{\rho}_{\widetilde{v}}}^{\square} \to \mathfrak{X}_{S \circ \overline{\rho}_{\widetilde{v}}}^{\square}$ and $\mathscr{T}_v^n \to \mathscr{T}_v^m$. This will also give part (1) for every $v \in \Sigma$, since only the first of the two maps appears there. The second map is analytic because it is polynomial. We show that the first map is also analytic. By definition the spaces $\mathfrak{X}^{\square}_{\overline{\rho}_{\widetilde{v}}}$ and $\mathfrak{X}^{\square}_{S\,\circ\,\overline{\rho}_{\widetilde{v}}}$ represent the framed deformation functors $D^{\square}_{\overline{\rho}_{\widetilde{v}}}$ and $D^{\square}_{S\,\circ\,\overline{\rho}_{\widetilde{v}}}$, respectively. The representation S defines a morphism of functors $S^{\square}\colon D^{\square}_{\overline{\rho}_{\widetilde{v}}} \to D^{\square}_{S\,\circ\,\overline{\rho}_{\widetilde{v}}}$: for an object A of $\mathcal{C}_{k_L}$ and an object $\rho'\colon G_{F_{\widetilde{v}}} \to \mathrm{GL}_n(A)$ of $D^{\square}_{\overline{\rho}_{\widetilde{v}}}(A)$, we define $S^{\square}(\rho')$ as $S\circ\rho'$. Since both $D^{\square}_{\overline{\rho}_{\widetilde{v}}}$ and $D^{\square}_{S\,\circ\,\overline{\rho}_{\widetilde{v}}}$ are representable, $S^{\square}$ induces a morphism of representing spaces $\mathfrak{X}^{\square}_{\overline{\rho}_{\widetilde{v}}} \to \mathfrak{X}^{\square}_{S\,\circ\,\overline{\rho}_{\widetilde{v}}}$.

It follows from Corollary 4.3(1) that the morphism defined in the previous paragraph maps $U^{\square}_{\mathrm{tri}}(\overline{\rho}_{\widetilde{v}})^{\mathrm{reg}}$ to $U^{\square}_{\mathrm{tri}}(S\circ\overline{\rho}_{\widetilde{v}})^{\mathrm{reg}}$, hence induces a morphism between the Zariski-closures $X^{\square}_{\mathrm{tri}}(\overline{\rho}_{\widetilde{v}})$ and $X^{\square}_{\mathrm{tri}}(S_{\mathrm{tri}}\circ\overline{\rho}_{\widetilde{v}})$ of the two sets. □

We set $S_p=\prod_{v\in\Sigma_p}S_v$, $S^p=\prod_{v\in\Sigma-\Sigma_p}S_v$ and $S_{\mathrm{tri},p}=\prod_{v\in\Sigma_p}S_{\mathrm{tri},v}$.

A weight $\kappa\in\mathcal{W}^n_L(L)$ is an n-tuple $(\kappa_1,\ldots,\kappa_n)$ of characters $\mathcal{O}^{\times}_{F_{\widetilde{v}}}\to L^{\times}$. We define a character $\eta\colon\mathbb{G}_m\to T_n(L)$ as $\prod_{1\le i\le n}\kappa_i\circ\beta'_{n,i}$, recalling that β'_i is the embedding of $\mathbb{G}_m$ into the i-th diagonal entry of T_n. Then the composition $S\circ\eta$ is a character $\mathbb{G}_m\to T_m(L)$, that we can write as $S\circ\eta=\prod_{1\le i\le m}(S\circ\kappa)_i\beta'_{m,i}$ for an m-tuple $((S\circ\kappa)_1,\ldots,(S\circ\kappa)_n)$ of characters $\mathcal{O}^{\times}_{F_{\widetilde{v}}}\to L^{\times}$. This m-tuple defines an element of $\mathcal{W}^m_L$ that we denote by $S\circ\kappa$.

