EXAMPLES OF m-ALGEBRAS

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ABSTRACT. A few examples of m-algebras are described. Two of them are semisimple bialgebras , one is a semisimple 3-algebra.

1. FIRST DEFINITIONS AND INTRODUCTION

(1.0) Conventions. In the text, the ground field K is always of characteristic zero and algebraically closed.

Let W be K-vector space ,

$$W^m = \overbrace{W \times W \times ... \times W}^{m},$$

W^* be the dual space of W. Sometimes, elements of W (denoted as $a, b, c...$) will be called linear forms, elements of W^* (denoted as $p, q, r...$) will be called vectors. Therefore , elements of $\mathbb{P}(W)$ are hyperplanes (or lines, if W is three-dimensional), elements of $\mathbb{P}(W^*)$ are points.

Let

$$W \times W^* \to K,\ (a,p) \mapsto < a, p > \in K,$$

$$W^* \times W \to K,\ (p,a)\ \mapsto < p, a > \in K$$

be two natural pairings denoted with the same symbol $<,>$.

We will say that two bases $\{e_1, ..., e_n\} \subset W,\ \ \{f_1, ..., f_n\} \subset W^*$ are projectively dual , if $< e_i, f_j > = 0$ for $i \neq j,\ < e_i, f_i > \neq 0$. Here, projective point of view means that we do not try to normalize elements of the bases and consider them up to proportionality.

(1.1) Definition. *m-algebra.* Let m be a positive integer. m-multiplication law on W (defining an m-algebra structure on W) consists of two multilinear maps (both of which will be denoted by square brackets)

$$W^m \to W^*, \qquad (a_1, a_2, ..., a_m) \mapsto [a_1, a_2, ..., a_m] \in W^*,$$

$$(W^*)^m \to W, \qquad (p_1, p_2, ..., p_m) \mapsto [p_1, p_2, ..., p_m] \in W.$$

The number $\dim(W)$ is dimension of the m-algebra $(W, [\ \ ])$.

m-algebra $(W, [\ \ ])$ is commutative if

$$[a_{i_1}, a_{i_2}, ..., a_{i_m}] = [a_1, a_2, ..., a_m],$$

$$[p_{i_1}, p_{i_2}, ..., p_{i_m}] = [p_1, p_2, ..., p_m]$$

for any $a_1, a_2, ..., a_m \in W$, $p_1, p_2, ..., p_m \in W^*$ and for any permutation $(i_1, i_2, ..., i_m)$ of the indices $(1, 2, ..., m)$.

(1.2) Notation. *Reciprocal of* x. Let $(W, [\ ])$ be an m-algebra . For $x \in W \cup W^*$, we denote

$$\widehat{x} = [\overbrace{x, x, ..., x}^{m}]/(m!),$$

the element $\widehat{x}$ is the reciprocal of x.

Moreover, for commutative m-algebra, we introduce the following polynomial function I on $W \cup W^*$.

$$I(x) =< x, \overbrace{[x, x, ..., x]}^{m} > /((m+1)!).$$

(1.3) Definition. *Projective idempotent.* An element $x \in W \cup W^*$ is an idempotent (precisely, a projective idempotent) if

$$I(x) \neq 0, \text{ and } \ \widehat{\widehat{x}} = kx$$

for some nonzero $k \in K$.

(1.4) Definition. *Semisimplicity.* A commutative m-algebra $(W, [\ ])$ is said to be semisimple, if there exists a basis $\{e_1, ..., e_n\}$ $(n = \dim W\ )$ for W such that e_i are *idempotents*, elements $\widehat{e}_1, ..., \widehat{e}_n$ constitute a basis for W^* and these two bases are mutually dual.

Some preliminary definitions are reproduced from previous MPIM preprint [5] on the subject. Some examples of bialgebras were described in [5], where it was indicated that the bialgebra of ternary quadratic forms is semisimple. In the present text, our goal is to describe some other examples of m-algebras and to show that some of them are semisimple. I attempted to make the present text independent on [5].

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2. POTENTIALS

(2.1) Definition. *Symmetric and skew-symmetric m-algebras, their potentials.* The m-multiplication law [ ] is symmetric (resp. skew) if both the multilinear functions

$$\phi(a_1, .., a_{m+1}) =< [a_1, ..., a_m], a_{m+1} >,$$

$$\psi(p_1, ..., p_{m+1}) =< [p, ..., p_m], p_{m+1} >,$$

$$(\ \phi : W^{m+1} \to K, \quad \psi : (W^*)^{m+1} \to K\ )$$

are symmetric (resp. alternative).

In both these cases, the pair (ϕ, ψ) of these multilinear functions is called bipotential for the m-multiplication, components of the bipotential are potentials. For the case of symmetric bialgebra, the homogeneos functions $i(a) = \phi(a, ..., a)$, $i'(p) = \psi(p, .., p)$ are called potentials also, because the latter pair of functions determins bipotential for symmetric bialgebra.

