SUMS OF SQUARES OF $|\zeta(\frac{1}{2}+it)|$ OVER SHORT INTERVALS

Aleksandar Ivić

Abstract. Sums of squares of $|\zeta(\frac{1}{2}+it)|$ over short intervals are investigated. Known upper bounds for the fourth and twelfth moment of $|\zeta(\frac{1}{2}+it)|$ are derived. A discussion concerning other possibilities for the estimation of higher power moments is given.

1. Introduction

Let as usual $\zeta(s)$ denote the Riemann zeta-function. The aim of this paper is to estimate the sum

$$\sum_{r\le R}\int_{t_r-2G}^{t_r+2G}\varphi_r(t)|\zeta(\tfrac{1}{2}+it)|^2\,\mathrm{d}t, \tag{1.1}$$

where $\varphi_r\in C^\infty$ is a non-negative function supported in $[t_r-2G,\, t_r+2G]$ that equals unity in $[t_r-G,\, t_r+G]$, $T\le t_1<\cdots<t_R\le 2T$, $t_{r+1}-t_r\ge 5G\,(r=1,\dots,R-1)$, and $T^\varepsilon\le G\le T^{1/3}$ (here and later ε denotes positive, absolute constants, not necessarily the same ones at each occurrence). This sum majorizes the classical sum

$$\sum_{r\le R}\int_{t_r-G}^{t_r+G}|\zeta(\tfrac{1}{2}+it)|^2\,\mathrm{d}t, \tag{1.2}$$

which is of great importance in zeta-function theory (see K. Matsumoto [18] for an extensive account on mean square theory involving $\zeta(s)$). One can treat the sum in (1.1) by at least the following methods.

a) Using exponential averaging (or some other smoothing like φ_r above), namely the Gaussian weight $\exp(-\frac{1}{2}x^2)$, in connection with (γ is Euler's constant)

$$E(T):=\int_0^T|\zeta(\tfrac{1}{2}+it)|^2\,\mathrm{d}t-T\left(\log\frac{T}{2\pi}+2\gamma-1\right),$$

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since there exists a well-known explicit formula of F.V. Atkinson [1] for $E(T)$ (see also the author's monographs [5] and [6]). This was done by D.R. Heath-Brown [2], who obtained

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^{12}\,\mathrm{d}t \ \ll\ T^2\log^{17}T, \tag{1.3}$$

which is hitherto the best estimate of its kind, with many applications to multiplicative number theory.

b) One can use the Voronoi summation formula (e.g., see [5, Chapter 3]) for the explicit expression (approximate functional equation) for $|\zeta(\frac{1}{2}+it)|^2 = \chi^{-1}(\frac{1}{2}+it)\zeta^2(\frac{1}{2}+it)$, where $\zeta(s) = \chi(s)\zeta(1-s)$, namely

$$\chi(s) \ =\ 2^s\pi^{s-1}\sin(\tfrac{1}{2}\pi s)\Gamma(1-s).$$

Voronoi's formula is present indirectly in Atkinson's formula, so that this approach is more direct. The effect of the smoothing function φ_r in (1.1) is to shorten the sum approximating $|\zeta|^2$ to the range $\frac{T}{2\pi}(1-G^{-1}T^\varepsilon) \le n \le \frac{T}{2\pi}\,(T=t_r)$. After this no integration is needed, and proceeding as in [5, Chapters 7-8] one obtains that the sum in (1.1) equals $O(RGT^\varepsilon)$ plus a multiple of

$$\sum_{r\le R}\int_{t_r-2G}^{t_r+2G}\varphi_r(t)\sum_{k\le T^{1+\varepsilon}G^{-2}}(-1)^k d(k)k^{-1/2}\left(\frac{1}{4}+\frac{t}{2\pi k}\right)^{-1/4}\sin f(t,k)\,\mathrm{d}t, \tag{1.4}$$

where $d(k)$ is the number of positive divisors of k and

$$f(t,k) := 2t\operatorname{arsinh}\sqrt{\frac{\pi k}{2t}}+\sqrt{2\pi kt+\pi^2k^2}-\tfrac{1}{4}\pi,\ \operatorname{arsinh}x = \log(x+\sqrt{1+x^2}). \tag{1.5}$$

