ON SOME ARITHMETIC QUANTITIES OF WEIGHTED QUASI-DIAGONAL SURFACES

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ABSTRACT. We investigate the minimal resolutions of certain weighted quasi-diagonal surfaces over finite fields. We show that their zeta-functions can be described in terms of twisted Jacobi sums. Using this information, we compute several arithmetic quantities of the minimal resolutions. Our main purpose is to calculate the Picard numbers of such surfaces over finite fields and give a formula for the orders of their Brauer groups in some special cases.

1. INTRODUCTION

Let $k = \mathbf{F}_q$ be a finite field of q elements of characteristic p (≥ 0). Fix an algebraic closure, $\overline{k}$, of k. Let $Q = (q_0, q_1, q_2, q_3)$ be a quadruplet of positive integers such that

$$p \not| q_i \ \text{ for } \ 0 \leq i \leq 3 \tag{1}$$

and

$$\gcd(q_\alpha, q_\beta, q_\gamma) = 1 \ \text{ for every } \ \{\alpha, \beta, \gamma\} \subset \{0, 1, 2, 3\}. \tag{2}$$

Let $k[x_0, x_1, x_2, x_3]$ be a polynomial algebra graded by the condition $\deg(x_i) = q_i$ for $0 \leq i \leq 3$. The projective variety $\mathbf{P}^3_k(Q) := \operatorname{Proj} k[x_0, x_1, x_2, x_3]$ is called the *weighted projective 3-space over k of type Q* (cf. [5], [8]). Throughout the paper, we assume conditions (1) and (2) (Condition (2) is not a restriction; see [5], Proposition 1.3).

Choose a positive integer, m, such that $p \not| m$ and

$$m = q_0 m_0 = q_0 + q_1 m_1 = q_2 m_2 = q_3 m_3 \tag{3}$$

for some positive integers m_i ($0 \leq i \leq 3$) with $p \not| m_i$. Put $\mathbf{c} := (c_0, c_1, c_2, c_3) \in (k^\times)^4$. Let X_k be a surface in $\mathbf{P}^3_k(Q)$ defined by the equation:

$$c_0 x_0^{m_0} + c_1 x_0 x_1^{m_1} + c_2 x_2^{m_2} + c_3 x_3^{m_3} = 0 \ \subset \ \mathbf{P}^3_k(Q). \tag{4}$$

We call X_k a *weighted quasi-diagonal surface in $\mathbb{P}^3_k(Q)$ of degree m with twist $\mathbf{c}$*. We may also call X_k a *weighted Delsarte surface with matrix*

$$\begin{bmatrix} m_0 & 0 & 0 & 0 \\ 1 & m_1 & 0 & 0 \\ 0 & 0 & m_2 & 0 \\ 0 & 0 & 0 & m_3 \end{bmatrix}$$

(cf. [16], [23]). In this paper, we use the former terminology. The latter, on the other hand, suggests that (4) should be regarded as a special case of a more general equation. In fact, our

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investigation started with more general forms of equations. We aimed at finding surfaces of which both algebraic and geometric properties can be described explicitly. Our choice of (4) is a result of such trial and error. To illustrate this point, consider the surface, X', defined by

$$c_0x_0^{m_0} + c_1x_0^a x_1^{m_1} + c_2x_2^{m_2} + c_3x_3^{m_3} = 0$$

with $a \geq 0$. As we will see in Section 3, the zeta-function of X' can be computed in terms of twisted Jacobi sums; thus, our algebraic requirement is satisfied. The geometry of it, however, is not so straightforward. In particular, X' is quasi-smooth if and only if $a = 0$ or 1 (cf. Section 2); when X' is quasi-smooth, it has only cyclic quotient singularities. If $a \geq 2$, we do not have (at this moment) a systematic way of describing and resolving the singularities of X'. Thus, we ought to choose either $a = 0$ or 1. But if $a = 0$, then X' is a weighted diagonal surface of degree m with twist $\boldsymbol{c}$. Such a surface has been studied in various articles (cf. [14], [17], [25]). Therefore we look into the case $a = 1$.

The purposes of this paper are to compute the zeta-function of the minimal resolution of X_k and to calculate some of its arithmetic quantities explicitly. We recall briefly what quantities are known to be related to the zeta-function.

Let X_k be a smooth projective surface over k. Denote by $Z(X_k, T)$ the (congruence) zeta-function of X_k with T an indeterminate. It is known ([9], [18]) that $Z(X_k, T)$ is a rational function of the form

$$Z(X_k, T) = \frac{P_1(X_k, T)P_3(X_k, T)}{(1-T)P_2(X_k, T)(1-q^2T)}$$

where $P_i(X_k, T) = \det(1 - \Phi T \mid H^2_{et}(X_{\overline{k}}, \mathbf{Q}_\ell))$ is the characteristic polynomial of the endomorphism, Φ, induced from the Frobenius automorphism of X_k and acting on the ℓ-adic ($\ell \neq p$) étale cohomology $H^2_{et}(X_{\overline{k}}, \mathbf{Q}_\ell)$. Deligne [4] has proved that $P_i(X_k, T)$ has integer coefficients and its reciprocal roots have absolute value $q^{i/2}$. In particular,

$$P_2(X_k, T) = \prod_{j=1}^{B_2}(1 - \epsilon_j qT)$$

with $|\epsilon_j| = 1$ for $1 \leq j \leq B_2 := \dim H^2_{et}(X_{\overline{k}}, \mathbf{Q}_\ell)$. Put

$$\rho'(X_k) = \#\{1 \leq j \leq B_2 \mid \epsilon_j = 1\}$$
$$\mathfrak{T}(X_k) = \{1 \leq j \leq B_2 \mid \epsilon_j \neq 1\}.$$

Then

$$P_2(X_k, T) = (1 - qT)^{\rho'(X_k)} \prod_{j \in \mathfrak{T}(X_k)} (1 - \epsilon_j qT).$$

On the other hand, let $\mathrm{NS}(X_{\overline{k}})$ be the Néron-Severi group of $X_{\overline{k}}$. Denote by $\mathrm{NS}(X_k)$ the image of $\mathrm{Pic}(X_k)$ in $\mathrm{NS}(X_{\overline{k}})$. As $\mathrm{NS}(X_{\overline{k}})$ is finitely generated over $\mathbf{Z}$, so is $\mathrm{NS}(X_k)$. The $\mathbf{Z}$-rank of $\mathrm{NS}(X_k)$ may be called the *Picard number of* X_k and written as $\rho(X_k)$; i.e. $\mathrm{NS}(X_k) \cong \mathbf{Z}^{\rho(X_k)} \oplus \mathrm{NS}(X_k)_{tor}$, where $\mathrm{NS}(X_k)_{tor}$ is the torsion subgroup of $\mathrm{NS}(X_k)$. About the Picard number, the Tate conjecture (cf. [28]) asserts that

$$\rho(X_k) = \rho'(X_k).$$

(An equivalent formulation is that the $\mathrm{Gal}(\overline{k}/k)$-invariant subspace of $H^2_{et}(X_{\overline{k}}, \mathbf{Q}_\ell(1))$ is spanned by algebraic cycles.)

Let $\mathrm{Br}(X_k)$ be the Brauer group of X_k: $\mathrm{Br}(X_k) = H^2(X_{et}, \mathbf{G}_m)$, where $\mathbf{G}_m$ denotes the sheaf of multiplicative groups on X_{et}. In [29], Artin and Tate conjecture that the order of $\mathrm{Br}(X_k)$ is finite. If we assume $p \neq 2$ and the validity of the Tate conjecture for X_k, then $\mathrm{Br}(X_k)$ is indeed finite and the order is either a square or twice a square (cf. [21], [29]). Furthermore there is a formula, which we may call the *Artin-Tate formula*, about the residue of $P_2(X_k, T)$ at $T = q^{-1}$:

$$\prod_{j \in \mathfrak{T}(X_k)} (1 - \epsilon_j) = \frac{(-1)^{\rho(X_k)-1} \# \mathrm{Br}(X_k) \mathrm{discNS}(X_k)}{q^{\alpha(X_k)} \# \mathrm{NS}_{tor}(X_k)^2}$$

where $\mathrm{discNS}(X_k)$ denotes the discriminant of $\mathrm{NS}(X_k)$ and $\alpha(X_k) = P_g - \dim H^1(X_{\overline{k}}, \mathcal{O}_X) + \dim \mathrm{PicVar}(X_{\overline{k}})$ (P_g is the geometric genus of X_k, $\mathcal{O}_X$ is the structure sheaf on $X_{\overline{k}}$, and $\mathrm{PicVar}(X_{\overline{k}})$ is the Picard variety of $X_{\overline{k}}$); see [29], Theorem 5.2 and [21], Theorem 6.1. If $\mathfrak{T}(X_k) = \emptyset$, then we assume that the left-hand side of the Artin-Tate formula is equal to 1.

In this paper, we describe the zeta-functions of the minimal resolutions of weighted quasi-diagonal surfaces in terms of twisted Jacobi sums. We then calculate the Picard numbers and orders of the Brauer groups of the minimal resolutions, using the validity of the Tate conjecture and Artin-Tate formula.

The paper is organized as follows. In Section 2, we show that a weighted quasi-diagonal surface over $\overline{k}$ has only cyclic quotient singularities of type $A_{n,\alpha}$. We determine its singular locus and find the type of each singularity. Let $\widetilde{X}_k$ be the minimal resolution of X_k. In Section 3, we describe the zeta-function of $\widetilde{X}_k$ in terms of twisted Jacobi sums. Using this property, in particular, we calculate the Betti numbers of $\widetilde{X}_k$. In Section 4, we prove that the Tate conjecture holds for $\widetilde{X}_k$. As a consequence, we give formulae for the Picard numbers of $\widetilde{X}_k$ and of $\widetilde{X}_{\overline{k}}$. In Section 5, we compute the order of the Brauer group of $\widetilde{X}_k$ in two cases where we can calculate the discriminant of the Néron-Severi group of $\widetilde{X}_k$. Our method is to use the Artin-Tate formula. In Section 6, we consider weighted quasi-diagonal surfaces which are birational to K3 surfaces. There are 85 such surfaces. We give a formula for the Picard numbers of their minimal resolutions. For several K3 surfaces, we also compute the orders of their Brauer groups.

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2. SINGULARITIES OF WEIGHTED QUASI-DIAGONAL SURFACES

In this section, we describe the singular locus of a weighted quasi-diagonal surface over $\overline{k}$.

It is known that the set of $\overline{k}$-rational points in $\mathbb{P}^3_{\overline{k}}(Q)$ can be identified with the set

$$\mathbf{P}^3_{\overline{k}}(Q) = (\mathbf{A}^4_{\overline{k}} \setminus \{O\}) / \overline{k}^{\times} \tag{5}$$

where $\overline{k}^{\times}$ acts on $\mathbf{A}^4_{\overline{k}} \setminus \{O\}$ by

$$t \cdot (x_0, \cdots, x_3) = (t^{q_0} x_0, \cdots, t^{q_3} x_3) \tag{6}$$

if $t \in \overline{k}^{\times}$ and $(x_0, \cdots, x_3) \in \mathbf{A}^4_{\overline{k}} \setminus \{O\}$ (cf. [8], §1.2). This induces a projection

$$\iota : \mathbf{A}^4_{\overline{k}} \setminus \{O\} \longrightarrow \mathbb{P}^3_{\overline{k}}(Q). \tag{7}$$

A (weighted) projective variety, V, in $\mathbf{P}^3_{\overline{k}}(Q)$ is said to be *quasi-smooth* if the Zariski closure of $\iota^{-1}(V)$ in $\mathbf{A}^4_{\overline{k}}$ is smooth outside the origin (cf. [8]). For instance, a weighted quasi-diagonal surface defined by the equation (4) are quasi-smooth.

Remark 2.1. As we mentioned in the introduction, it is more general to consider a quasi-diagonal surface of the form

$$c_0x_0^{m_0} + c_1x_0^a x_1^{m_1} + c_2x_2^{m_2} + c_3x_3^{m_3} = 0$$

where a is a non-negative integer and m_i's are chosen so that the equation becomes homogenous. However, this surface becomes quasi-smooth if and only if $a = 0$ or 1.