We call *S-weight space* and denote by $\mathcal{W}^S_L$ the image of the closed morphism of rigid analytic spaces

$$\begin{aligned} S_{\mathcal{W}}\colon \mathcal{W}^n_L &\to \mathcal{W}^m_L\\ \kappa &\mapsto S\circ\kappa. \end{aligned}$$

The space $\mathcal{W}^S_L$ is a Zariski-closed subspace of $\mathcal{W}^m_L$. Since $S\colon T_n(L)\to T_m(L)$ is an isogeny, $\mathcal{W}^S_L$ is equidimensional of dimension $n[F^+_v\colon\mathbb{Q}_p]+n-1$. It contains an accumulation and Zariski-dense subset of algebraic weights, for instance the images via $S_{\mathcal{W}}$ of the algebraic weights in $\mathcal{W}^n_L$.

We add subscripts m and n to distinguish the tori of GL_n and GL_m and their spaces of characters. Let $(x,\underline{\delta})$ be a point of $X^{\square}_{\mathrm{tri}}(\overline{\rho}_{\widetilde{v}})$ and let $\kappa\in\widehat{T}_{n,v,L}$ be its weight. Then $(S\circ x, S\circ\underline{\delta})$ defines a point of of weight $S\circ\kappa\in\widehat{T}_{m,v,L}$. For $n_0\in\mathbb{Z}_{\ge1}$ and an element $\mathbf{k}=(k_{i,\tau})\in(\mathbb{Z}^{n_0})^{\mathrm{Hom}(F_{\widetilde{v}},L)}$, we also denote by $x^{\mathbf{k}}$ the element of $\mathcal{W}^{n_0}_{v,L}$ defined by $x\mapsto\prod_{1\le i\le n_0,\tau\in\mathrm{Hom}(F_{\widetilde{v}},L)}x^{k_{i,\tau}}$. When $n_0=n$, we denote by $S\circ\mathbf{k}=((S\circ\mathbf{k})_{i,\tau})$ the unique element of $(\mathbb{Z}^m)^{\mathrm{Hom}(F_{\widetilde{v}},L)}$ such that $S\circ x^{\mathbf{k}}=x^{S\,\circ\,\mathbf{k}}$. With an abuse of notation, we will also write $\mathbf{k}$ for the weight $x^{\mathbf{k}}$. If $\mathbf{k}\in(\mathbb{Z}^{n_0}_{\ge0})^{\mathrm{Hom}(F_{\widetilde{v}},L)}$ and $(k_{i,\tau})$ is decreasing for fixed τ, we call $\mathbf{k}$ a classical weight.

Lemma 7.13. *Let $\tau\in\mathrm{Hom}(F_{\widetilde{v}},L)$. Let $(x,\underline{\delta})$ be a point of $X^{\square}_{\mathrm{tri}}(\overline{\rho}_{\widetilde{v}})$ of classical weight $\mathbf{k}\in\mathcal{W}^n_{v,L}(\overline{\mathbb{Q}}_p)$. Then:*

(1) the slopes of $S(x,\underline{\delta})$ satisfy $\max_{1\le i\le m}|\mathrm{sl}_{i,S(x,\underline{\delta})}|\le q\cdot\max_{1\le i\le n}|\mathrm{sl}_{i,(x,\underline{\delta})}|$;

(2) if $\kappa=(x^{\mathbf{k}_1},\ldots,x^{\mathbf{k}_n})$ with $(\mathbf{k}_{1,\tau},\ldots,\mathbf{k}_{n,\tau})$ decreasing, then:

- *(i) $\max_{1\le i\le m}|(S\circ\mathbf{k})_{i,\tau}|\le q\cdot\max_{1\le i\le n}|k_{i,\tau}|$;*
- *(ii) $\min_{1\le i\le m}|(S\circ\mathbf{k})_{i,\tau}-(S\circ\mathbf{k})_{i+1,\tau}|\ge\min_{1\le i\le n}|\mathbf{k}_{i,\tau}-\mathbf{k}_{i+1,\tau}|$;*
- *(iii) $\min_{1\le i\le m}|(S\circ\mathbf{k})_{i-1,\tau}-2(S\circ\mathbf{k})_{i,\tau}+(S\circ\mathbf{k})_{i+1,\tau}|\ge\min_{1\le i\le n}|\mathbf{k}_{i-1,\tau}-2\mathbf{k}_{i,\tau}+\mathbf{k}_{i+1,\tau}|$.*