c) Instead of the Voronoi summation formula one can use the (simpler) Poisson summation formula, namely

$$\sum_{n=1}^\infty f(n) = \int_0^\infty f(x)\,\mathrm{d}x + 2\sum_{n=1}^\infty\int_0^\infty f(x)\cos(2\pi nx)\,\mathrm{d}x, \tag{1.6}$$

provided that $f(x)$ is smooth and compactly supported in $(0,\infty)$. We shall give now a sketch of this approach. The integral in (1.1) (with $t_r = T, \varphi_r(t) = \varphi(t)$) is majorized by $O(\log T)$ integrals of the type

$$I := \int_{T-2G}^{T+2G}\varphi(t)\Bigl|\sum_{N<n\le N'\le 2N} n^{-1/2-it}\Bigr|^2\,\mathrm{d}t \quad (G^{1+\varepsilon}\le N\ll\sqrt{T}, \varphi^{(j)}(t)\ll_j G^{-j}),$$

since the contribution of $N \le G^{1+\varepsilon}$ is $\ll G^{1+\varepsilon}$ by the mean value theorem for Dirichlet polynomials ([5, Theorem 5.2]). Squaring out the above sum, estimating trivially the diagonal terms, and integrating by parts sufficiently many times the contribution of the non-diagonal terms we are led to the estimation of the expression

$$\int_{T-2G}^{T+2G}\varphi(t)\sum_{1\le\ell\le N^{1+\varepsilon}G^{-1}}\;\sum_{N<n\le N'\le 2N}(n(n+\ell))^{-1/2}\left(1+\frac{\ell}{n}\right)^{it}\,\mathrm{d}t.$$

Using Taylor's formula we can replace $(n+\ell)^{-1/2}$ by $n^{-1/2}$, and then we write

(1.7)
$$\sum_{N<n\le N'\le 2N}n^{-1}\left(1+\frac{\ell}{n}\right)^{it}=\sum_{N-G\le n\le N'+G}\Phi(n)n^{-1}\left(1+\frac{\ell}{n}\right)^{it}+O(GN^{-1}),$$

where $\Phi(x)$ (≥ 0) is a smooth function, supported in $[N-G, N_1+G]$, equal to unity in $[N, N_1]$. This function facilitates truncation when the Poisson summation formula (1.6) is applied; once with $f(x)=\Phi(x)\cos\left(t\log\left(1+\frac{\ell}{x}\right)\right)$, and once with the sine instead of cosine. The former gives the major contribution equal to

(1.8)
$$\begin{aligned}&\int_{T-2G}^{T+2G}\varphi(t)\sum_{\ell\le N^{1+\varepsilon}G^{-1}}\;\sum_{m\le T^{1+\varepsilon}\ell N^{-2}}\times\\ &\times\int_{N-G}^{N_1+G}\Phi(x)\cos\left(t\log\left(1+\frac{\ell}{x}\right)\right)\cos(2\pi mx)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

The integral over x is a linear combination of four exponential integrals, one of which is

(1.9)
$$\int_{N-G}^{N_1+G}\Phi(x)e^{iF(x)}\,\mathrm{d}x,\quad F(x):=t\log\left(1+\frac{\ell}{x}\right)+2\pi mx,$$

so that

$$F'(x)=t\left(\frac{1}{\ell+x}-\frac{1}{x}\right)+2\pi m,\quad F''(x)=t\left(\frac{1}{x^2}-\frac{1}{(\ell+x)^2}\right).$$

The main contribution to (1.9) will come from the saddle point x_0 (see e.g., [5, Chapter 2]) and will be a multiple of $|F''(x_0)|^{-1/2}e^{iF(x_0)}$. This is the solution of $F'(x_0)=0$, hence