It is evident that symmetric bialgebra is commutative : for any permutation $\sigma = (i_1, ..., i_m)$ of (1,2,..., m),

$$[a_{i_1}, ..., a_{i_m}] = [a_1, ..., a_m], \quad [p_{i_1}, ..., p_{i_m}] = [p_1, ..., p_m],$$

skew bialgebra is anti-commutative:

$$[a_{i_1}, ..., a_{i_m}] = \epsilon(\sigma)[a_1, ..., a_m], \quad [p_{i_1}, ..., p_{i_m}] = \epsilon(\sigma)[p_1, ..., p_m],$$

where $\epsilon(\sigma)$ is the sign of σ.

(2.2) Remark. Note that for symmetric or skew-symmetric m-algebra, the bipotential determines the m-multiplication : element $p = [a_1, ..., a_m] \in W^*$ is defined by the condition $< p, x >= \phi(a_1, ..., a_m, x)$ for all $x \in W$, similarly, element $a = [p_1, ..., p_m] \in W$ is defined by the condition $< a, y >= \psi(p_1, ..., p_m, y)$ for all $y \in W^*$.

(2.3) Definition. *Special m-algebra.* A symmetric or skew-symmetric m-algebra $(W, [\,])$ is said to be special, if there is a pair of projectively dual bases such that the linear isomorphism $W \to W^*$, defined with the help of the bases induces a coincidence (up to proportionality) of the bipotential components.

(2.4) Definition. *Projectively equivariant m-multiplication.*

(1) If $g : W \to W$ is a linear transformation , then $g^t : W^* \to W^*$ is its contragredient transformation, that is g^t is defined by the identity

$$< g(a), p >=< a, g^t(p) > .$$

(2) Suppose that a group G acts on W. We say that an m-algebra structure on W is projectively G-equivariant (or $\mathbb{P}(G)$-equivariant), if for some multiplicative character χ of G and for any $a, b \in W$, we have $[g(a_1), ..., g(a_m)] = \chi(g)(g^t)^{-1}([a_1, .., a_m])$.

(3) If the m-multiplication on W is projectively GL(W)-equivariant, then we say that it is projectively equivariant or PGL(W)-equivariant.

In the case of a symmetric projectively equivariant m-multiplication, the potential components ϕ and ψ are simultaneous invariants of the argument $(m + 1)$-tuple:

$$\phi(g(a_1), ..., g(a_{m+1})) = \chi_1(g)\phi(a_1, ..., a_{m+1}), \quad \psi(g(p_1), ..., g(p_{m+1})) = \chi_2(g)\phi(p_1, ..., p_{m+1})$$

for some characters χ_1, χ_2.

(2.5) Remark. For projectively equivariant m-multiplication, the function $I(x)$ from (1.2) is an invariant . The reciprocal $\widehat{x}$ of x is a contravariant of x.

3. EXTENSION OF BIMULTIPLICATION ON SYMMETRIC POWERS

(3.1) Notation. We will denote by $<,>$ the extension of the duality pairing $<,>$ to the symmetric powers $S^l(W)$, precisely,

$$<.,.>: S^l(W) \times S^l(W^*) \to K$$

is such that $< a^l, p^l >=< a, p >^l$, and therefore

$$< a^l, p_1p_2...p_l >=< a, p_1 >< a, p_2 > ... < a, p_l > .$$

(3.2) Theorem. Let $(W, [\ \ ])$ be an m-bialgebra with either skew-symmetric or symmetric PGL(W)-equivariant m-multiplication . Then for any natural l , there exists a unique PGL(W)-equivariant m-multiplication on $S^l(W)$ such that

$$[a_1^l, ..., a_m^l] = [a_1, ..., a_m]^l, \quad [p_1^l, ..., p_m^l] = [p_1, ..., p_m]^l,$$

where $a_1, ..., a_m \in W, \quad p_1, ..., p_m \in W^*$.

If the initial m-multiplication is symmetric (resp. special), then new m-multiplications are also symmetric (resp. special).

If the initial m-multiplication is skew, then the m-multiplication on $S^l(W)$ is skew by odd l and symmetric by even l.

Proof. An explanation (which is natural with the point of view of invariantists of XIX-th century) is based on the symbolic method : if $\phi(a_1, .., a_m, a_{m+1})$ is a potential for the m-multiplication in $(W, [\ ])$, and elements $f_1, ..., f_m, f_{m+1} \in S^l(W)$ are written symbolically as $f_1 = a_1^l, ..., f_m = a_m^l, f_{m+1} = a_{m+1}^l$, then the potential ϕ_l for the m-multiplication on $S^l(W)$ is defined symbolically by $\phi_l(f_1, ..., f_m, f_{m+1}) = (\phi(a_1, ..., a_m, a_{m+1}))^l$.

