Atoms of Topology

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We can think of Topology as the theory of spaces. A priori, one has for each $n = 1, 2, 3, \dots$ an Euclidean space $\mathbb{R}^n$ of dimension n. For $n = 1$ this is the real line $\mathbb{R}$ whose points are real numbers. Moreover $\mathbb{R}^2$ is the plane. The 3–dimensional space $\mathbb{R}^3$ is closest to our intuition and figures which we can see in the real world can be described by subsets of $\mathbb{R}^3$. In general $\mathbb{R}^n$ consists of all tuples $x = (x_1, \dots, x_n)$ where $x_1, \dots, x_n$ are real numbers in $\mathbb{R}$.

In the following a *space* means a subset of an Euclidean space $\mathbb{R}^n$ of arbitrary dimension so that spaces are very general geometrical figures. Spaces or figures are like topological molecules in which, as we show, a plan of molecular structure is deeply hidden. We use the analogy between spaces and molecules as a helpful tool to explain the mathematical results in this paper. In nature molecules have the well known properties in table 1.

Atoms in Nature

(a) Each *molecule* splits into atoms; in order to obtain the splitting one needs *forces* of chemistry.

(b) Atoms are *indecomposable* with respect to such forces.

(c) There is a list of atoms ordered by the number of *protons* inside an atom. The atom with exactly one proton is the hydrogen atom denoted by H.

table 1

In topology "finite spaces" have also a kind of molecular structure. In fact, we introduce the notion of an "atom" in topology and we show that finite spaces have the properties in table 2 analogous to properties of molecules in table 1.

Atoms in Topology

(A) Each *finite space* splits via suspension into atoms; here *suspension* is a kind of force which allows the splitting.

(B) Atoms are *indecomposable*.

(C) We describe a list of atoms of low dimension ordered by the number of *cells* inside an atom. The atom with exactly one cell is the 1–sphere denoted by S^1.

table 2

Such properties of finite spaces are fundamental results which can be explained easily by use of a few elementary notions of topology.

Some people in physics now say that the dimension of the real world is not 3 or 4 but 11. As a coincidence it turns out that in topology dimension 11 is the highest dimension for which the number of torsion free atoms is finite. In fact, there exist exactly 394 torsion free atoms of dimension ≤ 11. Such atoms have at most 6 cells. In dimension > 11 the number of torsion free atoms is infinite.

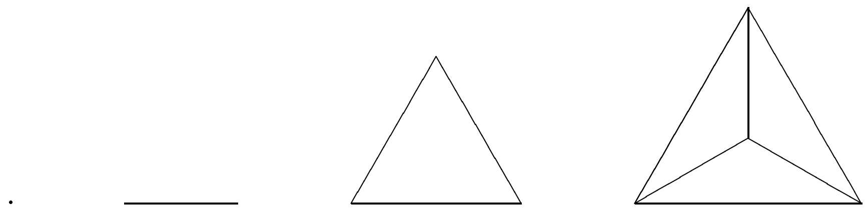
The determination of atoms in topology leads to two outstanding problems of Mathematics: On the one hand the *computation of stable homotopy groups of spheres* which is equivalent to the computation of a list of atoms with 2 cells, and on the other hand the *universal problem of representation theory* (see table 3) which is equivalent to the determination of a complete list of atoms of dimension 9.

1 Finite spaces

Two spaces X and Y are *homeomorphic* if there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that the composites $fg = 1_Y$ and $gf = 1_X$ are the identity maps. A class of homeomorphic spaces is called a *homeomorphism type.* We identify homeomorphic spaces and we write $X \approx Y$ if there is a homeomorphism $X \to Y$.

Homotopy types are equivalence classes of spaces which are considerably larger than homeomorphism types. To this end we use the notion of deformation or homotopy. Two maps $f, g : X \to Y$ are *homotopic*, $f \simeq g$, if there is a family of maps $f_t : X \to Y, 0 \leq t \leq 1$, with $f_0 = f, f_1 = g$ such that the map $(x, t) \mapsto f_t(x)$ is continuous as a function of two variables. The principal idea of the notion of homotopy type is to consider "nearby" objects (that is, objects, which are "deformed" or "perturbed" continuously a little bit) as being similar. This idea of perturbation is a common one in Mathematics and Science; properties which remain valid under small perturbations are considered to be the stable and essential features of an object. The equivalence relation generated by "slight continuous perturbations" has its precise definition by the notion of homotopy equivalence: Two spaces X and Y are *homotopy equivalent*, $X \simeq Y$, if there are continuous maps $f : X \to Y$ and $g : Y \to X$ such that the composites fg and gf are homotopic to the identities, $fg \simeq 1_Y$ and $gf \simeq 1_X$. A class of homotopy equivalent spaces is called a *homotopy type.*

For each number $n = 0, 1, 2, \ldots$ one has the *simplex* Δ^n which is the convex hull of the basic unit vectors $e_0, e_1, \ldots, e_n$ regarded as points in the Euclidean $(n+1)$–space $\mathbb{R}^{n+1}$. Hence Δ^0 is a point, Δ^1 an interval, Δ^2 a triangle, Δ^3 a tetrahedron, and so on:



Δ^0 Δ^1 Δ^2 Δ^3

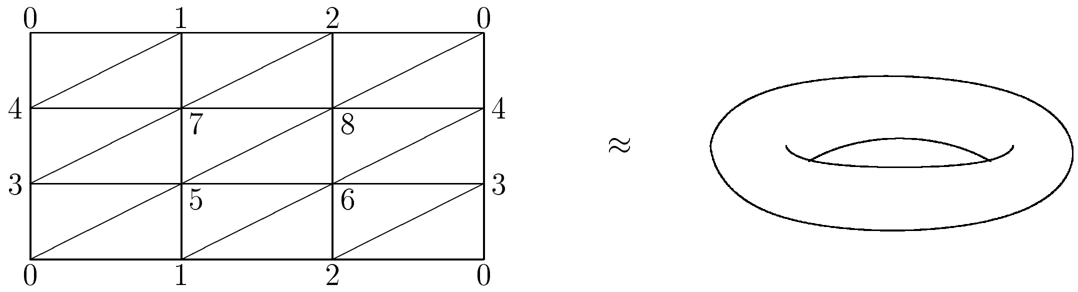
The dimension of Δ^n is n. The name "simplex" describes an object which is supposed to be very simple: Yet once the simplex was defined, the universe of polyhedra had to emerge:

