Surface Singularities on Cyclic Coverings and an Inequality for the Signature

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Introduction.

Let (B, P_0) be a germ of a plane curve singularity $g(x,y) = 0$ at the origin of $\mathbf{C}^2$. By a germ (V,P) of an n-fold cyclic cover $(n \geq 2)$ branched along (B, P_0), we mean a germ of 2-dimensional isolated hypersurface singularity defined by the equation $f(x,y,z) := z^n + g(x,y)$ at the origin of $\mathbf{C}^3$, which we simply call *a singularity of cyclic type*. The aim of this paper is to describe analytic and topological invariants of (V,P) via a certain algebro-geometric method. Especially, for the signature of the Milnor fiber $\sigma(V,P)$, we prove a certain inequality, which naturally induce an affirmative answer to the negativity conjecture for the signature posed by Durfee [D, p.96] in the case of a singularity of cyclic type.

In §1, we investigate the invariants of a complete surface S which is a cyclic covering over a nonsingular surface, where we admit non-normal singularities on S. Especially we define the "Milnor number" of such a non-normal surface in some sense, and prove a Noether type formula in Proposition 1.5.

In §2, we first realize a singularity of cyclic type on a complete surface, and resolve it by the method of Hirzebruch [Hi1]. Next we express the Milnor number and the geometric genus of (V,P) in terms of the non-reduced divisor which naturally appears at the final step of the resolution process by using Proposition 1.5 and the Esnault-Viehweg formula [E, V].

We note that, by the reason that these invariants are essentially described in A'Campo [Ac2] and in Esnault[E], our contribution in §§$1,2$ is only to propose another formalism. Our motivation of this formulation is to generalize to the arbitorary covering degree of Horikawa's method [Ho, §2] for double covering in some sence.

In §3, we define the improved singularity $(V^\vee, P^\vee)$ of (V,P) which has the following property: The number of times needed of blow-ups of the branch curve of $(V^\vee, P^\vee)$ such that the reduced scheme of its total transform has normal crossing is one less than that of (V,P) in general. We also calculate the difference of the invariants between (V,P) and $(V^\vee, P^\vee)$

The method of the construction of the improved singularity is a combinatorial interpretation of the "local perturbation method" on the way of the embedded resolution process of a plane curve singularity appeared in A'Campo [Ac1], and has some natural connection to A'Campo and Gusein-Zade theory [Ac1, G]. We will publish the topological meaning of this process as well as another application in our forthcoming paper.

In §4, by using Hirzebruch-Durfee-Laufer's formula [Hi2, D, L] and our previous result, we prove a certain formula for $\sigma(V,P)$ in Proposition 4.7.

§§5 and 6 are devoted to estimate the signature. Our method is, roughly speaking, to show that the signature of the improved singularity is not less than that of the original singularity. We note that, after producing our process inductively, we finally reach the singularity (W,P) whose branch curve is an ordinary singularity. Therefore we can compare $\sigma(V,P)$ with $\sigma(W,P)$. Our main result is the following:

THEOREM. *Let (B,P_0) be a germ of a plane curve singularity. Let (C,P_0) be a germ of an ordinary plane curve singularity whose multiplicity coincides with the multiplicity of (B,P_0). Let (V,P) and (W,P) be germs of n-fold cyclic cover branched along (B,P_0) and (C,P_0), respectively. Let $\sigma(V,P)$ (resp. $\sigma(W,P)$) be the signature of the Milnor fiber of (V,P) (resp. (W,P)). Then we have*

$$\sigma(V,P) \leq \sigma(W,P).$$

Furtherover, the equality $\sigma(V,P) = \sigma(W,P)$ holds if and only if (B,P_0) itself is an ordinary singularity.

We remark that $\sigma(W,P)$ is explicitely calculated and its negativity is well-known. Therefore the above theorem induce an answer to the problem of Durfee [D,p.96] in the case of a singularity of cyclic type as follows:

COROLLARY. *Let (V,P) be a germ of analytic function $f(x,y,z) = z^n + g(x,y)$ such that (V,P) defines at most an isolated singularity. Then we have*

$$\sigma(V,P) \leq 0.$$

Furthermore, the equality $\sigma(V,P) = 0$ holds if and only if (V,P) is a germ of a nonsingular point.

As a topological approach to the signature of a singularity of cyclic type, Neumann and Wahl [NW] show that, under the assumption that the link $\mathcal{L}$ of the singularity is a **Z**-homology sphere, σ coincides with the $(1/8)$-time the Casson invariant of $\mathcal{L}$. As they proved in [NW, Proposition 2.1], the assumption for $\mathcal{L}$ to be a homology sphere is somewhat strong, and so it seems an interesting problem to avoid it by extending the notion of the Casson invariant. We note that, in the double point case $z^2 + g(x,y)$, it is classically known that σ coincides with the signature of the symmetrized Seifert form of the classical link of the branch curve $g(x,y)$, which is the compound torus link (cf.[Sh]). In this case, since the signature of the Seifert form changes sign when the link changes into its mirror image, the negativity of σ means that the mirror image of a classical algebraic link cannot be algebraic. Therefore one can imagine that our result corresponds to some feature of the "one-sided chirality" of the non-classical link arising from such a surface singularity.

From another point of view, the Durfee problem is directly connected with the upper bound problem of the geometric genus of the singularity as is mentioned in [D, p.97]. For instance, we refer to [XY1, XY2, To1, To2, As] for some types of singularities, whereas Wahl [W] found a non-complete intersection singularity whose signature of the Milnor fiber is positive.

On the other hand, general and powerful approachs for the signature of singularity have been done via mixed Hodge theory and Seifert geometry ([Ne, St1, St2, SSS] etc.).

For the signature of weighted homogeneous singularities, many work have been done ([Br, HZ, HM, St2, E, XY2, FMS, NW] etc.).

For recent global study of cyclic coverings of surfaces, we refer Sakai[Sa]. For singularities on Galois coverings, we refer Tsuchihashi [Ts].

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§ 1. Cyclic coverings.

1.1 Let W be a complete nonsingular analytic surface and $L \longrightarrow W$ a line bundle on W. For an integer n greater than 1, let B be a divisor on W which is linearly equivalent to nL. We construct an *n-fold cyclic cover S of W branched along B* in the total space $V(L)$ of L in a usual manner as follows:

Suppose B is defined by the equation $b_i = 0$ on a chart U_i of W. Let ξ_i be the fiber coordinate of $V(L) \longrightarrow W$ on a local trivialization $U_i \times \mathbf{C}$. Then the equations $\xi_i^n + b_i = 0$ on $U_i \times \mathbf{C}$ for all i patched and define the surface S on $V(L)$.

We remark that if B is reduced, then S is normal.

Let $\overline{\pi} : \overline{V(L)} = \mathbf{P}(\mathcal{O}_W(L) \bigoplus \mathcal{O}_W) \longrightarrow W$ be the associated $\mathbf{P}^1$-bundle. It is convienient to consider the above S as a hypersurface on $\overline{V(L)}$. The dualizing sheaf ω_S is isomorphic to $\mathcal{O}_{\overline{V(L)}}(K_{\overline{V(L)}} + S) \bigotimes \mathcal{O}_S$ and the self-intersection number ω_S^2 is defined as $(K_{\overline{V(L)}} + S)^2 S$ on $\overline{V(L)}$.

LEMMA 1.2. *Let S be an n-fold cyclic cover of W on $V(L)$. Then we have*

(i) $\chi(\mathcal{O}_S) = \frac{1}{4}n(n-1)K_W L + \frac{1}{12}n(n-1)(2n-1)L^2 + n\chi(\mathcal{O}_W)$,

(ii) $\omega_S^2 = n\{(n-1)L + K_W\}^2$.

Proof We first prove the assertion (ii). We set $X = \overline{V(L)}$ and let T be the tautological line bundle on X. Since K_X is linearly equivalent to $-2T + \overline{\pi}^*(K_W + L)$, we have

$$\begin{aligned} \omega_S^2 &= \{(n-2)T + \overline{\pi}^*(K_W + L)\}^2 \cdot nT \quad (\text{on } X) \\ &= (n-2)^2 nL^2 + 2n(n-2)K_W L + n(K_W + L)^2 \quad (\text{on } W) \\ &= n\{(n-1)L + K_W\}^2 \end{aligned}$$

by the formula $T^3 = T(\overline{\pi}^* L)^2$.

Next we prove the assertion (i). Since we have

$$\begin{aligned} R^1\overline{\pi}_* \mathcal{O}_X(K_X+S) &= 0, \\ \overline{\pi}_* \mathcal{O}_X(K_X+S) &\simeq \mathrm{Symm}^{n-2}(\mathcal{O} \bigoplus \mathcal{O}(L)) \bigotimes \mathcal{O}(K_W+L) \simeq \bigoplus_{j=1}^{n-1} \mathcal{O}(K_W+jL), \end{aligned}$$

it follows from Leray's spectral sequence and the Riemann-Roch formula that

$$\begin{aligned} \chi(X, \mathcal{O}(K_X+S)) &= \textstyle\sum_{j=1}^{n-1} \chi(W, \mathcal{O}(K_W+jL)) \\ &= \tfrac{1}{2} \textstyle\sum_{j=1}^{n-1} (jK_W L + j^2 L^2) + (n-1)\chi(\mathcal{O}_W) \\ &= \tfrac{1}{4} n(n-1) K_W L + \tfrac{1}{12} n(n-1)(2n-1) L^2 + (n-1)\chi(\mathcal{O}_W). \end{aligned} \quad (1.2.1)$$

On the other hand, it follows from the exact sequence $0 \longrightarrow \mathcal{O}(K_X) \longrightarrow \mathcal{O}(K_X+S) \longrightarrow \mathcal{O}(\omega_S) \longrightarrow 0$ that

$$\begin{aligned} \chi(\mathcal{O}_S) &= \chi(\omega_S) = \chi(\mathcal{O}(K_X+S)) - \chi(\mathcal{O}(K_X)) \\ &= \chi(\mathcal{O}(K_X+S)) + \chi(\mathcal{O}_W). \end{aligned} \quad (1.2.2)$$

Therefore the assertion (i) follows. Q.E.D.

DEFINITION 1.3. (i) *For a divisor B on W, we define the number $\mu(B)$ by*

$$\mu(B) = -2\chi(\mathcal{O}_B) + e(B),$$

where $e(B)$ is the topological Euler number of the reduced scheme B_{red} of B.
(ii) *For an n-fold cyclic cover S of W branched along B, we define the number $\mu(S)$ by*

$$\mu(S) = (n-1)\mu(B)$$

LEMMA 1.4. (S.L.Tan) *If B is a reduced divisor on W, then the number $\mu(B)$ coincides with the total Milnor number of B, i.e. the sum of the Milnor numbers of all the isolated singularities on B.*

Moreover in this case, for an n-fold cyclic cover S of W branched along B (which is automatically normal), the number $\mu(S)$ coincides with the total Milnor number of S.

PROOF The first assertion is proved in [Ta, Lemma 1.1] by Milnor's formula [M, p.85]. The second assertion is clear from the first assertion because the equation of an isolated singularity $\overline{P}$ on S is written as $f(x,y,z) = z^n + g(x,y)$ and we have

$$\begin{aligned} \mu(\overline{P}) &= \dim \mathbf{C}\{x,y,z\}/(f_x, f_y, f_z) \\ &= (n-1)\dim \mathbf{C}\{x,y\}/(f_x, f_y) = (n-1)\mu(P), \end{aligned}$$

where P is the singularity defined by $g(x,y) = 0$. Q.E.D.

PROPOSITION 1.5. (The Noether type formula)
Let S be an n-fold cyclic cover of W with branch locus B on $V(L)$. Then we have

$$\chi(\mathcal{O}_S) = \frac{1}{12}(\omega_S^2 + e(S) + \mu(S)).$$

PROOF Since S is topologically obtained by patching n copies of W along B for $n-1$ times, it follows from the Mayer-Vietoris exact sequence that

$$e(S) = ne(W) - (n-1)e(B). \tag{1.5.1}$$

On the other hand, by the Riemann-Roch formula and the duality theorem, it follows from the exact sequence $0 \longrightarrow \mathcal{O}(K_W) \longrightarrow \mathcal{O}(K_W + B) \longrightarrow \mathcal{O}(\omega_B) \longrightarrow 0$ that

$$\chi(\mathcal{O}_B) = \chi(\mathcal{O}(\omega_B)) = -\chi(\mathcal{O}(K_W + B)) + \chi(\mathcal{O}(K_W)) = -\frac{1}{2}(K_W + B)B.$$

Therefore it follows from Lemma 1.2 and the usual Noether formula for W that

$$\begin{aligned} \chi(\mathcal{O}_S) - \tfrac{1}{12}(\omega_S^2 + e(S)) &= \tfrac{1}{12}n^2(n-1)L^2 + \tfrac{1}{12}n(n-1)K_W L + \tfrac{1}{12}(n-1)e(W) \\ &= \tfrac{1}{12}(n-1)(K_W B + B^2 + e(B)) \\ &= \tfrac{1}{12}(n-1)\{-2\chi(\mathcal{O}_B) + e(B)\} = \tfrac{1}{12}\mu(S). \end{aligned}$$

Q.E.D.

REMARK 1.6. In the situation of Proposition 1.5, we add the assumption that S is normal. Then the assertion directly follows from Laufer's formula [L] (see Brenton [B]).

§ 2. The invariant formula via the Hirzebruch-Jung resolution.

Let (V, P) be a germ of 2-dimensional isolated hypersurface singularity defined at the origin of $\mathbf{C}^2$ by the equation

$$f(x, y, z) = z^n + g(x, y),$$

where $g(x, y)$ is an analytic function with respect to the variables x, y, and n is an integer greater than 1. Such a singularity (V, P) is said to be *of cyclic type* in this paper. The aim of this section is to describe the geometric genus $p_g(V.P)$ and the Milnor number $\mu(V, P)$ in terms of the data on a branch curve which naturally appears in the process of the Hirzebruch-Jung resolution of (V, P). We note that this resolution process is a special case of the original one introduced by Hirzebruch [Hi1] for the solution of the resolution problem of any 2-dimensional analytic space.

LEMMA 2.1. *Let (V, P) be as above. Then there exist a nonsingular surface W_0, a line bundle L_0 on W_0 and an n-fold cyclic cover $\pi_0 : S_0 \longrightarrow W_0$ on $V(L_0)$ such that*
(i) there exists a point $\overline{P_0} \in S_0$ such that the germ of the singularity at $\overline{P_0}$ of S_0 coincides with (V, P),
(ii) the reduced scheme of the branch locus B_0 of π_0 has normal crossing except in a neighborhood of $P_0 = \pi_0(\overline{P_0})$.

PROOF By Artin's theorem [Ar], we may assume that $g(x, y)$ is algebraic. Let D be the closure in $\mathbf{P}^2$ of the affine curve $g(x, y) = 0$. The curve D is reduced since (V, P) is an isolated singularity. Let $Q_0 \in \mathbf{P}^2$ be the point corresponding to the origin of $\mathbf{C}^2$ in the open immersion $\mathbf{C}^2 \hookrightarrow \mathbf{P}^2$.

We choose general hyperplanes $H_1, \cdots, H_t$ of $\mathbf{P}^2$ which do not pass through Q_0 in such a way that the degree of the reduced divisor

$$D^\sharp := D + H_1 + \cdots + H_t$$

is a multiple of n. We set deg $D^\sharp = n\alpha$ and $L^\sharp = \mathcal{O}_{\mathbf{P}^2}(\alpha)$ for $\alpha \in \mathbf{Z}$. Let $\pi^\sharp : S^\sharp \longrightarrow \mathbf{P}^2$ be the n-fold cyclic cover branched along $D^\sharp$ on $V(L^\sharp)$. Let $\{Q_0, Q_1, \cdots, Q_s\}$ $(s \geq 0)$ be the set of all isolated singularities on $D^\sharp$. The fiber $(\pi^\sharp)^{-1}(Q_i)$ consists of one point for a fixed i $(0 \leq i \leq s)$, which we denote by $\overline{Q_i}$. Then $S^\sharp$ is a normal surface whose singularities are on $\overline{Q_0}, \cdots, \overline{Q_s}$. Moreover, the germ of the singularity of $S^\sharp$ at $\overline{Q_0}$ coincides with (V, P).

Now let $\tau^\sharp : W_0 \longrightarrow \mathbf{P}^2$ be the succession of blow-ups whose centers are infinitely near points of $Q_1, \cdots, Q_s$ such that the reduced scheme of the divisor

$$B_0 := (\tau^\sharp)^* D^\sharp$$

has normal crossing except in a neighborhood of $P_0 := (\tau^\sharp)^{-1}(Q_0)$. Set $L_0 = (\tau^\sharp)^* \mathcal{O}_{\mathbf{P}^2}(\alpha)$. Let $\pi_0 : S_0 \longrightarrow W_0$ be the n-fold cyclic cover branched along B_0 on $V(L_0)$, and we put $\overline{P_0} = \pi_0^{-1}(P_0)$. Then the desired properties are satified. Q.E.D.