One can generalize Theorem (3.2) by the following way.

(3.3) Theorem. Let k be a natural number. Suppose that it is given a PGL(W)-equivariant (with respect to the representation in k-th symmetric power of W) m-multiplication

$$[...] : (S^k(W))^m \to S^k(W^*), \quad [...] : (S^k(W^*))^m \to S^k(W).$$

Suppose that this m-multiplication is either skew-symmetric or symmetric. Then for any natural l , there exists a unique PGL(W)-equivariant m-multiplication

$$[...] : (S^{lk}(W))^m \to S^{lk}(W^*), \quad [...] : (S^{lk}(W^*))^m \to S^{lk}(W),$$

such that

$$[(a_{11}a_{12}...a_{1k})^l, .., (a_{m1}a_{m2}...a_{mk})^l] = [a_{11}a_{12}...a_{1k}, ..., a_{m1}a_{m2}...a_{mk}]^l,$$

where

$$a_{ij} \in W, \quad p_{ij} \in W^*, \quad 1 \le i \le m, \quad 1 \le i \le m.$$

If the initial m-multiplication is symmetric (resp. special), then new m-multiplications are also symmetric (resp. special).

If the initial m-multiplication is skew, then the m-multiplication on $S^{lk}(W)$ is skew by odd l and symmetric by even l.

Proof is almost the same as for Theorem (3.2), therefore we omit it.

4. EXAMPLES

(4.1) Example . *Example of a bialgebra constructed with the help of a semisimple Lie algebra .* Let W be a finite-dimensional semisimple Lie algebra over the ground field K , $\{,\}$ be the bracket in W, M be an exact finite-dimensional W-module a_M be the operator corresponding to $a \in W$ by the representation . The Killing form of the representation is $(a,b)_M = \text{Trace}(\ (a_M) \circ (b_M))$. The Killing form for the regular representation will be denoted by (a,b). Let W^* be the dual space of W, $f : W \to W^*$ be the isomorphism, corresponding to the Killing form $(.,.)$, that is $< f(a), x >= (a,x)$. This is the so-called Killing isomorphism, see (1.2.21) ,(1.5.14) from the book [1] by J. Dixmier. Then the homomorphisms

$$W \times W \to W^*, \quad (a,b) \mapsto f(\{a,b\}),$$

$$W^* \times W^* \to W, \quad (p,q) \mapsto \{f^{-1}(p), f^{-1}(q)\}$$

define a structure of a special skew bialgebra $(W, [,])$.

If G is an algebraic group such that its Lie algebra is W and the above representation is induced by a representation of G and

$$\text{Ad} : G \to \text{Aut}(W)$$

is the adjoint representation, then the bimultiplication $[,]$ is $\mathbb{P}(\text{Ad})$-equivariant.

Thus we have a structure of a symmetric 1-algebra on the the space W of the Lie algebra W,

$$f : W \to W^*, \quad f^{-1} : W^* \to W,$$

the extension of the structure to the symmetric powers

$$S^l(f) : S^l(W) \to S^l(W^*), \quad S^l(f^{-1}) : S^l(W^*) \to S^l(W),$$

is the extended Killing isomorphism (see of J.Dixmier, [1], (2.4.10)). Also, according to Theorem (3.2), we can extend the bimultiplication on every symmetric power $S^l(W)$ of the Lie algebra (in other words, on every homogeneous component $U^l(W)$ of the universal enveloping algebra of the Lie algebra W.).

(4.2) Example . *An example of a skew $(m+1)$-dimensional m-algebra.* Let W be $(m+1)$-dimensional vector space, Ω be a fixed non-zero element of $\Lambda^{m+1}(W)$, Ω^* be a fixed non-zero element of $\Lambda^{m+1}(W^*)$, (the elements are defined up to a constant multiplier). The m-multiplication is defined by the following formulas.

$$[a_1, ..., a_m] = (a_1 \wedge ... \wedge a_m) \lrcorner \Omega^*,$$

$$[p_1, ..., p_m] = (p_1 \wedge ... \wedge p_m) \lrcorner \Omega.$$

It is evident that it is a skew m-multiplication. If $m = 2$, then we get a particular case of the construction (4.1), where W is sl(2).

Up to natural proportionality , for a fixed m, there is only one $(m+1)$-dimensional skew m-algebra with nontrivial (that is with both non-zero brackets) m-multiplication. Indeed, there exits only one possible skew-symmetric multilinear function on W^{m+1} , and only one skew function on $(W^*)^{m+1}$.

As a corollary of Theorem (3.2), we get an m-multiplication on every space $S^l(W)$, the extended m-multiplication is $\mathrm{PGL}(m+1)$-equivariant, skew for odd l and symmetric for even l.