It is known that quasi-smooth weighted projective surfaces have only cyclic quotient singularities of type $A_{n,\alpha}$ (cf. [2], [8]). The resolution of cyclic quotient singularities was obtained by Hirzebruch for complex surfaces (cf. [19], see also [12]). By virtue of condition (1), Hirzebruch's resolution is valid also over $\overline{k}$.

Let $X(= X_{\overline{k}})$ be a weighted quasi-diagonal surface in $\mathbb{P}^3_{\overline{k}}(q_0, q_1, q_2, q_3)$ defined by the equation (4). (We often omit to specify the field of definition if it is defined over $\overline{k}$.) We are going to describe the singular locus, X_{sing}, of X. For each $P = [x_0, x_1, x_2, x_3] \in \mathbb{P}^3_{\overline{k}}(Q)$, let

$$I_P := \{i \mid 0 \leq i \leq 3, x_i \neq 0\}.$$

From Proposition 7 of [7], we have

$$\mathbb{P}^3_{\overline{k}}(Q)_{sing} = \{P \in \mathbb{P}^3_{\overline{k}}(Q) \mid \gcd(q_i \mid i \in I_P) \geq 2\}.$$

Let $\mathcal{I} := \{(0,1), (1,2), (1,3), (2,3)\}$. For each $(i,j) \in \mathcal{I}$, put

$$\mathcal{P}_{ij} := \{P = [x_0, x_1, x_2, x_3] \in X \mid \ x_ix_j \neq 0, \ x_h = 0 \text{ for } h \neq i,j\}$$

and

$$d_{ij} := \gcd(q_i, q_j).$$

(Note $\mathcal{P}_{ij} = \emptyset$ if $(i,j) = (0,2)$ or $(0,3)$.) Then as a set,

$$X = \{(0:1:0:0)\} \cup \bigcup_{(i,j)\in\mathcal{I}} \mathcal{P}_{ij}.$$

We define

$$\mathcal{I}_1 := \{(i,j) \in \mathcal{I} \mid d_{ij} \geq 2\}.$$

Proposition 2.2. *Let X be a weighted quasi-diagonal surface in $\mathbb{P}^3_{\overline{k}}(q_0, q_1, q_2, q_3)$ defined by the equation (4). Put $e_{ij} = \operatorname{lcm}(q_i, q_j)$. Then the following assertions hold.*

(a)

$$X_{sing} = \begin{cases} \bigcup_{(i,j)\in\mathcal{I}_1} \mathcal{P}_{ij} & \text{if } q_1 = 1 \\ \{(0:1:0:0)\} \cup \bigcup_{(i,j)\in\mathcal{I}_1} \mathcal{P}_{ij} & \text{if } q_1 \geq 2. \end{cases}$$

(b) *For $(i,j) \in \mathcal{I}$,*

$$\#\mathcal{P}_{ij} = \begin{cases} \frac{m-q_0}{e_{01}} & \text{if } (i,j) = (0,1) \\ \frac{m}{e_{ij}} & \text{otherwise.} \end{cases}$$

Proof. (a) As $\operatorname{codim}_X(X \cap \mathbb{P}^3_{\overline{k}}(Q)_{sing}) \geq 2$, Proposition 8 of [6] implies that

$$X_{sing} = X \cap \mathbb{P}^3_{\overline{k}}(Q)_{sing}.$$

It follows from this that $\mathcal{P}_{ij} \subset X_{sing}$ if and only if $d_{ij} \geq 2$ and that $(0:1:0:0)$ is a singularity if and only if $q_1 \geq 2$. Hence X_{sing} can be described as above.

(b) Note that $(m - q_0)/e_{01}$ is an integer since $m - q_0$ is divisible by q_0 and q_1. As we work over $\overline{k}$, we may disregard the coefficients c_i $(0 \leq i \leq 3)$.

Take $(i,j)=(0,1)$. Then the points in $\mathcal{P}_{01}$ satisfy the equation

$$\prod_{u=1}^{(m-q_0)/e_{01}} (x_0^{e_{01}/q_1} + \theta^u x_1^{e_{01}/q_1}) = 0$$

where θ is a primitive $(m-q_0)/e_{01}$-th root of unity. Each factor $x_0^{e_{01}/q_0} + \theta^u x_1^{e_{01}/q_1} = 0$ gives a point on X. Hence $\#\mathcal{P}_{01} = (m-q_0)/e_{01}$.

Assume $(i,j) \neq (0,1)$. Then the points in $\mathcal{P}_{ij}$ satisfy the equation

$$\prod_{u=1}^{m/e_{ij}} (x_i^{e_{ij}/q_i} + \theta^u x_j^{e_{ij}/q_j}) = 0$$

where θ is a primitive m/e_{ij}-th root of unity. As in the case $(i,j)=(0,1)$, each factor gives a point on X. Therefore $\#\mathcal{P}_{ij} = m/e_{ij}$. □

Corollary 2.3. *Let X be a weighted quasi-diagonal surface in $\mathbb{P}^3_{\overline{k}}(Q)$ defined by the equation (4). If $(0,1) \in \mathcal{I}_1$, put $\mathcal{I}_2 := \mathcal{I}_1 \setminus \{(0,1)\}$. Then*

$$\#X_{sing} = \begin{cases} 1 + \frac{m-q_0}{e_{01}} + \sum_{(i,j)\in\mathcal{I}_2} \frac{m}{e_{ij}} & \text{if } (0,1) \in \mathcal{I}_1 \\ 1 + \sum_{(i,j)\in\mathcal{I}_1} \frac{m}{e_{ij}} & \text{if } (0,1) \notin \mathcal{I}_1 \text{ and } q_1 \geq 2 \\ \sum_{(i,j)\in\mathcal{I}_1} \frac{m}{e_{ij}} & \text{otherwise.} \end{cases}$$

Next we determine the types of singularities of X. For an integer $n \geq 1$, let μ_n denote the group of n-th roots of unity in $\overline{k}^\times$.

Lemma 2.4. *With the assumptions (2) and (3), we have* $\gcd(q_1,q_2) = \gcd(q_1,q_3) = 1$.

Proof. From (3), we have $m = q_0 + m_1 q_1 = m_2 q_2 = m_3 q_3$. Suppose that $\gcd(q_1,q_2) > 1$ or $\gcd(q_1,q_3) > 1$. Then $\gcd(q_0,q_1,q_2) > 1$ or $\gcd(q_0,q_1,q_3) > 1$. But, this contradicts (2). □

Proposition 2.5. *Let X be a weighted quasi-diagonal surface in $\mathbb{P}^3_{\overline{k}}(q_0,q_1,q_2,q_3)$ defined by the equation (4).*

(a) *Fix $(i,j) \in \mathcal{I}_1$. Let $\{i_*, j_*\}$ be the complement of $\{i,j\}$ in $\{0,1,2,3\}$ (we fix their order once and for all). Let α_{ij} be a unique positive integer such that*

$$q_{i_*}\alpha_{ij} \equiv q_{j_*} \pmod{d_{ij}} \quad \text{and} \quad 1 \leq \alpha_{ij} < d_{ij}.$$

Then every point in $\mathcal{P}_{ij}$ is a cyclic quotient singularity of type $A_{d_{ij},\alpha_{ij}}$.

(b) *Assume $q_1 \geq 2$. Let α_1 be a unique positive integer such that*

$$q_2\alpha_1 \equiv q_3 \pmod{q_1} \quad \text{and} \quad 1 \leq \alpha_1 < q_1.$$

Then $(0:1:0:0) \in X$ is a cyclic quotient singularity of type A_{q_1,α_1}.

Proof. (a) By (2), $\gcd(d_{ij}, q_{i_*}) = \gcd(d_{ij}, q_{j_*}) = 1$. Hence α_{ij} is determined uniquely. Choose an arbitrary point $P = (a_0 : \cdots : a_3) \in \mathcal{P}_{ij}$. There is a covering of X by 4 affine quotient spaces

$$X = \bigcup_{u=0}^{3} V_u/\mu_{q_u}$$

where $V_u := \operatorname{Spec} A/(x_u - 1)$ with $A := \overline{k}[x_0,\cdots,x_3]/(c_0x_0^{m_0} + c_1x_0x_1^{m_1} + c_2x_2^{m_2} + c_3x_3^{m_3})$ and μ_{q_u} acts on V_u by $x_v \longmapsto \zeta^{q_v}x_v$ for $0 \leq v \leq 3, v \neq u$ (ζ ranges over μ_{q_u}). Since $a_i \neq 0$, P is on V_i/μ_{q_i}. At the inverse image of P on V_i, (x_{i_*}, x_{j_*}) gives a local coordinate system. Put $V := \operatorname{Spec}\overline{k}[x_{i_*}, x_{j_*}]$. Then the action of μ_{q_i} on V_i induces an action of $\mu_{d_{ij}}$ on V by

$$(x_{i_*}, x_{j_*}) \longmapsto (\zeta^{q_{i_*}}x_{i_*}, \zeta^{q_{j_*}}x_{j_*})$$

where $\zeta \in \mu_{d_{ij}}$. As $\gcd(q_{i\bullet}, d_{ij}) = 1$, $\zeta^{q_{i\bullet}}$ ranges over $\mu_{d_{ij}}$. Changing ζ to $\zeta^{q_{i\bullet}}$, we can write this action as

$$(z_{i\bullet}, z_{j\bullet}) \longmapsto (\zeta z_{i\bullet}, \zeta^{\alpha_{ij}} z_{j\bullet})$$

where α_{ij} is the integer defined above. Therefore P is of type $A_{d_{ij},\alpha_{ij}}$.

(b) The point $(0:1:0:0)$ can be identified with the image of $(0,0,0) \in V_1$ in V_1/μ_{q_1}. On V_1, (x_2, x_3) gives a local coordinate system at $(0,0,0)$. Put $V := \operatorname{Spec} \overline{k}[x_2, x_3]$. Since μ_{q_1} fixes $(0,0,0) \in V_1$, μ_{q_1} acts on V by

$$(x_2, x_3) \longmapsto (\zeta^{q_2} x_2, \zeta^{q_3} x_3)$$

where $\zeta \in \mu_{q_1}$. As $\gcd(q_1, q_2) = 1$ (see Lemma 2.4), ζ^{q_2} also ranges over μ_{q_1}. Hence we may express this action as

$$(x_2, x_3) \longmapsto (\zeta x_2, \zeta^{\alpha_1} x_3)$$

where α_1 is the positive integer defined above. Therefore $(0:1:0:0)$ is a cyclic quotient singularity of type A_{n,α_1}. $\square$

To describe the zeta-function of X_k, we need to know the minimal field of definition for each singularity of X. We shall determine it by assuming that $k^\times$ contains all m-th roots of unity in $\overline{k}^\times$. As $(0:1:0:0)$ is obviously defined over k, we may discuss only singularities in $\mathcal{P}_{ij}$ with $(i,j) \in \mathcal{I}_1$. For $P \in \mathcal{P}_{ij}$, we write $k(P)$ for the minimal field of definition of P over k.

Proposition 2.6. *Let X_k be a weighted quasi-diagonal surface of degree m in $\mathbb{P}^3_k(q_0, q_1, q_2, q_3)$ defined by the equation (4). Assume that $k^\times$ contains all m-th roots of unity in $\overline{k}^\times$. Let P be an arbitrary point in $\mathcal{P}_{ij}$ with $(i,j) \in \mathcal{I}_1$. Choose $\gamma_{ij} \in \overline{k}^\times$ satisfying*

$$\begin{cases} c_0 \gamma_{01}^{(m-q_0)/e_{01}} + c_1 = 0 & \text{if } (i,j) = (0,1) \\ c_i \gamma_{ij}^{m/e_{ij}} + c_j = 0 & \text{otherwise.} \end{cases}$$

Then $k(\gamma_{ij})$ does not depend on the choice of γ_{ij} and $k(P) = k(\gamma_{ij})$ for all $P \in \mathcal{P}_{ij}$.

Proof. Since $(m - q_0)/e_{01}$ and m/e_{ij} are divisors of m, the solutions to each equation are of the form $\zeta\gamma_{ij}$, where ζ is a m-th root of unity. From our assumption, ζ is in k. Hence $k(\gamma_{ij})$ does not depend on the choice of γ_{ij}.