2.2. Let $V(L_0) \supset S_0 \longrightarrow W_0$ be the n-fold cyclic cover branched along B_0 as in Lemma 2.1. Let $W_0 \xleftarrow{\tau_1} W_1 \xleftarrow{\tau_2} \cdots \xleftarrow{\tau_r} W_r$ be the succession of blow-ups at infinitely near points of P_0 such that the reduced scheme $(B_r)_{\text{red}}$ of the total transform B_r of B_0 by $\tau_1 \circ \cdots \circ \tau_r$ has normal crossing. We take the smallest such number r which enjoys the above property and fix it from now on.

We note that, if r coincides with 0, then B_0 defines an ordinary double point at P_0. In this case, (V, P) is a rational double point of type A_{n-1}, whose properties concerning our problem are well-known. From now on, we assume $r \geq 1$ in this section.

For $1 \leq i \leq r$, let S_i be the fiber product of S_{i-1} and W_i over W_{i-1}, and let $\widehat{\tau_i}$: $S_i \longrightarrow S_{i-1}$ and $\pi_i : S_i \longrightarrow W_i$ be the natural maps. We set $L_i = \tau_i^* L_{i-1}$. Then S_i is realized on $V(L_i)$ and the map $\widehat{\tau_i}$ is also induced by the restriction of the natural map $V(L_i) \longrightarrow V(L_{i-1})$. Let $\rho' : \widetilde{S} \longrightarrow S_r$ be the normalization. We set $\rho = \pi_r \circ \rho'$. Then the surface $\widetilde{S}$ has at most isolated cyclic quotient singulatities. Let $\rho'' : S^* \longrightarrow S$ be the resolution of such singularities. We note that this process ρ'' is performed by using the so-called Hirzebruch-Jung string ([Hi1], [BPV, Chap.III]). Therefore we obtain the following diagram:

$$\begin{array}{ccccccccccccc}
V(L^\sharp) & & V(L_0) & & V(L_1) & & & & V(L_r) & & & & \\
\cup & & \cup & & \cup & & & & \cup & & & & \\
S^\sharp & \xleftarrow{\widehat{\tau^\sharp}} & S_0 & \xleftarrow{\widehat{\tau_1}} & S_1 & \longleftarrow & \cdots & \xleftarrow{\widehat{\tau_r}} & S_r & \xleftarrow{\rho'} & \widetilde{S} & \xleftarrow{\rho''} & S^* \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow & & & & \\
\mathbf{P}^2 & \xleftarrow{\tau^\sharp} & W_0 & \xleftarrow{\tau_1} & W_1 & \longleftarrow & \cdots & \xleftarrow{\tau_r} & W_r & & & &
\end{array}$$

The nonsingular surface S^* itself is not uniquely determined by the germ (V, P) since it depends on the compactification $S^\sharp$ in Lemma 2.1. Although locally, i.e. over an open neighborhood which is the inverse image of the affine neighborhood of the original singularity, the structure of S^* is uniquely determined. Moreover, the above process is also uniquely determined up to order in the selection of the centers of blow-ups $\tau_1, \cdots, \tau_r$. We call this process *the Hirzebruch-Jung resolution of* (V, P).

For later use, we introduce some definitions and notation. For $0 \leq j < i \leq r$, we put $\tau_{j,i} = \tau_{j+1} \circ \cdots \circ \tau_i : W_i \longrightarrow W_j$. For $1 \leq i \leq r$, let B_i (resp. $\widetilde{B_i}$) be the total transform (resp. the proper transform) of B_0 by $\tau_{0,i}$. Then B_i coincides with the branch locus of the n-fold cyclic cover $\pi_i : S_i \longrightarrow W_i$. Let $P_{i-1} \in W_{i-1}$ be the center of the blow-up τ_i, and let $m_i = \mathrm{mult}_{P_{i-1}} B_{i-1}$ be the multiplicity of B_{i-1} at P_{i-1}. Let $E_{i,i} = \tau_i^{-1}(P_{i-1})$ be the exceptional curve of τ_i, and let $E_{i,j}$ be the proper transform of $E_{i,i}$ by $\tau_{i,j}$ for $i < j \leq r$. Let $\mathbf{E}_i$ be the (scheme-theoretic) exceptional divisor for $\tau_{0,i}$. Then we have the irreducible decomposition

$$\mathbf{E}_i = \sum_{j=1}^{i} m_j E_{j,i}.$$

We put $\mathcal{E}_i = (\mathbf{E}_i)_{\mathrm{red}}$. Moreover we decompose B_0 into $B_0' + B_0''$, where B_0' is the divisor consisting of the components of B_0 which pass through P_0 and $B_0'' = B_0 - B_0'$. Note that B_0' is a reduced divisor since P_0 is an isolated singularity. We formally write the irreducible decompositions as $B_0' = \sum_{j=-r'}^{0} E_{j,0}$ and $B_0'' = \sum_{j=-r''}^{-r'-1} m_j E_{j,0}$. For $1 \leq i \leq r$ and $-r'' \leq j \leq 0$, let $E_{j,i}$ be the proper transform of $E_{j,0}$ by $\tau_{0,i}$. We also formally put $m_j = 1$ for $-r' \leq j \leq 0$. Then we have the irreducible decomposition

$$B_i = \sum_{j=-r''}^{i} m_j E_{j,i}, \qquad \widetilde{B_i} = \sum_{j=-r''}^{0} m_j E_{j,i}.$$

Now we look at the branch curve $B_r = \sum_{j=-r''}^{r} m_j E_{j,r}$ on W_r. The following lemma essentially comes from the Esnault-Viehweg formula:

LEMMA 2.3. *In the above situation, we have*

(i) $\chi(\mathcal{O}_{S_r}) = \chi(\mathcal{O}_{S_0}) = \chi(\mathcal{O}_{S^\sharp})$, $\quad \omega_{S_r}^2 = \omega_{S_0}^2 - nr$,

(ii) $\chi(\mathcal{O}_{\widetilde{S}}) - \chi(\mathcal{O}_{S_r}) = \dfrac{1}{2} \displaystyle\sum_{i=1}^{n-1} \sum_{j=-r''}^{r} \left[\frac{m_j i}{n}\right] \left\{ 2 + E_{j,r}^2 + \sum_{k=-r''}^{r} \left[\frac{m_k i}{n}\right] E_{k,r} E_{j,r} \right\}$,

(iii) $\chi(\mathcal{O}_{S^*}) = \chi(\mathcal{O}_{\widetilde{S}})$,

where $[m_j i/n]$ is the greatest integer not exceeding the number $m_j i/n$.

PROOF Since $K_{W_i} = \tau_i^* K_{W_{i-1}} + E_{i,i}$ and $L_i = \tau_i^* L_{i-1}$ for $1 \leq i \leq r$, we easily have $\chi(\mathcal{O}_{S_i}) = \chi(\mathcal{O}_{S_{i-1}})$ and $\omega_{S_i}^2 = \omega_{S_{i-1}}^2 - n$ by Lemma 1.2. Therefore we have $\chi(\mathcal{O}_{S_r}) = \chi(\mathcal{O}_{S_0})$ and $\omega_{S_r}^2 = \omega_{S_0}^2 - nr$. We remark that the birational morphism $S_0 \longrightarrow S^\sharp$ is similarly constructed as for $S_r \longrightarrow S_0$ with respect to the isolated singularities $\{Q_i\}_{1 \leq i \leq s}$ which appear in the proof of Lemma 2.1. Therefore we have $\chi(\mathcal{O}_{S_0}) = \chi(\mathcal{O}_{S^\sharp})$ similarly.

The assertion (iii) is clear because ρ'' is the resolution of isolated rational singularities.

It remains to prove (ii). For $0 \leq i \leq n-1$, let $\mathcal{L}^{(i)}$ be the line bundle on W_r defined by

$$\mathcal{L}^{(i)} = L_r^{\otimes i} \bigotimes \mathcal{O}_{W_r}\Big(- \sum_{j=-r''}^{r} \left[\frac{m_j i}{n}\right] E_{j,r}\Big).$$

Since $(B_r)_{\rm red}$ has normal crossing, it follows from [E, Lemma 2] or [V, §1] that

$$\rho_*\mathcal{O}_{\widetilde{S}} \simeq \bigoplus_{i=0}^{n-1}(\mathcal{L}^{(i)})^\vee,$$

where $(\mathcal{L}^{(i)})^\vee$ is the dual bundle of $\mathcal{L}^{(i)}$.

Since $R^q\rho_*\mathcal{O}_{\widetilde{S}}$ vanishes for $q \geq 1$, it follows from the Riemann-Roch formula that

$$\begin{aligned}
\chi(\mathcal{O}_{\widetilde{S}}) &= \chi(\rho_*\mathcal{O}_{\widetilde{S}}) = \sum_{i=0}^{n-1}\chi(W_r, \mathcal{O}(-iL_r + \sum_j \left[\frac{m_j i}{n}\right] E_{j,r})) \\
&= \sum_{i=0}^{n-1}\left\{\frac{1}{2}(-iL_r + \sum_j \left[\frac{m_j i}{n}\right] E_{j,r})(-iL_r + \sum_j \left[\frac{m_j i}{n}\right] E_{j,r} - K_{W_r}) + \chi(\mathcal{O}_{W_r})\right\} \\
&= \frac{1}{2}\sum_{i=0}^{n-1} iL_r(iL_r + K_{W_r}) + n\chi(\mathcal{O}_{W_r}) \\
&\quad + \frac{1}{2}\sum_{i=1}^{n-1}\sum_{j=-r''}^{r}\left[\frac{m_j i}{n}\right]\left\{-\frac{2i}{n}B_r E_{j,r} + 2 + E_{j,r}^2 + \sum_{k=-r''}^{r}\left[\frac{m_k i}{n}\right] E_{k,r}E_{j,r}\right\},
\end{aligned}$$

On the other hand, by (1.2.1) and (1.2.2), we have

$$\chi(\mathcal{O}_{S_r}) = \frac{1}{2}\sum_{i=0}^{n-1} iL_r(iL_r + K_{W_r}) + n\chi(\mathcal{O}_{W_r}).$$

Moreover if $E_{j,r}$ is an exceptional curve for $\tau^\sharp \circ \tau_1 \circ \cdots \circ \tau_r$, then we have $B_r E_{j,i} = 0$. If $E_{j,r}$ is not an exceptional curve for $\tau^\sharp \circ \tau_1 \circ \cdots \circ \tau_r$, then we have $[m_j i/n] = 0$ $(1 \leq i \leq n-1)$ by $m_j = 1$. Therefore we obtain the assertion (ii). Q.E.D.

COROLLARY 2.4.

$$p_g(V,P) = -\frac{1}{2}\sum_{i=1}^{n-1}\sum_{j=1}^{r}\left[\frac{m_j i}{n}\right]\left\{2 + E_{j,r}^2 + \sum_{k=1}^{r}\left[\frac{m_k i}{n}\right] E_{k,r}E_{j,r}\right\}.$$

PROOF We put $\tau = \tau^\sharp \circ \tau_{0,r} : W_r \longrightarrow \mathbf{P}^2$. We first decompose the set of integers $J = \{-r'', -r''+1, \cdots, r\}$ into the disjoint union $J_E \amalg J_B$, where

$$J_E = \{j \in J \mid m_j \geq 2\}, \quad J_B = \{j \in J \mid m_j = 1\}.$$

Since $D^\sharp$ is a reduced divisor on $\mathbf{P}^2$, $\sum_{j\in J_B} E_{j,r}$ coincides with the proper transform $\widetilde{B_r}$ of $D^\sharp$ by τ, and $\sum_{j \in J_E} m_j E_{j,r}$ coincides with the exceptional divisor of $\cup_{\ell=0}^s Q_\ell$ for τ. We next decompose J_E into the disjoint union $\amalg_{0\leq \ell \leq s} J_\ell$, where

$$J_\ell = \{j \in J_E \mid \tau(E_{j,r}) = Q_\ell\}.$$

It follows from Lemma 2.3 that

$$\begin{aligned}
\textstyle\sum_{\ell=0}^s p_g(\overline{Q_\ell}) &= \chi(\mathcal{O}_{S^\sharp}) - \chi(\mathcal{O}_{S^*}) \\
&= -\frac{1}{2}\sum_{i=1}^{n-1}\sum_{j=-r''}^{r}\left[\frac{m_j i}{n}\right]\left\{2 + E_{j,r}^2 + \sum_{k=-r''}^{r}\left[\frac{m_k i}{n}\right] E_{k,r}E_{j,r}\right\} \\
&= -\frac{1}{2}\sum_{\ell=0}^{s}\sum_{i=1}^{n-1}\sum_{j\in J_\ell}\left[\frac{m_j i}{n}\right]\left\{2 + E_{j,r}^2 + \sum_{k\in J_\ell}\left[\frac{m_k i}{n}\right] E_{k,r}E_{j,r}\right\}.
\end{aligned}$$

We remark that, for any isolated singularity, its geometric genus is a local invariant and is independent of the compactification of its ambient space. Hence $p_g(\overline{Q_\ell})$ is determined by the resolution data of $\overline{Q_\ell}$ themselves, i.e. we have

$$p_g(\overline{Q_\ell}) = -\frac{1}{2}\sum_{i=1}^{n-1}\sum_{j\in J_\ell}\left[\frac{m_j i}{n}\right]\left\{2+E_{j,r}^2+\sum_{k\in J_\ell}\left[\frac{m_k i}{n}\right]E_{k,r}E_{j,r}\right\}.$$

Since we have $p_g(\overline{P}) = p_g(\overline{Q_0})$ and $J_0 = \{1,\cdots,r\}$, the assertion is proved. Q.E.D.

Lemma 2.5.

$$\begin{aligned}\mu(V,P) &= (n-1)\left\{-r+\sum_{i=1}^{r}(m_i-1)(m_iE_{i,r}^2-2)\right.\\ &\quad\left.+\sum_{-r'\le i\le r,1\le j\le r,i<j,E_{i,r}\cap E_{j,r}\ne\emptyset}(2m_im_j-1)\right\}.\end{aligned}$$

Proof For brevity, we write $E_i := E_{i,r}$ for $-r''\le i\le r$.

Since $e(W_r) = e(W_0)+r$ and $e(B_r) = e(B_0)+r$, we have $e(S_r) = e(S_0)+r$ by (1.5.1). Therefore it follows from Proposition 1.5 and Lemma 2.3 that

$$\mu(S_r)-\mu(S_0) = 12\{\chi(\mathcal{O}_{S_r})-\chi(\mathcal{O}_{S_0})\}-(\omega_{S_r}^2-\omega_{S_0}^2)-\{e(S_r)-e(S_0)\} = (n-1)r. \tag{2.4.1}$$

Now for an integer i with $0\le i\le r$, we define a divisor $\mathbf{B}^{(i)}$ on W_r by

$$\mathbf{B}^{(i)} = \sum_{j=-r''}^{-r'-1}m_jE_j+\sum_{j=-r'}^{i}E_j+\sum_{j=i+1}^{r}m_jE_j.$$

Then we have $B_r = \mathbf{B}^{(0)}$ and $\widetilde{B_r}+\mathcal{E}_r = \mathbf{B}^{(r)}$. Now we claim

$$\mu(B_r)-\mu(\widetilde{B_r}+\mathcal{E}_r) = 2\sum_{i=1}^{r}(m_i-1)(E_i\mathbf{B}^{(i)}-1)+\sum_{i=1}^{r}(m_i-1)(m_i-2)E_i^2. \tag{2.4.2}$$

Indeed, if $m_i=1$, then we clearly have $\chi(\mathcal{O}_{\mathbf{B}^{(i-1)}}) = \chi(\mathcal{O}_{\mathbf{B}^{(i)}})$. Let us assume $m_i\ge 2$. Then for any integer k $(1\le k\le m_i-1)$, we consider the exact sequence

$$0\longrightarrow\mathcal{O}_{E_i}(-\mathbf{B}^{(i)}-(k-1)E_i)\longrightarrow\mathcal{O}_{\mathbf{B}^{(i)}+kE_i}\longrightarrow\mathcal{O}_{\mathbf{B}^{(i)}+(k-1)E_i}\longrightarrow 0.$$

We note that E_i is isomorphic to $\mathbf{P}^1$ for $1\le i\le r$. Therefore, by using the above sequence inductively and by the Riemann-Roch formula, we have

$$\begin{aligned}\chi(\mathcal{O}_{\mathbf{B}^{(i-1)}}) &= \chi(\mathcal{O}_{\mathbf{B}^{(i)}+(m_i-1)E_i}) = \chi(\mathcal{O}_{\mathbf{B}^{(i)}})+\sum_{k=0}^{m_i-2}\chi(\mathcal{O}_{E_i}(-\mathbf{B}^{(i)}-kE_i))\\ &= \chi(\mathcal{O}_{\mathbf{B}^{(i)}})+\sum_{k=0}^{m_i-2}\{E_i(-\mathbf{B}^{(i)}-kE_i)+1\}\\ &= \chi(\mathcal{O}_{\mathbf{B}^{(i)}})-(m_i-1)(E_i\mathbf{B}^{(i)}-1)-\frac{1}{2}(m_i-1)(m_i-2)E_i^2.\end{aligned}$$

From this, we obtain (2.4.2).