For example, there are special symmetric tri-multiplications on quadric, quartic, etc.. surfaces of the three-dimensional projective space .

W. Fr. Meyer, in [8], page 1562 writes the following. Let equations of three quartic surfaces be written symbolically as

$$F_4 = (ax)^4 = 0, \quad G_4 = (bx)^4 = 0, \quad H_4 = (cx)^4 = 0.$$

Then $< abcu >^4 = 0$ is the symbolic equation of a quartic surface enveloping the planes, cutting three plane quartics (on the three surfaces), whose tri-linear simultaneous invariant vanishes. The mentioned quartic surface is a result of tri-multiplication of the initial three surfaces.

(4.3) Example. *The polarized Cayley curve of three cubic curves.* There is structure of a symmetric 3-algebra on the space of ternary cubics. If f, g, h are three ternary cubic forms, whose coefficients are a_{ijk}, b_{ijk}, c_{ijk} respectively, then

$$[f,g,h] = \sum_{(f,g,h)} \begin{vmatrix} a_{000} & a_{011} & a_{022} & a_{012} & a_{020} & a_{001} \\ b_{100} & b_{111} & b_{122} & b_{112} & b_{120} & b_{101} \\ c_{200} & c_{211} & c_{222} & c_{212} & c_{220} & c_{201} \\ 2u_0 & 0 & 0 & 0 & u_2 & u_1 \\ 0 & 2u_1 & 0 & u_2 & 0 & u_0 \\ 0 & 0 & 2u_2 & u_1 & u_0 & 0 \end{vmatrix}$$

where (f, g, h) runs all the six permutations. Note that $\widehat{f}$ is the so-called Cayley contravariant $\mathrm{Ca}(f)$ of f. Use of the Cayley form allows to restitute the main invariants and cubic (with respect to the variables) covariants and contravariants of a general ternary cubic form. Indeed, the invariant $S(f)$ of the forth degree with respect to the coefficients is

$$S(f) = - < f, \mathrm{Ca}\ (f) > .$$

Further, the form $\mathrm{Ca}(\mathrm{Ca}(f)) + 12S(f)^2 f$ is a reducible polynomial (as a polynomial from the ring over $\mathbb{Z}$ generated by the coefficients and the cogredient variables), precisely,

$$\mathrm{Ca}(\mathrm{Ca}(f)) + 12S(f)^2 f = 4T(f) \cdot \mathrm{He}(f),$$

where $T(f)$ depends on the coefficients (that is $T(f)$ is an invariant), degree of $T(f)$ is equal to 6, $\mathrm{He}(f)$ is a cubic form whose coefficients are homogeneous polynomials of degree 3 from the initial coefficients. $\mathrm{He}(f)$ is the Hessian of f. Note that

$$T(f) = - < \mathrm{Ca}(f) \cdot He(f) > .$$

The discriminant of f is

$$\mathrm{disc}(f) = T(f)^2 - S(f)^3.$$

The idempotents with respect to the operation (bilinearization of Ca) are curves projectively equivalent to one of the following .

$$x_0 x_1 x_2,$$

$$(1 \pm \sqrt{3})(x_0^3 + x_1^3 + x_2^3) + 6x_0x_1x_2.$$

(4.4) Example. The Eisenstein tri-algebra . *A special symmetric 3-multiplication on the space V_3 of binary cubic forms.* Because of symmetry, it is enough to indicate formulas for the reciprocal contragredient cubic form

$$\widehat{f} = \widehat{a_0}u_0^3 + 3\widehat{a_1}u_0^2u_1 + 3\widehat{a_2}u_0u_1^2 + \widehat{a_3}u_1^3$$

of the form

$$f = a_0x_0^3 + 3a_1x_0^2x_1 + 3a_2x_0x_1^2 + a_3x_1^3.$$

The reciprocity transformation is described by the following formulas.

$$\widehat{a_0} = 2a_2^3 - 3a_1a_2a_3 + a_0a_3^2,$$

$$\widehat{a_1} = 2a_1^2a_3 - a_0a_2a_3 - a_1a_2^2,$$
$$\widehat{a_2} = 2a_0a_2^2 - a_0a_1a_3 - a_1^2a_2,$$
$$\widehat{a_3} = 2a_1^3 - 3a_0a_1a_2 + a_0^2a_3.$$

The potential coincides with the discriminant

$$D(f) = a_0^2a_3^2 - 3a_1^2a_2^2 - 6a_0a_1a_2a_3 + 4a_0a_2^3 + 4a_3a_1^3.$$

The above formulas for the reciprocity transformation can be written as follows.