For any q_i and q_j, we see $\mathbf{P}^1_k(q_i, q_j) \cong \mathbf{P}^1_k$ by composing the following two isomorphisms:

$$\begin{array}{ccccc} \mathbf{P}^1_k(q_i, q_j) & \longrightarrow & \mathbf{P}^1_k(q_i/d_{ij}, q_j/d_{ij}) & \longrightarrow & \mathbf{P}^1_k \\ (x_i, x_j) & \longmapsto & (x_i, x_j) & \longmapsto & (x_i^{q_j/d_{ij}}, x_j^{q_i/d_{ij}}). \end{array}$$

These isomorphisms are defined over k. If $P = (a_0 : a_1 : a_2 : a_3) \in \mathcal{P}_{ij}$, then the non-zero coordinates are a_i and a_j. Hence P and $(a_i : a_j) \in \mathbf{P}^1_k(q_i, q_j)$ have the same field of definition. From the isomorphism $\mathbf{P}^1_k(q_i, q_j) \cong \mathbf{P}^1_k$, we obtain

$$k(P) = k(a_i^{q_j/d_{ij}} / a_j^{q_i/d_{ij}}).$$

Furthermore $a_i^{q_j/d_{ij}} / a_j^{q_i/d_{ij}}$ and γ_{ij} satisfy the same equation. Therefore $k(P) = k(\gamma_{ij})$. $\square$

The equation for γ_{ij} is not necessarily minimal over k; so, the extension degree $[k(\gamma_{ij}) : k]$ is a divisor of

$$\begin{cases} (m - q_0)/e_{01} & \text{if } (i,j) = (0,1) \\ m/e_{ij} & \text{otherwise.} \end{cases}$$

Let k' be the composite of $k(P)$ over all singularities of $X_{\overline{k}}$. Write $\widetilde{X}_{\overline{k}}$ for the minimal resolution of $X_{\overline{k}}$. Since each monoidal transformation is defined over the field of definition for its center, $\widetilde{X}_{\overline{k}}$ is defined over k'. (In fact, $\tilde{X}_{\bar{k}}$ can be defined over k as X_{sing} is closed under the action of $\mathrm{Gal}(\bar{k}/k)$.) In what follows, we always assume that k is large so that it contains k'.

3. Zeta-functions

In this section, we show that the zeta-function of a weighted quasi-diagonal surface can be described in terms of twisted Jacobi sums. For integers $\nu \geq 1$, we write $k_\nu := \mathbf{F}_{q^\nu}$.

Lemma 3.1. *Let $m_1, \cdots, m_r$ be r positive integers such that $q \equiv 1 \pmod{m_i}$ for $1 \leq i \leq r$. Let W_k be an affine variety in $\mathbf{A}_k^r$ defined by the equation*

$$b_0 + b_1 x_1^{m_1} + \cdots + b_r x_r^{m_r} = 0 \quad \subset \mathbf{A}_k^r$$

with $b_i \in k^\times$ $(0 \leq i \leq r)$. Define

$$M = \mathrm{lcm}\,(m_1, \cdots, m_r)$$
$$M_i = M/m_i$$

for $1 \leq i \leq r$. Assume $q \equiv 1 \pmod{M}$. Fix a character, χ, of $k^\times$ of exact order M. For each $\nu \geq 1$, let $N_\nu(W)$ denote the number of k_ν-rational points on $W_{\bar{k}}$. Then

$$N_\nu(W) = q^{\nu(r-1)} - \sum_{\boldsymbol{a}' \in \mathfrak{U}_1} \mathcal{J}(\boldsymbol{b}', \boldsymbol{a}')^\nu + \sum_{\boldsymbol{a}'' \in \mathfrak{U}_2} \mathcal{J}(\boldsymbol{b}'', \boldsymbol{a}'')^\nu$$

where

$$\boldsymbol{b}' = (b_1, \cdots, b_r), \ \boldsymbol{b}'' = (b_0, \cdots, b_r)$$

$$\mathfrak{U}_1 = \left\{ \boldsymbol{a}' = (a_1', \cdots, a_r') \middle| \ a_i' \in M_i\mathbf{Z}/M\mathbf{Z}, \ a_i' \neq 0 \ (1 \leq i \leq r), \ \sum_{i=1}^r a_i' = 0 \right\}$$

$$\mathfrak{U}_2 = \left\{ \boldsymbol{a}'' = (a_0'', a_1'', \cdots, a_r'') \middle| \begin{array}{l} a_0'' \in \mathbf{Z}/M\mathbf{Z}, \ a_i'' \in M_i\mathbf{Z}/M\mathbf{Z} \ (1 \leq i \leq r) \\ a_i'' \neq 0 \ for \ 0 \leq i \leq r, \ \sum_{i=0}^r a_i'' = 0 \end{array} \right\}$$

$$\mathcal{J}(\boldsymbol{b}', \boldsymbol{a}') = \chi^{-1}(b_1^{a_1'} \cdots b_r^{a_r'}) j(\boldsymbol{a}')$$

$$\mathcal{J}(\boldsymbol{b}'', \boldsymbol{a}'') = \chi^{-1}(b_0^{a_1'} \cdots b_r^{a_r'}) j(\boldsymbol{a}'')$$

$$j(\boldsymbol{a}') = \sum_{\substack{v_i \in k^\times \ (1 \leq i \leq r) \\ v_1 + \cdots + v_r = 0}} \chi^{a_1'}(v_1) \cdots \chi^{a_r'}(v_r)$$

$$j(\boldsymbol{a}'') = \sum_{\substack{v_i \in k^\times \ (0 \leq i \leq r) \\ v_0 + \cdots + v_r = 0}} \chi^{a_0'}(v_0) \cdots \chi^{a_r'}(v_r)$$

Proof. The idea of proof is entirely due to Weil [31]; we sketch the proof since there seems to be no article giving a complete formula for $N_\nu(W)$.

Fix $\nu \geq 1$. Consider the affine varieties:

$$W_{k_\nu}' : \ b_1 x_1^{m_1} + \cdots + b_r x_r^{m_r} = 0 \quad \subset \mathbf{A}_{k_\nu}^r$$

$$W_{k_\nu}'' : \ b_0 x_0^{q^\nu - 1} + b_1 x_1^{m_1} + \cdots + b_r x_r^{m_r} = 0 \quad \subset \mathbf{A}_{k_\nu}^{r+1}.$$

Since $x_0^{q^\nu - 1} = 1$ for $x_0 \in k_\nu{}^\times$, we have

$$N_\nu(W'') = N_\nu(W') + (q^\nu - 1) N_\nu(W). \tag{8}$$

It follows from [31] that

$$N_\nu(W') = q^{\nu(r-1)} + (q^\nu - 1) \sum_{a' \in \mathfrak{U}_1} \mathcal{J}(b', a')^\nu$$

and

$$N_\nu(W'') = q^{\nu r} + (q^\nu - 1) \sum_{a'' \in \mathfrak{U}_2} \mathcal{J}(b'', a'')^\nu.$$

Therefore substituting these formulae into (8), we obtain the formula for $N_\nu(W)$. □

Remark 3.2. The algebraic integer $\mathcal{J}(b', a')$ may be called a *twisted Jacobi sum associated with* b' *and* a' *relative to* χ (cf. [17]).

Lemma 3.3. *The set of k-rational points in* $\mathbf{P}^3_k(Q)$ *can be identified with the set*

$$(\mathbf{A}^4_k \setminus \{O\})/\sim$$

where "$\sim$" denotes the equivalence relation:

$$(x_0, \cdots, x_3) \sim (y_0, \cdots, y_3) \Leftrightarrow \exists t \in \overline{k}^\times \text{ such that } x_i = t^{q_i} y_i \ (0 \le i \le 3).$$

Proof. The result follows from (5) by taking the $\mathrm{Gal}(\overline{k}/k)$-invariant subset of $\mathbf{P}^3_{\overline{k}}(Q)$ (cf. [13]). □

Corollary 3.4. *Each equivalence class of* $(\mathbf{A}^4_k \setminus \{O\})/\sim$ *consists of* $q-1$ *elements of* $\mathbf{A}^4_k \setminus \{O\}$. *Consequently, if* W_k *is a projective variety in* $\mathbf{P}^3_k(Q)$ *and* $\overline{W}_k$ *is the Zariski closure of* $\iota^{-1}(W_k)$ *in* $\mathbf{A}^4_k$ *(cf. (7)), then for* $\nu \ge 1$,

$$N_\nu(\overline{W}) = 1 + (q^\nu - 1)N_\nu(W).$$

Proof. Given $x := (x_0, \cdots, x_3) \in \mathbf{A}^4_k \setminus \{O\}$, we see, by using (2), that there are exactly $q-1$ values for $t \in \overline{k}^\times$ such that tx are distinct. Since every class has the same cardinality, we obtain the asserted formula. Details may be found in [13]. □

We compute the zeta-function of a weighted quasi-diagonal surface applying Lemma 3.1 and Corollary 3.4.

Theorem 3.5. *Let* X_k *be a weighted quasi-diagonal surface in* $\mathbf{P}^3_k(Q)$ *of degree* m *with twist* $\mathbf{c} := (c_0, c_1, c_2, c_3)$ *defined by the equation:*

$$c_0 x_0^{m_0} + c_1 x_0 x_1^{m_1} + c_2 x_2^{m_2} + c_3 x_3^{m_3} = 0.$$

Define

$$M = \mathrm{lcm}\,(m_1, m_2, m_3)$$
$$M_i = M/m_i$$

for $1 \le i \le 3$. *Assume* $q \equiv 1 \pmod{M}$. *Fix a character,* χ, *of* $k^\times$ *of exact order* M. *Then the zeta-function of* X_k *can be described as:*

$$Z(X_k, T) = \frac{1}{(1-T)P_2(X_k, T)(1-q^2T)}$$

where

$$P_2(X_k,T) = (1-qT)\prod_{a\in\mathfrak{V}}(1-q\chi^{aM_3}(-c_2/c_3)T)\prod_{\boldsymbol{a}\in\mathfrak{W}}\Big(1-\mathcal{J}(\boldsymbol{c},\boldsymbol{a})T\Big)$$

$$\mathfrak{V} = \{a\in\mathbf{Z}/M\mathbf{Z}\mid a\neq 0,\ q_3a\equiv 0 \pmod{q_2}\}$$

$$\mathfrak{W} = \left\{\boldsymbol{a}=(a_0,a_1,a_2,a_3)\,\middle|\, \begin{array}{l} a_0\in\mathbf{Z}/M\mathbf{Z},\ a_i\in M_i\mathbf{Z}/M\mathbf{Z}\ (1\le i\le 3),\\ a_i\neq 0\ (0\le i\le 3),\ \sum_{i=0}^3 a_i = 0 \text{ and } \ m_0a_0+a_1=0\end{array}\right\}$$

$\mathcal{J}(\boldsymbol{c},\boldsymbol{a})$ is the twisted Jacobi sum associated to $\boldsymbol{c}$ and $\boldsymbol{a}$ relative to χ.