Next we compare $\mu(B_0)$ with $\mu(\widetilde{B_r}+\mathcal{E}_r)$. Let U_0 be a suitable affine neighborhood of P_0 in W_0 and set $U_r=\tau_{0,r}^{-1}(U_0)$. Then the restricted curves $B_0|_{W_0\setminus U_0}$ and $(\widetilde{B_r}+\mathcal{E}_r)|_{W_r\setminus U_r}$ are mutually isomorphic. On the other hand, $\mathcal{E}_r$ and $\widetilde{B_r}$ meet transversally at $\lambda:=\mathcal{E}_r\widetilde{B_r}$ points. Moreover $\mathcal{E}_r$ has just $r-1$ ordinally double points as its singularities. Therefore the set of singularities on $(\widetilde{B_r}+\mathcal{E}_r)|_{U_r}$ consists of $r+\lambda-1$ ordinally double points, while the set of singulariries on $B_0|_{U_0}$ consists of only P_0 itself. Hence it follows from Lemma 1.4 that

$$\mu(\widetilde{B_r}+\mathcal{E}_r)-\mu(B_0)=r+\lambda-1-\mu(P_0). \tag{2.4.3}$$

Therefore from (2.4.2) and (2.4.3), we have

$$\begin{aligned}
\mu(B_r)-\mu(B_0)&=\sum_{i=1}^{r}\{2(m_i-1)(E_i^2-1)+(m_i-1)(m_i-2)E_i^2\}\\
&\quad+2\sum_{i=1}^{r}(m_i-1)(\sum_{j=-r'}^{i-1}E_j+\sum_{j=i+1}^{r}m_jE_j)E_i+r+\lambda-1-\mu(P_0)\\
&=\sum_{i=1}^{r}(m_i-1)(m_iE_i^2-2)+\sum_{-r'\le i\le 0,1\le j\le r,E_i\cap E_j\ne\emptyset}\{(2m_j-1)+1\}\\
&\quad+\sum_{1\le i<j\le r,E_i\cap E_j\ne\emptyset}\{2(m_i-1)m_j+2(m_j-1)+1\}-\mu(P_0)\\
&=\sum_{i=1}^{r}(m_i-1)(m_iE_i^2-2)+\sum_{-r'\le i\le r,1\le j\le r,i<j,E_i\cap E_j\ne\emptyset}(2m_im_j-1)-\mu(P_0).
\end{aligned}$$

From this and (2.4.1), we obtain the assertion. Q.E.D.

Example 2.6. Let (V,P) be the singularity defined by $z^{45}+x(x^2+y^3)=0$. The total transform of the embedded resolution of the singularity $x(x^2+y^3)=0$ is written by $\sum_{i=-1}^{3}m_iE_i$ where $E_1^2=-3$, $E_2^2=-2$, $E_3^2=-1$, $E_3E_2=E_3E_1=E_2E_0=E_3E_{-1}=1$, $m_{-1}=m_0=1$, $m_1=3$, $m_2=5$ and $m_3=9$. Therefore we have $p_g(V,P)=35$ and $\mu(V,P)=308$ by Corollary 2.4 and Lemma 2.5.

§ 3. Improved singularities.

Let (V,P) be a germ of 2-dimensional isolated singularity of cyclic type. The aim of this section is to define a new singularity $(V^\vee,P^\vee)$ whose difficulty compared with (V,P) is improved in "a unit" in the following sense: If we complete the process of blow-ups of the base surfaces just r times in the Hirzebruch-Jung resolution of (V,P), then the corresponding process for $(V^\vee,P^\vee)$ is completed in at most $r-1$ times. Next we calculate the difference of invariants between (V,P) and $(V^\vee,P^\vee)$.

3.1. We go back to the situation of 2.2. We assume $r\ge 2$ in this section. For $2\le i\le r$, we consider the blow-up $\tau_i: W_i\longrightarrow W_{i-1}$ with center P_{i-1}. We set $\beta_{i-1}=\mathrm{mult}_{P_{i-1}}\widetilde{B_{i-1}}$. According to the position of P_{i-1} in $\mathcal{E}_{i-1}$, we classify τ_i into the following two types:

(A) Assume that P_{i-1} is a nonsingular point of $\mathcal{E}_{i-1}$. There exists an integer $\varphi_1(i)$ with $1 \leq \varphi_1(i) \leq i-1$ which is uniquely determined by i such that P_{i-1} is contained in $E_{\varphi_1(i),i-1}$. We have

$$m_i = m_{\varphi_1(i)} + \beta_{i-1}.$$

In this case, τ_i is said to be *of type A*.

(B) Assume that P_{i-1} is a double point of $\mathcal{E}_{i-1}$. There exist integers $\varphi_1(i)$ and $\varphi_2(i)$ with $1 \leq \varphi_1(i) < \varphi_2(i) \leq i-1$ determined by i such that P_{i-1} coincides with $E_{\varphi_1(i),i-1} \cap E_{\varphi_2(i),i-1}$. We have

$$m_i = m_{\varphi_1(i)} + m_{\varphi_2(i)} + \beta_{i-1}.$$

In this case, τ_i is said to be *of type B*.

3.2. We consider the blow up $\tau_r : W_r \longrightarrow W_{r-1}$. We define a new curve D_{r-1} on a neighborhood of $\mathcal{E}_{r-1}$ in W_{r-1} in the following way:

(A) Assume τ_r is of type A. Since τ_r is the last blow-up in order to obtain the normal crossing property for $(B_r)_{\mathrm{red}}$ near $\mathcal{E}_r$, the local branch of $\widetilde{B_{r-1}}$ at P_{r-1} consists of β_{r-1} nonsingular components $C_1, \cdots, C_{\beta_{r-1}}$ such that the tangent lines at P_{r-1} of $C_1, \cdots, C_{\beta_{r-1}}$ and $E_{\varphi_1(r)}$ are mutually distinct. Now we take any nonsingular analytic curves $C'_1, \cdots, C'_{\beta_{r-1}}$ locally defined in a suitable neighborhood U of $\mathcal{E}_{r-1}$ in W_{r-1} such that

(i) $C'_1, \cdots, C'_{\beta_{r-1}}$ do not intersect one another,

(ii) C'_i intersects $\mathcal{E}_{r-1}$ transversally at one point in $E_{\varphi_1(r),r-1} \setminus (E_{\varphi_1(r),r-1} \cap \mathrm{Sing}(\mathcal{E}_{r-1}))$ for $1 \leq i \leq \beta_{r-1}$.

Then we define a new reduced curve D_{r-1} on U by

$$D_{r-1} = \widetilde{B_{r-1}}|_U - \sum_{i=1}^{\beta_{r-1}} C_i + \sum_{i=1}^{\beta_{r-1}} C'_i,$$

where $\widetilde{B_{r-1}}|_U$ is the restriction of $\widetilde{B_{r-1}}$ to U. See Figure 1.

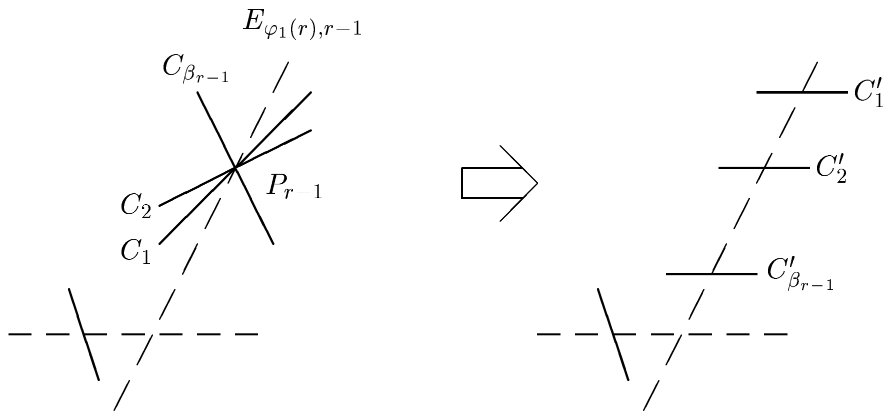


Figure 1:

(B) Assume τ_r is of type B. The local branch of $\widetilde{B_{r-1}}$ at P_{r-1} consists of β_{r-1} nonsingular components $C_1, \cdots, C_{\beta_{r-1}}$ such that the tangent lines at P_{r-1} of $C_1, \cdots, C_{\beta_{r-1}}$, $E_{\varphi 1(r)}$ and $E_{\varphi 2(r)}$ at P_{r-1} are mutually distinct. We take nonsingular analytic curves $C'_1, \cdots, C'_{\beta_{r-1}}$, $C''_1, \cdots, C''_{\beta_{r-1}}$ defined on U such that
(i) $C'_1, \cdots, C'_{\beta_{r-1}}$, $C''_1, \cdots, C''_{\beta_{r-1}}$ do not intersect one another,
(ii) C'_i intersects $\mathcal{E}_{r-1}$ transversally at one point in $E_{\varphi 1(r),r-1} \setminus (E_{\varphi 1(r),r-1} \cap \mathrm{Sing}(\mathcal{E}_{r-1}))$, and C''_i intersects $\mathcal{E}_{r-1}$ transversally at one point in $E_{\varphi 2(r),r-1} \setminus (E_{\varphi 2(r),r-1} \cap \mathrm{Sing}(\mathcal{E}_{r-1}))$ for $1 \leq i \leq \beta_{r-1}$.
Then we define a new reduced curve D_{r-1} on U by

$$D_{r-1} = \widetilde{B_{r-1}}|_U - \sum_{i=1}^{\beta_{r-1}} C_i + \sum_{i=1}^{\beta_{r-1}} C'_i + \sum_{i=1}^{\beta_{r-1}} C''_i.$$

as in Figure 2.

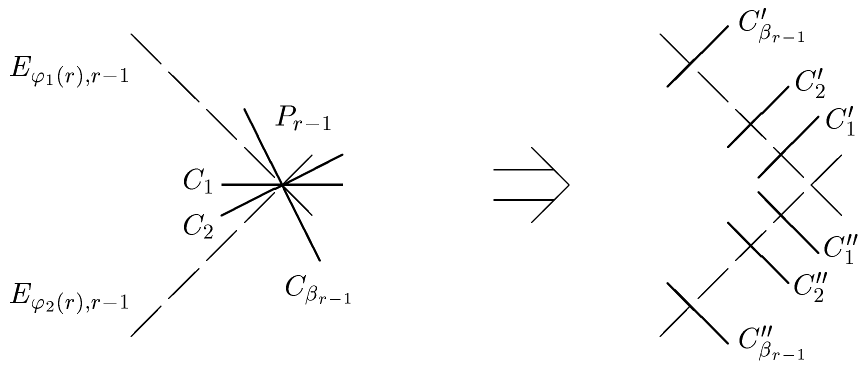


Figure 2:

For $0 \leq i \leq r-2$, we put

$$D_i = (\tau_{i,r-1})_* D_{r-1}.$$

The curve D_i is defined in a neighborhood of $\mathcal{E}_i$ in W_i for $1 \leq i \leq r-1$, and is reduced by our construction. The curve D_0 is defined in a neighborhood of $P_0 \in W_0$ and is also reduced.

REMARK 3.3. The above construction of D_{r-1} is a combinatorial interpretation of the following: If we could perturb the germ at P_{r-1} of $\widetilde{B_{r-1}}$ to the transversal direction against $\mathcal{E}_{r-1}$, we would obtain D_{r-1}.

This idea is already seen in A'Campo [Ac1]. Furthermore this is directly related to the method of T.S.Röbbecke [R]. (This was pointed out to the author by Professor Brieskorn.) For more general treatment of the argument of this type, we refer Oka [O].

DEFINITION 3.4. *As in the above situation, we put $g^\vee(x,y) = 0$ to be the defining equation of D_0 at P_0. We also define a germ of 2-dimensional isolated singularity $(V^\vee, P^\vee)$ of cyclic type to be the germ at the origin of $\mathbf{C}^3$ defined by the equation*

$$f^\vee(x,y,z) := z^n + g^\vee(x,y) = 0.$$

We call $(V^\vee, P^\vee)$ an improved singularity *of* (V,P).

EXAMPLE 3.5. If $g(x,y)=x^2+y^5$, then we have $g^\vee(x,y)=x^2+y^4$ as in Figure 3.

EXAMPLE 3.6. Since the definition of $g^\vee(x,y)$ depends on the order of blow-ups to resolve $g(x,y)$, if $g(x,y)$ has many local analytic branches, then $g^\vee(x,y)$ is not uniquely determined. For example, if $g(x,y)=(x^2+y^3)\{(x+y)^3+y^6\}$, then we easily have $g^\vee(x,y)=(x^2+y^2)\{(x+y)^3+y^6\}$ or $g^\vee(x,y)=(x^2+y^3)\{(x+y)^3+y^3\}$.

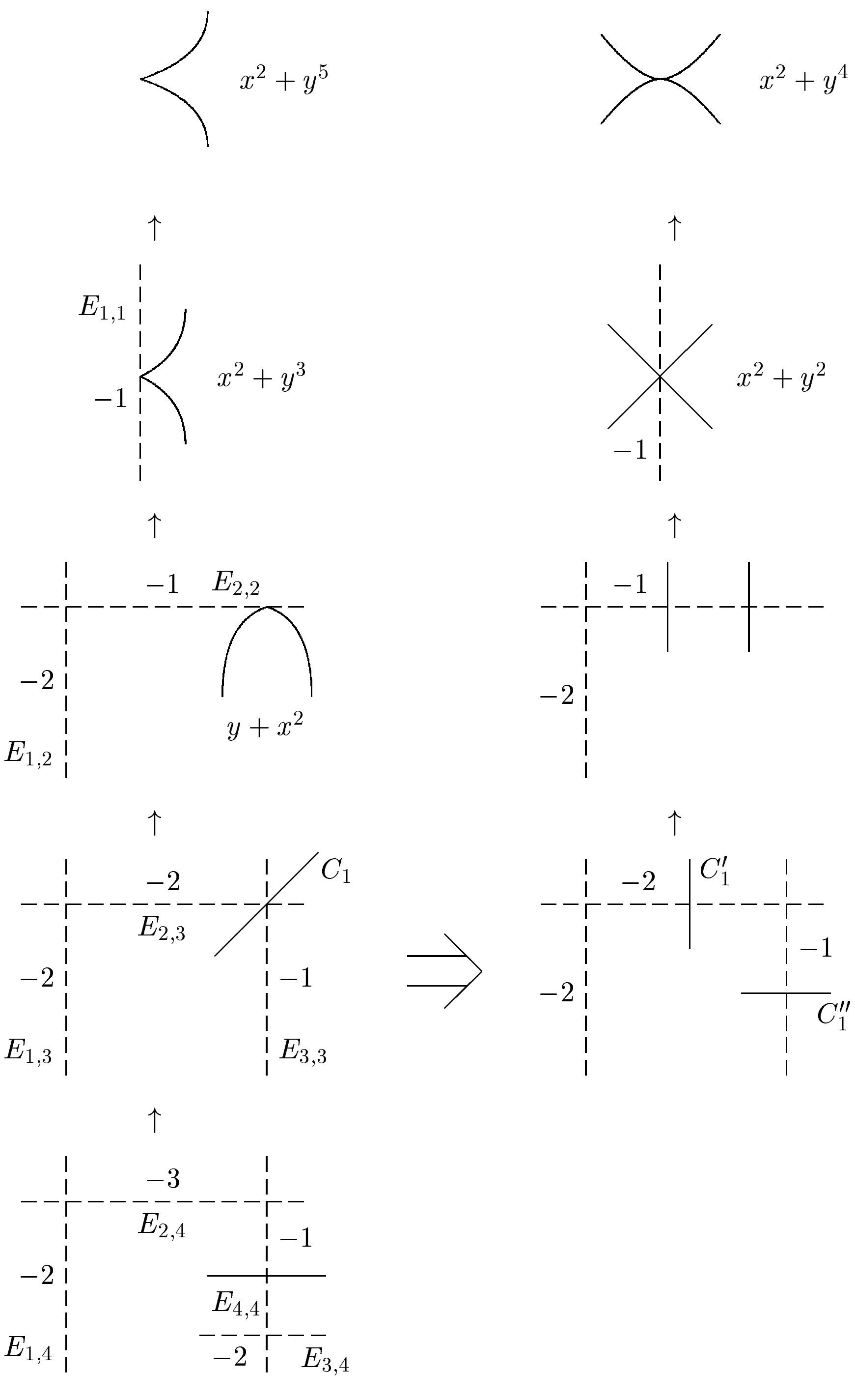
3.7. The Hirzebruch-Jung resolution of $(V^\vee,P^\vee)$ is produced in the following way: We may assume that $g^\vee(x,y)$ is algebraic as a germ of an isolated singularity by Artin [Ar]. Then by the same argument as in Lemma 2.1, there is an n-fold cyclic cover $\pi'_0 : S'_0 \longrightarrow W'_0$ on $V(L'_0)$ for some line bundle L'_0 with the branch locus $B_0^\vee$ such that
(i) there exist a point $P'_0 \in W'_0$ and an open neighborhood U' of P'_0 in W'_0 such that the restricted curve $B_0^\vee|_{U'}$ coincides with $D_0|_U$ for some open neighborhood U of P_0 in W_0. Especially the germ of the singularity of S'_0 at $\overline{P'_0}=\pi_0'^{-1}(P'_0)$ coincides with $(V^\vee,P^\vee)$.
(ii) $(B_0^\vee)_{\mathrm{red}}$ has normal crossing on the locus $W'_0\setminus U'$.
Since D_0 is the image of the local curve D_{r-1} induced by the succession of blow-downs of the ambient spaces $\tau_{0,r-1}: W_{r-1}\longrightarrow W_0$, the embedded resolution process for the singularity P'_0 is produced by the same process locally isomorphic to these $(r-1)$-fold blow-ups. In other words, there exists a succession of blow-ups $W'_0 \xleftarrow{\tau'_1} W'_1 \xleftarrow{\tau'_2} \cdots \xleftarrow{\tau'_{r-1}} W'_{r-1}$ which satisfies the following properties: For $1\le i<j\le r-1$, we set $\tau'_{i,j}=\tau'_{i+1}\circ\cdots\circ\tau'_j$. Then:
(i) For any i $(1\le i\le r-1)$, $(\tau'_{0,i})^{-1}(U')$ is isomorphic to $\tau_{0,i}^{-1}(U)$.
(ii) Let $\widetilde{B_i^\vee}$ be the proper transform of $B_0^\vee$ by $\tau'_{0,i}$. Then $\widetilde{B_i^\vee}\mid_{(\tau'_{0,i})^{-1}(U')}$ is isomorphic to $D_i\mid_{\tau_{0,i}^{-1}(U)}$.