$$\widehat{a_0} = \frac{1}{2}\frac{\partial D}{\partial a_0}, \quad \widehat{a_1} = \frac{1}{6}\frac{\partial D}{\partial a_1}, \quad \widehat{a_2} = \frac{1}{6}\frac{\partial D}{\partial a_2}, \quad \widehat{a_3} = \frac{1}{2}\frac{\partial D}{\partial a_3}.$$

This transformation is a birational involution of the three-dimensional projective space $\mathbb{P}^3 = \mathbb{P}(V_3)$ of binary cubics , precisely,

$$\widehat{\widehat{f}} = D^2(f) \cdot f$$

(4.4.2) Theorem. The symmetric tri-algebra of binary cubics is semisimple. In the space of binary cubics, there exist a basis $\{e_1, e_2, e_3, e_4\}$ whose elements have non-zero discriminants, and this basis is the dual of $\{\widehat{e_1}, \widehat{e_2}, \widehat{e_3}, \widehat{e_4}\}$.

Proof. It is enough to write down four mutually apolar binary cubics. Let us introduce the following numbers ω and λ.

$$\omega = \frac{-1 + i\sqrt{3}}{2}, \quad \lambda = \frac{-2 + i\sqrt{2}}{2}.$$

Let us consider the following four cubic forms.

$$e_1 = x_0^3 - x_1^3,$$

$$e_2 = \lambda(x_0^3 + x_1^3) + 3(x_0^2x_1 + x_0x_1^2),$$
$$e_3 = \lambda(x_0^3 + x_1^3) + 3(\omega x_0^2x_1 + \omega^2x_0x_1^2),$$
$$e_4 = \lambda(x_0^3 + x_1^3) + 3(\omega^2x_0^2x_1 + \omega x_0x_1^2).$$

Then

$$\widehat{e_1} = u_0^3 - u_1^3,$$
$$\widehat{e_2} = (\lambda-1)^2[(\lambda+2)(u_0^3+u_1^3) - 3(u_0^2u_1 + u_0u_1^2],$$
$$\widehat{e_3} = (\lambda-1)^2[(\lambda+2)(u_0^3+u_1^3) - 3(\omega^2u_0^2u_1 + \omega u_0u_1^2],$$
$$\widehat{e_4} = (\lambda-1)^2[(\lambda+2)(u_0^3+u_1^3) - 3(\omega u_0^2u_1 + \omega^2 u_0u_1^2].$$

It is not hard to see that if $i \neq j$, then $< e_i \cdot \widehat{e_j} >= 0$, and that $< e_i \cdot \widehat{e_i} >\neq 0$.

The theorem is proved.

(4.5). Remarks . *Additional remarks, historical remarks and a conjecture.* By the transformation $f \mapsto \widehat{f}$, the differential form

$$a_0da_3 - a_3da_0 + 3(a_2da_1 - a_1da_2)$$

is invariant (up to a variable multiplier, a power of the discriminant D). This property means that the transformation is the so-called birational contact transformation of the space. See Fano [3], no.6 , page 42. Fano gives a geometric description of this birational transformation, lists some geometers which studied the transformation, but he misses G. Eisenstein.

It seems that now nobody knows that the first discoverer of the transformation was G. Eisenstein [2]. He marked both the properties

$$\widehat{\widehat{f}} = D^2(f)\cdot f, \quad D(\widehat{f}) = (D(f))^3. \tag{4.5.1}$$

Eisenstein indicated that the pair of identities (4.5.1) is an analog of the behavior of the determinant of a symmetric 3×3 matrix and its adjoint matrix. Moreover , he noticed that there is a parallelism between representations of a number as a discriminant and as a sum of four squares , that is (if we use more modern language) he saw a parallelism between the quaternion norm and the discriminant. Maybe our consideration of the discriminant as a potential (potential is a variant of the notion of norm) of the symmetric tri-multiplication is a partial explanation of the analogy marked by Eisenstein. In the time of writing of the short article [2], his age was 20 years , the editor of the Journal indicates that the author is a student. The article belongs to Eisenstein's large series of papers published in the issue.

Further, the properties (4.5.1) indicate that $\mathbb{P}(\widehat{\ })$ is an automorphism of the complement to the discriminant hypersurface in $\mathbb{P}^3$. There are other automorphisms of the complement. Of course, the group PGL(2) acts on the complement. Moreover, if $\Phi_m(s,t)$ is a binary form of degree m , and $\phi = \Phi_m(a_0^4, D)$, then the following transformation is an automorphism of the complement of the discriminant surface.

$$a_0{}' = a_0D^{3m},$$

$$a_1{}' = a_1D^{3m} + a_0D^{2m}\phi,$$
$$a_2{}' = a_2D^{3m} + 2a_1D^{2m}\phi + a_0D^m\phi^2,$$
$$a_3{}' = a_3D^{3m} + 3a_2D^{2m}\phi + 3a_1D^m\phi^2 + a_0\phi^3.$$

This family is an infinite-dimensional subgroup of the group of automorphisms.