(1.10)
$$x_0=\frac{-2\pi m\ell+\sqrt{8\pi m\ell t+4\pi^2m^2\ell}}{4\pi m}\sim\sqrt{\frac{\ell t}{2\pi m}}\qquad(T\to\infty),$$

where $x_0\in[N,\,N_1]$, giving $m\asymp \ell TN^{-2}$, which is then the summation condition in (1.8). Further we have $F''(x_0)\sim t\ell x_0^{-3}$, hence the major contribution to (1.4) will come from a multiple of

$$\begin{aligned}&\int_{T-2G}^{T+2G}\varphi(t)t^{-1/4}\sum_{\ell\le N^{1+\varepsilon}G^{-1}}\ell^{-1/4}\\ &\times\sum_{m\asymp\ell TN^{-2}}m^{-1/4}\exp\left(it\left(\frac{\ell}{x_0}-\frac12\frac{\ell^2}{x_0^2}+\frac13\frac{\ell^3}{x_0^3}+\ldots\right)+2\pi imx_0\right)\mathrm{d}t.\end{aligned}$$

Using (1.10) and grouping together the terms with $\ell m = k$, we have $k \ll T^{1+\varepsilon}G^{-2}$, and after some transformations we arrive essentially at the expression (1.4) with $R = 1, t_r = T$ and $\left(\frac14 + \frac{t}{2\pi k}\right)^{-1/4}$ replaced by $\left(\frac{t}{2\pi k}\right)^{-1/4}$.

2. The twelfth moment of $|\zeta(\frac12 + it)|$

The most direct way to treat (1.4) is first to develop $(\frac14 + \frac{t}{2k\pi})^{-1/4}$ by Taylor's formula. After this the mean value theorem for integrals is applied. The essential part of (1.4) is then

$$G\sum_{r\le R}\varphi_r(\tau_r)\tau_r^{-1/4}\sum_{k\le T^{1+\varepsilon}G^{-2}} d(k)k^{-1/4}\sin f(\tau_r, k), \tag{2.1}$$

where $\tau_r \in [t_r - 2G,\, \tau_r + 2G]$, and (by considering separately points with even and odd indices) we can assume the spacing condition

$$T/3 < \tau_1 < \cdots < \tau_R < 8T/3, \quad \tau_{r+1} - \tau_r \ge 5G \;\; (r = 1, \dots R-1).$$

The important feature of (2.1) is that it contains no absolute value signs, so that one can change the order of summation and use the Cauchy-Schwarz inequality. Thus the expression in (2.1) equals

$$\begin{aligned}
&G\sum_{k\le T^{1+\varepsilon}G^{-2}} d(k)k^{-1/4}\sum_{r\le R}\varphi_r(\tau_r)\tau_r^{-1/4}\sin f(\tau_r, k)\\
&\ll G\Bigl(\sum_{k\le T^{1+\varepsilon}G^{-2}} d^2(k)k^{-1/2}\Bigr)^{1/2}\Bigl(\sum_{k\le T^{1+\varepsilon}G^{-2}}\Bigl|\sum_{r\le R}\varphi_r(\tau_r)\tau_r^{-1/4}e^{if(\tau_r,k)}\Bigr|^2\Bigr)^{1/2}.
\end{aligned}$$

After squaring and changing the order of summation, the last sum above becomes

$$\begin{aligned}
&\sum_{r,s\le R}\varphi_r(\tau_r)\varphi_s(\tau_s)(\tau_r\tau_s)^{-1/4}\sum_{k\le T^{1+\varepsilon}G^{-2}}\exp(if(\tau_r,k) - if(\tau_s,k))\\
&\ll T^{-1/2}\sum_{r,s\le R}\Bigl|\sum_{k\le T^{1+\varepsilon}G^{-2}}\exp(if(\tau_r,k) - if(\tau_s,k))\Bigr|\\
&\ll_\varepsilon RT^{\frac12+\varepsilon}G^{-2} + T^{-1/2}\sum_{r,s\ne R}\Bigl|\sum_{k\le T^{1+\varepsilon}G^{-2}}\exp(if(\tau_r,k) - if(\tau_s,k))\Bigr|.
\end{aligned}$$