Proof. Statements (1) and (2.i) are an immediate consequence of Remark 4.8(2). Statements (2.ii-iii) follow from the fact that $(\mathbf{k}_{i,\tau})_{1\le i\le n}$ is a decreasing n-tuple and that the ordered n-tuple of exponents of $(\mathbf{k}_{1,\tau},\ldots,\mathbf{k}_{n,\tau})$ in the monomial defining $(S\circ\mathbf{k})_{i+1,\tau}$ is strictly smaller than that of the monomial defining $(S\circ\mathbf{k})_{i,\tau}$ with respect to the lexicographic order. $\square$

7.2. **Schur functors on Hecke-Taylor-Wiles varieties.** We add subscripts m and n to most objects depending on the dimension of the relevant Galois representations. Let $\overline{\rho}_m\colon G_F\to \mathrm{GL}_m(\overline{\mathbb{F}}_p)$ be a residual representation of automorphic origin. Let H be an algebraic subgroup of GL_m.

Definition 7.14. *We call H-*congruence locus *on the eigenvariety $X_p(\overline{\rho}_{m,p})$ the locus of its $\overline{\mathbb{Q}}_p$-points x satisfying* $\mathrm{Im}\,\rho_{x,v}\subset H(\overline{\mathbb{Q}}_p)$ *for every $v\in\Sigma$. We denote it by $X_p(\overline{\rho}_m)^H$. When $H=S(\mathrm{GL}_n(\overline{\mathbb{Q}}_p))$ with the above notations, we also call $X_p(\overline{\rho}_m)^H$ the S-*congruence locus *and we denote it by $X_p(\overline{\rho}_m)^S$.*

Remark 7.15. *Consider the morphism $\omega_{m,v}\circ\mathrm{res}_{m,v}\colon X_p(\overline{\rho}_m)\to\mathcal{W}_L^m$. Under this map, the image of $X_p(\overline{\rho}_m)^S$ is contained in $\mathcal{W}_L^S$.*

Lemma 7.16. *Assume that $\overline{\rho}_m$ is absolutely irreducible. The H-congruence locus is Zariski-closed in $X_p(\overline{\rho}_m)(\overline{\mathbb{Q}}_p)$. In particular it admits a unique structure of reduced rigid $\overline{\mathbb{Q}}_p$-analytic subspace of $X_p(\overline{\rho}_m)$.*

Proof. Let $\{U_i\}_{i\in\mathbb{N}}$ be an admissible covering of X_p in wide open subdomains with the properties required by Lemma 7.6(2). It is sufficient to show that the intersection of the H-congruence locus with each of the U_i's is Zariski-closed. Let $i\in\mathbb{N}$. By Lemma 7.6(2) there is a continuous representation $\rho_{U_i}\colon G_F\to\mathrm{GL}_m(\mathcal{O}(U_i))$ that specializes to ρ_x for every $\overline{\mathbb{Q}}_p$-point x of U_i. Let $\{f_j\}_{j\in J}$ be a set of equations for the subgroup $H(\overline{\mathbb{Q}}_p)$ of $G(\overline{\mathbb{Q}}_p)$. Then the H-congruence locus intersects U_i on the zero-locus of the ideal of $\mathcal{O}^\circ(U_i)$ generated by $\{f_j(\rho_{U_i}(g))\}_{j\in J,g\in G_{\mathbb{Q}}}$ (a finite number of these generators will be sufficient since $\mathcal{O}^\circ(U_i)$ is Noetherian), hence on a Zariski-closed subspace of U_i. $\square$

In what follows we assume that $\overline{\rho}_m$ satisfies the condition:

$$\text{(Type } S\text{)}\qquad \begin{gathered}\text{there exists a continuous representation } \overline{\rho}_n\colon G_F\to\mathrm{GL}_n(\overline{\mathbb{Q}}_p)\\ \text{of automorphic origin such that } \overline{\rho}_m=S\circ\overline{\rho}_n.\end{gathered}$$

If (Type S) is not true, the S-congruence locus on $X_p(\overline{\rho}_m)$ is clearly empty. We fix from now on a representation $\overline{\rho}_n$ satisfying assumption (Type S).