Proof. Let $\overline{X}_k$ be the affine variety in $\mathbf{A}_k^4$ defined by the same equation as X_k. By Corollary 3.4, we have $N_\nu(\overline{X}) = 1+(q^\nu-1)N_\nu(X)$. Let Z_k be the closed subset of $\overline{X}_k$ defined by $x_0=0$; i.e.

$$Z_k:\ c_2x_2^{m_2}+c_3x_3^{m_3}=0 \ \text{ with } x_1 \text{ free } \ \subset \mathbf{A}_k^3.$$

Write U_k for the open subset $\overline{X}_k\setminus Z_k$:

$$U_k:\ c_0x_0^{m_0}+c_1x_0x_1^{m_1}+c_2x_2^{m_2}+c_3x_3^{m_3}=0 \ \text{ with } x_0\neq 0 \ \subset \mathbf{A}_k^4. \tag{9}$$

Then $\overline{X}_{k_\nu} = Z_{k_\nu}\cup U_{k_\nu}$ (disjoint union) for every $\nu\ge 1$. Regarding x_0 as a constant, write $U(x_0)$ for the affine surface in $\mathbf{A}_k^3$ defined by (9). Then for $\nu\ge 1$, we have

$$N_\nu(\overline{X}) = N_\nu(Z)+\sum_{x_0\in k_\nu^\times} N_\nu(U(x_0)).$$

It follows from [31] that

$$N_\nu(Z) = q^\nu\Big(q^\nu+(q^\nu-1)\sum_{(u_2,u_3)}\chi^{-1}((-c_2)^{u_2}c_3^{u_3})^\nu\Big)$$

where (u_2,u_3) ranges over the set

$$\{(u_2,u_3)\mid u_i\in M_i\mathbf{Z}/M\mathbf{Z},\ u_i\neq 0\ (i=2,3),\ u_2+u_3=0\}.$$

By substituting $u_3=aM_3$, the summation over (u_2,u_3) can be transformed into

$$\sum_{(u_2,u_3)}\chi^{-1}((-c_2)^{u_2}c_3^{u_3})^\nu = \sum_{a\in\mathfrak{V}}\chi^{aM_3}(-c_2/c_3)^\nu.$$

Hence

$$N_\nu(Z) = q^\nu\Big(q^\nu+(q^\nu-1)\sum_{a\in\mathfrak{V}}\chi^{aM_3}(-c_2/c_3)^\nu\Big).$$

Applying Lemma 3.1 to the case $r=3$, $b_0=c_0x_0^{m_0}$, $b_1=c_1x_0$, $b_2=c_2$ and $b_3=c_3$, we find

$$N_\nu(U(x_0)) = q^{2\nu}-\sum_{\boldsymbol{a}'\in\mathfrak{U}_1}\mathcal{J}(\boldsymbol{b}',\boldsymbol{a}')^\nu+\sum_{\boldsymbol{a}''\in\mathfrak{U}_2}\mathcal{J}(\boldsymbol{b}'',\boldsymbol{a}'')^\nu$$

where

$$b' = (c_1x_0, c_2, c_3),\ b'' = (c_0x_0^{m_0}, c_1x_0, c_2, c_3)$$
$$\mathfrak{U}_1 = \left\{ a' = (a_1', a_2', a_3') \middle| \ a_i' \in M_i\mathbf{Z}/M\mathbf{Z},\ a_i' \neq 0\ (1 \leq i \leq 3),\ \sum_{i=1}^{3} a_i' = 0 \right\}$$
$$\mathfrak{U}_2 = \left\{ a'' = (a_0'', a_1'', a_2'', a_3'') \middle| \ \begin{array}{l} a_0'' \in \mathbf{Z}/M\mathbf{Z},\ a_i'' \in M_i\mathbf{Z}/M\mathbf{Z}\ (1 \leq i \leq 3) \\ a_i'' \neq 0 \text{ for } 0 \leq i \leq 3,\ \sum_{i=0}^{3} a_i'' = 0 \end{array} \right\}$$
$$\mathcal{J}(b', a') = \chi^{-1}(c_1^{a_1'} c_2^{a_2'} c_3^{a_3'} x_0^{a_1'}) j(a')$$
$$\mathcal{J}(b'', a'') = \chi^{-1}(c_0^{a_0'} c_1^{a_1'} c_2^{a_2'} c_3^{a_3'} x_0^{a_0' m_0 + a_1'}) j(a'').$$

Since $a_1' \neq 0$, we obtain

$$\begin{aligned} \sum_{x_0 \in k_\nu^\times} \sum_{a' \in \mathfrak{U}_1} \mathcal{J}(b', a')^\nu &= \sum_{a' \in \mathfrak{U}_1} \sum_{x_0 \in k_\nu^\times} \mathcal{J}(b', a')^\nu \\ &= \sum_{a' \in \mathfrak{U}_1} \chi^{-1}(c_1^{a_1'} c_2^{a_2'} c_3^{a_3'})^\nu j(a')^\nu \sum_{x_0 \in k_\nu^\times} \chi^{-1}(x_0^{a_1'})^\nu \\ &= 0. \end{aligned}$$

On the other hand,

$$\begin{aligned} \sum_{x_0 \in k_\nu^\times} \sum_{a' \in \mathfrak{U}_2} \mathcal{J}(b'', a'')^\nu &= \sum_{a'' \in \mathfrak{U}_2} \chi^{-1}(c_0^{a_0''} \cdots c_3^{a_3''})^\nu j(a'')^\nu \sum_{x_0 \in k_\nu^\times} \chi^{-1}(x_0^{a_0'' m_0 + a_1''})^\nu \\ &= (q^\nu - 1) \sum_{\substack{a'' \in \mathfrak{U}_2 \\ a_0'' m_0 + a_1'' = 0}} \chi^{-1}(c_0^{a_0''} \cdots c_3^{a_3''})^\nu j(a'')^\nu. \end{aligned}$$

Hence

$$\sum_{x_0 \in k_\nu^\times} N_\nu(U(x_0)) = q^{2\nu}(q^\nu - 1) + (q^\nu - 1) \sum_{a \in \mathfrak{W}} \mathcal{J}(c, a)^\nu.$$

Combining this with $N_\nu(Z)$, we conclude

$$N_\nu(\overline{X}) = q^\nu \left\{ q^\nu + (q^\nu - 1) \sum_{a \in \mathfrak{V}} \chi^{aM_3}(-c_2/c_3)^\nu \right\} + (q^\nu - 1) \sum_{a \in \mathfrak{W}} \mathcal{J}(c, a)^\nu.$$

Therefore by Corollary 3.4,

$$N_\nu = 1 + q^\nu + q^\nu \sum_{a \in \mathfrak{V}} \chi^{aM_3}(-c_2/c_3)^\nu + \sum_{a \in \mathfrak{W}} \mathcal{J}(c, a)^\nu.$$

This gives rise to the zeta-function of X_k. □

Since $\mathfrak{W}$ is a subset of the group of characters of $\mu_M \times \mu_M \times \mu_M \times \mu_M$, we may call $a \in \mathfrak{W}$ a *character*.

Remark 3.6. Put $e_{23} := \operatorname{lcm}(q_2, q_3)$. Then the cardinality of $\mathfrak{V}$ in Theorem 3.5 is equal to

$$\#\mathfrak{V} = \frac{m}{e_{23}} - 1.$$

Remark 3.7. Let H_0 be a closed subscheme of X_k defined by the equation $x_0 = 0$ (compare with Z_k in the proof of Theorem 3.5). Then H_0 is a union of m/e_{23} rational lines. These lines intersect to each other at $(1:0:0)$ (usually not transversally). All the singularities of $X_{\bar{k}}$, except for those in $\mathcal{P}_{0,1}$, are on these lines.

Let α_{ij}, α_1 and d_{ij} be the integers defined in Proposition 2.5. For $(i,j) \in \mathcal{I}_1$, denote by r_{ij} the length of the continued fraction expansion of d_{ij}/α_{ij}. If $q_1 \geq 2$, then write r_1 for the length of the continued fraction expansion of q_1/α_1; if $q_1 = 1$, put $r_1 = 0$.

Proposition 3.8. *Let X_k be a weighted quasi-diagonal surface in $\mathbf{P}^3_k(q_0, q_1, q_2, q_3)$ of degree m with twists $\mathbf{c}$ defined by the equation (4). Let M be the integer defined in Theorem 3.5. Assume that k is large so that all the singularities of $X_{\bar{k}}$ are defined over k (i.e. $k \supset k(P)$ for all $P \in X_{sing}$); assume also that $q \equiv 1 \pmod{M}$. If $(0,1) \in \mathcal{I}_1$, then put $\mathcal{I}_2 := \mathcal{I}_1 \setminus \{(0,1)\}$. Let*

$$e = \begin{cases} r_1 + \frac{m-q_0}{e_{01}} r_{01} + \sum_{(i,j)\in\mathcal{I}_2} \frac{m}{e_{ij}} r_{ij} & \text{if } (0,1) \in \mathcal{I}_1 \\ r_1 + \sum_{(i,j)\in\mathcal{I}_1} \frac{m}{e_{ij}} r_{ij} & \text{if } (0,1) \notin \mathcal{I}_1. \end{cases}$$

Let $\widetilde{X}_{\bar{k}}$ be the minimal resolution of $X_{\bar{k}}$; it is defined over k. Then the zeta-function of $\widetilde{X}_k$ has the following form:

$$Z(\widetilde{X}_k, T) = \frac{1}{(1-T)P_2(\widetilde{X}_k, T)(1-q^2T)}$$

where

$$P_2(\widetilde{X}_k, T) = (1-qT)^{1+e} \prod_{\mathbf{a}\in\mathfrak{V}} (1 - q\chi^{aM_3}(-c_2/c_3)T) \prod_{\mathbf{a}\in\mathfrak{W}} \Big(1 - \mathcal{J}(\mathbf{c}, \mathbf{a})T\Big).$$

(For the notation, see Theorem 3.5.)

Proof. X_k and $\widetilde{X}_k$ are isomorphic over k outside of the exceptional locus. Hence $Z(\widetilde{X}_k, T)$ can be computed from $Z(X_k, T)$ by counting the number of k_ν-rational points on the exceptional divisors on $\widetilde{X}_{\bar{k}}$ for every $\nu \geq 1$. We know from Hirzebruch's resolution [19] that there are e exceptional lines on $\widetilde{X}_k$ (cf. Corollary 2.3) each of which is isomorphic to $\mathbf{P}^1_k$ and that these lines intersect transversally. Hence $\widetilde{X}_{k_\nu}$ acquires eq^ν more points than X_ν. Therefore

$$Z(\widetilde{X}_k, T) = Z(X_k, T)/(1-qT)^e.$$

Applying Theorem 3.5, we complete the proof. □

Corollary 3.9. *Let X_k be a weighted quasi-diagonal surface in $\mathbf{P}^3_k(q_0, q_1, q_2, q_3)$ of degree m defined by the equation (4). Let $\widetilde{X}_{\bar{k}}$ be the minimal resolution of $X_{\bar{k}}$. Put $e_{23} := \operatorname{lcm}(q_2, q_3)$. Then the following assertions hold.*

(a) *The Betti numbers of $\widetilde{X}_{\bar{k}}$ are equal to $B_0 = B_4 = 1$, $B_1 = B_3 = 0$ and*

$$B_2 = e + \frac{m}{e_{23}} + \#\mathfrak{W}.$$

(b) *The self-intersection number of the canonical divisor, K, of $\widetilde{X}_{\bar{k}}$ is equal to*

$$K^2 = 10 + 12P_g(\widetilde{X}_{\bar{k}}) - e - \frac{m}{e_{23}} - \#\mathfrak{W}.$$

Proof. (a) The Betti numbers can be computed from the degrees of $P_i(\widetilde{X}_k, T)$ $(0 \leq i \leq 4)$ using Proposition 3.8 and Remark 3.6.

(b) This follows from the Riemann-Roch Theorem. □

Given m and Q, we can calculate $\#\mathfrak{W}$ directly from the definition of $\mathfrak{W}$; but, we do not have a closed formula for this.

4. The Picard numbers

In this section, we prove the Tate conjecture for the minimal resolution of a weighted quasi-diagonal surface over a finite field. Consequently, we give a formula for the Picard number of the minimal resolution.

Proposition 4.1. *Let X_k be a weighted quasi-diagonal surface in $\mathbf{P}^3_k(Q)$ defined by the equation:*

$$X_k: \quad c_0x_0^{m_0} + c_1x_0x_1^{m_1} + c_2x_2^{m_2} + c_3x_3^{m_3} = 0.$$

Denote by $\widetilde{X}_{\overline{k}}$ the minimal resolution of $X_{\overline{k}}$. Assume that k is large so that every singularity of $X_{\overline{k}}$ is defined over k. Then $\widetilde{X}_{\overline{k}}$ is defined over k and the Tate conjecture holds for $\widetilde{X}_k$.

Proof. For the field of definition of $\widetilde{X}_{\overline{k}}$, see the remark after Proposition 2.6. It follows from a cohomological formulation of the Tate conjecture (cf. [29]) that if the conjecture is true for X over some finite extension of k, then so is it over k. Thus, without loss of generality, we may assume that X_k has coefficients 1; i.e. $c_i = 1$ for $0 \le i \le 3$.