The effect of this procedure is that there are no divisor coefficients in the last sum over k. If the term $RT^{\frac12+\varepsilon}G^{-2}$ (coming from the diagonal terms $r = s$) dominates, we would obtain not only (1.3), but the stronger bound

$$\int_0^T |\zeta(\tfrac12 + it)|^6\,\mathrm{d}t \;\ll_\varepsilon\; T^{1+\varepsilon}, \tag{2.2}$$

which is not known to hold yet. For a conditional proof of (2.2), which rests on a hypothesis involving the ternary additive divisor problem, see the author's work [8]. However, the estimation of the crucial exponential sum

$$S := \sum_{k\le T^{1+\varepsilon}G^{-2}} \exp(if(\tau_r,k) - if(\tau_s,k)) \tag{2.3}$$

is limited by the scope of the present-day exponential sum techniques (see e.g., M.N. Huxley [4]), and it does not seem that (2.2) can be reached (unconditionally) in this fashion. Observing that

$$\frac{\partial f(x,k)}{\partial x} = 2\mathrm{arsinh}\sqrt{\frac{\pi k}{2x}} \sim \sqrt{\frac{2\pi k}{x}} \quad (x \asymp T), \tag{2.4}$$

setting

$$f(u) := f(\tau_r,u) - f(\tau_s,u), \quad F := |\tau_r-\tau_s|(KT)^{-1/2},$$

we have (see [5, Chapters 1-2] for the relevant exponent pair technique) that S is split into $O(\log T)$ subsums of the type

$$\begin{aligned}\sum_{K<k\le K'\le 2K} \exp(if(k)) &\ll F^\kappa K^\lambda + F^{-1}\\ &\ll J^\kappa T^{-\kappa/2}K^{\lambda-\kappa/2} + (KT)^{1/2}|\tau_r-\tau_s|^{-1} \quad (K\le T^{1+\varepsilon}G^{-2}),\end{aligned}$$

provided that $|\tau_r-\tau_s| \le J(\ll T)$, and (κ,λ) is a (one-dimensional) exponent pair. Choosing $(\kappa,\lambda) = (\frac{1}{2},\frac{1}{2})$, $J = T^{-\varepsilon}G^3$ we obtain (1.3) (with T^ε in place of $\log^{17}T$). Namely with $J = T^{-\varepsilon}G^3$ the number of points $R = R_0$ to be estimated satisfies $R_0 \ll_\varepsilon T^{1+\varepsilon}G^{-3}$, hence dividing $[T/2,\,5T/2]$ into subintervals of length not exceeding J one obtains

$$R \ll R_0(1+T/T_0) \ll_\varepsilon T^{2+\varepsilon}G^{-6} \ll_\varepsilon T^{2+\varepsilon}V^{-12},$$

which easily yields (1.3) (with T^ε in place of $\log^{17}T$). This analysis was carried in detail in [5, Chapter 8], where the possibilities of choosing other exponent pairs besides $(\kappa,\lambda) = (\frac{1}{2},\frac{1}{2})$ were discussed. The method resembles the method used originally by Heath-Brown [2] in his proof of (1.3).

3. The fourth moment of $|\zeta(\frac{1}{2}+it)|$

We now present another approach. Let us start from the sum

$$\sum := \sum_{r\le R}\int_{t_r-2G}^{t_r+2G} \varphi_r(t)t^{-1/4} \sum_{k\le T^{1+\varepsilon}G^{-2}} (-1)^k d(k)k^{-1/4}\sin f(t,k)\,\mathrm{d}t, \tag{3.1}$$

which is the dominant part of (1.4) after the removal of $(\frac{1}{4}+\frac{t}{2\pi k})^{-1/4}$ by Taylor's formula. By Hölder's inequality we obtain, if $M \in \mathbb{N}$ is fixed,
(3.2)

$$\begin{aligned}\sum &\ll \sum_{r\le R} T^{-1/4}\left(\int_{t_r-2G}^{t_r+2G}\varphi_r(t)\Bigl|\sum_{k\le T^{1+\varepsilon}G^{-2}}(-1)^k\ldots\Bigr|^{2M}\,\mathrm{d}t\right)^{\frac{1}{2M}} G^{1-\frac{1}{2M}}\\
&\ll T^{-1/4}G^{1-\frac{1}{2M}}\left(\sum_{r\le R}\int_{t_r-2G}^{t_r+2G}\Bigl|\sum_{k\le T^{1+\varepsilon}G^{-2}}(-1)^k\ldots\Bigr|^{2M}\,\mathrm{d}t\right)^{\frac{1}{2M}} R^{1-\frac{1}{2M}}\\
&\ll T^{-1/4}(RG)^{1-\frac{1}{2M}}\left(\sum_{r\le R}\int_{T/2}^{5T/2}\varphi(t)\Bigl|\sum_{k\le T^{1+\varepsilon}G^{-2}}(-1)^k\ldots\Bigr|^{2M}\,\mathrm{d}t\right)^{\frac{1}{2M}},\end{aligned}$$