Conjecture. The group $\mathrm{Aut}(\mathbb{P}^3 - D)$ is generated by PGL(2), $\mathbb{P}(\widehat{\ })$ and by the mentioned infinite-dimensional subgroup.

(4.6) Example . For the space V_3 of binary cubics, there is a natural structure of a skew PGL(2)-equivariant 3-multiplication. If $F_{ij} = \partial^2 F/\partial x_i \partial x_j$, and f, g, h are three binary cubic forms, then we pose

$$[f, g, h] = N(f, g, h)(-u_1, u_0),$$

where

$$N(f, g, h) = \begin{vmatrix} f_{00} & f_{01} & f_{11} \\ g_{00} & g_{01} & g_{11} \\ h_{00} & h_{01} & h_{11} \end{vmatrix}$$

(4.7) Example. There is a PGL(2)-equivariant structure of a symmetric 3-algebra on the space V_{3m} of binary forms of degree $3m$. It is a corollary of the example (4.5) and of Theorem 3.3. For binary sextics the corresponding potential is $(i, i)^4$ (in the notations of [6] , page 156), the reciprocal $\widehat{f}$ coincides with $(i, f)^2(-u_1, u_0)$ (in the same notations). Also ,we can equip the same spaces with a structure of 3-algebra using the example (4.6) and Theorem 3.3. For binary sextics, both the structure are the same.

(4.8) Example. There is a PGL(2)-equivariant structure of symmetric 3-algebra on the space of binary quintics , the corresponding $\widehat{f}$ is the image of the covariant (i, f) (in notations of Grace and Young [6] , page 131) by the substitution $x_0 = -u_1, x_1 = u_0$. The potential is $A(f)$ (loc. cit.).

(4.9) Example. Let V be a threedimensional vector space over K, $W = V \otimes V^*$. If we fix a basis x_0, x_1, x_2 of V and the dual basis u_0, u_1, u_2 of V^*, then element of W has the form

$$a = \sum_{i,j} a_{ij} x_i u_j =$$

$$a_{00}x_0u_0 + a_{01}x_0u_1 + a_{02}x_0u_2 +$$

$$a_{10}x_1u_0 + a_{11}x_1u_1 + a_{12}x_1u_2 +$$

$$a_{20}x_2u_0 + a_{21}x_2u_1 + a_{22}x_2u_2.$$

The dual space is $W^* = V^* \otimes V$, its elements are

$$p = \sum_{i,j} p_{ij} u_i x_j =$$

$$p_{00}u_0x_0 + p_{01}u_0x_1 + p_{02}u_0x_2 +$$

$$p_{10}u_1x_0 + p_{11}u_1x_1 + p_{12}u_1x_2 +$$

$$p_{20}u_2x_0 + p_{21}u_2x_1 + p_{22}u_2x_2.$$

Of course ,

$$< a, p >= \sum_{i,j} a_{ij} p_{ij}.$$

Elements of W are 3×3 matrices. Let us consider the cubic invariant $I(a) = \det(a)$. The polarization of the invariant defines a potential for a special bimultiplication [,] on W.

The reciprocal $\widehat{a}$ is the adjugate matrix of a, therefore all the nondegenerate matrices are idempotents.

Theorem. The bialgebra $(W, [,])$ is semisimple.

Proof. Indeed, if we denote by ω a root of the equation $x^2 + x + 1 = 0$, then the following basis satisfies all the necessary conditions.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \omega & 0 \\ 0 & 0 & \omega^2 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \omega^2 & 0 \\ 0 & 0 & \omega \\ 1 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ \omega & 0 & 0 \\ 0 & \omega^2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ \omega^2 & 0 & 0 \\ 0 & \omega & 0 \end{pmatrix}.$$

(4.10) Example . Let us consider 15-dimensional vector space W of skew-symmetric 6×6 matrices. Such a matrix a is defined by its elements a_{ij}, where $1 \le i < j \le 6$. The dual space W^* consists of similar matrix, whose elements are p_{ij}. The pairing is

$$< a, p >= \sum_{i<j} a_{ij} p_{ij}.$$

Let us consider the Pfaffian cubic polynomial

$$\mathrm{Pf}(a) =$$

$$a_{16}a_{25}a_{34} + a_{14}a_{26}a_{35} + a_{15}a_{24}a_{36} +$$

$$a_{16}a_{23}a_{45} + a_{13}a_{25}a_{46} + a_{12}a_{34}a_{56} +$$

$$a_{14}a_{23}a_{56} + a_{12}a_{36}a_{45} - a_{15}a_{26}a_{34} -$$

$$a_{16}a_{24}a_{35} - a_{14}a_{25}a_{36} - a_{13}a_{26}a_{45} -$$

$$a_{15}a_{23}a_{46} - a_{12}a_{35}a_{46} - a_{13}a_{24}a_{56}.$$

This potential defines a special bimultiplication [,] on W. If $p = \widehat{a}$, then elements of p are defined by the following formulas.