We prove the assertion by showing that X_k is birational to a quotient of a Fermat surface. Put $n = m_0m_1m_2m_3$. Let Y_k be the Fermat surface in $\mathbb{P}^3_k$ of degree n:

$$Y_k: \quad y_0^n + y_1^n + y_2^n + y_3^n = 0.$$

Write μ_n for the group of n-th roots of unity in $\overline{k}^\times$. Let Γ be a subgroup of $\oplus_{i=0}^3\mu_n/(\text{diagonal elements})$ defined by

$$\Gamma = \{(\lambda_0^{m_0}, \lambda_0\lambda_1^{m_1}, \lambda_2^{m_2}, \lambda_3^{m_3}) \mid (\lambda_0, \lambda_1, \lambda_2, \lambda_3) \in \oplus_{i=0}^3\mu_n/(\text{diagonal elements})\}.$$

Since $p \not| n$, $\#\Gamma$ is coprime to p. We may let Γ act on Y_k by

$$(y_0 : y_1 : y_2 : y_3) \longrightarrow (\lambda_0^{m_0}y_0 : \lambda_0\lambda_1^{m_1}y_1 : \lambda_2^{m_2}y_2 : \lambda_3^{m_3}y_3).$$

Associated with this action, there is a dominant rational map $Y_k \longrightarrow X_k$ defined by

$$\begin{cases} x_0 &= y_0^{m_1m_2m_3} \\ x_1 &= y_1^{m_0m_2m_3}/y_0^{m_2m_3} \\ x_2 &= y_2^{m_0m_1m_3} \\ x_3 &= y_3^{m_0m_1m_2}. \end{cases}$$

Hence both $X_{\overline{k}}$ and $\widetilde{X}_{\overline{k}}$ are birational to $Y_{\overline{k}}/\Gamma$ (the quotient of $Y_{\overline{k}}$ by Γ). The birational maps are defined over some finite extension of k.

It is known that the Tate conjecture is true for Fermat surfaces over finite fields [26]. Hence it is also true for smooth open subspaces of $Y_{\overline{k}}/\Gamma$ and $X_{\overline{k}}$ over their fields of definition. Therefore the Tate conjecture holds for $\widetilde{X}_k$ since the cohomologies of X_k and $\widetilde{X}_k$ differ only by algebraic cycles over k (cf. [13], [30]). □

Corollary 4.2. *With the hypotheses of Proposition 4.1, let M be the integer defined in Theorem 3.5. Assume $q \equiv 1 \pmod{M}$. Then the following assertions hold.*

(a) *For $\nu \ge 1$, write*

$$\mathfrak{V}_{al}(\nu) := \{a \in \mathfrak{V} \mid \chi^{aM_3}(-c_2/c_3)^\nu = 1\}$$
$$\mathfrak{W}_{al}(\nu) := \{\boldsymbol{a} \in \mathfrak{W} \mid \mathcal{J}(\boldsymbol{c}, \boldsymbol{a})^\nu = q^\nu\}.$$

Then for $\nu \ge 1$,

$$\rho(\widetilde{X}_{k_\nu}) = 1 + e + \#\mathfrak{V}_{al}(\nu) + \#\mathfrak{W}_{al}(\nu).$$

(b) *Let e be the integer defined in Proposition 3.8. Put*

$$\mathfrak{W}_{al}(\infty) := \{\boldsymbol{a} \in \mathfrak{W} \mid \mathcal{J}(\boldsymbol{c}, \boldsymbol{a})/q \text{ is a root of unity}\}.$$

Then

$$\rho(\widetilde{X}_{\bar{k}}) = e + \frac{m}{e_{23}} + \#\mathfrak{W}_{al}(\infty).$$

Proof. (a) Replacing k with k_ν in Proposition 4.1, we see that the Tate conjecture holds for $\widetilde{X}_{k_\nu}$ for $\nu \geq 1$. (The conjecture holds for Fermat surfaces over arbitrary finite extensions of k.) Hence the formula for $\rho(\widetilde{X}_{k_\nu})$ is a direct implication of the Tate conjecture (cf. Introduction) and Proposition 3.8.

(b) $\rho(\widetilde{X}_{\bar{k}})$ is the maximal number among all $\rho(\widetilde{X}_{k_\nu})$ for $\nu \geq 1$. The maximal $\#\mathfrak{W}_{al}(\nu)$ is equal to $\#\mathfrak{W}$. Hence by Part (a) and Remark 3.6, we obtain the asserted formula. □

We can express $\#\mathfrak{W}_{al}(\infty)$ in terms of combinatorical data on characters. The details of what follows may be found in [26] and [27]. Let K be the M-th cyclotomic field over $\mathbf{Q}$. Write $G := \mathrm{Gal}(K/\mathbf{Q})$ for the Galois group of $K/\mathbf{Q}$. We may identify G with the group $U(\mathbf{Z}/M\mathbf{Z}) := \{t \in \mathbf{Z}/M\mathbf{Z} \mid \gcd(t, M) = 1\}$ by the correspondence

$$\begin{array}{ccc} U(\mathbf{Z}/M\mathbf{Z}) & \longrightarrow & G \\ t & \longmapsto & \sigma_t \end{array}$$

where σ_t is the automorphism of $K/\mathbf{Q}$ satisfying $\sigma_t(\xi) = \xi^t$ for every primitive M-th root of unity ξ in $\mathbf{C}$. Twisted Jacobi sums $\mathcal{J}(\boldsymbol{c}, \boldsymbol{a})$ are elements of K; G acts on them by $\sigma_t(\mathcal{J}(\boldsymbol{c}, \boldsymbol{a})) = \mathcal{J}(\boldsymbol{c}, t\boldsymbol{a})$, where $t\boldsymbol{a} = (ta_0, ta_1, ta_2, ta_3) \in \mathfrak{W}$. Corresponding to this action, we have an action of $U(\mathbf{Z}/M\mathbf{Z})$ on $\mathfrak{W}$ by $\boldsymbol{a} \longrightarrow t\boldsymbol{a}$. Let $O_G(\mathfrak{A})$ denote the set of $U(\mathbf{Z}/M\mathbf{Z})$-orbits in $\mathfrak{W}$. The orbit of $\boldsymbol{a} \in \mathfrak{W}$ will be written as

$$[\boldsymbol{a}] := \{t\boldsymbol{a} \in \mathfrak{W} \mid t \in U(\mathbf{Z}/M\mathbf{Z})\}.$$

Noting that p is coprime to M, let f be the order of p modulo M. Let H be the subgroup of $U(\mathbf{Z}/M\mathbf{Z})$ generated by p:

$$H = \{p^i \pmod{M} \mid 1 \leq i \leq f\} \subset U(\mathbf{Z}/M\mathbf{Z}).$$

We define

$$\|t\boldsymbol{a}\| = \sum_{i=0}^{3} \left\langle \frac{ta_i}{M} \right\rangle - 1$$

$$A_H(\boldsymbol{a}) = \sum_{t \in H} \|t\boldsymbol{a}\|$$

where $< a >$ denotes the fractional part of $a \in \mathbf{Q}$. For each $\boldsymbol{a} \in \mathfrak{W}$, $\|\boldsymbol{a}\|$ is equal to $0, 1$ or 2. It follows from Lemma 3.1 of [26] and Proposition 5.11 of [27] that $[\boldsymbol{a}] \subset \mathfrak{W}_{al}(\infty)$ if and only if

$$A_H(t\boldsymbol{a}) = f \text{ for every } t\boldsymbol{a} \in [\boldsymbol{a}] \text{ with } \|t\boldsymbol{a}\| = 0. \tag{10}$$

Note that if there is no t such that $\|t\boldsymbol{a}\| = 0$, then $[\boldsymbol{a}] \subset \mathfrak{W}_{al}(\infty)$. We put

$$O_G(\mathfrak{W})_{al} = \{[\boldsymbol{a}] \in O(\mathfrak{W}) \mid [\boldsymbol{a}] \text{ satisfies } (10)\}.$$

Then $\#\mathfrak{W}_{al}(\infty)$ can be expressed as follows.

Lemma 4.3.

$$\#\mathfrak{W}_{al}(\infty) = \sum_{[\boldsymbol{a}] \in O_G(\mathfrak{W})_{al}} \#[\boldsymbol{a}].$$

5. Orders of the Brauer groups

In this section, we compute the order of the Brauer group of the minimal resolution of a weighted quasi-diagonal surface over a finite field in two special cases.

Let X_k be a weighted quasi-diagonal surface in $\mathbf{P}^3_k(Q)$ defined by the equation (4). Let M be the integer defined in Theorem 3.5. Throughout the section, we assume $p \neq 2$, $q \equiv 1 \pmod M$ and that k is large so that every singularity of $X_{\overline{k}}$ is defined over k. Write $\widetilde{X}_k$ for the minimal resolution of X_k. We are going to consider the following two cases:

(i) $\mathrm{NS}(\widetilde{X}_k) = \mathrm{NS}(\widetilde{X}_{\overline{k}})$ and $\rho(\widetilde{X}_k) = B_2$, where B_2 is the second Betti number of $\widetilde{X}_{\overline{k}}$

(ii) $\mathrm{NS}(\widetilde{X}_k) = \mathrm{NS}(\widetilde{X}_{\overline{k}})$ and $\rho(\widetilde{X}_k) = e + m/e_{23}$ (cf. Corollary 4.2).

If $\rho(\widetilde{X}_k) = B_2$, then $\widetilde{X}_k$ may be said to be *supersingular* in the sense of [24]. The condition $\mathrm{NS}(\widetilde{X}_k) = \mathrm{NS}(\widetilde{X}_{\overline{k}})$ is equivalent to assuming that all the algebraic cycles of $\widetilde{X}_{\overline{k}}$ are defined over k. Since the Tate conjecture holds for $\widetilde{X}_k$ (Proposition 4.1), $\mathrm{NS}(\widetilde{X}_k) = \mathrm{NS}(\widetilde{X}_{\overline{k}})$ implies $\mathfrak{V}_{al}(\nu) = \mathfrak{V}$ and $\mathfrak{W}_{al}(\nu) = \mathfrak{W}_{al}(\infty)$ for all $\nu \geq 1$.

Lemma 5.1. *Let $\widetilde{X}_k$ be the minimal resolution of X_k. Write $\alpha(\widetilde{X}_k) = P_g - \dim H^1(\widetilde{X}_{\overline{k}}, \mathcal{O}) + \dim \mathrm{PicVar}(\widetilde{X}_{\overline{k}})$ as in the Artin-Tate formula. Then the Néron-Severi group of $\widetilde{X}_k$ is torsion-free and $\alpha(\widetilde{X}_k) = P_g(\widetilde{X}_k)$.*

Proof. Recall that $\widetilde{X}_{\overline{k}}$ is birational to a quotient of a Fermat surface by an action of a finite group (see Proposition 4.1). For a complete intersection, it is known that the Néron-Severi group is torsion free, the Picard variety is trivial and that the irregularity is equal to 0 (cf. [3]). Hence the same properties hold for $\widetilde{X}_{\overline{k}}$. □

Lemma 5.2. *Let $\widetilde{X}_k$ be the minimal resolution of X_k. Put $\mathfrak{T} = \mathfrak{W} \setminus \mathfrak{W}_{al}(\infty)$. Then the Artin-Tate formula for $\widetilde{X}_k$ can be described as*

$$\prod_{\mathbf{a}\in\mathfrak{T}} \left(1 - \frac{\mathcal{J}(\mathbf{c}, \mathbf{a})}{q}\right) = \frac{(-1)^{\rho(\widetilde{X}_k)-1} \# \mathrm{Br}(\widetilde{X}_k) \mathrm{discNS}(\widetilde{X}_k)}{q^{P_g(\widetilde{X}_k)}}.$$

Proof. Since the Tate conjecture holds for $\widetilde{X}_k$, the Artin-Tate formula is valid for $\widetilde{X}_k$. Hence the assertion follows from Lemma 5.1. □

Case (i): Assume $\mathrm{NS}(\widetilde{X}_k) = \mathrm{NS}(\widetilde{X}_{\overline{k}})$ and $\rho(\widetilde{X}_k) = B_2$.