since the intervals $[t_r-2G,\,t_r+2G]$ are non-overlapping. Here $\varphi(t)$ is a non-negative, smooth function supported in $[T/2, 5T/2]$ such that $\varphi(t)=1$ for $T\le t\le 2T$, hence $\varphi^{(m)}(t)\ll_m T^{-m}$.

Take now $M=1$ in (3.2) and split the sum over k into $\ll \log T$ subsums over $K<k\le K'\le 2K$, where $K\le T^{1+\varepsilon}G^{-2}$. Then

$$\begin{aligned}&\int_{T/2}^{5T/2}\varphi(t)\Bigl|\sum_{K<k\le K'}(-1)^k d(k)k^{-1/4}e^{if(t,k)}\Bigr|^2\,\mathrm{d}t\\
&= O(TK^{1/2}\log^4 K)\\
&+\sum_{K<k\ne m\le K'}(-1)^{k+m}d(k)d(m)(km)^{-1/4}\int_{T/2}^{5T/2}\varphi(t)e^{if(t,k)-if(t,m)}\,\mathrm{d}t\\
&\ll_\varepsilon TK^{1/2}\log^4 K + K^{\varepsilon-1/2}\sum_{K<k\ne m\le K'}\frac{K^{1/2}T^{1/2}}{|k-m|}\\
&\ll_\varepsilon T^\varepsilon(TK^{1/2}+T^{1/2}K)\ll_\varepsilon T^{1+\varepsilon}K^{1/2},\end{aligned}$$

where we used the first derivative test ([5, Lemma 2.1]) in conjunction with (2.4). Thus for $M=1$ we obtain

$$\sum \ll_\varepsilon T^{-1/4}(RG)^{1/2}T^{1/2+\varepsilon}K^{1/4}\ll_\varepsilon R^{1/2}T^{1/2+\varepsilon}. \tag{3.3}$$

If $|\zeta(\frac{1}{2}+i\bar t_r)|\ge V>T^\varepsilon$ for $\bar t_{r+1}-\bar t_r\ge 1$ $(r=1,\ldots,R-1)$ and $\bar t_r\in[T,2T]$, then on using (see e.g., [6, Theorem 1.2])

$$|\zeta(\tfrac{1}{2}+i\bar t_r)|^2\ll \log T\left(\int_{\bar t_r-1}^{\bar t_r+1}|\zeta(\tfrac{1}{2}+it)|^2\,\mathrm{d}t+1\right)$$

and grouping the points $\bar t_r$ into intervals of length $\ll G$ with center at t_r, choosing then $G=V^2T^{-\varepsilon}$, we obtain from (3.3)

$$RV^2\ll_\varepsilon R^{1/2}T^{1/2+\varepsilon},\quad R\ll_\varepsilon T^{1+\varepsilon}V^{-4},$$

which easily gives then

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^4 \,\mathrm{d}t \;\ll_\varepsilon\; T^{1+\varepsilon}. \tag{3.4}$$

This is a weakened form of the fourth moment of $|\zeta(\frac{1}{2}+it)|$, which is of the order $T\log^4 T$. For an extensive account on results on the fourth moment of $|\zeta(\frac{1}{2}+it)|$, the reader is referred to [2], [7], [9]–[17], [19]–[21]. In particular, the sum (1.2) with $|\zeta|^4$ instead of $|\zeta|^2$ is treated by spectral methods in [6], [15], [17] and [21]; the monograph [21] of Y. Motohashi contains a comprehensive account of spectral theory.