For each subset $a \subset \{0, 1, \ldots, n\}$ with $a = \{a_0 < \ldots < a_r\}$ one has the r–dimensional *face* $\Delta_a \subset \Delta^n$ which is the convex hull of the set of vertices $e_{a_0}, \ldots, e_{a_r}$. Hence the set of all subsets of the set $\{0, 1, \ldots, n\}$ can be identified with the set of all faces of the simplex Δ^n. There are "substructures" S of a simplex obtained by the union of several faces, that is,

$$S = \Delta_{a^1} \cup \Delta_{a^2} \cup \ldots \cup \Delta_{a^k} \subset \Delta^n \text{ with } a^1, \ldots, a^k \subset \{0, \ldots, n\}.$$

Finite polyhedra are topological spaces X homeomorphic to such substructures S of simplexes $\Delta^n, n \geq 0$. A homeomorphism $S \approx X$ is called a *triangulation* of X. Hence a polyhedron X is a space in which we do not see any simplexes. We can introduce simplexes in X via triangulation.

For example the *torus* is the product $S^1 \times S^1$ of two 1–spheres. A triangulation of the torus is obtained as in the following picture:



Here the corresponding subsets $a^i \subset \{0, \ldots, 8\}$ are defined by triangles. In the sketch we use the fact that the torus $S^1 \times S^1$ is obtained from the square by identifying sides in the boundary; for example the edge connecting vertices 0 and 3 on the left hand side is identified with the corresponding edge on the right hand side.

The initial problem of topology (Seifert and Threlfall [ST] called it the main problem) is the classification of homotopy types of finite polyhedra. A great deal of J. H. C. Whitehead's work focused on this problem. Compare the article of Hilton–James [HJ].

(1.1) Definition. A space X is *finite* if there exists a finite polyhedron homotopy equivalent to X.

The *dimension* of a polyhedron X is the maximal dimension of a simplex which can appear in a triangulation of X. The *homotopy dimension* $\mathsf{dim}(X)$ of a finite space X is the minimal dimension of a polyhedron homotopy equivalent to X. Hence the homotopy dimension of a finite space X is an invariant of the homotopy type of X.

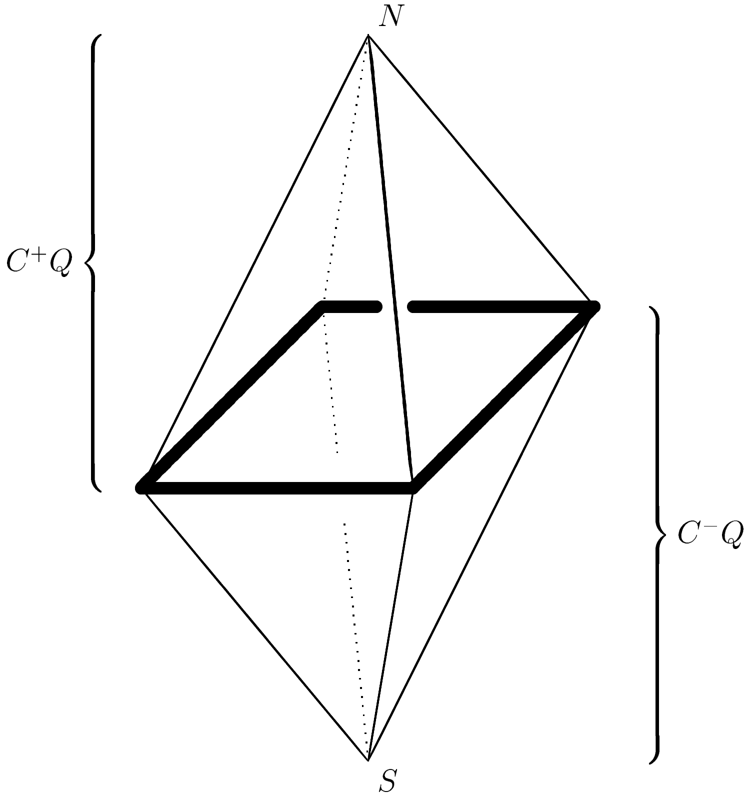
The *n–sphere* S^n is the set of all points x in $\mathbb{R}^{n+1}$ with $||x|| = 1$ and the *$(n+1)$–ball* B^{n+1} is the set of all $x \in \mathbb{R}^{n+1}$ with $||x|| \leq 1$. The sphere S^n is the boundary ∂B^{n+1} of B^{n+1}. A space X is $(n-1)$*–connected* if each continuous map $S^i \to X$ with $i \leq n-1$ admits a continuous extension $B^{i+1} \to X$. We call $n-1 = \mathsf{conn}(X)$ the *connectivity* of X if X is $(n-1)$–connected but not n–connected.

The complexity of a homotopy type in general is largely measured by the dimension and, for a given dimension, the complexity diminishes as the connectivity increases.

A space X is *contractible* if X is homotopy equivalent to a point. For a contractible space X we have $\mathsf{conn}(X) = \infty$ since X is n–connected for all n and $\mathsf{dim}(X) = 0$. All balls B^n are contractible. The torus $S^1 \times S^1$ is 0–connected but not 1–connected so that $\mathsf{conn}(S^1 \times S^1) = 0$.

We need the notion of suspension originally termed "Einhängung" by Freudenthal [F]. Given a space X we define the *suspension* ΣX of X as follows. For $X \subset \mathbb{R}^n$ we choose points N and S in $\mathbb{R}^{n+1}$, respectively above and below $\mathbb{R}^n$ identified with the space of points

x in $\mathbb{R}^{n+1}$ with $x_{n+1} = 0$. Then ΣX consists of all points $tN + (1-t)x$ and $tS + (1-t)x$ with $0 \le t \le 1$ and $x \in X$. The suspension ΣQ of a square $Q = \Box$ is sketched by the double–pyramid



Each suspension consists of the union of two cones C^+X and C^-X which are the parts of ΣX above $\mathbb{R}^n$ and below $\mathbb{R}^n$ respectively. Both cones C^+X and C^-X are contractible but ΣX in general is not contractible.

If X is a polyhedron then so is ΣX. For a map $f : X \to Y$ between spaces we have the obvious extension $\Sigma X \to \Sigma Y$ carrying N to N and S to S. Given f the suspension Σf is often denoted by $f : \Sigma X \to \Sigma Y$.