$$p_{12} = a_{34}a_{56} + a_{36}a_{45} - a_{35}a_{46},$$

$$p_{13} = a_{25}a_{46} - a_{26}a_{45} - a_{24}a_{56}.$$

$$p_{14} = a_{26}a_{35} + a_{23}a_{56} - a_{25}a_{36},$$

$$p_{15} = a_{24}a_{36} - a_{26}a_{34} - a_{23}a_{46},$$

$$p_{16} = a_{25}a_{34} + a_{23}a_{45} - a_{24}a_{35},$$

$$p_{23} = a_{16}a_{45} + a_{14}a_{56} - a_{15}a_{46},$$

$$p_{24} = a_{15}a_{36} - a_{16}a_{35} - a_{13}a_{56},$$

$$p_{25} = a_{16}a_{34} + a_{13}a_{46} - a_{14}a_{36},$$

$$p_{26} = a_{14}a_{35} - a_{15}a_{34} - a_{13}a_{45},$$

$$p_{34} = a_{16}a_{25} + a_{12}a_{56} - a_{15}a_{26},$$

$$p_{35} = a_{14}a_{26} - a_{16}a_{24} - a_{12}a_{46},$$

$$p_{36} = a_{15}a_{24} + a_{12}a_{45} - a_{14}a_{25},$$

$$p_{45} = a_{16}a_{23} + a_{12}a_{36} - a_{13}a_{26},$$

$$p_{46} = a_{13}a_{25} - a_{15}a_{23} - a_{12}a_{35},$$

$$p_{56} = a_{12}a_{34} + a_{14}a_{23} - a_{13}a_{24}.$$

Formulas for $[a, b]$ can be derived by the bilinearization of these formulas.

The map $a \mapsto \widehat{a}$ induces a birational map

$$\mathbb{P}(\widehat{\ }) : \mathbb{P}(W) \to \mathbb{P}(W^*),$$

precisely,

$$\widehat{\widehat{a}} = \mathrm{Pf}(a) \cdot a.$$

Therefore, every nondegenerate (that is with nonzero Pfaffian) element $a \in W$ is an idempotent.

Theorem. The fifteen-dimensional bialgebra $(W, [,])$ of 6×6 skew-symmetric matrices is semisimple.

Proof. For $1 \leq l < m \leq 6$, let $A(l, m)$ be the matrix whose elements a_{ij} are defined by the conditions

$$a_{ij} = 0, \text{ if } (i, j) \neq (l, m), \quad (i, j) \neq (m, l),$$

$$a_{lm} = 1, \qquad a_{ml} = -1.$$

Let us consider the following list of five triples of ordered pairs

$$(\{1, 2\}, \{3, 6\}, \{4, 5\}),$$

$$(\{1, 3\}, \{2, 4\}, \{5, 6\}),$$

$$(\{1, 4\}, \{3, 5\}, \{2, 6\}),$$

$$(\{1, 5\}, \{4, 6\}, \{2, 3\}),$$

$$(\{1, 6\}, \{2, 5\}, \{3, 4\}).$$

The fifteen matrices

$$A(i, j) + A(k, l) + A(m, n),$$

$$A(i, j) + \omega A(k, l) + \omega^2 A(m, n),$$

$$A(i, j) + \omega^2 A(k, l) + \omega A(m, n),$$

where $(\{i,j\},\{k,l\},\{m,n\})$ is a triple of the list, form the basis of W with the necessary properties. Precisely, the fifteen matrices are

$$A(1,2)+A(3,6)+A(4,5)=\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix},$$

$$A(1,2)+\omega A(3,6)+\omega^2 A(4,5)=\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega \\ 0 & 0 & 0 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & -\omega^2 & 0 & 0 \\ 0 & 0 & -\omega & 0 & 0 & 0 \end{pmatrix},$$

$$A(1,2)+\omega^2 A(3,6)+\omega A(4,5)=\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega^2 \\ 0 & 0 & 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & -\omega & 0 & 0 \\ 0 & 0 & -\omega^2 & 0 & 0 & 0 \end{pmatrix},$$

$$A(1,3)+A(2,4)+A(5,6)=\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix},$$

$$A(1,3)+\omega A(2,4)+\omega^2 A(5,6)=\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\omega & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega^2 \\ 0 & 0 & 0 & 0 & -\omega^2 & 0 \end{pmatrix},$$

$$A(1,3)+\omega^2 A(2,4)+\omega A(5,6)=\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega^2 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\omega^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega \\ 0 & 0 & 0 & 0 & -\omega & 0 \end{pmatrix},$$