4. Higher moments

We start from (3.1), observing that we have, for $t \asymp T$, $k \le T^{1+\varepsilon}G^{-2}$,

$$f(t,k) = -\tfrac{1}{4}\pi + 2\sqrt{2\pi kt} + \tfrac{1}{6}\sqrt{2\pi^3}k^{3/2}t^{-1/2} + a_5k^{5/2}t^{-3/2} + a_7k^{7/2}t^{-5/2} + \ldots \tag{4.1}$$

with effectively computable constants $a_{2\ell-1}$ $(\ell \ge 3)$. Noting that for $G = V^2T^{-\varepsilon}$

$$k^{5/2}t^{-3/2} \;\ll\; T^{1+\varepsilon}G^{-5} \;\le\; T^{-\varepsilon}$$

for

$$V \;\ge\; T^{\frac{1}{10}+\varepsilon}, \tag{4.2}$$

it follows that we shall obtain (provided that (4.2) holds) (3.1) with $f(t,k)$ replaced by

$$-\tfrac{1}{4}\pi + 2\sqrt{2\pi kt} + \tfrac{1}{6}\sqrt{2\pi^3}k^{3/2}t^{-1/2}$$

times a series whose terms are of descending order of magnitude. The main contribution will thus come from the above term, and proceeding as in the proof of (3.4) we obtain

THEOREM 1. *Let $T \le t_1 < \ldots < t_R \le 2T$ be points such that $t_{r+1} - t_r \ge V \ge T^{\frac{1}{10}+\varepsilon}$ for $r = 1, \ldots, R-1$. Then, for any fixed integer $M \ge 1$,*

$$R \ll_\varepsilon T^{\varepsilon - M/2}V^{-2} \max_{K \le T^{1+\varepsilon}V^{-4}} \times$$
$$\times \int_{T/2}^{5T/2} \varphi(t)\Bigl|\sum_{K \le k \le K' \le 2K} (-1)^k d(k)k^{-1/4}\exp(2i\sqrt{2\pi kt} + cik^{3/2}t^{-1/2})\Bigr|^{2M} \,\mathrm{d}t, \tag{4.3}$$

where $c = \sqrt{2\pi^3}/6$ and $\varphi(t)$ is a non-negative, smooth function supported in $[T/2, 5T/2]$ such that $\varphi(t) = 1$ for $T \le t \le 2T$.

In Section 3 we investigated the case $M = 1$ of (4.3), and it was shown that this leads to the known bound (3.4). The next case $M = 2$ is probably the most promising one, and perhaps could lead to a sharper bound than (1.3). It reduces to the evaluation of the integral

$$\begin{aligned}(4.4)\qquad &\int_{T/2}^{5T/2} \varphi(t) \sum_{K<m,n,k,\ell\le K'} (-1)^{m+n+k+\ell}\times\\ &\times d(m)d(n)d(k)d(\ell)(mnk\ell)^{-1/4}\exp(iDt^{1/2}+iEt^{-1/2})\,\mathrm{d}t,\end{aligned}$$

where we have set

$$\begin{aligned} D &= D(m,n,k,l) := 2\sqrt{2\pi}(\sqrt{m}+\sqrt{n}-\sqrt{k}-\sqrt{\ell}),\\ E &= E(m,n,k,l) := \tfrac{1}{6}\sqrt{2\pi^3}(m\sqrt{m}+n\sqrt{n}-k\sqrt{k}-\ell\sqrt{\ell}).\end{aligned}$$

This problem bears resemblance to the evaluation of

$$\int_1^X \Delta^4(x)\,\mathrm{d}x,\ \Delta(x) = \sum_{n\le x} d(n) - x(\log x + 2\gamma - 1),$$

where $\Delta(x)$ represents the error term in the classical Dirichlet divisor problem. This was investigated by K.-M. Tsang [22], and the connection between the fourth moments of $\Delta(x)$ and $E(T)$ was considered by the author [12]. On one hand the problem considered by Tsang was more difficult, since he aimed at an asymptotic formula, and not only an upper bound. On the other hand, the presence of $iEt^{-1/2}$ in the exponential in (4.4) induces extra difficulties.