We set $X_p(\overline{\rho}_m)^{S,\mathrm{str}}=X_p(\overline{\rho}_m)^S\cap X_p(\overline{\rho}_m)^{\mathrm{str}}$.

Proposition 7.17. *Suppose that the "only if" part of Conjecture 7.4 is true for $U(m)$. Then there exists a morphism of rigid analytic spaces $S_{TW}\colon X_p(\overline{\rho}_n)\to X_p(\overline{\rho}_m)$ fitting in the commutative diagram*

$$\begin{array}{ccc} X_p(\overline{\rho}_n) & \xrightarrow{\mathrm{res}_p} & X_{\mathrm{tri}}^{\square}(\overline{\rho}_{n,p})\times\mathfrak{X}^{\square}_{\overline{\rho}_n^p} \\ \Big\downarrow{\scriptstyle S_{TW}} & & \Big\downarrow{\scriptstyle S_{\mathrm{tri}}} \\ X_p(\overline{\rho}_m) & \xrightarrow{\mathrm{res}_p} & X_{\mathrm{tri}}^{\square}(\overline{\rho}_{m,p})\times\mathfrak{X}^{\square}_{\overline{\rho}_m^p} \end{array}$$

Proof. Consider an irreducible component I_n of $X_p(\overline{\rho}_n)$. By [BHS17, Théorème 3.18] the component I_n contains a classical $\overline{\mathbb{Q}}_p$-point x. The representation $\rho_{x,v}$ is crystalline for every $v\in\Sigma_p$. Let y_{tri} denote the $\overline{\mathbb{Q}}_p$-point $S_{\mathrm{tri}}\circ\prod\mathrm{res}_v(x)$ of $X_{\mathrm{tri}}^{\square}(\overline{\rho}_{m,p})$. The representation $\rho_{y_{\mathrm{tri}},v}$ is isomorphic to $S\circ\rho_{x,v}$ at every $v\in\Sigma_p$ by definition of S_{tri}. In particular it is crystalline by Theorem 4.1(3). By the "only if" part of Conjecture 7.4 for $U(m)$, the component I_m is in the image of res_p. $\square$

Conditionally on a part of Conjecture 7.4, we give a description of the space $X_p(\overline{\rho}_m)^{S,\mathrm{str}}$. This result is an analogue of [Cont16b, Theorem 10.10].

In the following g_m is the integer given by [BHS17, Theorem 3.4] for the representation $\overline{\rho}_m$. We denote by $\operatorname{res}_\Sigma$ the morphism $X_p(\overline{\rho}_m)^{\mathrm{str}} \hookrightarrow \mathfrak{X}^\square_{\overline{\rho}_{m,p}} \times \mathfrak{X}^\square_{\overline{\rho}_{m,p}}$ given by the embedding (7.1) composed with the projection onto the first two factors, and we use the same notation when m is replace by n.

Theorem 7.18. *Suppose that the "if" part of Conjecture 7.4 is true for $U(n)$. Let x be a $\overline{\mathbb{Q}}_p$-point of $X_p(\overline{\rho}_m)^{S,\mathrm{str}}$ such that:*

(1) the embedding

$$X_p(\overline{\rho}_m)^{\mathrm{str}} \hookrightarrow \iota(X^\square_{\mathrm{tri}}(\overline{\rho}_{m,p})) \times \mathfrak{X}^\square_{\overline{\rho}_{m,p}} \times \mathbb{U}^{g_m}$$

is a local isomorphism at x;

(2) the trianguline parameters $(\mu_{x,v})_{v\in\Sigma_p} \in \mathscr{T}^n_{v,L}(\overline{\mathbb{Q}}_p)$ of $(\rho_{x,v})_{v\in\Sigma_p}$, lift to cocharacters $(\mu_{x,v,n})_{v\in\Sigma_p} \in \mathscr{T}^m_{v,L}(\overline{\mathbb{Q}}_p)$ via the isogeny $S\colon \mathscr{T}^m_{v,L} \to \mathscr{T}^n_{v,L}$.