Let $B_i^\vee$ be the total transform of $B_0^\vee$ by $\tau'_{0,i}$ $(0\le i\le r-1)$. Then the reduced scheme of $B_{r-1}^\vee$ has normal crossing. Let $\pi'_{r-1}: S'_{r-1}\longrightarrow W'_{r-1}$ be the n-fold cyclic cover branched along $B_{r-1}^\vee$ on $V(\tau'^{*}_{0,r-1}L'_0)$. Let $S'_{r-1}\xleftarrow{\rho'''}\widehat{S}\xleftarrow{\rho''''} S^{**}$ be the composite of the normalization and the resolution of cyclic quotient singularities on $\widehat{S}$. This complete the process of the Hirzebruch-Jung resolution of $(V^\vee,P^\vee)$.

Incidentally in the resolution process for $(V^\vee,P^\vee)$, the minimal number of times of blow-ups so that the reduced scheme of the proper transform of the branch curve has normal crossing need not coincide with $r-1$ in general. For instance in Example 3.5, we have $r=4$ for (V,P) while $r=2$ for $(V^\vee,P^\vee)$. However, it does not matter for our argument, because we do not need the minimality of the resolution.

From now on, since we only consider local properties satisfied in the neighborhoods of the exceptional sets of P_0 and P'_0, we identify W'_i with W_i and τ'_i with τ_i, respectively, for $0\le i\le r-1$, and use the symbols P_i, E_i and $E_{i,j}$ as in the previous case. That is, P_{i-1} and E_i are respectively the center and the exceptional set of the blow-up $\tau'_i=\tau_i$, and $E_{i,j}$ $(1\le i<j\le r-1)$ is the proper transform of E_i by $\tau'_{i,j}=\tau_{i,j}$. We set $\mathbf{E}_{r-1}^\vee = B_{r-1}^\vee - \widetilde{B_{r-1}^\vee}$. The support of $\mathbf{E}_{r-1}^\vee$ coincides with $\mathcal{E}_{r-1}=\sum_{k=1}^{r-1}E_{k,r-1}$. We set $\mathbf{E}^\vee := (\pi'_{r-1}\circ\rho'''\circ\rho'''')^{-1}(\mathcal{E}_{r-1})$, which is the exceptional set of the singularity $(V^\vee,P^\vee)$ by this resolution.

Now the proof of the following lemma is due to Masataka Tomari.



$x^2 + y^5$
$x^2 + y^4$
$E_{1,1}$
-1
$x^2 + y^3$
$x^2 + y^2$
-1
-1
$E_{2,2}$
-1
-2
-2
$y + x^2$
$E_{1,2}$
-2
C_1
$E_{2,3}$
-2
C_1'
-1
-2
-1
-2
C_1''
$E_{1,3}$
$E_{3,3}$
-3
$E_{2,4}$
-1
-2
$E_{4,4}$
$E_{1,4}$
-2
$E_{3,4}$

Figure 3:

LEMMA 3.8. (Tomari) (i) *For $0 \leq i \leq r-2$, we have*

$$\mathrm{mult}_{P_i}\widetilde{B_i^\vee} = \mathrm{mult}_{P_i}\widetilde{B}_i.$$

(ii) *The divisor $\mathbf{E}_{r-1}^\vee$ coincides with the divisor $\mathbf{E}_{r-1}$.*

PROOF We prove (i). Let

$$\tau_{i,r-1}^*\mathbf{m}_{P_i} = \mathcal{O}(\sum_{j=i+1}^{r-1} n_j E_{j,r-1})$$

be the pull-back of the maximal ideal $\mathbf{m}_{P_i}$ at P_i. Since $\widetilde{B_{r-1}}$ (resp. $\widetilde{B_{r-1}^\vee}$) is the proper transform of $\widetilde{B}_i$ (resp. $\widetilde{B_i^\vee}$) by $\tau_{i,r-1}$, it is well-known (and easily proved by induction) that

$$\mathrm{mult}_{P_i}\widetilde{B}_i = \widetilde{B_{r-1}} \cdot \tau_{i,r-1}^*\mathbf{m}_{P_i}, \quad \mathrm{mult}_{P_i}\widetilde{B_i^\vee} = \widetilde{B_{r-1}^\vee} \cdot \tau_{i,r-1}^*\mathbf{m}_{P_i}.$$

Therefore if τ_r is of type A, then we have

$$\begin{aligned}\mathrm{mult}_{P_i}\widetilde{B}_i - \mathrm{mult}_{P_i}\widetilde{B_i^\vee} &= \textstyle\sum_{j=i+1}^{r-1} n_j E_{j,r-1}(\widetilde{B}_i - \widetilde{B_i^\vee}) \\ &= n_{\varphi_1(r)}E_{\varphi_1(r),r-1}(\textstyle\sum_{k=1}^{\beta_{r-1}} C_k - \sum_{k=1}^{\beta_{r-1}} C_k') = 0.\end{aligned}$$

If τ_r is of type B, then we also have

$$\begin{aligned}&\mathrm{mult}_{P_i}\widetilde{B}_i - \mathrm{mult}_{P_i}\widetilde{B_i^\vee} \\ &= (n_{\varphi_1(r)}E_{\varphi_1(r),r-1} + n_{\varphi_2(r)}E_{\varphi_2(r),r-1})(\textstyle\sum_{k=1}^{\beta_{r-1}} C_k - \sum_{k=1}^{\beta_{r-1}} C_k' - \sum_{k=1}^{\beta_{r-1}} C_k'') = 0.\end{aligned}$$

Hence the assertion (i) is proved.

Next we prove (ii). Let $\mathbf{E}_{r-1}^\vee = \sum_{i=1}^{r-1} m_i' E_{i,r-1}$ be the irreducible decomposition. It suffices to prove

$$m_i = m_i' \tag{3.8.1}$$

for $1 \leq i \leq r-1$. We prove this by induction on i. If $i = 1$, it follows from the assertion (i) that

$$m_1' = \mathrm{mult}_{P_0}\widetilde{B_0^\vee} = \mathrm{mult}_{P_0}\widetilde{B}_0 = m_1.$$

Assume (3.8.1) is satisfied for any j with $1 \leq j \leq i-1$. If τ_i is of type A, then it follows from the assertion (i) that

$$m_i' = m_{\varphi_1(i)}' + \mathrm{mult}_{P_{i-1}}\widetilde{B_{i-1}^\vee} = m_{\varphi_1(i)} + \mathrm{mult}_{P_{i-1}}\widetilde{B_{i-1}} = m_i.$$

If τ_i is of type B, we also have

$$m_i' = m_{\varphi_1(i)}' + m_{\varphi_2(i)}' + \mathrm{mult}_{P_{i-1}}\widetilde{B_{i-1}^\vee} = m_{\varphi_1(i)} + m_{\varphi_2(i)} + \mathrm{mult}_{P_{i-1}}\widetilde{B_{i-1}} = m_i.$$

Thus the assertion (ii) is also proved. Q.E.D.

Now we compare the invariants of the improved singularity with those of the original one. For a rational number α, we set $\langle\alpha\rangle = \alpha - [\alpha]$, and call it *the fractional part* of α.

LEMMA 3.9. (a) *Assume τ_r is of type A. Then we have*

$$
\begin{aligned}
p_g(V,P) &- p_g(V^\vee,P^\vee) = \frac{\beta_{r-1}(n-1)}{12n}\{\beta_{r-1}(2n-1)-3n\} \\
&-\frac{\beta_{r-1}}{n}\sum_{i=1}^{n-1} i\left(\left\langle\frac{m_r i}{n}\right\rangle-\left\langle\frac{m_{\varphi_1(r)} i}{n}\right\rangle\right)-\frac{1}{4}\{\gcd(m_r,n)-\gcd(m_{\varphi_1(r)},n)\} \\
&+\frac{1}{2}\sum_{i=1}^{n-1}\left(\left\langle\frac{m_r i}{n}\right\rangle-\left\langle\frac{m_{\varphi_1(r)} i}{n}\right\rangle\right)^2.
\end{aligned}
$$

(b) *Assume τ_r is of type B. Then we have*

$$
\begin{aligned}
p_g(V,P)-p_g(V^\vee,P^\vee) &= \frac{\beta_{r-1}(n-1)}{12n}\{\beta_{r-1}(2n-1)-3n\} \\
&-\frac{\beta_{r-1}}{n}\sum_{i=1}^{n-1} i\left(\left\langle\frac{m_r i}{n}\right\rangle-\left\langle\frac{m_{\varphi_1(r)} i}{n}\right\rangle-\left\langle\frac{m_{\varphi_2(r)} i}{n}\right\rangle\right) \\
&-\frac{1}{4}\left\{n+\gcd(m_r,n)-\gcd(m_{\varphi_1(r)},n))-\gcd(m_{\varphi_2(r)},n)\right\} \\
&+\frac{1}{2}\sum_{i=1}^{n-1}\left(\left\langle\frac{m_r i}{n}\right\rangle-\left\langle\frac{m_{\varphi_1(r)} i}{n}\right\rangle-\left\langle\frac{m_{\varphi_2(r)} i}{n}\right\rangle\right)^2.
\end{aligned}
$$

PROOF For brevity, we write $\beta=\beta_{r-1}$, $\varphi_i=\varphi_i(r)$ $(i=1,2)$, $E_r=E_{r,r}$ $d_r=\gcd(m_r,n)$, $d_{\varphi_1}=\gcd(m_{\varphi_1(r)},n)$ and so on.

We first assume τ_r is of type A. It follows from Corollary 2.4 and Lemma 3.8 that

$$
\begin{aligned}
p_g(V,P)-p_g(V^\vee,P^\vee) &= -\frac{1}{2}\sum_{i=1}^{n-1}\left[\frac{m_{\varphi_1} i}{n}\right]\left\{E_{\varphi_1,r}^2-E_{\varphi_1,r-1}^2+\left[\frac{m_{\varphi_1} i}{n}\right](E_{\varphi_1,r}^2-E_{\varphi_1,r-1}^2)\right. \\
&\left.+\left[\frac{m_r i}{n}\right]E_{\varphi_1,r}E_r\right\}-\frac{1}{2}\sum_{i=1}^{n-1}\left[\frac{m_r i}{n}\right]\left(E_r^2+2+\left[\frac{m_{\varphi_1} i}{n}\right]E_rE_{\varphi_1,r}+\left[\frac{m_r i}{n}\right]E_r^2\right) \\
&=\frac{1}{2}\sum_{i=1}^{n-1}\left(\left[\frac{m_r i}{n}\right]-\left[\frac{m_{\varphi_1} i}{n}\right]\right)\left(\left[\frac{m_r i}{n}\right]-\left[\frac{m_{\varphi_1} i}{n}\right]-1\right) \\
&=\frac{1}{2}\sum_{i=1}^{n-1}\left\{\frac{\beta i}{n}-\left(\left\langle\frac{m_r i}{n}\right\rangle-\left\langle\frac{m_{\varphi_1} i}{n}\right\rangle\right)\right\}\left\{\frac{\beta i}{n}-1-\left(\left\langle\frac{m_r i}{n}\right\rangle-\left\langle\frac{m_{\varphi_1} i}{n}\right\rangle\right)\right\} \\
&=\frac{\beta(n-1)}{12n}\{\beta(2n-1)-3n\}-\frac{\beta}{n}\sum_{i=1}^{n-1} i\left(\left\langle\frac{m_r i}{n}\right\rangle-\left\langle\frac{m_{\varphi_1} i}{n}\right\rangle\right) \\
&+\frac{1}{2}\sum_{i=1}^{n-1}\left(\left\langle\frac{m_r i}{n}\right\rangle-\left\langle\frac{m_{\varphi_1} i}{n}\right\rangle\right)+\frac{1}{2}\sum_{i=1}^{n-1}\left(\left\langle\frac{m_r i}{n}\right\rangle-\left\langle\frac{m_{\varphi_1} i}{n}\right\rangle\right)^2.
\end{aligned}
$$

We set $n=d_r\widetilde{n}$ and $m_r=d_r\widetilde{m_r}$. Since $\widetilde{n}$ and $\widetilde{m_r}$ are mutually prime, we have

$$
\sum_{i=1}^{n-1}\left\langle\frac{m_r i}{n}\right\rangle=\sum_{i=1}^{d_r\widetilde{n}-1}\left\langle\frac{\widetilde{m_r} i}{n}\right\rangle=d_r\sum_{j=1}^{\widetilde{n}-1}\frac{j}{\widetilde{n}}=\frac{n-d_r}{2}.
$$

By the above argument, we have

$$
\sum_{i=1}^{n-1}\left(\left\langle\frac{m_r i}{n}\right\rangle-\left\langle\frac{m_{\varphi_1} i}{n}\right\rangle\right)=-\frac{1}{2}\left(d_r-d_{\varphi_1}\right).
$$

Hence the assertion (a) is proved.

In the case of type B, we have the assertion (b) by the similar calculation.

Q.E.D.

LEMMA 3.10. (a) *Assume τ_r is of type A. Then we have*

$$\mu(V,P)-\mu(V^{\vee},P^{\vee})=(n-1)\beta_{r-1}(\beta_{r-1}-1).$$

(b) *Assume τ_r is of type B. Then we have*

$$\mu(V,P)-\mu(V^{\vee},P^{\vee})=(n-1)\beta_{r-1}^2.$$

PROOF We use the shortened symbol as in Lemma 3.9. We use Lemmas 2.5 and 3.8. Assume τ_r is of type A. Since m_r is equal to $m_{\varphi_1}+\beta$, we have

$$\begin{aligned}\tfrac{1}{n-1}\{\mu(V,P)-\mu(V^{\vee},P^{\vee})\} &= (m_r-1)(m_rE_r^2-2)+(m_{\varphi_1}-1)m_{\varphi_1}(E_{\varphi_1,r}^2-E_{\varphi_1,r-1}^2)\\ &+(2m_r\beta-1)+(2m_rm_{\varphi_1}-1)-(2m_{\varphi_1}\beta-1)-1 \;=\; \beta^2-\beta.\end{aligned}$$

Thus the assertion (a) is proved.

In the case of type B, we have the assertion (b) by the similar calculation. Q.E.D.

Next we calculate the difference of the first Betti number of the exceptional sets. We consider $\mathbf{E}_r$ on W_r. For each i $(1\le i\le r)$, we set

$$\mathbf{I}(i)=\{j|-r_0\le j\le r,\ j\ne i,\ E_j\cap E_i\ne\emptyset\}.$$

Let c_i be the great common divisor of n, m_i and all of m_j $(j\in\mathbf{I}(i))$.

LEMMA 3.11. (a) *Assume τ_r is of type A. Then we have*

$$\begin{aligned}b_1(\widetilde{\mathbf{E}})-b_1(\mathbf{E}^{\vee}) &= (\beta_{r-1}-1)\{\gcd(m_r,n)-\gcd(m_{\varphi_1(r)},n)\}\\ &+c_{\varphi_1(r)}-\gcd(m_r,m_{\varphi_1(r)},n).\end{aligned}$$

(b) *Assume τ_r is of type B. Then we have*

$$\begin{aligned}b_1(\widetilde{\mathbf{E}})-b_1(\mathbf{E}^{\vee}) &= \beta_{r-1}\{\gcd(m_r,n)-\textstyle\sum_{k=1}^2\gcd(m_{\varphi_k(r)},n)+1\}+\sum_{k=1}^2 c_{\varphi_k(r)}\\ &-1-\textstyle\sum_{k=1}^2\gcd(m_r,m_{\varphi_k(r)},n)+\gcd(m_{\varphi_1(r)},m_{\varphi_2(r)},n).\end{aligned}$$

PROOF Adding to the simplified notation as in Lemma 3.9, we further write $d_{\varphi_1,r}=\gcd(m_{\varphi_1},m_r,n)$ and so on.