In this case, the left-hand side of the formula in Lemma (5.2) is equal to 1. Thus

$$q^{P_g(\widetilde{X}_k)} = (-1)^{B_2(\widetilde{X}_k)-1} \# \mathrm{Br}(\widetilde{X}_k) \mathrm{discNS}(\widetilde{X}_k). \tag{11}$$

Hence $\mathrm{discNS}(\widetilde{X}_k)$ is a power of p. More precisely, one knows that

$$\mathrm{discNS}(\widetilde{X}_k) = (-1)^{B_2-1} p^{2\sigma_0(\widetilde{X}_k)}$$

for some non-negative integer $\sigma_0(\widetilde{X}_k)$. If $\widetilde{X}_k$ is a K3 surface, then σ_0 is called the Artin invariant (cf. [1], [24]). In general, if W denotes the ring of Witt vectors over $\overline{k}$ and $H^2_{cris}(\widetilde{X}_{\overline{k}}/W)$ denotes the second crystalline cohomology of $\widetilde{X}_{\overline{k}}$, then σ_0 is equal to the W-length of the cokernel of the Chern class map

$$c_1 : \mathrm{NS}(\widetilde{X}_{\overline{k}}) \otimes_{\mathbf{Z}} W \longrightarrow H^2_{cris}(\widetilde{X}_{\overline{k}}/W)$$

(see [20], II.5). To compute σ_0, we may use an algorithm of Ekedahl [10]. We explain the algorithm briefly in the case of Fermat surfaces and their quotients.

Let Y_k be a Fermat surface or a quotient of a Fermat surface by an action of a finite group which is compatible with the diagonal action of $\mu_M \times \mu_M \times \mu_M \times \mu_M$. Then $H^2_{cris}(Y_{\bar{k}}/W)$ is decomposed into a product of W-modules of rank 1:

$$H^2_{cris}(Y_{\bar{k}}/W) = V(0) \oplus \bigoplus_{\boldsymbol{a}\in\mathfrak{A}} V(\boldsymbol{a})$$

where $\boldsymbol{a} = (a_0, a_1, a_2, a_3)$ and $\mathfrak{A}$ is a subset of characters of $\mu_M \times \mu_M \times \mu_M \times \mu_M$. As in Section 4, write f for the order of p modulo M and let $H = \{p^i \pmod M) \mid 1 \le i \le f\}$. Let $O_H(\mathfrak{A})$ be the set of H-orbits in $\mathfrak{A}$. The H-orbit of $\boldsymbol{a} \in \mathfrak{A}$ will be denoted by $[\boldsymbol{a}]_p$: i.e., if h is the smallest positive integer such that $p^h\boldsymbol{a} = \boldsymbol{a}$, then

$$[\boldsymbol{a}]_p = \{\boldsymbol{a}, p\boldsymbol{a}, \cdots, p^{h-1}\boldsymbol{a}\}$$

(the length h depends on $\boldsymbol{a}$). Given an H-orbit $[\boldsymbol{a}]_p$ of length h, we define

$$b_i = \|p^i\boldsymbol{a}\| - 1 \ (0 \le i \le h-1)$$

$$n = -\min\left\{\sum_{i=0}^{j} b_i \mid 0 \le j \le h-1\right\}$$

$$\sigma_0([\boldsymbol{a}]_p) = \sum_{j=0}^{h-1}\left(n + \sum_{i=0}^{j} b_i\right).$$

In particular, if $\|p^i\boldsymbol{a}\| = 1$ for all i, then $\sigma_0([\boldsymbol{a}]_p) = 0$. Let $\widetilde{Y}_{\bar{k}}$ be an arbitrary smooth surface over $\bar{k}$ that is birational to $Y_{\bar{k}}$. Then Ekedahl's algorithm gives

$$\sigma_0(Y_{\bar{k}}) = \sum_{[\boldsymbol{a}]_p \in O_H(\mathfrak{A})} \sigma_0([\boldsymbol{a}]_p).$$

Proposition 5.3. *Let X_k be a weighted quasi-diagonal surface defined by the equation (4). Let $\widetilde{X}_k$ be the minimal resolution of X_k. Put $\sigma_0(\widetilde{X}_k) = \sum_{[\boldsymbol{a}]_p \in O_H(\mathfrak{W})} \sigma_0([\boldsymbol{a}]_p)$. Assume that there exists a positive integer κ such that $p^\kappa \equiv -1 \pmod M$. Then $\widetilde{X}_k$ is supersingular and*

$$\operatorname{discNS}(\widetilde{X}_k) = (-1)^{B_2-1} p^{2\sigma_0(\widetilde{X}_k)}$$

$$\#\operatorname{Br}(\widetilde{X}_k) = \frac{q^{P_g(\widetilde{X}_k)}}{p^{2\sigma_0(\widetilde{X}_k)}}.$$

Proof. The result is a direct application of Ekedahl's algorithm. We just note that if $p^\kappa \equiv -1 \pmod M$ for some κ, then $\mathfrak{W}_{al}(\infty) = \mathfrak{W}$ by (10). Hence $\widetilde{X}_k$ is supersingular. From Theorem 3.5, we have

$$H^2_{cris}(\widetilde{X}_{\bar{k}}/W) = V(0) \oplus \bigoplus_{a\in\mathfrak{V}} V(a) \oplus \bigoplus_{\boldsymbol{a}\in\mathfrak{W}} V(\boldsymbol{a}) \oplus E$$

where E is a submodule of $H^2_{cris}(\widetilde{X}_{\bar{k}}/W)$ associated to the exceptional cycles arising from the resolution $\widetilde{X}_{\bar{k}} \longrightarrow X_{\bar{k}}$. All the submodules of $H^2_{cris}(\widetilde{X}_{\bar{k}}/W)$, except for $V(\boldsymbol{a})$ with $\boldsymbol{a} \in \mathfrak{W}$, contribute to 0 in $\sigma_0(\widetilde{X}_k)$ since they are algebraic cycles of type $(1,1)$. Therefore $\sigma_0(\widetilde{X}_k)$ is given as above and $\#\operatorname{Br}(\widetilde{X}_k)$ can be computed from (11). □

Case (ii): Assume $\operatorname{NS}(\widetilde{X}_k) = \operatorname{NS}(\widetilde{X}_{\bar{k}})$ and $\rho(\widetilde{X}_k) = e + m/e_{23}$.

It follows from $\rho(\widetilde{X}_k) = e + m/e_{23}$ that $\operatorname{NS}(\widetilde{X}_k) \otimes_{\mathbf{Z}} \mathbf{Q}$ is spanned by the exceptional cycles and m/e_{23} lines obtained by letting $x_0 = 0$ in (4) (see Remark 3.7); in general, they do not give a $\mathbf{Z}$-basis for $\operatorname{NS}(\widetilde{X}_k)$. In what follows, we consider the case where they are indeed a $\mathbf{Z}$-basis for $\operatorname{NS}(\widetilde{X}_k)$.

Let H_0 be a subscheme of $X_{\overline{k}}$ defined by the equation $x_0 = 0$. Write U_0 for the complement of H_0 in $X_{\overline{k}}$: $U_0 = X_{\overline{k}} \setminus H_0$. By Theorem 3.1.6 of [8], we find $U_0 = \mathrm{Spec}R$, where

$$R = (\overline{k}[x_1, x_2, x_3]/(1 + x_1^{m_1} + x_2^{m_2} + x_3^{m_3}))^{\mu_{q_0}}$$

and μ_{q_0} acts on the quotient ring by $(x_1, x_2, x_3) \longmapsto (\zeta^{q_1}x_1, \zeta^{q_2}x_2, \zeta^{q_3}x_3)$ for $\zeta \in \mu_{q_0}$.

Lemma 5.4. *Let X_k be a weighted quasi-diagonal surface defined by the equation (4). Let $\widetilde{X}_k$ be the minimal resolution of X_k. Denote by $E_1, \cdots, E_e$ the exceptional curves on $\widetilde{X}_k$ arising from the singularities of X_k and by $C_1, \cdots, C_{m/e_{23}}$ the irreducible components of H_0. Write $\widetilde{C}_i$ for the strict transform of C_i via $\widetilde{X}_{\overline{k}} \longrightarrow X_{\overline{k}}$. Then with the notation above, $\{E_1, \cdots, E_e, \widetilde{C}_1, \cdots, \widetilde{C}_{m/e_{23}}\}$ gives a $\mathbf{Z}$-basis for $\mathrm{NS}(\widetilde{X}_k)$ if and only if R is a unique factorization domain.*

Proof. There are two exact sequences

$$0 \longrightarrow E_1\mathbf{Z} \oplus \cdots \oplus E_e\mathbf{Z} \longrightarrow \mathrm{Pic}(\widetilde{X}_{\overline{k}}) \longrightarrow \mathrm{Pic}(X_{\overline{k}}) \longrightarrow 0$$
$$0 \longrightarrow \widetilde{C}_1\mathbf{Z} \oplus \cdots \oplus \widetilde{C}_{m/e_{23}}\mathbf{Z} \longrightarrow \mathrm{Pic}(X_{\overline{k}}) \longrightarrow \mathrm{Pic}(U_0) \longrightarrow 0.$$

By Lemma 5.1, we have $\mathrm{NS}(\widetilde{X}_{\overline{k}}) = \mathrm{Pic}(\widetilde{X}_{\overline{k}})$. Clearly, R is normal. Hence the equivalence follows from the fact that $\mathrm{Pic}(U_0) = 0$ if and only if R is a unique factorization domain. □

We give a criterion for $\{E_1, \cdots, E_e, \widetilde{C}_1, \cdots, \widetilde{C}_{m/e_{23}}\}$ to form a $\mathbf{Z}$-basis for $NS(\widetilde{X}_k)$ (but this method does not work very often).

Lemma 5.5. *Let E_i $(1 \leq i \leq e)$ and $\widetilde{C}_j$ $(1 \leq j \leq m/e_{23})$ be as in Lemma 5.4. Let Δ be the determinant of the intersection matrix of E_i's and $\widetilde{C}_j$'s. Assume that Δ is square-free. Then $\{E_1, \cdots, E_e, \widetilde{C}_1, \cdots, \widetilde{C}_{m/e_{23}}\}$ gives a $\mathbf{Z}$-basis for $\mathrm{NS}(\widetilde{X}_k)$.*

Proposition 5.6. *Let X_k be a weighted quasi-diagonal surface defined by the equation (4). Let $\widetilde{X}_k$ be the minimal resolution of X_k. Define E_i $(1 \leq i \leq e)$, $\widetilde{C}_j$ $(1 \leq j \leq m/e_{23})$ and Δ as in Lemmas 5.4 and 5.5. Assume that R is a unique factorization domain. Then $\mathrm{discNS}(\widetilde{X}_k) = \Delta$ (up to sign) and*

$$\#\mathrm{Br}(\widetilde{X}_k) = \left|\frac{q^{P_g(\widetilde{X}_k)}}{\Delta} \prod_{a \in \mathcal{I}} \left(1 - \frac{\mathcal{J}(c, a)}{q}\right)\right|.$$

Proof. Since R is a unique factorization domain, Lemma 5.4 implies $\mathrm{discNS}(\widetilde{X}_k) = \Delta$ (up to sign). Hence the formula for the order of the Brauer group follows from Lemma 5.2. □

6. Weighted quasi-diagonal K3 surfaces

In this section, we consider weighted quasi-diagonal surfaces of which the minimal resolutions are K3. We determine all the possible weights and degrees that produce such K3 surfaces. We compute the Picard numbers of the K3 surfaces over $\overline{k}$. Furthermore we calculate the orders of the Brauer groups of several K3 surfaces.

Proposition 6.1. *Let X_k be a weighted quasi-diagonal surface in $\mathbb{P}_k^3(Q)$ of degree m defined by the equation (4). Let $\widetilde{X}_{\overline{k}}$ be the minimal resolution of $X_{\overline{k}}$. Then there exist 51 pairs of m and Q for which $\widetilde{X}_{\overline{k}}$ becomes K3; they produce 85 weighted quasi-diagonal K3 surfaces (see Table 1).*

Proof. As we noted in Section 2, $X_{\bar{k}}$ is quasi-smooth. Moreover, $X_{\bar{k}}$ is a weighted complete intersection (cf. [8]). Hence by Theorems 3.2.4 and 3.3.4 of [8], $\widetilde{X}_{\bar{k}}$ is K3 if and only if $m = q_0+q_1+q_2+q_3$. Recall that m and Q also satisfy conditions (2) and (3). Solving these conditions, we find that there are 51 such m and Q. (More generally, there are 95 pairs of m and Q which satisfy $m = q_0+q_1+q_2+q_3$ ([22]); a list of these pairs can be found in [11]. We may also use his list to find all the 51 pairs.) □

Given a weighted quasi-diagonal surface, its Picard number over $\bar{k}$ depends only on the characteristic of k. As a special case of [23] (but we need to be a little more careful with weights), we give the following formula.