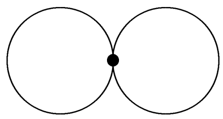
We also use the iterated suspension $\Sigma^n X$ with $\Sigma(\Sigma^n X) = \Sigma^{n+1} X$. We have the well known formulas

$$\begin{aligned} \mathsf{dim}(\Sigma^n X) &\le n + \mathsf{dim}(X), \\ \mathsf{conn}(\Sigma^n X) &\ge n + \mathsf{conn}(X). \end{aligned} \tag{1.2}$$

Here equality holds if X is 1–connected. For the spheres S^n one gets $S^{n+1} \approx \Sigma S^n$ and $\mathsf{dim}(S^n) = n$ and $\mathsf{conn}(S^n) = n + 1$. Moreover for the ball B^n one has a homeomorphism $B^{n+1} \approx \Sigma B^n$.

2 Atoms

We consider the *decomposition* of spaces. The *one point union* $A \vee B$ of two spaces A and B is obtained by choosing points $a_0 \in A$ and $b_0 \in B$ and by identifying a_0 and b_0; i.e. by gluing the point a_0 to the point b_0. The following picture shows the one point union $S^1 \vee S^1$ of two 1–spheres S^1



We have inclusions $i_1 : A \to A \vee B$ and $i_2 : B \to A \vee B$. If A and B are 0–connected polyhedra then the homotopy type of $A \vee B$ does not depend on the choice of the points $a_0 \in A$ and $b_0 \in B$. The space X is *decomposable* if X is homotopy equivalent to a one point union $A \vee B$ where A and B are non–contractible; otherwise X is *indecomposable*. We also call $X \simeq A \vee B$ a *splitting* of X.

For example the sphere $S^n, n \geq 1$, is indecomposable. We are now ready to introduce the notion of an atom in topology.

(2.1) Definition. An *atom* is a finite space X which is indecomposable and satisfies

$$\mathsf{dim}(X) = 2\mathsf{conn}(X) + 1.$$

Each atom X yields the sequence of *suspended atoms* $X, \Sigma X, \Sigma^2 X, \ldots, \Sigma^n X, \ldots$ [1].

The 1–sphere S^1 is an atom. The sphere S^n with $n \geq 2$ is not an atom but a suspended atom $S^n = \Sigma^{n-1} S^1$. The next lemma yields property B in table 2.

(2.2) Lemma. *All suspended atoms are indecomposable. If suspended atoms $\Sigma^k X \simeq \Sigma^k Y$ are homotopy equivalent then the atoms $X \simeq Y$ are homotopy equivalent.*

Hence an atom X is indecomposable in the strong sense that X is indecomposable and also all iterated suspensions of X are indecomposable. In addition each iterated suspension of X determines the homotopy type of X. Such a property does not hold for an indecomposable space with $\mathsf{dim}(X) > 2\mathsf{conn}(X) + 1$. For example the product $S^n \times S^n$ of two spheres S^n has $\mathsf{conn}(S^n \times S^n) = n - 1$ and $\mathsf{dim}(S^n \times S^n) = 2n$. Such a product is indecomposable as a space but this property is not preserved by suspension since there is a homotopy equivalence

$$\Sigma(S^n \times S^n) \simeq S^{n+1} \vee S^{n+1} \vee S^{2n+1}$$

showing that $\Sigma(S^n \times S^n)$ is decomposable.

(2.3) Proposition. *A finite space Y with*

$$\mathsf{dim}(Y) \leq 2\mathsf{conn}(Y) + 1 \qquad (*)$$

is homotopy equivalent to a finite one point union $Y_1 \vee \ldots \vee Y_r$ of suspended atoms Y_i with

$$\mathsf{conn}(Y) \leq \mathsf{conn}(Y_i) < \mathsf{dim}(Y_i) \leq \mathsf{dim}(Y)$$

for $i = 1, \ldots, r$. (Here the empty one point union with $r = 0$ denotes a point.)

The next corollary yields for each finite space X atoms $A_1, \ldots, A_r$ *associated to* X. This corresponds to property (A) in table 2.

[1] Such sequences of suspended spaces lead to the notion of *"spectrum"* in stable homotopy theory.

(2.4) Corollary. *Let X be any finite space. Then there are atoms $A_1, \dots, A_r$ such that for appropriate natural numbers $k, k_1, \dots, k_r \geq 0$ one has a homotopy equivalence*

$$\Sigma^k X \simeq \Sigma^{k_1} A_1 \vee \ldots \vee \Sigma^{k_r} A_r$$

In fact, we can choose $k \geq \mathsf{dim}(X) - 2\mathsf{conn}(X) - 1$. Then (1.2) shows that $Y = \Sigma^k X$ satisfies (*) in (2.3) so that the suspended atom $Y_i = \Sigma^{k_i} A_i$ in (2.3) determines A_i and k_i by (2.2).

The results above show that atoms as defined in (2.1) are indeed distinguished spaces. According to (2.3) a list of atoms yields a classification of all homotopy types of finite spaces X with $\mathsf{dim}(X) \leq 2\mathsf{conn}(X)+1$. We describe such a list as follows. Recall that the dimension of an atom is always an odd number.

(2.5) Theorem.

- *The 1–sphere S^1 is an atom and this is the only atom of dimension 1.*[2]
- *For each prime power $q = p^j, j \geq 1$, one has the* Moore atom $M_q = M(\mathbb{Z}/q, 2)$ *and this is a complete list of atoms of dimension 3.*
- *The* Chang atoms *form a complete list of atoms of dimension 5. [Ch, B1, B2, H1, H2].*
- *The* Baues–Hennes atoms *form a complete list of atoms of dimension 7. [BH, B1, B2].*
- *The computation of all atoms of dimension 9 requires the solution of the* universal problem of representation theory *[BD4]; see table 3.*

We shall describe the cell structure of the atoms in section 4 below.

The universal problem of representation theory

For $n = 1, 2, \dots$ we consider $n \times n$–matrices A with entries in $\mathbb{R}$ (or in a field $\mathbb{K}$, for example the field $\mathbb{K} = \mathbb{Z}/2$ consisting of two elements). Given an $m \times m$–matrix B we obtain the $(n+m) \times (n+m)$–matrix

$$A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

We say that a pair (D_1, D_2) of $k \times k$–matrices D_1, D_2 is *decomposable* if there exists a pair (A_1, A_2) of $n \times n$–matrices and a pair (B_1, B_2) of $m \times m$–matrices with $n + m = k$ such that (D_1, D_2) is equivalent to $(A_1 \oplus B_1, A_2 \oplus B_2)$, that is

$$D_1 = U(A_1 \oplus B_1)U^{-1}$$
$$D_2 = U(A_2 \oplus B_2)U^{-1}$$

where U is an invertible $k \times k$-matrix. Otherwise (D_1, D_2) is *indecomposable.* The computation of a complete list of equivalence classes of indecomposable pairs of matrices (D_1, D_2) is the *universal* (or *wild*) problem or representation theory.

table 3

[2] Here S^1 is an atom as a space and its uniqueness refers to an atom as a homotopy type.