Assume τ_r is of type A. We first consider $S^*\longrightarrow W_r$. The number of components in $\widetilde{\mathbf{E}}$ which dominate E_{φ_1} is c_{φ_1}. (e.g. Tsuchihashi [Ts,§3].) We write these components by $\widetilde{E_{\varphi_1}^{(1)}},\cdots,\widetilde{E_{\varphi_1}^{(c_{\varphi_1})}}$. For each i $(1\le i\le c_{\varphi_1})$, the natural map $\widetilde{E_{\varphi_1}^{(i)}}\longrightarrow E_{\varphi_1}$ is a cyclic covering of degree $d_{\varphi_1}/c_{\varphi_1}$ whose ramification points are on the inverse image of $E_{\varphi_1} \cap E_j$ $(j \in \mathbf{I}(\varphi_1))$. Since the number of the ponts on the inverse image of $E_{\varphi_1} \cap E_j$ is $d_{\varphi_1,j}/c_{\varphi_1}$ and the ramification index at these points is $d_{\varphi_1}/d_{\varphi_1,j}$, it follows from the Riemann-Hurwiz formula that

$$\sum_{i=1}^{c_{\varphi_1}} b_1(\widetilde{E_{\varphi_1}^{(i)}}) = 2c_{\varphi_1} - d_{\varphi_1} - d_{\varphi_1,r} + \sum_{j \in \mathbf{I}(\varphi_1), j \neq r} (d_{\varphi_1} - d_{\varphi_1,j}).$$

Similarly there is an unique component $\widetilde{E_r}$ of $\mathbf{E}$ which dominate E_r and we have

$$b_1(\widetilde{E_r}) = 2 - d_r - d_{\varphi_1,r} + \beta(d_r - 1).$$

Moreover each $\widetilde{E_{\varphi_1}^{(i)}}$ $(1 \leq i \leq c_{\varphi_1})$ and $\widetilde{E_r}$ is connected by $d_{\varphi_1,r}/c_{\varphi_1}$ Hirzebruch-Jung strings. Therefore they contribute

$$c_{\varphi_1}\left(\frac{d_{\varphi_1,r}}{c_{\varphi_1}} - 1\right) = d_{\varphi_1,r} - c_{\varphi_1}$$

to $b_1(\mathbf{E})$ as loops of the dual graph of $\mathbf{E}$.

Next we consider $S^{**} \longrightarrow W_{r-1}$. There is a unique component $\widetilde{E_{\varphi_1,r-1}}$ of $\mathbf{E}^\vee$ which dominate $E_{\varphi_1,r-1}$ and we have

$$b_1(\widetilde{E_{\varphi_1,r-1}}) = (\beta - 2)(d_{\varphi_1} - 1) + \sum_{j \in \mathbf{I}(\varphi_1), j \neq r} (d_{\varphi_1} - d_{\varphi_1,j}).$$

It is verified that the other components of $\mathbf{E}$ and $\mathbf{E}^\vee$ do not contribute to the difference of the first Betti numbers. Therefore the assertion (a) easily follows.

In the case of type B, we have the assertion (b) by the similar calculation. Q.E.D.

§ 4. The signature of the Milnor fiber.

For a germ of 2-dimensional isolated hypersurface singularity $(\bar{V}, \bar{P})$, the signature of smoothing $\sigma(\bar{V}, \bar{P})$ is defined as follows: Let F and ∂F be the Milnor fiber of $(\bar{V}, \bar{P})$ and its natural real boundary, respectively. Then $\sigma(\bar{V}, \bar{P})$ is defined to be the signature of the quadratic form

$$H^2(F, \partial F; \mathbf{R}) \times H^2(F, \partial F; \mathbf{R}) \xrightarrow{\alpha_1} H^2(F, \partial F; \mathbf{R}) \times H^2(F; \mathbf{R}) \xrightarrow{\alpha_2} H^4(F, \partial F; \mathbf{R}) \simeq \mathbf{R},$$

where the map α_1 is induced by the natural map between the second factors and α_2 is induced by the cup product. By using essentially the Hirzebruch theory of the signature defect [Hi2], Durfee [D] proposed some formula for σ. By combining the formulas [D, Corollary 2.1] and Laufer [L], we have:

THEOREM 4.1. (Hirzebruch, Durfee, Laufer)

$$\sigma(\bar{V}, \bar{P}) = -\mu(\bar{V}, \bar{P}) + 4p_g(\bar{V}, \bar{P}) - b_1(\bar{\mathbf{E}}),$$

where $\bar{\mathbf{E}}$ is the exceptional set of a good resolution of $(\bar{V}, \bar{P})$.

Let (V,P) be a germ of 2-dimensional isolated singularity of cyclic type. The aim of this section is to propose a formula for $\sigma(V,P)$ by using the method in the previous sections.

We consider the Hirzebruch-Jung resolution of (V,P), and use the same notation as in §§ 2 and 3. We first formally set $(V_{[r]},P_{[r]})=(V,P)$. If $r\geq 2$, then for any integer i with $1\leq i\leq r-1$, we inductively define a new germ $(V_{[i]},P_{[i]})$ of 2-dimensional isolated singularity of cyclic type to be the improved singularity $(V_{[i]},P_{[i]}):=(V^{\vee}_{[i+1]},P^{\vee}_{[i+1]})$ of $(V_{[i+1]},P_{[i+1]})$. We note that the blow-up τ_i is of type A (resp. type B) as a step of the Hirzebruch-Jung resolution of (V,P) if and only if the i-th blow up of the Hirzebruch-Jung resolution of $(V_{[i]},P_{[i]})$ is of type A (resp. type B). Therefore by Lemmas 3.9, 3.10 and 3.11 and Theorem 4.1, we easily obtain the following:

Lemma 4.2. *We simply write $d_i=\gcd(m_i,n)$, $d_{i,\varphi_1(i)}=\gcd(m_i,m_{\varphi_1(i)},n)$, $c_{\varphi_1(i)}=\gcd(m_{\varphi_1(i)},m_{j_1},\cdots,m_{j_k},n)$ where $\mathbf{I}(\varphi_1(i))=\{j_1,\cdots,j_k\}$ and so on.*
For $2\leq i\leq r$, we have:
(a) *If τ_i is of type A, then*

$$\begin{aligned}\sigma(V_{[i]},P_{[i]})-\sigma(V_{[i-1]},P_{[i-1]}) &= -\left(\frac{n}{3}-\frac{1}{3n}\right)\beta_{i-1}^2-\beta_{i-1}(d_i-d_{\varphi_1(i)})+d_{i,\varphi_1(i)}\\ &+2\sum_{j=1}^{n-1}\left(\left\langle\frac{m_ij}{n}\right\rangle-\left\langle\frac{m_{\varphi_1}j}{n}\right\rangle\right)^2-\frac{4\beta_{i-1}}{n}\sum_{j=1}^{n-1}j\left(\left\langle\frac{m_ij}{n}\right\rangle-\left\langle\frac{m_{\varphi_1}j}{n}\right\rangle\right)-c_{\varphi_1(i)}.\end{aligned}$$

(b) *If τ_i is of type B, then*

$$\begin{aligned}\sigma(V_{[i]},P_{[i]})-\sigma(V_{[i-1]},P_{[i-1]}) &= -\left(\frac{n}{3}-\frac{1}{3n}\right)\beta_{i-1}^2-(\beta_{i-1}+1)(n+d_i-d_{\varphi_1(i)}\\ &-d_{\varphi_2(i)})+d_{\varphi_1(i),i}+d_{\varphi_2(i),i}-d_{\varphi_1(i),\varphi_2(i)}+2\sum_{j=1}^{n-1}\left(\left\langle\frac{m_ij}{n}\right\rangle-\left\langle\frac{m_{\varphi_1}j}{n}\right\rangle-\left\langle\frac{m_{\varphi_2}j}{n}\right\rangle\right)^2\\ &-\frac{4\beta_{i-1}}{n}\sum_{j=1}^{n-1}j\left(\left\langle\frac{m_ij}{n}\right\rangle-\left\langle\frac{m_{\varphi_1}j}{n}\right\rangle-\left\langle\frac{m_{\varphi_2}j}{n}\right\rangle\right)-c_{\varphi_1(i)}-c_{\varphi_2(i)}+1.\end{aligned}$$

Next we consider $(V_{[1]},P_{[1]})$. We note that $\mathrm{mult}(V_{[1]},P_{[1]})=\mathrm{mult}(V,P)=\beta_0$ by Lemma 3.7.

Lemma 4.3.

$$\sigma(V_{[1]},P_{[1]})=-\frac{1}{3}\left(\beta_0^2-3\beta_0-2\right)n-\beta_0d_{\beta_0}-1+\frac{\beta_0^2+d_{\beta_0}^2}{3n}-\frac{4\beta_0}{n}\sum_{j=1}^{n-1}j\left\langle\frac{\beta_0j}{n}\right\rangle.$$

Proof By Lemma 2.1, we construct an n-fold cyclic cover $\pi_{1,0}:S_{1,0}\longrightarrow W_{1,0}$ such that the germ $(S_{1,0},\pi_{1,0}^{-1}(P_0))$ ($P_0\in W_{1,0}$) coincides with $(V_{[1]},P_{[1]})$. Let

$$\begin{array}{ccccccc} S_{1,0} & \longleftarrow & S_{1,1} & \xleftarrow{\rho_1'} & \widetilde{S_1} & \xleftarrow{\rho_1''} & S_1^* \\ \downarrow & & \downarrow & & & & \\ W_{1,0} & \xleftarrow{\tau_{1,1}} & W_{1,1} & & & & \end{array}$$

be the Hirzebruch-Jung resolution of $(V_{[1]}, P_{[1]})$. Then the branch divisor $B_{1,1}$ of $\pi_{1,1}$: $S_{1,1} \longrightarrow W_{1,1}$ is written in a neighborhood of $E_1 = \tau_{1,1}^{-1}(P_0)$ on $W_{1,1}$ as

$$B_{1,1} = \beta_0 E_1 + C_1 + \cdots + C_{\beta_0} \qquad \text{(locally)}$$

such that C_i $(1 \le i \le \beta_0)$ do not intersect one another and each of C_i $(1 \le i \le \beta_0)$ intersects E_1 transversally at one point. Therefore by Corollary 2.4 and Lemma 2.5, we easily have

$$\begin{aligned} p_g(V_{[1]}, P_{[1]}) &= \frac{\beta_0^2(n-1)(2n-1)}{12n} - \frac{\beta_0(n-1)}{4} - \frac{\beta_0}{n}\sum_{j=1}^{n-1} j\left\langle \frac{\beta_0 j}{n}\right\rangle \\ &\quad + \frac{n-d_{\beta_0}}{4} + \frac{(n-d_{\beta_0})(2n-d_{\beta_0})}{12n}. \\ \mu(V_{[1]}, P_{[1]}) &= (n-1)(\beta_0-1)^2. \end{aligned}$$

We set $\widetilde{E_1} = (\pi_1 \circ \rho_1' \circ \rho_1'')^{-1}(E_1)$, which is the exceptional set of $(V_{[1]}, P_{[1]})$ by this resolution. Then we have

$$b_1(\widetilde{E_1}) = 2 - 2d_{\beta_0} + \beta_0(d_{\beta_0} - 1).$$

Therefore from the above equalities and Theorem 4.1, the assertion follows. Q.E.D.

4.4. We go back to the Hirzebruch-Jung resolution of (V, P) in 2.2. We further introduce some definitions and notation. For any integer i with $1 \le i \le r$, let $T(i) = \{R \in E_{i,i} \cap \widetilde{B_i}\}$ be the set of points at which $E_{i,i}$ and $\widetilde{B_i}$ intersect. We decompose $T(i)$ into a disjoint union $T_1(i) \amalg T_2(i)$ where

$$\begin{aligned} T_1(i) &= \{R \in T(i) \mid R \notin E_{j,i}\ (1 \le \forall j \le i-1) \text{ and } I_R(\widetilde{B_i}, E_{i,i}) = 1\}, \\ T_2(i) &= \{R \in T(i) \mid R \in E_{j,i}\ (1 \le \exists j \le i-1) \text{ or } I_R(\widetilde{B_i}, E_{i,i}) \ge 2\}. \end{aligned}$$

Here $I_R(\widetilde{B_i}, E_{i,i})$ is the intersection number of $\widetilde{B_i}$ and $E_{i,i}$ at R. We denote the cardinality of the set $T_1(i)$ by

$$\theta_i = \sharp T_1(i).$$

Since the multiplicity β_{i-1} coincides with $\sum_{R \in T(i)} I_R(\widetilde{B_i}, E_{i,i})$, we have

$$0 \le \theta_i \le \beta_{i-1}.$$

Moreover if $r \ge 2$, then we decompose the set $\{2, \cdots, r\}$ into a disjoint union $\mathbf{A} \amalg \mathbf{B}$ where

$$\mathbf{A} = \{2 \le i \le r \mid \tau_i \text{ is of type A }\}, \quad \mathbf{B} = \{2 \le i \le r \mid \tau_i \text{ is of type B }\}.$$

LEMMA 4.5. *We have*

(i) $$\beta_0 \sum_{j=1}^{n-1} j\left\langle\frac{m_1 j}{n}\right\rangle + \sum_{i\in\mathbf{A}} \beta_{i-1}\sum_{j=1}^{n-1} j\left(\left\langle\frac{m_i j}{n}\right\rangle - \left\langle\frac{m_{\varphi_1(i)} j}{n}\right\rangle\right)$$
$$+ \sum_{i\in\mathbf{B}} \beta_{i-1}\sum_{j=1}^{n-1} j\left(\left\langle\frac{m_i j}{n}\right\rangle - \left\langle\frac{m_{\varphi_1(i)} j}{n}\right\rangle - \left\langle\frac{m_{\varphi_2(i)} j}{n}\right\rangle\right) = \sum_{i=1}^{r}\theta_i\sum_{j=1}^{n-1} j\left\langle\frac{m_i j}{n}\right\rangle,$$

(ii) $$\beta_0 d_1 + \sum_{i\in\mathbf{A}}\beta_{i-1}(d_i - d_{\varphi_1(i)}) + \sum_{i\in\mathbf{B}}\beta_{i-1}(d_i - d_{\varphi_1(i)} - d_{\varphi_2(i)}) = \sum_{i=1}^{r}\theta_i d_i.$$

Proof We set $G(x) = \sum_{j=1}^{n-1} j \left\langle \frac{xj}{n} \right\rangle$ for an integer x. For $1 \le i \le r$, we define subsets $J_{\mathbf{A}}(i)$ and $J_{\mathbf{B}}(i)$ of $\{1, \cdots, r\}$ by

$J_{\mathbf{A}}(i) = \{j \in \mathbf{Z}, i+1 \le j \le r \mid \ \tau_j$ is of type A and $\varphi_1(j) = i\}$,

$J_{\mathbf{B}}(i) = \{j \in \mathbf{Z}, i+1 \le j \le r \mid \ \tau_j$ is of type B and $\varphi_1(j) = i$ or $\varphi_2(j) = i\}$.

We put

$$a_i = \beta_{i-1} - \sum_{j \in J_{\mathbf{A}}(i) \cup J_{\mathbf{B}}(i)} \beta_{j-1}.$$

Then it is clear that

$$\beta_0 G(m_1) + \textstyle\sum_{i \in \mathbf{A}} \beta_{i-1}\{G(m_i) - G(m_{\varphi_1(i)})\} + \sum_{i \in \mathbf{B}} \beta_{i-1}\{G(m_i) - G(m_{\varphi_1(i)}) \\ -G(m_{\varphi_2(i)})\} = \sum_{i=1}^{r} a_i G(m_i).$$

Therefore it suffices to prove

$$a_i = \theta_i \quad (1 \le i \le r).$$

We note that

$$\theta_i = \beta_{i-1} - \sum_{R \in T_2(i)} I_R(\widetilde{B_i}, E_{i,i}) \ .$$

For any fixed $R \in T_2(i)$, let

$$R_0 = R, R_1 = E_{i,i_1} \cap \widetilde{B_{i_1}}, \cdots, R_s = E_{i,i_s} \cap \widetilde{B_{i_s}} \quad (i < i_1 < \cdots < i_s \le r)$$

be all the infinitely near points of R. Then by Max Noether's theorem (e.g.,[BK,§8]), we have

$$I_R(\widetilde{B_i}, E_{i,i}) = \sum_{k=0}^{s} \mathrm{mult}_{R_k} \widetilde{B_{i_k}} \cdot \mathrm{mult}_{R_k} E_{i,i_k} = \sum_{k=0}^{s} \beta_{i_s}.$$

From this, we have

$$\sum_{R \in T_2(i)} I_R(\widetilde{B_i}, E_{i,i}) = \sum_{j \in J_A(i) \cup J_B(i)} \beta_{j-1}.$$

Hence we obtain $a_i = \theta_i$, and the assertion (i) is proved.

To prove (ii), we just replace the definition of $G(x)$ by $G(x) = \gcd(x, n)$.

Q.E.D.