Then:

(i) there exists a $\overline{\mathbb{Q}}_p$-point y of $X_p(\overline{\rho}_n)^{\mathrm{str}}$ such that $\rho_x = S \circ \rho_y$ and $\mu_y = \mu_{x,n}$:

(ii) every $\overline{\mathbb{Q}}_p$-point y_Σ of $\mathfrak{X}^\square_{\overline{\rho}_{n,p}} \times \mathfrak{X}^\square_{\overline{\rho}_{m,p}}$ such that $S_p \times S^p(y_{\mathrm{tri}}) = \operatorname{res}_\Sigma(x)$ belongs to $\operatorname{res}_\Sigma(X_p(\overline{\rho}_m)^{\mathrm{str}})$.

Note that condition (1) is trivially satisfied for all smooth $\overline{\mathbb{Q}}_p$-points of $X_p(\overline{\rho}_m)^{S,\mathrm{str}}$.

Proof. Consider the following diagram:

$$\begin{array}{ccc} X_p(\overline{\rho}_n)^{\mathrm{str}} & \xrightarrow{\prod_{v\in\Sigma}\operatorname{res}_v} & \iota(X^\square_{\mathrm{tri}}(\overline{\rho}_{n,p})) \times \mathfrak{X}^\square_{\overline{\rho}^p_n} \\ & & \big\downarrow {\scriptstyle S_{\mathrm{tri},p}\times S^p} \\ X_p(\overline{\rho}_m)^{\mathrm{str}} & \xrightarrow{\prod_{v\in\Sigma}\operatorname{res}_v} & \iota(X^\square_{\mathrm{tri}}(\overline{\rho}_{m,p})) \times \mathfrak{X}^\square_{\overline{\rho}^p_m} \end{array}$$

Since $x \in X_p(\overline{\rho}_m)^{S,\mathrm{str}}(\overline{\mathbb{Q}}_p)$. By Corollary 5.4 and the assumption on x, there exists for every $v \in \Sigma$ a representation $\rho_{y,v}\colon G_{F_{\tilde{v}}} \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ such that:

(1) $\rho_{x,v} = S \circ \rho_{y,v}$;

(2) if $v \in \Sigma_p$, then $\rho_{y,v}$ is strictly trianguline of parameter $\mu_{x,n}$.

Let y_{tri} be the point of $\iota(X^\square_{\mathrm{tri}}(\overline{\rho}_{n,p})) \times \mathfrak{X}^\square_{\overline{\rho}^p_n}$ whose corresponding Galois representations (with parameters at $v \in \Sigma_p$) are $(\rho_{y,v}, \mu_{x,v,n}))$ for $v \in \Sigma_p$ and $(\rho_{y,v})$ for $v \in \Sigma - \Sigma_p$. It is clear from the properties (1-3) that $S_{\mathrm{tri}}(y_{\mathrm{tri}}) = x_{\mathrm{tri}}$. We will show that y_{tri} is in the image of the map $\prod_{v\in\Sigma}\operatorname{res}_v\colon X_p(\overline{\rho}_n)^{\mathrm{str}} \to \iota(X^\square_{\mathrm{tri}}(\overline{\rho}_{n,p})) \times \mathfrak{X}^\square_{\overline{\rho}_{n,p}}$. Every inverse image y of y_{tri} via $\prod_{v\in\Sigma}\operatorname{res}_v$ then satisfies statement (i) in the theorem because of the properties (1,2) above. We will also obtain statement (ii) of the theorem in the following way. If y_Σ is a $\overline{\mathbb{Q}}_p$-point of $\mathfrak{X}^\square_{\overline{\rho}_{n,p}} \times \mathfrak{X}^\square_{\overline{\rho}_{m,p}}$ satisfying condition (1) above, it belongs to $\iota(X^\square_{\mathrm{tri}}(\overline{\rho}_{m,p})) \times \mathfrak{X}^\square_{\overline{\rho}_{m,p}}$ by Corollary 5.4, and its trianguline parameters $\mu_{y_{\mathrm{tri}},v}$ are lifts of the parameters $\mu_{x,v}$. In particular y_Σ is the image of a $\overline{\mathbb{Q}}_p$-point y_{tri} of $\iota(X^\square_{\mathrm{tri}}(\overline{\rho}_{n,p})) \times \mathfrak{X}^\square_{\overline{\rho}^p_n}$ that satisfies (1) and (2). By showing that such a point is in the image of $\prod_{v\in\Sigma}\operatorname{res}_v\colon X_p(\overline{\rho}_n)^{\mathrm{str}} \to \iota(X^\square_{\mathrm{tri}}(\overline{\rho}_{n,p})) \times \mathfrak{X}^\square_{\overline{\rho}_{n,p}}$, we obtain (ii).