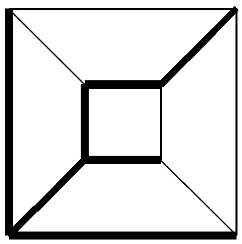
According to proposition (2.3) we have the following applications of theorem (2.5) on the classification of homotpy types.

(2.6) Corollary. *A 0–connected 1–dimensional finite space is homotopy equivalent to a finite one point union of 1–spheres* S^1.

This result is classical and has the following well known proof. Let X be a 0–connected 1–dimensional space so that X is homotopy equivalent to a connected graph Y. We can choose in Y a maximal tree T and by contracting the tree T to a point we obtain a homotopy equivalence

$$X \simeq Y \simeq S^1 \vee \ldots \vee S^1$$

between X and a one point union of 1–spheres. For example the edges of a cube form a graph which is homotopy equivalent to a one point union of five 1–spheres:



The bold edges in the sketch describe a maximal tree.

Property (A) of atoms in table 2 describes a higher dimensional analogue of the old result (2.6) on connected graphs. As a next application of theorem (2.5) we get the following well known fact.

(2.7) Corollary. *A 1–connected 3–dimensional finite space is homotopy equivalent to a finite one point union of spheres* S^2, S^3 *and Moore atoms.*

This result can be used to show that a 1–connected 3–dimensional manifold M is homotopy equivalent to a 3–sphere (the Poincaré conjecture claims that M is actually homeomorphic to the 3–sphere). Moreover corollary (2.7) is crucial for the classification of 1–connected 5–dimensional manifolds N (see Barden [Ba] and Stoecker [St]) since the complement of a point in N is a 1–connected 3–dimensional finite space.

The next application of theorem (2.5) is originally due to J. H. C. Whitehead [W1] and Chang [Ch].

(2.8) Corollary. *A 2–connected 5–dimensional finite space is homotopy equivalent to a finite one point union of spheres* S^3, S^4, S^5*, suspended Moore atoms, and Chang atoms.*

Moreover the next corollary of theorem (2.5) is proved in [BH].

(2.9) Corollary. *A 3–connected 7–dimensional finite space is homotopy equivalent to a finite one point union of spheres* S^4, S^5, S^6, S^7*, suspended Moore atoms, suspended Chang atoms and Baues–Hennes atoms.*

The complexity of atoms of dimension 9 involves the universal problem of representation theory in table 3. Therefore we cannot classify atoms of dimension ≥ 9 in general. We can, however, look at subclasses of atoms satisfying for example certain "homological" conditions.

In 1895 Poincaré introduced the *homology* $H_n(X)$ of a polyhedron X which is an abelian group depending only on the homotopy type of X. Moreover this group is compatible with suspension, that is $H_n(X) = H_{n+1}(\Sigma X)$. In 1932 P. Alexandroff [A] wrote an introductory essay “Einfachste Grundbegriffe der Topologie” on polyhedra and homology. Nowadays homology is one of the most important mathematical concepts.

The Moore atom M_q can be characterized by homology groups since M_q up to homotopy equivalence is the unique finite space which is 1–connected and satisfies $H_2(M_q) = \mathbb{Z}/q\mathbb{Z}$ and $H_n(M_q) = 0$ for $n \neq 2$.

(2.10) Definition. We say that a finite space X (or an atom X) is *torsion free* if all homology groups $H^n(X) \cong \mathbb{Z}^{\beta_n}$ are finitely generated free abelian groups, $n \geq 0$. Here the rank β_n is the n–th *Betti number* of X.

It is not possible to compute all atoms of dimension 9. If we restrict, however, to torsion free atoms we are able to classify them up to dimension 11, see [BD1, BD2, BD3].

(2.11) Theorem. *For $n \leq 13$ the number $A(n)$ of all torsion free atoms of dimension n is given by the following table 4.*

n	1	3	5	7	9	11	13
$A(n)$	1	0	1	1	67	324	infinite

table 4

Moreover the computation of all torsion free atoms of dimension 21 requires the solution of the universal problem of representation theory, see table 3.

We shall describe the *Baues–Drozd atoms* of theorem (2.11) more explicitly in table 10 below. Theorem (2.11) shows that up to dimension 11 there exists only a finite number of torsion free atoms and this number is 394. In dimension 13, however, there is an infinite number of such atoms. It still might be possible to compute all torsion free atoms of dimension 13, 15, 17 and 19. As an application of theorem (2.11) and proposition (2.3) we get the corollary:

(2.12) Corollary. *For $n \leq 6$ each $(n-1)$–connected $(2n-1)$–dimensional torsion free finite space is homotopy equivalent to a finite one point union of suspended Baues–Drozd atoms of dimension $\leq 2n-1$ and connectivity $\geq n-1$. Such suspended atoms form a finite list which is explicitly computed. See table 10 below.*

Using real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$ and quaterionic numbers $\mathbb{H}$ one obtains the according *projective planes* denoted by $\mathbb{R}P_2, \mathbb{C}P_2$ and $\mathbb{H}P_2$ respectively. The suspension of these planes yields examples of atoms. In fact $\Sigma\mathbb{R}P_2 = M_2$ is a Moore–atom, $\Sigma\mathbb{C}P_2$ is a Chang atom, and $\Sigma\mathbb{H}P_2$ is a Baues–Drozd atom of dimension 9.