4.6. We introduce some definitions. We first set

$$F(1) = -\frac{1}{3}(\beta_0^2 - 3\beta_0 - 2)n - 1 + \frac{\beta_0^2 + d_{\beta_0}^2}{3n} - \frac{4\theta_1}{n} \sum_{j=1}^{n-1} j \left\langle \frac{\beta_0 j}{n} \right\rangle - \theta_1 d_{\beta_0}.$$

For $i \in \mathbf{A}$, set

$$\begin{aligned} F_A(i) \quad = \quad & -\frac{1}{3}\left(n - \frac{1}{n}\right)\beta_{i-1}^2 + d_{i,\varphi_1(i)} + 2\sum_{j=1}^{n-1}\left(\left\langle \frac{m_i j}{n} \right\rangle - \left\langle \frac{m_{\varphi_1(i)} j}{n} \right\rangle\right)^2 \\ & -\frac{4\theta_i}{n}\sum_{j=1}^{n-1} j \left\langle \frac{m_i j}{n} \right\rangle - \theta_i d_i - c_{\varphi_1(i)}. \end{aligned}$$

For $i \in \mathbf{B}$, set

$$\begin{aligned} F_B(i) &= -\frac{1}{3}\left(n-\frac{1}{n}\right)\beta_{i-1}^2 - (\beta_{i-1}+1)n - d_i + d_{\varphi_1(i)} + d_{\varphi_2(i)} + d_{i,\varphi_1(i)} \\ &\quad + d_{i,\varphi_2(i)} - d_{\varphi_1(i),\varphi_2(i)} + 2\sum_{j=1}^{n-1}\left(\left\langle\frac{m_i j}{n}\right\rangle - \left\langle\frac{m_{\varphi_1(i)} j}{n}\right\rangle - \left\langle\frac{m_{\varphi_2(i)} j}{n}\right\rangle\right)^2 \\ &\quad -\frac{4\theta_i}{n}\sum_{j=1}^{n-1} j\left\langle\frac{m_i j}{n}\right\rangle - \theta_i d_i - c_{\varphi_1(i)} - c_{\varphi_2(i)} + 1. \end{aligned}$$

PROPOSITION 4.7. *For a 2-dimensional isolated singularity (V,P) of cyclic type with $r \geq 1$, we have*

$$\sigma(V,P) = F(1) + \sum_{i\in\mathbf{A}} F_A(i) + \sum_{i\in\mathbf{B}} F_B(i).$$

PROOF Since we have

$$\sigma(V,P) = \sigma(V_{[1]},P_{[1]}) + \sum_{i=2}^{r}\{\sigma(V_{[i]},P_{[i]}) - \sigma(V_{[i-1]},P_{[i-1]})\},$$

the assertion follows from Lemmas 4.2, 4.3 and 4.5. Q.E.D.

§ 5. The estimate concerning the fractional part symbol.

In order to estimate the formula in Proposition 4.7, we preprare some calculations concerning the fractional part symbol $\langle\quad\rangle$.

LEMMA 5.1. *Let n_1, n_2 be natural numbers which are not less than 2, and we set $s = \gcd(n_1,n_2)$ and $N = \mathrm{lcm}(n_1,n_2)$ (the least common multiple). Let s_1, s_2 be any natural numbers with $(n_1,s_1)=1$ and $(n_2,s_2)=1$. Then we have*

$$\sum_{i=0}^{N-1}\left\langle\frac{s_1 i}{n_1}\right\rangle\left\langle\frac{s_2 i}{n_2}\right\rangle = \frac{1}{4s}\{(n_1-1)(n_2-1)-(s-1)^2\} + \sum_{i=0}^{s-1}\left\langle\frac{s_1 i}{s}\right\rangle\left\langle\frac{s_2 i}{s}\right\rangle.$$

PROOF We set $\widetilde{n_1} = n_1/s$ and $\widetilde{n_2} = n_2/s$. Since $N = \widetilde{n_1}\widetilde{n_2}s = n_1\widetilde{n_2}$, we have

$$\begin{aligned} \sum_{i=0}^{N-1}\left\langle\frac{s_1 i}{n_1}\right\rangle\left\langle\frac{s_2 i}{n_2}\right\rangle &= \sum_{i=0}^{n_1-1}\left\langle\frac{s_1 i}{n_1}\right\rangle\sum_{j=0}^{\widetilde{n_2}-1}\left\langle\frac{s_2(i+jn_1)}{n_2}\right\rangle \\ &= \sum_{i=0}^{n_1-1}\left\langle\frac{s_1 i}{n_1}\right\rangle\sum_{j=0}^{\widetilde{n_2}-1}\left\langle\frac{1}{\widetilde{n_2}}\left(s_2\widetilde{n_1}j + \left[\frac{s_2 i}{s}\right]\right) + \frac{1}{\widetilde{n_2}}\left\langle\frac{s_2 i}{s}\right\rangle\right\rangle. \end{aligned}$$

Since $s_2\widetilde{n_1}$ and $\widetilde{n_2}$ are mutually prime, the numbers $s_2\widetilde{n_1}j + [s_2 i/s]$ $(0 \leq j \leq \widetilde{n_2}-1)$ define mutually distinct equivalence classes in $\mathbf{Z}/\widetilde{n_2}\mathbf{Z}$, i.e. the set $\{s_2\widetilde{n_1}j + [s_2 i/s] | 0 \leq j \leq \widetilde{n_2}-1\}$ is congruent modulo $\widetilde{n_2}$ to $\{0,1,\cdots,\widetilde{n_2}-1\}$ neglecting its order. The sets $\{s_1 i | 0 \le i \le n_1 - 1\}$ and $\{0,1,\cdots,n_1-1\}$ are congruent modulo n_1 in the same sence. Therefore we have

$$\begin{aligned}
&\sum_{i=0}^{N-1} \left\langle \frac{s_1 i}{n_1} \right\rangle \left\langle \frac{s_2 i}{n_2} \right\rangle = \sum_{i=0}^{n_1-1} \left\langle \frac{s_1 i}{n_1} \right\rangle \sum_{k=0}^{\widetilde{n_2}-1} \left\langle \frac{k}{\widetilde{n_2}} + \frac{1}{\widetilde{n_2}} \left\langle \frac{s_2 i}{s} \right\rangle \right\rangle \\
&= \sum_{i=0}^{n_1-1} \left\langle \frac{s_1 i}{n_1} \right\rangle \sum_{k=0}^{\widetilde{n_2}-1} \left(\frac{k}{\widetilde{n_2}} + \frac{1}{\widetilde{n_2}} \left\langle \frac{s_2 i}{s} \right\rangle \right) = \sum_{i=0}^{n_1-1} \left\langle \frac{s_1 i}{n_1} \right\rangle \left(\frac{\widetilde{n_2}-1}{2} + \left\langle \frac{s_2 i}{s} \right\rangle \right) \\
&= \frac{(n_1-1)(\widetilde{n_2}-1)}{4} + \sum_{i=0}^{\widetilde{n_1}s-1} \left\langle \frac{s_1 i}{\widetilde{n_1}s} \right\rangle \left\langle \frac{s_2 i}{s} \right\rangle .
\end{aligned}$$

Since $(\widetilde{n_1}s, s_1) = (s, s_2) = 1$, by the same argument as above, we have

$$\sum_{i=0}^{\widetilde{n_1}s-1} \left\langle \frac{s_1 i}{\widetilde{n_1}s} \right\rangle \left\langle \frac{s_2 i}{s} \right\rangle = \frac{(s-1)(\widetilde{n_1}-1)}{4} + \sum_{i=0}^{s-1} \left\langle \frac{s_1 i}{s} \right\rangle \left\langle \frac{s_2 i}{s} \right\rangle .$$

From these, the assertion follows. Q.E.D.

LEMMA 5.2. *Let s be any natural number and let s_1, s_2 be natural numbers with $(s, s_1) = (s, s_2) = 1$. Then we have*

$$\frac{(s+1)(s-1)}{6s} \le \sum_{i=0}^{s-1} \left\langle \frac{s_1 i}{s} \right\rangle \left\langle \frac{s_2 i}{s} \right\rangle \le \frac{(s-1)(2s-1)}{6s}.$$

Moreover the left hand side equality holds if and only if $s_1 + s_2 \equiv 0 \pmod{s}$, and the right hand side equality holds if and only if $s_1 \equiv s_2 \pmod{s}$.

PROOF For a permutation

$$\tau = \begin{pmatrix} 1, & 2, & \cdots, & s-1 \\ \tau(1), & \tau(2), & \cdots, & \tau(s-1) \end{pmatrix},$$

we define the function

$$F(\tau) := \sum_{i=1}^{s-1} i\tau(i).$$

We easily obtain that, when τ moves all the permutations of $(1, 2, \cdots, s-1)$, the minimal value of $F(\tau)$ attains if and only if τ coincides with

$$\begin{pmatrix} 1, & 2, & \cdots, & s-1 \\ s-1, & s-2, & \cdots, & 1 \end{pmatrix},$$

and the maximal value of $F(\tau)$ attains if and only if τ coincides with the identity permutation.

Now for any natural number α, we write $\ll \alpha \gg$ as the remainder divided by s. We put

$$\sigma_{s_1,s_2} = \begin{pmatrix} \ll s_1 \gg, & \ll 2s_1 \gg, & \cdots, & \ll (s-1)s_1 \gg \\ \ll s_2 \gg, & \ll 2s_2 \gg, & \cdots, & \ll (s-1)s_2 \gg \end{pmatrix}.$$

By $(s, s_1) = (s, s_2) = 1$, σ_{s_1,s_2} is considered as a permutation of $(1, 2, \cdots, s-1)$. Moreover we have

$$\sum_{i=0}^{s-1} \left\langle \frac{s_1 i}{s} \right\rangle \left\langle \frac{s_2 i}{s} \right\rangle = \frac{F(\sigma_{s_1,s_2})}{s^2}.$$

Therefore the assertion easily follows. Q.E.D.

LEMMA 5.3. *Let n be an integer greater than 2 and let M_1, M_2 be natural numbers with $M_1 \not\equiv 0, M_2 \not\equiv 0 \pmod n$. Then we have*

$$\sum_{i=1}^{n-1} \left(\left\langle \frac{M_1 i}{n} \right\rangle - \left\langle \frac{M_2 i}{n} \right\rangle \right)^2 \leq \frac{(n-1)(n-2)}{3n}.$$

PROOF For $j = 1, 2$, we set $d_j = \gcd(M_j, n)$, $n_j = n/d_j$ and $s_j = M_j/d_j$. We have $n_j \geq 2$ by $M_j \not\equiv 0 \pmod n$. Moreover we set $s = \gcd(n_1, n_2)$, $N = \operatorname{lcm}(n_1, n_2)$ and $\widetilde{n_j} = n_j/s$. We have $N = \widetilde{n_1}\widetilde{n_2}s$. We set $\tilde{d} = n/N$. From $d_1 = \tilde{d}\widetilde{n_2}$ and $d_2 = \tilde{d}\widetilde{n_1}$, we have

$$\sum_{i=1}^{n-1} \left(\left\langle \frac{M_1 i}{n} \right\rangle - \left\langle \frac{M_2 i}{n} \right\rangle \right)^2 = \tilde{d} \sum_{i=1}^{N-1} \left(\left\langle \frac{\widetilde{n_2} s_1 i}{N} \right\rangle - \left\langle \frac{\widetilde{n_1} s_2 i}{N} \right\rangle \right)^2.$$

Now we claim

$$\sum_{i=1}^{N-1} \left(\left\langle \frac{\widetilde{n_2} s_1 i}{N} \right\rangle - \left\langle \frac{\widetilde{n_1} s_2 i}{N} \right\rangle \right)^2 \leq \frac{(N-1)(N-2)}{3N}. \tag{5.3.1}$$

The assertion (5.3.1) implies the assertion of the Lemma, because we have

$$\frac{(n-1)(n-2)}{3n} - \frac{\tilde{d}(N-1)(N-2)}{3N} = \frac{(\tilde{d}-1)(3\tilde{d}N - \tilde{d} - 1)}{3\tilde{d}N} \geq 0.$$

It suffices to prove (5.3.1). Since $(n_1, s_1) = (n_2, s_2) = 1$, we have

$$\begin{aligned}
&\sum_{i=1}^{N-1} \left(\left\langle \frac{\widetilde{n_2} s_1 i}{N} \right\rangle - \left\langle \frac{\widetilde{n_1} s_2 i}{N} \right\rangle \right)^2 = \widetilde{n_2} \sum_{i=1}^{n_1-1} \left\langle \frac{s_1 i}{n_1} \right\rangle^2 + \widetilde{n_1} \sum_{i=1}^{n_2-1} \left\langle \frac{s_2 i}{n_2} \right\rangle^2 - 2 \sum_{i=1}^{N-1} \left\langle \frac{s_1 i}{n_1} \right\rangle \left\langle \frac{s_2 i}{n_2} \right\rangle \\
&= \frac{\widetilde{n_2}(n_1-1)(2n_1-1)}{6n_1} + \frac{\widetilde{n_1}(n_2-1)(2n_2-1)}{6n_2} - 2 \sum_{i=1}^{N-1} \left\langle \frac{s_1 i}{n_1} \right\rangle \left\langle \frac{s_2 i}{n_2} \right\rangle.
\end{aligned}$$

Therefore it easily follows from Lemmas 5.1 and 5.2 that

$$\begin{aligned}
&\frac{(N-1)(N-2)}{3N} - \sum_{i=1}^{N-1} \left(\left\langle \frac{\widetilde{n_2} s_1 i}{N} \right\rangle - \left\langle \frac{\widetilde{n_1} s_2 i}{N} \right\rangle \right)^2 \\
&= \left(\frac{(N-1)(2N-1)}{6N} - \frac{\widetilde{n_2}(n_1-1)(2n_1-1)}{6n_1} \right) + \left(\frac{(N-1)(2N-1)}{6N} \right. \\
&\quad \left. - \frac{\widetilde{n_1}(n_2-1)(2n_2-1)}{6n_2} \right) + 2 \left\{ \sum_{i=1}^{N-1} \left\langle \frac{s_1 i}{n_1} \right\rangle \left\langle \frac{s_2 i}{n_2} \right\rangle - \frac{(N-1)(N+1)}{6N} \right\} \\
&\geq \frac{1}{6N} \{ (3N - \widetilde{n_2} - 1)(\widetilde{n_2} - 1) + (3N - \widetilde{n_1} - 1)(\widetilde{n_1} - 1) \} \\
&\quad + 2 \left\{ \frac{(n_1-1)(n_2-1) - (s-1)^2}{4s} + \frac{(s-1)(s+1)}{6s} - \frac{(N-1)(N+1)}{6N} \right\} \\
&= \frac{\widetilde{n_1}\widetilde{n_2}(\widetilde{n_1}\widetilde{n_2} - 1)s^2 - (\widetilde{n_1} + \widetilde{n_2})^2 + 4}{6\widetilde{n_1}\widetilde{n_2}s} \geq \frac{(\widetilde{n_1}^2 - 1)(\widetilde{n_2}^2 - 1) - 3\widetilde{n_1}\widetilde{n_2} + 3}{6\widetilde{n_1}\widetilde{n_2}} \geq 0.
\end{aligned}$$

by $\widetilde{n_1} \geq 2$ and $\widetilde{n_2} \geq 2$. Hence the assertion is proved. Q.E.D.

We slightly generalize a part of Lemma 5.2.

LEMMA 5.4 *Let n, M_1, M_2 be any natural numbers. Then we have*

$$\sum_{i=1}^{n-1} \left\langle \frac{M_1 i}{n} \right\rangle \left\langle \frac{M_2 i}{n} \right\rangle \leq \frac{(n-1)(2n-1)}{6n} .$$

PROOF If $M_1 \equiv 0$ or $M_2 \equiv 0 \pmod n$, then the assertion is clear. Assume $M_1 \not\equiv 0$ and $M_2 \not\equiv 0$. We denote by $d_j, s_j, n_j, \widetilde{n_j}$, $(j = 1, 2)$ s, N and $\widetilde{d}$ as in the proof of Lemma 5.3. We easily have

$$\begin{aligned} \frac{(n-1)(2n-1)}{6n} &\geq \frac{\widetilde{d}(N-1)(2N-1)}{6N} , \\ \sum_{i=1}^{n-1} \left\langle \frac{M_1 i}{n} \right\rangle \left\langle \frac{M_2 i}{n} \right\rangle &= \widetilde{d} \sum_{i=1}^{N-1} \left\langle \frac{s_1 i}{n_1} \right\rangle \left\langle \frac{s_2 i}{n_2} \right\rangle . \end{aligned}$$

Moreover it follows from Lemmas 5.1 and 5.2 that

$$\begin{aligned} & \frac{(N-1)(2N-1)}{6N} - \sum_{i=1}^{N-1} \left\langle \frac{s_1 i}{n_1} \right\rangle \left\langle \frac{s_2 i}{n_2} \right\rangle \\ \geq & \frac{(s\widetilde{n_1}\widetilde{n_2}-1)(2s\widetilde{n_1}\widetilde{n_2}-1)}{6s\widetilde{n_1}\widetilde{n_2}} - \frac{(s\widetilde{n_1}-1)(s\widetilde{n_2}-1)}{4s} - \frac{(s-1)(2s-1)}{6s} \\ = & \frac{\widetilde{n_1}\widetilde{n_2}-1}{12} s + \frac{\widetilde{n_1}+\widetilde{n_2}-2}{4} + \left(\frac{1}{6\widetilde{n_1}\widetilde{n_2}} - \frac{1}{6} \right) \frac{1}{s} \geq 0. \end{aligned}$$

Hence the assertion follows. Q.E.D.