Consider an irreducible component $\mathfrak{I}_n$ of $\iota(X_{\mathrm{tri}}^{\square}(\overline{\rho}_{n,p})) \times \mathfrak{X}_{\overline{\rho}_n^p}^{\square}$ containing y_{tri}. Let $\mathfrak{I}_m$ be an irreducible component of $\iota(X_{\mathrm{tri}}^{\square}(\overline{\rho}_{m,p})) \times \mathfrak{X}_{\overline{\rho}_m^p}^{\square}$ containing $S_{\mathrm{tri},p} \times S^p(\mathfrak{I}_m)$. By assumption (1), $\mathfrak{I}_m$ is contained in the image of $X_p(\overline{\rho}_m)$.

The component $\mathfrak{I}_n$ is a product $\mathfrak{I}_{n,p} \times \mathfrak{I}_n^p$ of an irreducible component $\mathfrak{I}_{n,p}$ of $\iota(X_{\mathrm{tri}}^{\square}(\overline{\rho}_{n,p}))$ and an irreducible component $\mathfrak{I}_n^p$ of $\mathfrak{X}_{\overline{\rho}_n^p}^{\square}$. We show that the component $\mathfrak{I}_{n,p}$ is automorphic. This will imply that the point y_{tri} of $\mathfrak{I}_{n,p} \times \mathfrak{I}_n^p$ is in the image of $X_p(\overline{\rho}_n)$ via $\prod_{v\in\Sigma} \mathrm{res}_v$, giving the conclusion of the theorem.

By the "if" part of Conjecture 7.4 for $U(n)$, it is sufficient to show that $\mathfrak{I}_{n,p} \cap U_{\mathrm{tri}}^{\square}(\overline{\rho}_p)$ contains a crystalline point. By Remark 7.1(3) the image of $\mathfrak{I}_{n,p}$ in the weight space $\mathcal{W}_L^n$ contains a weight $\mathbf{k} \in (\mathbb{Z}^n)^{\mathrm{Hom}(F_v^+,L)}$. Let $y_{\mathrm{tri},\mathbf{k}}$ be a point of $\mathfrak{I}_{n,p}$ of weight $\mathbf{k}$. Consider an affinoid subdomain U_n of $\mathfrak{I}_{n,p}$ containing both y_{tri} and $y_{\mathrm{tri},\mathbf{k}}$. The absolute values of the slopes sl_i are bounded on U_n by a common constant C. We claim that there exists a weight $\mathbf{k}'$ and a point $y_{\mathrm{tri},\mathbf{k}'}$ of U_n such that, for every embedding $\tau\colon F^+ \to \overline{\mathbb{Q}}_p$:

(1) $k_{1,\tau} - k_{2,\tau} > 2q(C+1)$;

(2) $k_{i,\tau} - k_{i+1,\tau} > k_{i-1,\tau} - k_{i,\tau} + q(C+1)$ for $i \geq 2$.