3 Cell attachments

For a space X and a map $f : S^{m-1} \to X$ we obtain the cell attachment

$$X \cup_f e^m = (X \cup B^m)/_{y \sim f(y)} \tag{3.1}$$

by identifying points $y \in S^{m-1} \subset B^m$ with $f(y) \in X$. We call the subspace $e^m = B^m - S^{m-1} \subset X \cup_f e^m$ a *cell* and the image $f(S^{m-1})$ in X the *boundary* of the cell e^m. We shall consider spaces

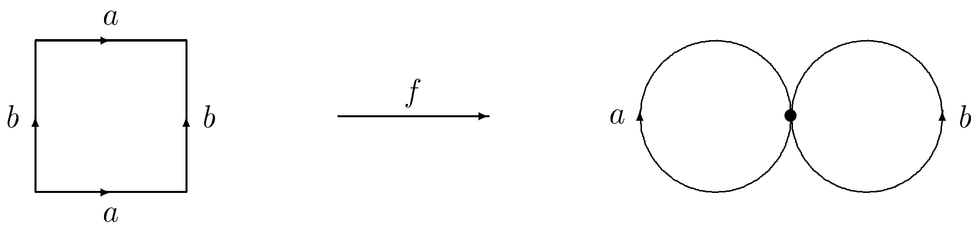
$$Y = (S^{n_1} \vee \ldots \vee S^{n_k}) \cup e^{m_1} \cup \ldots \cup e^{m_r} \tag{3.2}$$

obtained from a one point union $S^{n_1} \vee \ldots \vee S^{n_k}$ of spheres by inductive cell attachments where $n_1, \ldots, n_k, m_1, \ldots, m_r \geq 1$. The sphere S^n is considered to be a space with one n–cell obtained by the cell attachment $* \cup e^n$. Here $*$ is the space consisting of a point. Therefore the space Y in (3.2) has $k + r$ cells. Each finite *cell complex* Y as defined in (3.2) is a finite space in the sense of definition (1.1).

It is a lot more economical to use cell complexes instead of polyhedra. For example the torus $S^1 \times S^1$ can be described by the cell attachment

$$S^1 \times S^1 \approx (S^1 \vee S^1) \cup_f e^2$$

where the attaching map $f : S^1 \to S^1 \vee S^1$ is the commutator map sketched by



Hence the torus has 3 cells while the triangulation of the torus in section 1 uses 30 1–simplexes and 18 2–simplexes.

The cell attachment for a map $\alpha : S^{m-1} \to S^n$ beetween spheres yields the cell–complex

$$S^n \cup_\alpha e^m. \tag{3.3}$$

If α is homotopic to a map $\beta : S^{m-1} \to S^n$ then there is a homotopy equivalence $S^n \cup_\alpha e^m \simeq S^n \cup_\beta e^m$. The homotopy type of $S^n \cup_\alpha e^m$ is therefore defined by the element $\alpha \in \pi_{m-1}(S^n)$ where $\pi_{m-1}(S^n)$ is the set of homotopy classes[3] of maps $S^{m-1} \to S^n$. This set which canonically has the structure of an abelian group is termed a *homotopy group of a sphere.*

For example $\pi_1(S^1) = \mathbb{Z}$. Here the map $q : S^1 \to S^1$ with $q \in \mathbb{Z}$ carries $x \in S^1 \subset \mathbb{C}$ to the q–th power $x^q = x \cdot \ldots \cdot x \in S^1$ where we use the multiplication of complex numbers.[4] The suspension $\Sigma^{n-1} q = q : S^n \to S^n$ is the map of Brouwer degree q [Br].

The suspension Σ yields a homomorphism

$$\Sigma : \pi_{n+k}(S^n) \to \pi_{n+k+1}(S^{n+1}) \tag{3.4}$$

which by the Freudenthal suspension theorem is an isomorphism if $k \leq n - 2$. Therefore the group $\pi_{2n-2}(S^n)$ for $n \geq 2$ is termed a *stable* homotopy group of spheres. Table 5 shows a list of homotopy groups $\pi_{n+k}(S^n)$ for $k \leq 9$ and $n \leq 11$ as computed by Toda [T]. In this list the diagonal indicates the stable groups $\pi_{2n-2}(S^n)$.

[3]The *homotopy class* of a map f is the set of all maps g which admit a homotopy $g \simeq f$.

table 5

Homotopy classes of maps between spheres, $S^{n+k} \to S^n$

$\pi_{n+k}(S^n)$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$	$n=11$
$k=0$	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
$k=1$	∞	2	2	2	2	2	2	2	2	2
$k=2$	2	2	2	2	2	2	2	2	2	2
$k=3$	2	12	$\infty + 12$	24	24	24	24	24	24	24
$k=4$	12	2	$(2)^2$	2	0	0	0	0	0	0
$k=5$	2	2	$(2)^2$	2	∞	0	0	0	0	0
$k=6$	2	3	$24+3$	2	2	2	2	2	2	2
$k=7$	3	15	15	30	60	120	$\infty + 120$	240	240	240
$k=8$	15	2	2	2	$24+2$	$(2)^3$	$(2)^4$	$(2)^3$	$(2)^2$	$(2)^2$
$k=9$	2	$(2)^2$	$(2)^3$	$(2)^3$	$(2)^3$	$(2)^4$	$(2)^5$	$(2)^4$	$\infty + (2)^3$	$(2)^3$

For a prime power $q = p^i \in \pi_2 S^2 = \mathbb{Z}$ the cell complex $M_q = S^2 \cup_q e^2$ is the Moore atom in theorem (2.5). Moreover for $n \geq 3$ each non trivial map α in $\pi_{2n-2}(S^n)$ yields a torsion free atom $S^n \cup_\alpha e^{2n-1}$ with 2–cells.

Hence the computation of all atoms with 2–cells requires the computation of stable homotopy groups of spheres. It remains one of the greatest challenges of Mathematics to compute and determine the structure of these homotopy groups. Serre [S] showed the surprising result that $\pi_{2n-2}(S^n)$ is finite for all $n \geq 3$.

The distinguished nature of elements in homotopy groups of spheres is exemplified by the classical Hopf maps. The Hopf maps $2 : S^1 \to S^1, \eta : S^3 \to S^3$ and $\nu : S^7 \to S^4$ are attaching maps for the projective planes

$$\begin{aligned} \mathbb{R}P_2 &= S^1 \cup_2 e^2 \\ \mathbb{C}P_2 &= S^2 \cup_\eta e^4 \\ \mathbb{H}P_2 &= S^4 \cup_\nu e^8 \end{aligned}$$

The suspension of these projective planes are atoms as mentioned above. Here the suspension of $S^n \cup_\alpha e^m$ satisfies $\Sigma(S^n \cup_\alpha e^m) = S^{n+1} \cup_{\Sigma\alpha} e^{m+1}$ where $\Sigma\alpha$ is the suspension of the map α. The suspensions $\Sigma\eta$ and $\Sigma\nu$ generate the the stable groups $\pi_4 S^3 = \mathbb{Z}/2\mathbb{Z}$ and $\pi_8 S^4 = \mathbb{Z}/24\mathbb{Z}$ respectively.