LEMMA 5.5. *Let n, M be natural numbers with $n \geq 2$. Then we have*

$$\frac{4}{n} \sum_{i=1}^{n-1} i \left\langle \frac{Mi}{n} \right\rangle + \gcd(M, n) \geq \frac{2n}{3} + 1 - \frac{2}{3n} .$$

PROOF The calculation is similar as above, and we omit it.

REMARK 5.6. The statement of lemmas in this section can be translated into the language of the Dedekind sum (e.g. Hirzebruch-Zagier [HZ]).

§ 6. The estimate of the signature.

The aim of this section is to prove the following:

THEOREM 6.1. *Let (C, P_0) be a germ of an isolated plane curve singularity of multiplicity m. Let $\{C_1, \cdots, C_t\}$ be the set of local analytic branches of C at P_0. Let $\{C_{i_1}, \cdots, C_{i_\theta}\}$ be the subset of $\{C_1, \cdots, C_t\}$ such that;*

($\star$) *For any j $(1 \le j \le \theta)$, C_{i_j} is nonsingular at P_0 and the tangent line of C_{i_j} at P_0 is mutually distinct from the tangent line of any other one of $\{C_1, \cdots, C_t\}$.*

The number $\theta = \theta(C, P_0)$ is defined to be the maximal number which enjoy the property ($\star$). If such a subset $\{C_{i_1}, \cdots, C_{i_\theta}\}$ is empty, then we set $\theta = 0$.

Now let (V, P) be the germ of the singularity on n-fold cyclic cover ($n \ge 2$) branched along (C, P_0), i.e. (V, P) is defined by the equation $f(x, y, z) = z^n + g(x, y)$ where $g(x, y)$ is the defining equation of (C, P_0). Let $\sigma(V, P)$ be the signature of the Milnor fiber of (V, P). Then we have

$$\begin{aligned} \sigma(V,P) \quad \le \quad & -\frac{1}{3}(m^2 - 3m - 2)n - 1 + \frac{m^2 + \gcd(m,n)^2}{3n} \\ & -\frac{4\theta}{n}\sum_{i=1}^{n-1} i\left\langle \frac{mi}{n} \right\rangle - \theta \cdot \gcd(m,n). \end{aligned} \tag{6.1.1}$$

where $\langle mi/n \rangle = mi/n - [mi/n]$ is the fractional part of mi/n.

Furthermore, the equality of (6.1.1) holds if and only the singularity (V, P) satisfies the property $\theta = m$, that is, (C, P_0) is an ordinary singularity.

It is clear that Theorem 6.1 implies the Theorem in the introduction.

Corollary 6.2. *Let (V, P) be a germ of analytic function $f(x, y, z) = z^n + g(x, y)$ such that (V, P) defines at most an isolated singularity. Then we have*

$$\sigma(V, P) \le 0.$$

Furthermore, the equality $\sigma(V, P) = 0$ holds if and only if (V, P) is a germ of a nonsingular point.

Proof If the multiplicity of (V, P) is less than 4, the assertion is known ([To1][A]). Therefore we may assume $n \ge 4$ and $m \ge 4$. Then we have $\sigma(V, P) < 0$ by an easy calculation. Q.E.D.

6.3. We start to prove Theorem 6.1. We produce the Hirzebruch-Jung resolution of (V, P). We first assume $r = 0$. Then (C, P_0) is an ordinary double point and (V, P) is a rational double point of type A_{n-1}. By $p_g = 0$, $\mu = n - 1$ and $b_1 = 0$, Theorem 4.1 imply $\sigma = -n + 1$ in this case. On the other hand, we have

$$\sum_{j=1}^{n-1} j\left\langle \frac{2j}{n} \right\rangle = \begin{cases} (n-1)(7n-5)/24, & \text{if } n \text{ is odd} \\ (n-2)(7n-4)/24, & \text{if } n \text{ is even.} \end{cases}$$

by an easy calculation. Since we have $m = \theta = 2$, the right hand side of (6.1.1) also coincides with $-n + 1$, which prove the assertion in this case. (Another proof: Although the branch curve has normal crossing, we blow up the base surface at one time and use Proposition 4.7 by settig $r = 1$.)

Next we assume $r = 1$. In the notation of § 4, we note that $\theta = \theta_1$ and $m = \beta_0$. Moreover, $F(1)$ in Proposition 4.7 coincides with the right hand side term of (6.1.1). Therefore the equality of (6.1.1) holds. We remark that the condition $r=1$ occurs if and only if θ coincides with m.

From now on, we assume $r \geq 2$. By Proposition 4.7, it suffices to prove

$$\sum_{i \in \mathbf{A}} F_A(i) + \sum_{i \in \mathbf{B}} F_B(i) < 0. \tag{6.3.1}$$

We prepare several lemmas to prove (6.3.1).

LEMMA 6.4. *Assume $i \in \mathbf{A}$. If $m_i \not\equiv m_{\varphi_1(i)}$ (mod n), then*

$$F_A(i) \leq -\frac{1}{3}\left(n-\frac{1}{n}\right)(\beta_{i-1}-2)(\beta_{i-1}+2) + \mathrm{Max}\left\{-\frac{5}{12}n-3+\frac{8}{3n}, -\frac{7}{12}n+\frac{4}{3n}\right\}.$$

If $m_i \equiv m_{\varphi_1(i)}$ (mod n), then there exists an integer $k \geq 1$ with $\beta_{i-1} = kn$ such that

$$F_A(i) \leq -\frac{1}{3}k^2n^2\left(n-\frac{1}{n}\right) + d_i - 1.$$

Especially if $\beta_{i-1} \geq 2$, then we always have $F_A(i) < 0$.

PROOF We first assume $m_i \not\equiv 0$, $m_{\varphi_1(i)} \not\equiv 0$ and $m_i \not\equiv m_{\varphi_1(i)}$ (mod n). Since we have $d_{i,\varphi_1(i)} \leq \gcd(n, n/2, n/4) = n/4$ and $c_{\varphi_1(i)} \geq 1$, it follows from Lemma 5.3 that

$$\begin{aligned} F_A(i) &\leq -\frac{4}{3}\left(n-\frac{1}{n}\right) - \frac{1}{3}\left(n-\frac{1}{n}\right)\left(\beta_{i-1}^2-4\right) + \frac{n}{4} - 1 + 2\left(\frac{n}{3}-1+\frac{2}{3n}\right) \\ &= -\frac{5}{12}n - 3 + \frac{8}{3n} - \frac{1}{3}\left(n-\frac{1}{n}\right)(\beta_{i-1}-2)(\beta_{i-1}+2). \end{aligned}$$

Next we assume $m_i \not\equiv 0$ and $m_{\varphi_1(i)} \equiv 0$ (mod n). We have $\sum_{j=1}^{n-1}\langle m_i j/n\rangle^2 = (n-d_i)(2n-d_i)/6n$. Therefore we have

$$\begin{aligned} F_A(i) &\leq -\frac{2}{3}n + \frac{4+d_i^2}{3n} - \frac{1}{3}\left(n-\frac{1}{n}\right)(\beta_{i-1}-2)(\beta_{i-1}+2) \\ &\leq -\frac{7}{12}n + \frac{4}{3n} - \frac{1}{3}\left(n-\frac{1}{n}\right)(\beta_{i-1}-2)(\beta_{i-1}+2) \end{aligned}$$

by $d_i \leq n/2$. In the case of $m_i \equiv 0$ and $m_{\varphi_1(i)} \not\equiv 0$ (mod n), we have the same assertion by a similar argument. Therefore we obtain the former assertion of the Lemma.

Assume $m_i \equiv m_{\varphi_1(i)}$ (mod n). Then we also obtain the latter assertion of the Lemma by a similar calculation. Q.E.D.

LEMMA 6.5. *We have*

$$\begin{aligned} F_B(i) \leq\ & -\frac{1}{3}\left(n-\frac{1}{n}\right)(\beta_{i-1}-2)(\beta_{i-1}+2) - (\beta_{i-1}-2)n + \frac{d_{\varphi_1(i)}^2}{3n} \\ & + \mathrm{Max}\left\{-n-4+\frac{2}{n}, -\frac{5n}{3}-1+\frac{4}{3n}\right\} \end{aligned}$$

for any $i \in \mathbf{B}$. Especially if $\beta_{i-1} \geq 2$, then we have $F_B(i) < 0$.

PROOF We set

$$
\begin{aligned}
F^\star &= -\frac{1}{3}\left(n-\frac{1}{n}\right)\beta_{i-1}^2-(\beta_{i-1}+1)n+d_{\varphi_1(i)}+d_{i,\varphi_2(i)}-1\\
&\quad +2\sum_{j=1}^{n-1}\left(\left\langle\frac{m_ij}{n}\right\rangle-\left\langle\frac{m_{\varphi_2(i)}j}{n}\right\rangle\right)^2+2\sum_{j=1}^{n-1}\left\langle\frac{m_{\varphi_1(i)}j}{n}\right\rangle^2,\\
F^{\star\star} &= -d_i+d_{\varphi_2(i)}+d_{i,\varphi_1(i)}-d_{\varphi_1(i),\varphi_2(i)}\\
&\quad -4\sum_{j=1}^{n-1}\left(\left\langle\frac{m_ij}{n}\right\rangle-\left\langle\frac{m_{\varphi_2(i)}j}{n}\right\rangle\right)\left\langle\frac{m_{\varphi_1(i)}j}{n}\right\rangle.
\end{aligned}
$$

Then we claim

$$
\begin{aligned}
F^\star &\leq -\frac{1}{3}\left(n-\frac{1}{n}\right)(\beta_{i-1}-1)(\beta_{i-1}+1)-(\beta_{i-1}-1)n-\frac{2}{3}n-1+\frac{1+d_{\varphi_1(i)}^2}{3n}, && (6.5.1)\\
F^{\star\star} &\leq \mathrm{Max}\left\{\frac{5}{3}n-3+\frac{2}{3n},\ n\right\}. && (6.5.2)
\end{aligned}
$$

Since we have $F_B(i) \leq F^\star + F^{\star\star}$, the assertions (6.5.1) and (6.5.2) easily imply the assertion of the Lemma.

We prove (6.5.1). Assume $d_{i,\varphi_2(i)} \leq n/3$. Then it follows from Lemma 5.3 that

$$
\begin{aligned}
F^\star &\leq -\frac{1}{3}\left(n-\frac{1}{n}\right)(\beta_{i-1}-1)(\beta_{i-1}+1)-\frac{1}{3}\left(n-\frac{1}{n}\right)-(\beta_{i-1}-1)n-2n\\
&\quad +d_{\varphi_1(i)}+\frac{n}{3}-1++\frac{2(n-1)(n-2)}{3n}+\frac{(n-d_{\varphi_1(i)})(2n-d_{\varphi_1(i)})}{3n}\\
&= -\frac{1}{3}\left(n-\frac{1}{n}\right)(\beta_{i-1}-1)(\beta_{i-1}+1)-(\beta_{i-1}-1)n-\frac{2}{3}n-3+\frac{5+d_{\varphi_1(i)}^2}{3n}.
\end{aligned}
$$

Assume $d_{i,\varphi_2(i)} > n/3$. Then one of the following cases occur:
(i) " $m_i \equiv 0$, $m_{\varphi_2(i)} \equiv n/2 \pmod{n}$ " or " $m_i \equiv n/2$, $m_{\varphi_2(i)} \equiv 0 \pmod{n}$ " ,
(ii) $m_i \equiv m_{\varphi_2(i)} \equiv n/2 \pmod{n}$,
(iii) $m_i \equiv m_{\varphi_2(i)} \equiv 0 \pmod{n}$.
We consider the case (i). Then the integer n is even, and we have

$$
\sum_{j=1}^{n-1}\left(\left\langle\frac{m_ij}{n}\right\rangle-\left\langle\frac{m_{\varphi_2(i)}j}{n}\right\rangle\right)^2=\sum_{j=1}^{n-1}\left\langle\frac{j}{2}\right\rangle^2=\frac{n}{8}.
$$

Therefore we obtain

$$
F^\star \leq -\frac{1}{3}\left(n-\frac{1}{n}\right)(\beta_{i-1}-1)(\beta_{i-1}+1)-(\beta_{i-1}-1)n-\frac{11}{12}n-1+\frac{1+d_{\varphi_1(i)}^2}{3n}.
$$

In the case of (ii), we have

$$
F^\star \leq -\frac{1}{3}\left(n-\frac{1}{n}\right)(\beta_{i-1}-1)(\beta_{i-1}+1)-(\beta_{i-1}-1)n-\frac{7}{6}n-1+\frac{1+d_{\varphi_1(i)}^2}{3n}.
$$

In the case of (iii), we have

$$F^\star \leq -\frac{1}{3}\left(n-\frac{1}{n}\right)(\beta_{i-1}-1)(\beta_{i-1}+1)-(\beta_{i-1}-1)n-\frac{2}{3}n-1+\frac{1+d_{\varphi_1(i)}^2}{3n}.$$

From these, we obtain the assertion (6.5.1).

Next we prove (6.5.2). First we have

$$F^{\star\star} \leq d_{\varphi_2(i)} - 1 + 4\sum_{j=1}^{n-1}\left\langle \frac{m_{\varphi_1(i)}j}{n}\right\rangle \left\langle \frac{m_{\varphi_2(i)}j}{n}\right\rangle.$$

We assume $d_{\varphi_2(i)} \leq n/3$. By Lemma 5.4, we have

$$F^{\star\star} \leq \frac{n}{3} - 1 + 4\cdot\frac{(n-1)(2n-1)}{6n} = \frac{5n}{3} - 3 + \frac{2}{3n}.$$

We assume $d_{\varphi_2(i)} = n/2$. We set $n = 2n'$. We have

$$F^{\star\star} \leq \frac{n}{2} - 1 + 2\sum_{k=1}^{n'}\left\langle \frac{m_{\varphi_1(i)}(2k-1)}{n}\right\rangle.$$

Since the number $m_{\varphi_1(i)}(2k-1)/n+m_{\varphi_1(i)}\{2(n'-k+1)-1\}/n$ is an integer for $1 \leq k \leq [n/2]$, we have

$$\left\langle \frac{m_{\varphi_1(i)}(2k-1)}{n}\right\rangle + \left\langle \frac{m_{\varphi_1(i)}\{2(n'-k+1)-1\}}{n}\right\rangle \leq 1.$$

Therefore we have

$$F^{\star\star} < \frac{n}{2} - 1 + 2\left(\left[\frac{n'}{2}\right]+1\right) \leq n+1.$$

If $d_{\varphi_2(i)} = n$, then we have

$$F^{\star\star} \leq n-1.$$

From these, we obtain the assetion (6.5.2). Q.E.D.

6.6 Assume τ_{i_0+1} is of type B and $\beta_{i_0} = 1$ for some i_0 $(2 \leq i_0 \leq r-1)$. By our construction, there are integers v and w with $1 \leq v < w \leq i_0$ such that τ_{i_0+1} is the blow-up at the center $P_{i_0} = E_{v,i_0} \cap E_{w,i_0}$, and the curve $\widetilde{B_{i_0}}$ is nonsingular at P_{i_0}. We denote by $\widetilde{C}$ the unique local branch of $\widetilde{B_{i_0}}$ at P_{i_0}. Then one of the following two conditions (I) and (II) is satisfied :

(I) i) $\widetilde{C}$ intersect both E_{v,i_0} and E_{w,i_0} transversally at P_{i_0}, or

ii) $\widetilde{C}$ is tangential to E_{v,i_0} at P_{i_0} of order $t \geq 2$.

(II) $\widetilde{C}$ is tangential to E_{w,i_0} at P_{i_0} of order $t \geq 2$.

From now on, we consider that the case i) of (I) is the special case of the case ii) of (I) by setting $t = 1$.

We consider the curve $\widetilde{B_{w-1}}$ on W_{w-1}. In both cases (I) and (II), there exists a local branch C of $\widetilde{B_{w-1}}$ at the center P_{w-1} such that the proper transform of C by τ_w is isomorphic to $\widetilde{C}$. More precisely, if i_0 is greater than w, then any of the centers of the blow-ups $\tau_{w+1}, \cdots, \tau_{i_0}$ does not coincide with the infinitely near point of P_{w-1}, and so these blow-ups do not matter the analysis of the singularity at P_{i_0}. Therefore by changing the order of the blow-ups if necessary, we may assume that the number w coincides with i_0.

Now assume that the case (I) occure. Then the following two conditions are satisfied (see Figure 4) :

I-a) C is nonsingular and contacts at P_{w-1} to $E_{v,w-1}$ of order $t+1$, and

I-b) If other local branches of $\widetilde{B_{w-1}}$ at P_{w-1} exist, then any tangent line of them does not coincides with $E_{v,w-1}$.

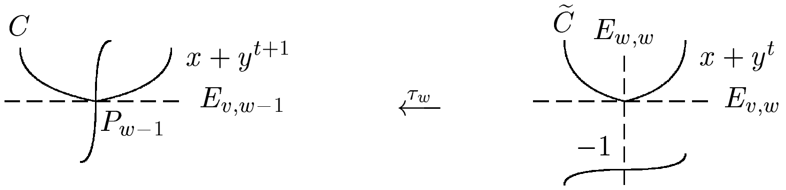


Figure 4:

On the other hand, if the case (II) occure, then the following conditions are satisfied (see Figure 5) :

II-a) C defines at P_{w-1} a *tangential $(t, t+1)$-cusp to $E_{v,w-1}$*, i.e. the local analytic equation of C at P_{w-1} is given by $x^t + y^{t+1} = 0$, where $E_{v,w-1}$ is locally given by $x = 0$, and

II-b) If other local branches of $\widetilde{B_{w-1}}$ at P_{w-1} exist, then any tangent line of them does not coincides with $E_{v,w-1}$.