Proposition 6.2. *Let X_k be one of the 85 weighted quasi-diagonal K3 surfaces in $\mathbf{P}^3_k(Q)$ of degree m obtained in Proposition* 6.1. *Let M be the integer defined in Theorem* 3.5. *Then the Picard number of $\widetilde{X}_{\bar{k}}$ is equal to*

$$\rho(\widetilde{X}_{\bar{k}}) = \begin{cases} 22 & \text{if } p^\kappa \equiv -1 \pmod{M} \text{ for some } \kappa \geq 1 \\ 22-\varphi(M) & \text{otherwise} \end{cases}$$

where φ denotes the Euler function.

Proof. We use the formula in Corollary 4.2 (b). Since $\widetilde{X}_{\bar{k}}$ is K3, its second Betti number is equal to 22. Hence

$$\rho(\widetilde{X}_{\bar{k}}) = 22 - (\mathfrak{W} - \mathfrak{W}_{al}(\infty)).$$

Since $\widetilde{X}_k$ is a K3 surface, there exits only one $\boldsymbol{a} \in \mathfrak{W}$ satisfying $\|\boldsymbol{a}\| = 0$ (for the definition of $\|\boldsymbol{a}\|$, see Section 5). Using $m = q_0 + q_1 + q_2 + q_3$ (cf. the proof of Proposition 6.1) and condition (3), we see that

$$\boldsymbol{a}_0 := ((M - M_1)/m_0, M_1, M_2, M_3)$$

is the one with this property. (The divisibility of $M - M_1$ by m_0 can be checked, for instance, by case-by-case analysis on all possible m and Q; see Table 1.) For $t \in H$, we find

$$\|t\boldsymbol{a}_0\| = \begin{cases} 0 & \text{if } t = 1 \\ 2 & \text{if } H \ni -1 \text{ and } t = -1 \\ 1 & \text{otherwise.} \end{cases}$$

Hence by (10) and Lemma 4.3,

$$\#\mathfrak{W}_{al}(\infty) = \begin{cases} \mathfrak{W} & \text{if } H \ni -1 \\ \mathfrak{W} - \#[\boldsymbol{a}_0] & \text{otherwise.} \end{cases}$$

This gives rise to

$$\rho(\widetilde{X}_{\bar{k}}) = \begin{cases} 22 & \text{if } H \ni -1 \\ 22 - \#[\boldsymbol{a}_0] & \text{otherwise.} \end{cases}$$

Here $H \ni -1$ holds if and only if $p^\kappa \equiv -1 \pmod{M}$ for some $\kappa \geq 1$. As $\gcd(M_1, M_2, M_3) = 1$, we have $\gcd((M - M_1)/m_0, M_1, M_2, M_3) = 1$. Hence $\#[\boldsymbol{a}_0] = \varphi(M)$. This completes the proof. □

Next we compute the order of the Brauer group of $\widetilde{X}_k$. When $\widetilde{X}_k$ is supersingular, our result is a special case of [16] and the idea of proof is the same as that of Shioda in [24] (but, we need to modify his method since it does not work very well for non-trivial weights).

Proposition 6.3. *Let X_k be one of the 85 weighted quasi-diagonal K3 surfaces obtained in Proposition 6.1. Let M and $\sigma_0(\widetilde{X}_k)$ be the integers defined in Theorem 3.5 and Section 5, respectively. Assume that there exists a positive integer κ such that $p^\kappa \equiv -1 \pmod{M}$. Then the following assertions hold.*

(a) *$\widetilde{X}_k$ is a supersingular K3 surface.*

(b) *If κ_0 is the smallest positive integer satisfying $p^{\kappa_0} \equiv -1 \pmod{M}$, then $\sigma_0(\widetilde{X}_k) = \kappa_0$. Consequently,*

$$\#\mathrm{Br}(\widetilde{X}_k) = \frac{q}{p^{2\kappa_0}}.$$

Proof. As we have seen in the proof of Proposition 6.2, $\boldsymbol{a}_0$ is the only character with $\|\boldsymbol{a}_0\| = 0$. From this, we find $\sigma_0(\widetilde{X}_k) = \sigma_0([\boldsymbol{a}_0]_p) = \kappa_0$. Hence the result follows from Proposition 5.3. □

We give two examples of the orders of the Brauer groups in non-supersingular cases. The first example appears originally in [24], where the focus is put rather on the supersingular case. In what follows, we discuss a non-supersingular case.

Example 6.4. Let $m = 12$ and $Q = (1,1,4,6)$. Let X_k be a weighted quasi-diagonal surface defined by the equation

$$c_0x_0^{12} + c_1x_0x_1^{11} + c_2x_2^3 + c_3x_3^2 = 0$$

with $c_i \in k^\times$. Assume $p \neq 2,3,11$ and $p \not\equiv 17,29,35,41,65 \pmod{66}$. Then X_k has exactly one singularity of type $A_{2,1}$ at $(0:0:1:\sqrt{-1})$; the singularity is defined over k. We have $\mathfrak{W}_{al}(\infty) = \emptyset$. Hence $\mathfrak{T} = \mathfrak{W}$. Moreover, $e = m/e_{23} = 1$ and $\rho(\widetilde{X}_k) = 2$. Hence the conditions of Case (ii) in Section 5 are satisfied. As $m/e_{23} = 1$, H_0 is irreducible. The singularity is on H_0. If $\widetilde{H}_0$ denotes the strict transform of H_0, then we find $\widetilde{H}_0^2 = 0$. Hence $\Delta = -1$. Therefore by Lemma 5.5 and Proposition 5.6,

$$\#\mathrm{Br}(\widetilde{X}_k) = q\prod_{\boldsymbol{a}\in\mathfrak{W}}\left(1 - \frac{\mathcal{J}(\boldsymbol{c},\boldsymbol{a})}{q}\right) = Norm_{\mathbb{Q}(\zeta_{66})/\mathbb{Q}}\left(1 - \frac{\mathcal{J}(\boldsymbol{c},\boldsymbol{a}_0)}{q}\right)$$

where ζ_{66} is a primitive 66-th root of unity in $\mathbb{C}$. Furthermore by Proposition 5.4, we conclude that

$$R = \bar{k}[x_1,x_2,x_3]/(c_0 + c_1x_1^{11} + c_2x_2^3 + c_3x_3^2)$$

is a unique factorization domain.

Example 6.5. Let $m = 42$ and $Q = (21,1,6,14)$. Let X_k be a weighted quasi-diagonal surface defined by the equation

$$c_0x_0^2 + c_1x_0x_1^{21} + c_2x_2^7 + c_3x_3^3 = 0$$

with $c_i \in k^\times$. Assume $p \neq 2,3,7$ and $p \not\equiv 5,17,20 \pmod{21}$. Then X_k has three singularities

$$\begin{aligned} P_1 &:= (0:0:*:*) && \text{of type } A_{2,1}\\ P_2 &:= (*:0:*:0) && \text{of type } A_{3,2}\\ P_3 &:= (*:0:0:*) && \text{of type } A_{7,6}\end{aligned}$$

where $*$ means some non-zero element of $\bar{k}$. Since each singularity has multiplicity 1, they are all defined over k. We have $\mathfrak{W}_{al}(\infty) = \emptyset$. Hence $\mathfrak{T} = \mathfrak{W}$. Moreover, $e = 9$, $m/e_{23} = 1$ and $\rho(\widetilde{X}_k) = 10$. Hence the conditions of Case (ii) in Section 5 are satisfied. We fix the order of the 9 exceptional curves as follows:

$$\begin{aligned} P_1 &\longleftarrow E_1\\ P_2 &\longleftarrow E_2 \cup E_3\\ P_3 &\longleftarrow E_4 \cup E_5 \cup E_6 \cup E_7 \cup E_8 \cup E_9\end{aligned}$$

where $\longleftarrow$ indicates blowings-up. We modify the method of Lemma 5.5 slightly to give a **Z**-basis for $\mathrm{NS}(\widetilde{X}_k)$. For $0 \le i \le 3$, let $\widetilde{H}_i$ be the strict transform of the hyperplane section H_i on $X_{\bar{k}}$ defined by $x_i = 0$. (Note that H_i are irreducible for $0 \le i \le 3$.) Then on $\widetilde{X}_{\bar{k}}$, we find $\widetilde{H}_0^2 = 10$ and $\widetilde{H}_3^2 = 4$. Further, from the linear equivalence

$$2\widetilde{H}_0 + E_1 \sim 7\widetilde{H}_2 + 6E_4 + 5E_5 + 4E_6 + 3E_7 + 2E_8 + E_9$$

we obtain $\widetilde{H}_2^2 = 0$. The determinant of the intersection matrix of $\{\widetilde{H}_i, E_1, \cdots, E_9\}$ is equal to

$$\begin{cases} -3^2 7^2 & \text{if } i = 0 \\ -2^2 3^2 & \text{if } i = 2 \\ -2^2 7^2 & \text{if } i = 3. \end{cases}$$

Since $\mathrm{discNS}(\widetilde{X}_k)$ must be a common divisor of these numbers, we conclude $\mathrm{discNS}(\widetilde{X}_k) = -1$. In fact, if we put $\widetilde{H} := \widetilde{H}_0 - \widetilde{H}_2 - \widetilde{H}_3$, then the determinant of the intersection matrix of $\{\widetilde{H}, E_1, \cdots, E_9\}$ is equal to -1. Hence this gives a **Z**-basis for $\mathrm{NS}(\widetilde{X}_k)$. Therefore by Proposition 5.6, we obtain

$$\#\mathrm{Br}(\widetilde{X}_k) = q \prod_{\mathbf{a} \in \mathfrak{W}} \left(1 - \frac{\mathcal{J}(\mathbf{c}, \mathbf{a})}{q}\right) = Norm_{\mathbf{Q}(\zeta_{21})/\mathbf{Q}}\left(1 - \frac{\mathcal{J}(\mathbf{c}, \mathbf{a}_0)}{q}\right)$$

where ζ_{21} is a primitive 21st root of unity in $\mathbb{C}$.

Table 1: Picard numbers of weighted quasi-diagonal K3 surfaces over $\overline{k}$