$$A(1,4)+A(3,5)+A(2,6)=\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A(1,4)+\omega A(3,5)+\omega^2 A(2,6)=\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega^2 \\ 0 & 0 & 0 & 0 & \omega & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\omega & 0 & 0 & 0 \\ 0 & -\omega^2 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A(1,4)+\omega^2A(3,5)+\omega A(2,6)=\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega \\ 0 & 0 & 0 & 0 & \omega^2 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\omega^2 & 0 & 0 & 0 \\ 0 & -\omega & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A(1,5)+A(4,6)+A(2,3)=\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix},$$

$$A(1,5)+\omega A(4,6)+\omega^2A(2,3)=\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \omega^2 & 0 & 0 & 0 \\ 0 & -\omega^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\omega & 0 & 0 \end{pmatrix},$$

$$A(1,5)+\omega^2A(4,6)+\omega A(2,3)=\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \omega & 0 & 0 & 0 \\ 0 & -\omega & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & \omega^2 \\ 0 & 0 & 0 & 0 & -\omega^2 & 0 \end{pmatrix},$$

$$A(1,6)+A(2,5)+A(3,4)=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A(1,6)+\omega A(2,5)+\omega^2 A(3,4)=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & \omega^2 & 0 & 0 \\ 0 & 0 & -\omega^2 & 0 & 0 & 0 \\ 0 & -\omega & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A(1,6)+\omega^2 A(2,5)+\omega A(3,4)=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & \omega & 0 & 0 \\ 0 & 0 & -\omega & 0 & 0 & 0 \\ 0 & -\omega^2 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(4.11) Example. Of course, if we have a non-degenerated symmetric 1-algebra, that is a symmetric isomorphism between W and W^*, then the potential is a nondegenerate quadratic form. Such a 1-algebra is semisimple, because for this case semisimplicity means existence of an orthogonal basis. We will indicate a very special example of such 1-algebra, in this example one can see an analog of six Gerbaldi's conics from [5] (in this case it is the set of three binary forms), and a simple analog of the Valentiner group (in this case it is the octahedron group).

Let W be three-dimensional space of binary forms. Three binary quadratic forms

$$2t_0t_1, \quad t_0^2-t_1^2, \quad t_0^2+t_1^2,$$

are mutually apolar. The three pairs of points on the projective line defined by the three quadratic forms are

$$\{(0:1),(1:0)\}, \quad \{(1:1),(1:-1)\}, \quad \{(1:i),(1:-i)\}.$$

There is a connection with the Bernulli lemniscate and with the octahedral group, see [5]. The octahedral group of fractional linear transformations is described in books [7], [4] by Klein and Ford. This set of 24 fractional linear transformations has the following description (where $t = t_1/t_0$ is the affine coordinate on the projective line, and k runs the set $\{0, 1, 2, 3\}$).

$$t' = i^k t, \quad t' = i^k/t,$$

$$t' = i^k \frac{t+1}{t-1}, \quad t' = i^k \frac{t-1}{t+1},$$

$$t' = i^k \frac{t+i}{t-i}, \quad t' = i^k \frac{t-i}{t+i}.$$

See Klein [7], page 137, formulas (4a), (4b), (4c), or Ford [4] , no. 55, page 124-125 . The transformations of the group G_{24} (isomorphic to the group of substitutions of four symbols) permutates the three pairs of points on the projective line.

References

[1] J. Dixmier, *Enveloping algebras*, American Mathematical Society, 1977, 1996.

[2] G. Eisenstein, *Ueber eine merkwürdige identishe Gleichung*, Journal für die reine und angewandte Mathematik **vol 27** (1844), 105-106; = *Mathematishe Werke*, vol. I, Chelsea, New York, 1975, 1989, pp. 26-27..

[3] G. Fano, *Trasformazioni di contatto birazionali del piano*, Atti del congresso Internazionale dei matematici. Bologna, 3-10 Settembre , 1928 (VI). Tomo IV, sezione II (A-B), Nicola Zanichelli, Bologna, pp. 35-42.

[4] L.R. Ford, *Automorphic functions*, Chelsea, 1929, 1951.

[5] M. Gizatullin, *Bialgebra and geometry of plane quartics. Preprint of MPIM.*

[6] J.H.Grace, A. Young, *The algebra of invariants*, Cambridge, 1903.

[7] F. Klein, Elementarmathematik von höheren Standpunkt aus **I** (1933, 1968), Springer, Berlin.

[8] W.Fr. Meyer, *Flächen vierter und höherer Ordnung*, Encyklopädie der mathematischen Wissenschaften, vol. III C 10 b, Teubner, Leipzig, 1898-1904, pp. 1533 -1779 .