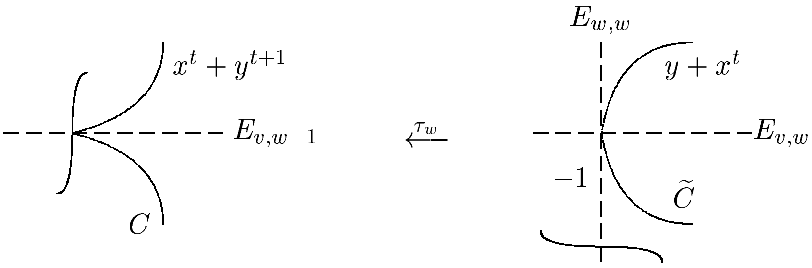


Figure 5:

Assume that the case (I) occure. By changing the order of the blow-ups if necessary, we may assume that the successive blow-ups $\tau_{w+1}, \cdots, \tau_{w+t}$ are produced at the infinitely near points of P_{w-1}, and therefore τ_{w+j} $(1 \leq j \leq t)$ satisfies the following:

(i) τ_{w+j} is the blow-up of type B with the center $P_{w+j} = E_{v,w+j} \cap E_{w+j,w+j}$,

(ii) $\widetilde{B_{w+j}}$ is nonsingular at P_{w+j} and is tangent to $E_{v,w+j}$ of order $t-j$ at P_{w+j}.

After these bow-ups, the curve $\widetilde{B_{w+t}} + \mathcal{E}_{w+t}$ has normal crossing in the pull back of a neighborhood of P_w.

If the case (II) occure, then we have the same argument by replacing the definition of P_{w+j} by $E_{w,w+j} \cap E_{w+j,w+j}$.

LEMMA 6.7 *In the above situation , we further assume that τ_w is of type A. If $\widetilde{C}$ satisfies the codition (I), then*

$$F_A(w)+\sum_{j=1}^{t}F_B(w+j)\le -\frac{1}{3}\left(n-\frac{1}{n}\right)(\beta_{w-1}-1)(\beta_{w-1}+1)$$
$$+\left(-\frac{2}{3}n-1+\frac{1+d_v^2}{3n}\right)(t-1)-\frac{1}{3}n-d_v-3+\frac{d_w^2+2d_v^2+4}{3n}<0.$$

If $\widetilde{C}$ satisfies the codition (II), then

$$F_A(w)+\sum_{j=1}^{t}F_B(w+j)\le -\frac{1}{3}\left(n-\frac{1}{n}\right)(\beta_{w-1}-1)(\beta_{w-1}+1)$$
$$+\left(-\frac{2}{3}n-1+\frac{1+d_w^2}{3n}\right)(t-1)-\frac{1}{3}n-d_w-3+\frac{d_v^2+2d_w^2+4}{3n}<0.$$

PROOF First we assume that $\widetilde{C}$ satisfies the condition (I). In this case, we have $\varphi_1(w+j)=v$ $(0\le j\le t)$, $\varphi_2(w+j)=w+j-1$ $(1\le j\le t)$, $\theta_{w+j}=0$ $(1\le j\le t-1)$, $\theta_{w+t}=1$, $\beta_{w+j-1}=1$ $(1\le j\le t)$ and $c_{\varphi_1(w+j)}=c_{\varphi_2(w+j)}=1$ $(1\le j\le t)$. We have

$$\begin{aligned} F_A(w) \quad \le \quad & -\frac{1}{3}\left(n-\frac{1}{n}\right)\beta_{w-1}^2+d_{v,w}-1+\frac{(n-d_w)(2n-d_w)}{3n} \\ & +\frac{(n-d_v)(2n-d_v)}{3n}-4\sum_{k=1}^{n-1}\left\langle\frac{m_wk}{n}\right\rangle\left\langle\frac{m_vk}{n}\right\rangle . \end{aligned}$$

On the other hand, it follows from (6.5.1) and Lemma 5.5 that

$$\begin{aligned} \sum_{j=1}^{t}F_B(w+j) &= \sum_{j=1}^{t}\left\{-\frac{1}{3}\left(n-\frac{1}{n}\right)-2n+d_v+d_{w+j-1,w+j}-1\right. \\ &\left.+2\sum_{k=1}^{n-1}\left(\left\langle\frac{m_{w+j}k}{n}\right\rangle-\left\langle\frac{m_{w+j-1}k}{n}\right\rangle\right)^2+2\sum_{k=1}^{n-1}\left\langle\frac{m_vk}{n}\right\rangle^2\right\} \\ &+\sum_{j=1}^{t}\left\{d_{w+j-1}-d_{w+j}-d_{w+j-1,v}+d_{w+j,v}-4\sum_{k=1}^{n-1}\left(\left\langle\frac{m_{w+j+1}k}{n}\right\rangle-\left\langle\frac{m_{w+j}k}{n}\right\rangle\right)\right. \\ &\left.\times\left\langle\frac{m_vk}{n}\right\rangle\right\}-\frac{4}{n}\sum_{k=1}^{n-1}k\left\langle\frac{m_{w+t}k}{n}\right\rangle-d_{w+t} \end{aligned}$$

$$\begin{aligned} &\le\left(-\frac{2}{3}n-1+\frac{1+d_v^2}{3n}\right)t+d_w-d_{w+t}-d_{v,w}+d_{w+t,v}+4\sum_{k=1}^{n-1}\left\langle\frac{m_wk}{n}\right\rangle\left\langle\frac{m_vk}{n}\right\rangle \\ &\quad -4\sum_{k=1}^{n-1}\left\langle\frac{m_{w+t}k}{n}\right\rangle\left\langle\frac{m_vk}{n}\right\rangle-\left(\frac{2}{3}n+1-\frac{2}{3n}\right) \\ &\le\left(-\frac{2}{3}n-1+\frac{1+d_v^2}{3n}\right)(t-1)+d_w-d_{v,w}+4\sum_{k=1}^{n-1}\left\langle\frac{m_wk}{n}\right\rangle\left\langle\frac{m_vk}{n}\right\rangle \\ &\quad -\frac{4}{3}n-2+\frac{d_v^2+3}{3n} . \end{aligned} \tag{6.7.1}$$

Therefore by an easy calculation, the left hand side inequality of the Lemma follows.

We prove the negativity. The claim is clear except for the case

$$\beta_{w-1} = 1 \text{ and } d_w^2 + 2d_v^2 > n^2. \tag{$*$}$$

Assume the codition $(*)$. The inequality $d_w^2 + 2d_v^2 > n^2$ holds if and only if "$d_w = n$ or $d_v = n$". Suppose $d_w = n$. It follows from $m_v = m_w - \beta_{w-1} = m_w - 1$ that $d_v = 1$. Therefore we have

$$-\frac{1}{3}n - d_v - 3 + \frac{d_w^2 + 2d_v^2 + 4}{3n} = -4 + \frac{2}{n} < 0.$$

The case $d_v = n$ is similar, and we omit it.

Secondly we assume that $\widetilde{C}$ satisfies the condition (II). In this case, we have $\varphi_1(w+j) = w$ $(1 \le j \le t)$, $\varphi_2(w+1) = v$, $\varphi_2(w+j) = w+j-1$ $(2 \le j \le t)$, $\theta_{w+j} = 0$ $(1 \le j \le t-1)$ and $\theta_{w+t} = 1$. By the similar calculation as above, we obtain the desired result. (We omit it.) Q.E.D.

Corollary 6.8. *Assume that τ_i is of type A and $\beta_{i-1} = 1$ for some i $(2 \le i \le r)$. Then $\widetilde{B_{i-1}}$ is nonsingular at P_{i-1} and is tangent to $E_{v,i-1}$ at P_{i-1} of order $t+1$ for some $t \ge 1$. Let $\tau_{i+1}, \cdots, \tau_{i+t}$ be the succession of blow-ups whose centers are infinitely near points of $P_i = E_{v,i} \cap E_{i,i}$ such that the curve $\widetilde{B_{i+t}} + \mathcal{E}_{i+t}$ has normal crossing in the pull back of a neighborhood of P_i. Then we have*

$$\begin{aligned} F_A(i) + \sum_{j=1}^{t} F_B(i+j) \quad \le \quad & \left(-\frac{2}{3}n - 1 + \frac{1+d_i^2}{3n}\right)(t-1) - \frac{1}{3}n \\ & -d_i - 3 + \frac{d_{i+1}^2 + 2d_i^2 + 4}{3n} < 0. \end{aligned}$$

Proof Under this situation, the blow-up τ_{i+1} is of type B and the unique local branch of $\widetilde{B_i}$ at P_i satisfies the condition (I) in 6.6. Therefore the assertion follows from Lemma 6.7. Q.E.D.

Lemma 6.9. *In the situation of 6.6, we further assume that τ_w is of type B. The point of the center P_{w-1} of τ_w is written as $P_{w-1} = E_{v,w-1} \cap E_{u,w-1}$ for some u $(1 \le u \le w-1$, $u \ne v)$. Moreover we assume $\beta_{w-1} \ge 2$. If $\widetilde{C}$ satisfies the codition (I), then we have*

$$\begin{aligned} &\sum_{j=0}^{t} F_B(w+j) \le -\frac{1}{3}\left(n - \frac{1}{n}\right)(\beta_{w-1} - 2)(\beta_{w-1} + 2) - (\beta_{w-1} - 2)n \\ &+ \left(-\frac{2}{3}n - 1 + \frac{1+d_v^2}{3n}\right)(t-1) - \frac{7}{3}n - d_{v,u} - 5 + \frac{d_w^2 + 2d_v^2 + d_u^2 + 9}{3n} < 0. \end{aligned}$$

If $\widetilde{C}$ satisfies the codition (II), then we have

$$\begin{aligned} &\sum_{j=0}^{t} F_B(w+j) \le -\frac{1}{3}\left(n - \frac{1}{n}\right)(\beta_{w-1} - 2)(\beta_{w-1} + 2) - (\beta_{w-1} - 2)n \\ &+ \left(-\frac{2}{3}n - 1 + \frac{1+d_w^2}{3n}\right)(t-1) - \frac{7}{3}n + d_v - d_w - d_{v,u} - 5 + \frac{d_v^2 + 2d_w^2 + d_u^2 + 9}{3n} < 0. \end{aligned}$$

PROOF First we assume that $\widetilde{C}$ satifies the condition (I). We have

$$\begin{aligned} F_B(w) \le\ & -\frac{1}{3}\left(n-\frac{1}{n}\right)\beta_{w-1}^2-(\beta_{w-1}+1)n+2n-2d_w+d_{w,v}+d_{w,u}-d_{v,u} \\ & -1+\frac{d_w^2+d_v^2+d_u^2}{3n}-4\sum_{k=1}^{n-1}\left\langle\frac{m_w k}{n}\right\rangle\left\langle\frac{m_v k}{n}\right\rangle-4\sum_{k=1}^{n-1}\left\langle\frac{m_w k}{n}\right\rangle\left\langle\frac{m_u k}{n}\right\rangle \\ & +4\sum_{k=1}^{n-1}\left\langle\frac{m_v k}{n}\right\rangle\left\langle\frac{m_u k}{n}\right\rangle . \end{aligned} \tag{6.9.1}$$

From (6.9.1), (6.7.1) and Lemma 5.4, we obtain the desired inequality by an easy calculation.

If the curve $\widetilde{C}$ satisfies the condition (II), then the calculation is similar as above, and we omit it. Q.E.D.

6.10. In the situation of 6.6, we further assume that τ_w is of type B, and we write the point $P_{w-1} := E_{v,w-1} \cap E_{u,w-1}$ as in the previous Lemma. Moreover we assume that there exist two local branches C and C' of $\widetilde{B_{w-1}}$ at P_{w-1} such that the following conditions (i) $\sim$ (iii) are satisfied:
(i) C is nonsingular at P_{w-1} and is tangent to $E_{v,w-1}$ at P_{w-1} of order $t+1$ for some $t \ge 1$, or C defines at P_{w-1} a tangential $(t, t+1)$-cusp to $E_{v,w-1}$,
(ii) C' is nonsingular at P_{w-1} and is tangent to $E_{u,w-1}$ at P_{w-1} of order $t'+1$ for some $t' \ge 1$, or C' defines at P_{w-1} a tangential $(t', t'+1)$-cusp to $E_{u,w-1}$,
(iii) if other local branches of $\widetilde{B_{w-1}}$ at P_{w-1} exist, then the tangent line of any component of them coincides with neither $E_{v,w-1}$ nor $E_{u,w-1}$.

Let $\widetilde{C}$ and $\widetilde{C'}$ be the proper transform of C and C' by the blow-up τ_w respectively. Then $\widetilde{C}$ pass through $P_w := E_{v,w} \cap E_{w,w}$, and $\widetilde{C'}$ pass through $P_{w+t} := E_{u,w} \cap E_{w,w}$. Moreover any other components of $\widetilde{B_w}$ except for C and C' pass through neither P_w nor P_{w+t}. By changing the order of the blow-ups if necesarry, we may assume that the succession of blow-ups $\tau_{w+1}, \cdots, \tau_{w+t}$ are produced at infinitely near points of P_w and the succession of blow-ups $\tau_{w+t+1}, \cdots, \tau_{w+t+t'}$ are produced at infinitely near points of P_{w+t}. After these blow-ups, the curve $\widetilde{B_{w+t+t'}} + \mathcal{E}_{w+t+t'}$ has normal crossing in the pull back of the union of neighborhood of P_w and P_{w+t}.

LEMMA 6.11. *In the above situation, if C and C' are nonsingular at P_{w-1}, then we have*

$$\begin{aligned} \sum_{j=0}^{t+t'} F_B(w+j) \le\ & -\frac{1}{3}\left(n-\frac{1}{n}\right)(\beta_{w-1}-2)(\beta_{w-1}+2)-(\beta_{w-1}-2)n \\ & +\left(-\frac{2}{3}n-1+\frac{1+d_v^2}{3n}\right)(t-1)+\left(-\frac{2}{3}n-1+\frac{1+d_u^2}{3n}\right)(t'-1) \\ & -\frac{11}{3}n-d_{v,u}-7+\frac{2d_u^2+2d_v^2+d_w^2+12}{3n} \ < \ 0. \end{aligned}$$

If C is nonsingular at P_{w-1} and C' defines a tangential cusp at P_{w-1}, then we have

$$\begin{aligned}\sum_{j=0}^{t+t'} F_B(w+j) \leq \ & -\frac{1}{3}\left(n-\frac{1}{n}\right)(\beta_{w-1}-2)(\beta_{w-1}+2)-(\beta_{w-1}-2)n \\ & +\left(-\frac{2}{3}n-1+\frac{1+d_v^2}{3n}\right)(t-1)+\left(-\frac{2}{3}n-1+\frac{1+d_w^2}{3n}\right)(t'-1) \\ & -\frac{11}{3}n+d_u-d_{u,w}-d_{u,v}-7+\frac{d_u^2+2d_v^2+2d_w^2+12}{3n} \ < \ 0.\end{aligned}$$

If C and C' also define tangential cusps at P_{w-1}, then we have

$$\begin{aligned}\sum_{j=0}^{t+t'} F_B(w+j) \leq \ & -\frac{1}{3}\left(n-\frac{1}{n}\right)(\beta_{w-1}-2)(\beta_{w-1}+2)-(\beta_{w-1}-2)n \\ & +\left(-\frac{2}{3}n-1+\frac{1+d_w^2}{3n}\right)(t+t'-2)-\frac{11}{3}n+d_u+d_v-2d_w \\ & -d_{v,u}-7+\frac{d_u^2+d_v^2+3d_w^2+12}{3n} \ < \ 0.\end{aligned}$$

Proof Assume C is nonsingular at P_{w-1} and C' defines a tangential cusp at P_{w-1}. Since $\widetilde{C'}$ is nonsingular at P_{w+t} and is tangent to $E_{w,w}$ of order t', by the same argument as (6.7.1), we have

$$\begin{aligned}\sum_{j=t+1}^{t+t'} F_B(w+j) \ \leq \ & \left(-\frac{2}{3}n-1+\frac{1+d_w^2}{3n}\right)(t'-1)+d_u-d_{u,w} \\ & +4\sum_{k=1}^{n-1}\left\langle\frac{m_w k}{n}\right\rangle\left\langle\frac{m_u k}{n}\right\rangle-\frac{4}{3}n-2+\frac{d_w^2+3}{3n}.\end{aligned}$$

On the other hand, we have the same estimates for $F_B(w)$ and $\sum_{j=1}^{t} F_B(w+j)$ as (6.9.1) and (6.7.1). Therefore by using Lemma 5.4, we easily obtain the desired inequality. We omit the other two cases. Q.E.D.

Proof of Theorem 6.1. From Lemmas 6.4, 6.5, 6.7, 6.9, 6.11 and Corollary 6.8, the assertion (6.3.1) follows. Hence we complete the proof of Theorem 6.1.

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