This is clear because $\omega_X(U_n)$ will contain an affinoid neighborhood of $\mathbf{k}$, and we can choose a weight $\mathbf{k}'$ arbitrarily p-adically close to $\mathbf{k}$ satisfying (1) and (2). Let $x'_{\mathrm{tri}} = S_{\mathrm{tri},p} \times S^p(y_{\mathrm{tri},\mathbf{k}'})$, which is a point of weight $S \circ \mathbf{k}'$ on the component $\mathfrak{I}_m$. Note that $\rho_{x'_{\mathrm{tri}},v} = S \circ \rho_{y_{\mathrm{tri},\mathbf{k}'},v}$ for every $v \in \Sigma$, by definition of $S_{\mathrm{tri},p}$. It follows from assumptions (1) and (2) and Lemma 7.13(2.ii-iii) that

(1S) $(S \circ \mathbf{k}')_{1,\tau} - (S \circ \mathbf{k}')_{2,\tau} > 2(C+1)$;

(2S) $(S \circ \mathbf{k}')_{i,\tau} - (S \circ \mathbf{k}')_{i+1,\tau} > (S \circ \mathbf{k}')_{i-1,\tau} - (S \circ \mathbf{k}')_{i,\tau} + C + 1$ for $i \geq 2$.

In particular x'_{tri} is a classical point by the argument in the proof [BHS17, Théorème 3.18] (it is shown there that the set W appearing in the statement can be taken to be the set of weights satisfying conditions (1S) and (2S)). In particular the representation $\rho_{x'_{\mathrm{tri}},v}$ is crystalline for every $v \in \Sigma_p$. By the fact that $\rho_{x'_{\mathrm{tri}},v} = S \circ \rho_{y_{\mathrm{tri},\mathbf{k}'},v}$ and a result of Di Matteo, the representation $\rho_{y_{\mathrm{tri},\mathbf{k}'},v}$ is also crystalline. We deduce that the component $\mathfrak{I}_{n,p}$ of $X_{\mathrm{tri}}^{\square}(\overline{\rho}_{n,p})$ contains the crystalline point $y_{\mathrm{tri},\mathbf{k}'}$, as desired. □

Let $X_p(\overline{\rho}_m)^{\mathrm{sm}}$ be the locus of smooth points of $X_p(\overline{\rho}_m)$.

Corollary 7.19. *Suppose that the "if" part Conjecture 7.4 is true for $U(n)$ and the "only if" part of the conjecture is true for $U(m)$. Let $\mathscr{F} = (\mathrm{Sp}(A), \rho_A)$ be a family of representations of $G_{F_{\tilde{v}}}$ appearing in $X_p(\overline{\rho}^m)^{S,\mathrm{sm}}$. Then $\mathscr{F}$ also appears in $S_{TW}(X_p(\overline{\rho}_n))$.*

Proof. Consider the following commmutative diagram:

$$
\begin{array}{ccc}
X_p(\overline{\rho}_n)^{\mathrm{str}} & \xrightarrow{\mathrm{res}_\Sigma} & \mathfrak{X}_{\overline{\rho}_{n,p}}^{\square} \times \mathfrak{X}_{\overline{\rho}_{n,p}}^{\square} \\
\Big\downarrow{\scriptstyle S_{TW}} & & \Big\downarrow{\scriptstyle S_p \times S^p} \\
X_p(\overline{\rho}_m)^{\mathrm{str}} & \xrightarrow{\mathrm{res}_\Sigma} & \mathfrak{X}_{\overline{\rho}_{m,p}}^{\square} \times \mathfrak{X}_{\overline{\rho}_{m,p}}^{\square}
\end{array}
\tag{7.5}
$$