In order to define the *Hopf map* $\eta : S^3 \to S^2$ (see [Ho]) we use the multiplication map

$$\mu : S^1 \times S^1 \to S^1$$

which carries (x, y) with $x, y \in S^1 \subset \mathbb{C}$ to the product $\mu(x, y) = x \cdot y$ of complex numbers.[4] For $0 \leq \lambda \leq 1$ and $x \in S^1$ we get $\lambda x \in B^2$. Now the Hopf map η is the composite

$$\eta : S^3 = \partial(B^2 \times B^2) = B^2 \times S^1 \cup S^1 \times B^2 \xrightarrow{\overline{\eta}} \Sigma S^1 = S^2$$

where $\overline{\eta}$ carries $(\lambda x, y)$ to $(1-\lambda)N + \lambda(x \cdot y) \in \Sigma S^1$ and carries $(x, \lambda y)$ to $(1-\lambda)S + \lambda(x \cdot y) \in \Sigma S^1$. In table 6 we show a perspective view of the Hopf map due to W. A. Dreckmann. This view describes for various points $x \in S^2$ the fibers $\eta^{-1}(x) \subset S^3 = * \cup e^3 \approx * \cup \mathbb{R}^3$.

The quaterionic *Hopf map* $\nu : S^7 \to S^4$ is defined by the multiplication map

$$\mu : S^3 \times S^3 \to S^3$$

which carries (x, y) with $x, y \in S^3 \subset \mathbb{H} = \mathbb{R}^4$ to the product $\mu(x, y) = x \cdot y$ of quaterionic

[4] Each element in $\mathbb{C} = \mathbb{R}^2$ can be written as $(x_0, x_1) = x_0 \cdot 1 + x_1 \cdot i$ and two such elements are multiplied according to the table

	1	i
1	1	i
i	i	-1

numbers.[5] We obtain ν by the composite

$$\nu : S^7 = \partial(B^4 \times B^4) = B^4 \times S^3 \cup S^3 \times B^4 \xrightarrow{\overline{\nu}} \Sigma S^3 = S^4$$

where $\overline{\nu}$ is defined in terms of μ similarly as $\overline{\eta}$ above.

[5] Each element in $\mathbb{H} = \mathbb{R}^4$ can be written as $x = (x_0, x_1, x_2, x_3) = x_0 \cdot 1 + x_1 \cdot i + x_2 \cdot j + x_3 \cdot k$ and two such elements are multiplied according to the table

	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

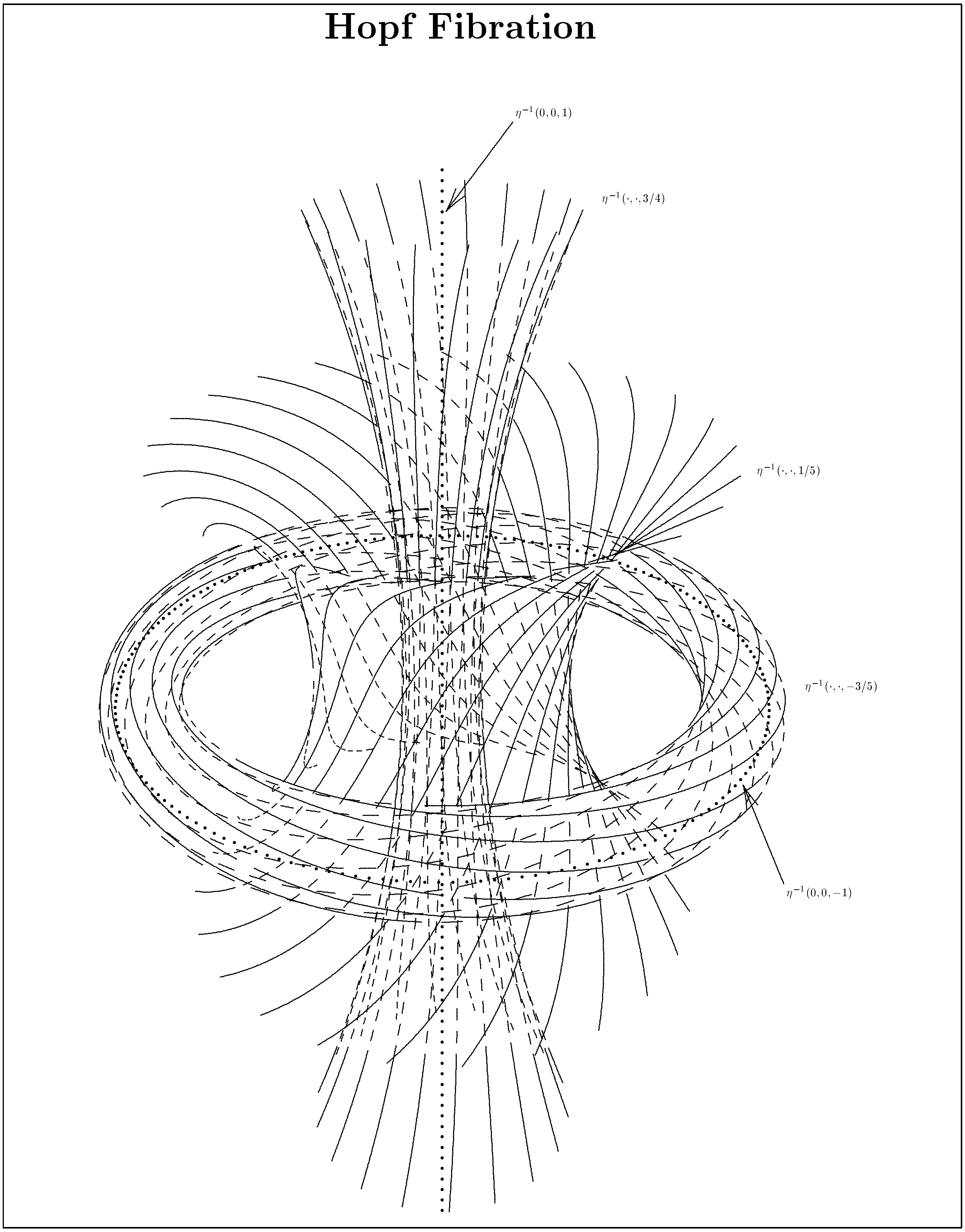

Hopf Fibration
$\eta^{-1}(0,0,1)$
$\eta^{-1}(\cdot,\cdot,3/4)$
$\eta^{-1}(\cdot,\cdot,1/5)$
$\eta^{-1}(\cdot,\cdot,-3/5)$
$\eta^{-1}(0,0,-1)$

table 6

4 The cell structure of atoms

The cell structure of atoms of low dimension can be described by certain connected graphs with vertices of level $n = 1, 2, \ldots$. Such a vertex of level n describes a cell of dimension n (which is a sphere S^n if the vertex is not connected by an edge to a vertex of lower level). The edges characterize the attaching maps of cells.