m	Q	(m_0, m_1, m_2, m_3)	M	M_1	a_0	$\rho(X_{\overline{k}})$
4	(1,1,1,1)	(4,3,4,4)	12	4	(2,4,3,3)	22 if $p \equiv 11 \pmod{12}$; 18 otherwise
5	(1,2,1,1)	(5,2,5,5)	10	5	(1,5,2,2)	22 if $p \equiv 3, 7, 9 \pmod{10}$; 18 otherwise
6	(1,1,1,3)	(6,5,6,2)	30	6	(4,6,5,15)	22 if $p \equiv 29 \pmod{30}$; 14 otherwise
	(3,1,1,1)	(2,3,6,6)	6	2	(2,2,1,1)	22 if $p \equiv 5 \pmod 6$; 20 otherwise
	(1,1,2,2)	(6,5,3,3)	15	3	(2,3,5,5)	22 if $p \equiv 14 \pmod{15}$; 14 otherwise
	(2,2,1,1)	(3,2,6,6)	6	3	(1,3,1,1)	22 if $p \equiv 5 \pmod 6$; 20 otherwise
	(2,1,1,2)	(3,4,6,3)	12	3	(3,3,2,4)	22 if $p \equiv 11 \pmod{12}$; 18 otherwise
8	(1,1,2,4)	(8,7,4,2)	28	4	(3,4,7,14)	22 if $p \equiv 3, 19, 27 \pmod{28}$; 10 otherwise
	(2,1,1,4)	(4,6,8,2)	24	4	(5,4,3,12)	22 if $p \equiv 23 \pmod{24}$; 14 otherwise
	(4,2,1,1)	(2,2,8,8)	8	4	(2,4,1,1)	22 if $p \equiv 7 \pmod 8$; 18 otherwise
	(4,1,1,2)	(2,4,8,4)	8	2	(3,2,1,2)	22 if $p \equiv 7 \pmod 8$; 18 otherwise
	(2,3,1,2)	(4,2,8,4)	8	4	(1,4,1,2)	22 if $p \equiv 7 \pmod 8$; 18 otherwise
9	(1,4,1,3)	(9,2,9,3)	18	9	(1,9,2,6)	22 if $p \equiv 5, 11, 17 \pmod{18}$; 16 otherwise
	(1,2,3,3)	(9,4,3,3)	12	3	(1,3,4,4)	22 if $p \equiv 11 \pmod{12}$; 18 otherwise
	(3,2,1,3)	(3,3,9,3)	9	3	(2,3,1,3)	22 if $p \equiv 2, 5, 8 \pmod 9$; 16 otherwise
10	(1,3,1,5)	(10,3,10,2)	30	10	(2,10,3,15)	22 if $p \equiv 29 \pmod{30}$; 14 otherwise
	(2,2,1,5)	(5,4,10,2)	20	5	(3,5,2,10)	22 if $p \equiv 19 \pmod{20}$; 14 otherwise
	(2,1,2,5)	(5,8,5,2)	40	5	(7,5,8,20)	22 if $p \equiv 39 \pmod{40}$; 6 otherwise
	(5,1,2,2)	(2,5,5,5)	5	1	(2,1,1,1)	22 if $p \equiv 2, 3, 4 \pmod 5$; 18 otherwise
12	(1,1,4,6)	(12,11,3,2)	66	6	(5,6,22,33)	22 if $p \equiv 17, 29, 35, 41, 65 \pmod{66}$; 2 otherwise
	(4,1,1,6)	(3,8,12,2)	24	3	(7,3,2,12)	22 if $p \equiv 23 \pmod{24}$; 14 otherwise
	(6,1,1,4)	(2,6,12,3)	12	2	(5,2,1,4)	22 if $p \equiv 11 \pmod{12}$; 18 otherwise
	(2,1,3,6)	(6,10,4,2)	20	2	(3,2,5,10)	22 if $p \equiv 19 \pmod{20}$; 14 otherwise
	(3,1,2,6)	(4,9,6,2)	18	2	(4,2,3,9)	22 if $p \equiv 5, 11, 17 \pmod{18}$; 16 otherwise
	(6,3,1,2)	(2,2,12,6)	12	6	(3,6,1,2)	22 if $p \equiv 11 \pmod{12}$; 18 otherwise
	(6,2,1,3)	(2,3,12,4)	12	4	(4,4,1,3)	22 if $p \equiv 11 \pmod{12}$; 18 otherwise
	(6,1,2,3)	(2,6,6,4)	12	2	(5,2,2,3)	22 if $p \equiv 11 \pmod{12}$; 18 otherwise
	(2,5,1,4)	(6,2,12,3)	12	6	(1,6,1,4)	22 if $p \equiv 11 \pmod{12}$; 18 otherwise
	(3,1,4,4)	(4,9,3,3)	9	1	(2,1,3,3)	22 if $p \equiv 2, 5, 8 \pmod 9$; 16 otherwise
	(4,4,1,3)	(3,2,12,4)	12	6	(2,6,1,3)	22 if $p \equiv 11 \pmod{12}$; 18 otherwise
	(4,1,3,4)	(3,8,4,3)	24	3	(7,3,6,8)	22 if $p \equiv 23 \pmod{24}$; 14 otherwise
	(2,5,2,3)	(6,2,6,4)	12	6	(1,6,2,3)	22 if $p \equiv 11 \pmod{12}$; 18 otherwise
	(3,3,2,4)	(4,3,6,3)	6	2	(1,2,1,2)	22 if $p \equiv 5 \pmod 6$; 20 otherwise
	(4,2,3,3)	(3,4,4,4)	4	1	(1,1,1,1)	22 if $p \equiv 3 \pmod 4$; 20 otherwise
14	(2,4,1,7)	(7,3,14,2)	42	14	(4,14,3,21)	22 if $p \equiv 5, 17, 41 \pmod{42}$; 10 otherwise
	(2,3,2,7)	(7,4,7,2)	28	7	(3,7,4,14)	22 if $p \equiv 3, 19, 27 \pmod{28}$; 10 otherwise
15	(3,6,1,5)	(5,2,15,3)	30	15	(3,15,2,10)	22 if $p \equiv 29 \pmod{30}$; 14 otherwise
	(3,2,5,5)	(5,6,3,3)	6	1	(1,1,2,2)	22 if $p \equiv 5 \pmod 6$; 20 otherwise
	(5,2,3,5)	(3,5,5,3)	15	3	(4,3,3,5)	22 if $p \equiv 14 \pmod{15}$; 14 otherwise
	(3,4,3,5)	(5,3,5,3)	15	5	(2,5,3,5)	22 if $p \equiv 14 \pmod{15}$; 14 otherwise
16	(1,5,2,8)	(16,3,8,2)	24	8	(1,8,3,12)	22 if $p \equiv 23 \pmod{24}$; 14 otherwise
	(1,3,4,8)	(16,5,4,2)	20	4	(1,4,5,10)	22 if $p \equiv 19 \pmod{20}$; 14 otherwise
	(4,3,1,8)	(4,4,16,2)	16	4	(3,4,1,8)	22 if $p \equiv 15 \pmod{16}$; 14 otherwise

(continued)

m	Q	(m_0, m_1, m_2, m_3)	M	M_1	a_0	$\rho(X_{\bar{k}})$
18	(2,1,6,9)	(9,16,3,2)	48	3	(5,3,16,24)	22 if $p \equiv 47 \pmod{48}$; 6 otherwise
	(6,2,1,9)	(3,6,18,2)	18	3	(5,3,1,9)	22 if $p \equiv 5, 11, 17 \pmod{18}$; 16 otherwise
	(6,1,2,9)	(3,12,9,2)	36	3	(11,3,4,18)	22 if $p \equiv 11, 23, 35 \pmod{36}$; 10 otherwise
	(9,1,2,6)	(2,9,9,3)	9	1	(4,1,1,3)	22 if $p \equiv 2, 5, 8 \pmod{9}$; 16 otherwise
	(3,5,1,9)	(6,3,18,2)	18	6	(2,6,1,9)	22 if $p \equiv 5, 11, 17 \pmod{18}$; 16 otherwise
	(2,4,3,9)	(9,4,6,2)	12	3	(1,3,2,6)	22 if $p \equiv 11 \pmod{12}$; 18 otherwise
20	(4,1,5,10)	(5,16,4,2)	16	1	(3,1,4,8)	22 if $p \equiv 15 \pmod{16}$; 14 otherwise
	(5,1,4,10)	(4,15,5,2)	30	2	(7,2,6,15)	22 if $p \equiv 29 \pmod{30}$; 14 otherwise
	(10,5,1,4)	(2,2,20,5)	20	10	(5,10,1,4)	22 if $p \equiv 19 \pmod{20}$; 14 otherwise
	(10,1,4,5)	(2,10,5,4)	20	2	(9,2,4,5)	22 if $p \equiv 19 \pmod{20}$; 14 otherwise
	(2,3,5,10)	(10,6,4,2)	12	2	(1,2,3,6)	22 if $p \equiv 11 \pmod{12}$; 18 otherwise
	(5,3,2,10)	(4,5,10,2)	10	2	(2,2,1,5)	22 if $p \equiv 3, 7, 9 \pmod{10}$; 18 otherwise
	(2,9,4,5)	(10,2,5,4)	20	10	(1,10,4,5)	22 if $p \equiv 19 \pmod{20}$; 14 otherwise
21	(1,10,3,7)	(21,2,7,3)	42	21	(1,21,6,14)	22 if $p \equiv 5, 17, 41 \pmod{42}$; 10 otherwise
24	(3,1,8,12)	(8,21,3,2)	42	2	(5,2,14,21)	22 if $p \equiv 5, 17, 41 \pmod{42}$; 10 otherwise
	(8,1,3,12)	(3,16,8,2)	16	1	(5,1,2,8)	22 if $p \equiv 15 \pmod{16}$; 14 otherwise
	(12,3,1,8)	(2,4,24,3)	24	6	(9,6,1,8)	22 if $p \equiv 23 \pmod{24}$; 14 otherwise
	(12,1,3,8)	(2,12,8,3)	24	2	(11,2,3,8)	22 if $p \equiv 23 \pmod{24}$; 14 otherwise
	(6,9,1,8)	(4,2,24,3)	24	12	(3,12,1,8)	22 if $p \equiv 23 \pmod{24}$; 14 otherwise
	(3,7,2,12)	(8,3,12,2)	12	4	(1,4,1,6)	22 if $p \equiv 11 \pmod{12}$; 18 otherwise
	(2,11,3,8)	(12,2,8,3)	24	12	(1,12,3,8)	22 if $p \equiv 23 \pmod{24}$; 14 otherwise
	(4,5,3,12)	(6,4,8,2)	8	2	(1,2,1,4)	22 if $p \equiv 7 \pmod{8}$; 18 otherwise
	(3,7,6,8)	(8,3,4,3)	12	4	((1,4,3,4)	22 if $p \equiv 11 \pmod{12}$; 18 otherwise
28	(1,9,4,14)	(28,3,7,2)	42	14	(1,14,6,21)	22 if $p \equiv 5, 17, 41 \pmod{42}$; 10 otherwise
	(4,3,7,14)	(7,8,4,2)	8	1	(1,1,2,4)	22 if $p \equiv 5, 17, 41 \pmod{42}$; 10 otherwise
	(7,3,4,14)	(4,7,7,2)	14	2	(3,2,2,7)	22 if $p \equiv 3, 5, 13 \pmod{14}$; 16 otherwise
30	(10,4,1,15)	(3,5,30,2)	30	6	(8,6,1,15)	22 if $p \equiv 29 \pmod{30}$; 14 otherwise
	(6,8,1,15)	(5,3,30,2)	30	10	(4,10,1,15)	22 if $p \equiv 29 \pmod{30}$; 14 otherwise
	(10,2,3,15)	(3,10,10,2)	10	1	(3,1,1,5)	22 if $p \equiv 3, 7, 9 \pmod{10}$; 18 otherwise
	(15,3,2,10)	(2,5,15,3)	15	3	(6,3,1,5)	22 if $p \equiv 14 \pmod{15}$; 14 otherwise
	(2,7,6,15)	(15,4,5,2)	20	5	(1,5,4,10)	22 if $p \equiv 19 \pmod{20}$; 14 otherwise
	(6,4,5,15)	(5,6,6,2)	6	1	(1,1,1,3)	22 if $p \equiv 5 \pmod{6}$; 20 otherwise
36	(1,5,12,18)	(36,7,3,2)	42	6	(1,6,14,21)	22 if $p \equiv 5, 17, 41 \pmod{42}$; 10 otherwise
	(3,11,4,18)	(12,3,9,2)	18	6	(1,6,2,9)	22 if $p \equiv 5, 17, 41 \pmod{42}$; 10 otherwise
40	(5,7,8,20)	(8,5,5,2)	10	2	(1,2,2,5)	22 if $p \equiv 3, 7, 9 \pmod{10}$; 18 otherwise
42	(6,1,14,21)	(7,36,3,2)	36	1	(5,1,12,18)	22 if $p \equiv 3, 7, 9 \pmod{10}$; 18 otherwise
	(14,1,6,21)	(3,28,7,2)	28	1	(9,1,4,14)	22 if $p \equiv 3, 19, 27 \pmod{28}$; 10 otherwise
	(21,1,6,14)	(2,21,7,3)	21	1	(10,1,3,7)	22 if $p \equiv 5, 17, 20 \pmod{21}$; 10 otherwise
	(2,5,14,21)	(21,8,3,2)	24	3	(1,3,8,12)	22 if $p \equiv 23 \pmod{24}$; 14 otherwise
	(14,4,3,21)	(3,7,14,2)	14	2	(4,2,1,7)	22 if $p \equiv 3, 5, 13 \pmod{14}$; 16 otherwise
48	(3,5,16,24)	(16,9,3,2)	18	2	(1,2,6,9)	22 if $p \equiv 5, 11, 17 \pmod{18}$; 16 otherwise
66	(6,5,22,33)	(11,12,3,2)	12	1	(1,1,4,6)	22 if $p \equiv 11 \pmod{12}$; 18 otherwise

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