Suppose that $\mathscr{F}$ appears in $X_p(\overline{\rho}^m)^{S,\mathrm{sm}}$ via $\alpha\colon \mathrm{Sp}(A) \to X_p(\overline{\rho}^m)^{S,\mathrm{sm}}$. Consider the image $\mathrm{res}_\Sigma(\alpha(\mathrm{Sp}(A)))$ in $\mathfrak{X}_{\overline{\rho}_{m,p}}^{\square} \times \mathfrak{X}_{\overline{\rho}_{m,p}}^{\square}$ and its inverse image $(S_p \times S^p)^{-1}(\mathrm{res}_\Sigma(\alpha(\mathrm{Sp}(A))))$ in $\mathfrak{X}_{\overline{\rho}_{n,p}}^{\square} \times \mathfrak{X}_{\overline{\rho}_{n,p}}^{\square}$. By Theorem 7.18(i), the morphism $S_p \times S^p$ induces a surjection $(S_p \times S^p)^{-1}(\mathrm{res}_\Sigma(\alpha(\mathrm{Sp}(A)))) \to \mathrm{res}_\Sigma(\alpha(\mathrm{Sp}(A)))$. This morphism is also continuous and closed, so there exists an affinoid subdomain $\mathrm{Sp}(B)$ of $\mathfrak{X}^{\square}_{\overline{\rho}_{n,p}} \times \mathfrak{X}^{\square}_{\overline{\rho}_{n,p}}$ such that $(S_p \times S^p)(\mathrm{Sp}(B)) = \mathrm{res}_{\Sigma}(\alpha(\mathrm{Sp}(A)))$. By Theorem 7.18(ii), the affinoid $\mathrm{Sp}(B)$ is contained in the image of $X_p(\overline{\rho}_n)$ via res_{Σ}. Since res_{Σ} is a closed embedding, we deduce that there exists an affinoid subdomain $\mathrm{Sp}(C)$ of $X_p(\overline{\rho}_n)$ such that $\mathrm{res}_{\Sigma}(\mathrm{Sp}(C)) = \mathrm{Sp}(B)$. By the commutativity of Diagram (7.5), $\mathrm{res}_{\Sigma}(S_{TW}(\mathrm{Sp}(C))) = \mathrm{res}_{\Sigma}(\alpha(\mathrm{Sp}(A)))$, which implies that the family of representations of $G_{F_{\tilde{v}}}$ carried by $\mathrm{Sp}(C)$ is the same as the family carried by $\alpha(\mathrm{Sp}(A))$, that is that the family $\mathscr{F}$ appears in $S_{TW}(X_p(\overline{\rho}_n))$ with support $S_{TW}(\mathrm{Sp}(C))$. ☐

Corollary 7.20. *Suppose that the "if" part Conjecture 7.4 is true for $U(n)$ and the "only if" part of the conjecture is true for $U(m)$. Let $S_p\colon X_p(\overline{\rho}_n) \to X_p(\overline{\rho}_m)$ be the morphism of rigid analytic spaces given by Proposition 7.17. Then the rigid analytic space $S_p(X_p(\overline{\rho}_n)) \cap X_p(\overline{\rho}_m)^{\mathrm{sm}}$ consists of the irreducible components of maximal dimension of $X_p(\overline{\rho}_m)^S \cap X_p(\overline{\rho}_m)^{\mathrm{sm}}$.*

Proof. Let d be the dimension of $\mathcal{W}_L^S$. Since $\omega(X_p(\overline{\rho}_m)^S)$ is Zariski-open in $\mathcal{W}_{v,L}^S$ and ω is of relative dimension 0, the irreducible components of $X_p(\overline{\rho}_m)^S$ have dimension at most d. This is also the dimension of every irreducible component $X_p(\overline{\rho}_n)$, hence of every irreducible component of $S_p(X_p(\overline{\rho}_n))$.

Let A be an affinoid $\mathbb{Q}_p$-algebra and let $\mathscr{F} = (\mathrm{Sp}(A), \rho_A)$ be a family of Galois representations appearing in $X_p(\overline{\rho}^m)^S$ via $\alpha\colon \mathrm{Sp}(A) \to X_p(\overline{\rho}^m)^S$. Assume that $\alpha(\mathrm{Sp}(A))$ is not contained in $S_p(X_p(\overline{\rho}_n))$. By Corollary 7.19 the family $\mathscr{F}$ also appears in $S_p(X_p(\overline{\rho}_n))$, so it appears at least twice on $X_p(\overline{\rho}^m)^S$. By Remark 7.10 the family $\mathscr{F}$ must have some constant weight in $\mathcal{W}_L^m$. If $\mathscr{F}$ were of dimension d then $\omega \circ \alpha(\mathrm{Sp}(A))$ would be Zariski-open in $\mathcal{W}_L^S$, hence could not have any constant weight in $\mathcal{W}_L^m$. We conclude that the dimension of $\mathscr{F}$ is strictly smaller than d. ☐

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