For example atoms of dimension ≤ 5 are described by the graphs in table 7. The Chang atom corresponding to the graph at the right hand side of table 7 is accordingly a cell complex of the form $S^3 \vee S^4 \cup e^4 \cup e^5$ with 4 cells.

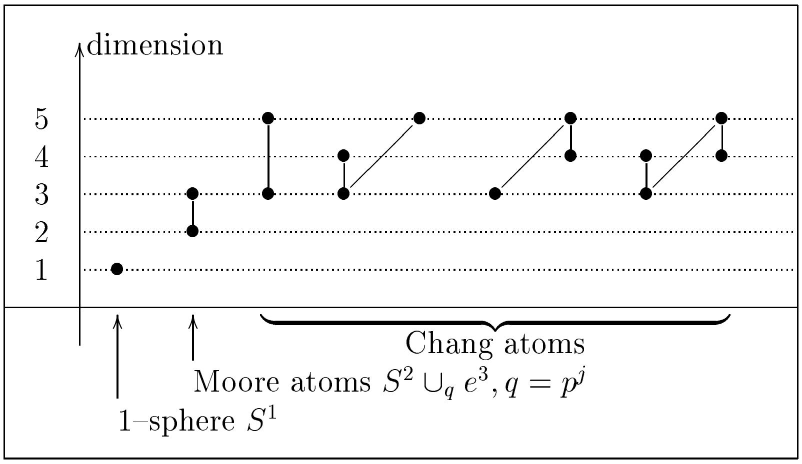


table 7

The Baues–Hennes atoms of dimension 7 can be described by connected finite subgraphs of maximal height in the infinite graphs of table 8.

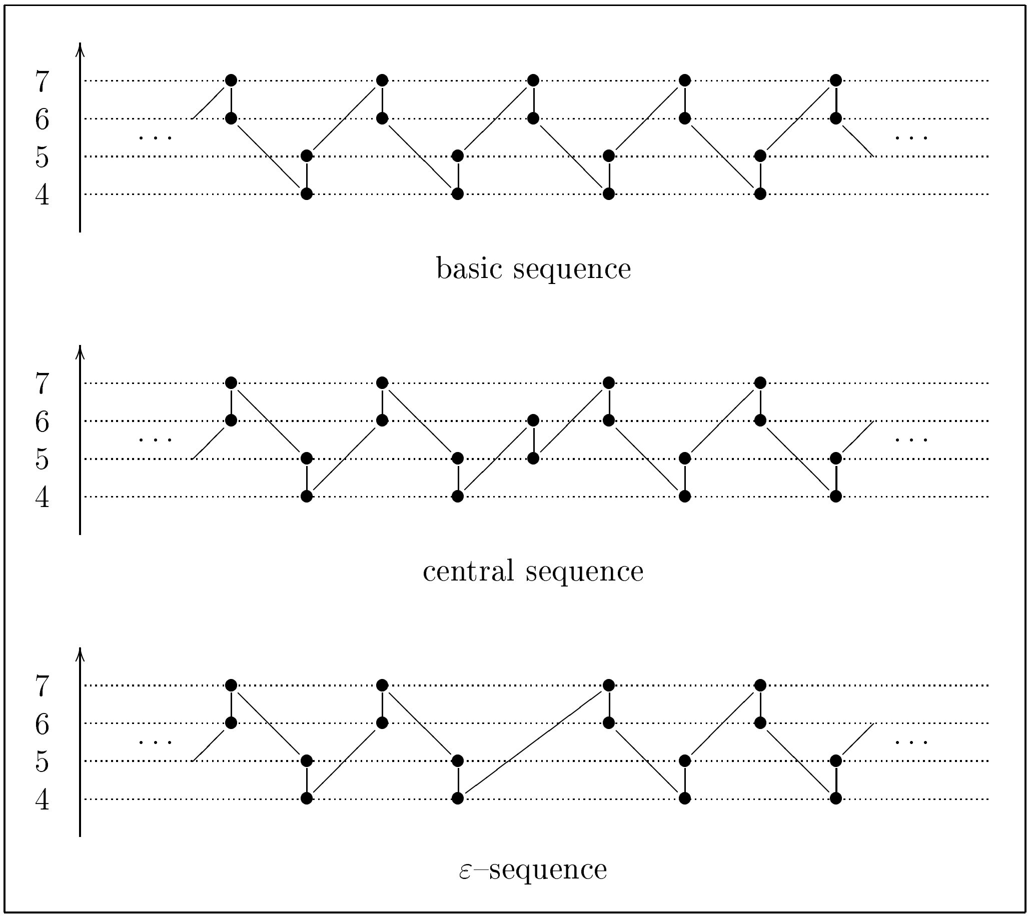


table 8

For example a complete list of all atoms of dimension 7 for which homology groups are cyclic torsion groups is described by the graphs in table 9.

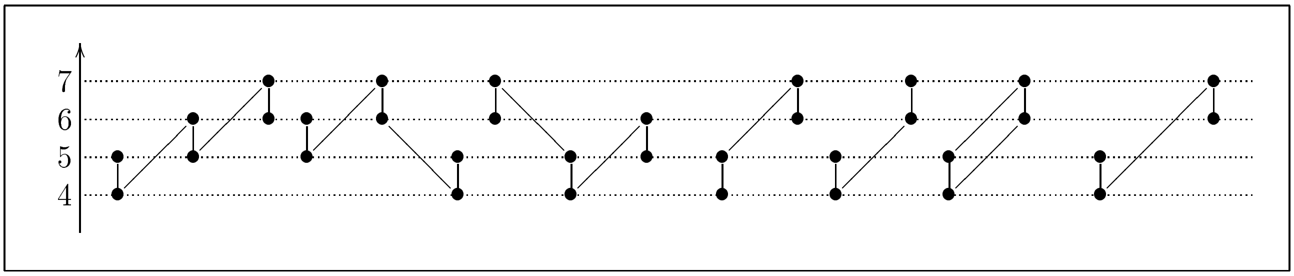
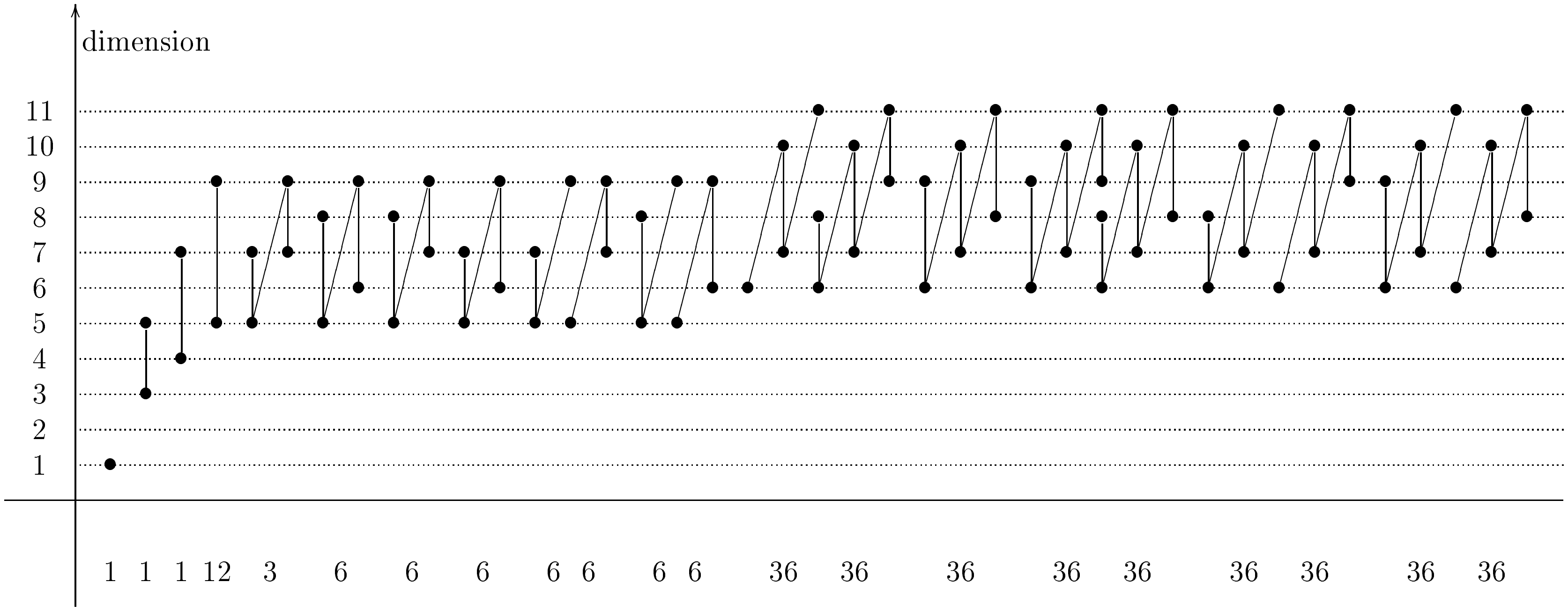


table 9

Finally the Baues–Drozd atoms in table 10 form a complete list of torsion free atoms of dimension ≤ 11. Such atoms have at most 6 cells. A number in the bottom line of table 10 indicates the number of possible realizations of the corresponding graph as a cell–complex according to the choice of attaching maps. For example, a cell complex given by the graph at the right hand side of table 10 has the form $S^6 \vee S^7 \vee S^8 \cup e^{10} \cup e^{11}$ and there are 36 possibilities to choose attaching maps. In a similar way one gets the cell structure of all atoms defined by the connected graphs in table 10.

table 10

The torsion free Atoms of Topology



In order to achieve results on the classification of atoms as described in theorem (2.5) and theorem (2.12) one first has to translate the purely topological problem into an algebraic problem. For this we use methods of Algebraic Topology as developed in the book [B1]; see also [U1], [U2] and [BG]. Then the highly sophisticated algebraic problem has to be transformed into an equivalent matrix problem which can be solved by methods of Representation Theory. For example the computation of torsion free atoms of dimension 11 in [BD3] leads to a ring R determined by the 21×21–matrix (M_{ij}) in table 11. The ring R consists of all 21×21–matrices (A_{ij}) for which $A_{ij} \in \mathbb{Z}$ is divisible by M_{ij}. The classification of indecomposable matrices over this ring R yields the result on 11-dimensional torsion free atoms in table 10.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	12	12
2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	12	6
3	1	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	12	12
4	1	2	2	1	2	1	2	2	2	1	2	1	2	1	2	1	2	2	2	12	12
5	1	2	2	1	1	1	2	2	2	1	1	1	2	1	1	1	2	2	2	12	12
6	1	2	2	2	2	1	2	2	2	2	2	1	2	2	2	1	2	2	2	12	12
7	1	2	2	1	1	1	1	2	1	1	1	1	2	1	1	1	1	2	1	12	12
8	1	2	2	1	1	1	1	1	1	1	1	1	2	1	1	1	1	1	1	12	12
9	1	2	2	1	1	1	2	2	1	1	1	1	2	1	1	1	2	2	1	12	12
10	1	2	2	1	2	1	2	2	2	1	2	1	2	1	2	1	2	2	2	12	12
11	1	2	2	1	1	1	2	2	2	1	1	1	2	1	1	1	2	2	2	12	12
12	1	2	2	2	2	1	2	2	2	2	2	1	2	2	2	1	2	2	2	12	12
13	1	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	12	12
14	1	2	2	1	2	1	2	2	2	1	2	1	2	1	2	1	2	2	2	12	12
15	1	2	2	1	1	1	2	2	2	1	1	1	2	1	1	1	2	2	2	12	12
16	1	2	2	2	2	1	2	2	2	2	2	1	2	2	2	1	2	2	2	12	12
17	1	2	2	1	1	1	1	2	1	1	1	1	2	1	1	1	1	2	1	12	12
18	1	2	2	1	1	1	1	1	1	1	1	1	2	1	1	1	1	1	1	12	12
19	1	2	2	1	1	1	2	2	1	1	1	1	2	1	1	1	2	2	1	12	12
20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
21	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

table 11

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