If $D = 0$ in (4.4), then the integral is estimated trivially as $O(T)$. One has (see [22]) $D = 0$ if $(m,n) = (k,\ell)$ or $(m,n) = (\ell,k)$ or $m = \alpha^2 h, n = \beta^2 h, k = \gamma^2 h, \ell = \delta^2 h, \alpha+\beta = \gamma+\delta$ and h is squarefree. Thus the contribution of (m,n,k,ℓ) for which $D = 0$ is

$$\begin{aligned}&\ll T\Bigl(\Bigl(K^{-1}\sum_{K<k\le K'} d^2(k)\Bigr)^2 + K^{\varepsilon-1}\sum_{h=1}^{\infty}\ \sum_{\alpha,\beta,\gamma\asymp(K/h)^{1/2}} 1\Bigr)\\ &\ll_\varepsilon T^{1+\varepsilon}K.\end{aligned}$$

When inserted in (4.4) this portion will yield ((4.3) with $M = 2$)

$$(4.5)\qquad R \ll_\varepsilon T^{\varepsilon-1}V^{-2}\cdot T^{1+\varepsilon}\max_{K\le T^{1+\varepsilon}V^{-4}} K \ll_\varepsilon T^{1+\varepsilon}V^{-6},$$

and (4.5) is the estimate that leads to (2.2). As in the discussion involving the twelfth moment estimate of $|\zeta(\frac{1}{2}+it)|$, it turns out that non-diagonal terms, i.e. those for which $D \ne 0$ in our case, are the difficult ones to estimate. We suppose now that $D > 0$ (the case $D < 0$ is analogous) and $E > 0$. Namely in case

$E \le 0$ the estimation is easier. When $E < 0$ the relevant exponential integral has no saddle point and the contribution of $D > T^{\varepsilon-1/2}$ is negligible. If $E = 0$ the estimation is much easier. Write now (4.4) as

$$\sum_{K<m,n,k,\ell\le K'} (-1)^{m+n+k+\ell} d(m)d(n)d(k)d(\ell)(mnk\ell)^{-1/4} I(T),$$

where

$$I(T) = I(T;m,n,k,\ell) := \int_{T/2}^{5T/2} \varphi(t) e^{iDt^{1/2}+iEt^{-1/2}}\,\mathrm{d}t.$$

Change of variable ($x = \sqrt{t}$, $\Phi(x) = 2x\varphi(x^2)$) gives

$$I(T) = \int_{\sqrt{T/2}}^{\sqrt{5T/2}} \Phi(x) e^{iDx+iE/x}\,\mathrm{d}x, \quad \Phi^{(j)}(x) \ll_j x^{1-j}\ (j = 0,1,2,\ldots), \tag{4.6}$$

and it is assumed that $D, E > 0$. If

$$D \le \eta K^{-1/2} \tag{4.7}$$

for a fixed, small $\eta > 0$, then from $\sqrt{\ell} = \sqrt{m} + \sqrt{n} - \sqrt{k} + O(D)$ it follows by squaring that

$$\ell = m + n + k + 2(\sqrt{mn} - \sqrt{mk} - \sqrt{nk}) + O(DK^{1/2}). \tag{4.8}$$

Hence for suitable small η the O-term in (4.8) will be $< 1/3$ in absolute value, so that there will be at most one ℓ for any given choice of m, n, k. Moreover, for fixed m, n it follows from (4.8) that we must have ($||x||$ is the distance of x to the nearest integer)

$$||\, 2\sqrt{k}(\sqrt{m}+\sqrt{n}) - 2\sqrt{mn}\, || \ll DK^{1/2} \tag{4.9}$$

for any k for which there will exist an ℓ satisfying (4.8).

Suppose that beside (4.7) one also has

$$D > T^{\varepsilon-1/2}. \tag{4.10}$$

From

$$\begin{aligned} I(T) &= i\int_{\sqrt{T/2}}^{\sqrt{5T/2}} \left(\frac{\Phi(x)}{D-Ex^{-2}}\right)' e^{iDx+iE/x}\,\mathrm{d}x \\ &= i\int_{\sqrt{T/2}}^{\sqrt{5T/2}} \left(\Phi'(x) - \frac{2Ex^{-3}}{D-Ex^{-2}}\Phi(x)\right) \frac{e^{iDx+iE/x}}{D-Ex^{-2}}\,\mathrm{d}x \end{aligned}$$

we see that, if $D > C_2E/T$ or $D < C_1E/T$ for suitable constants $0 < C_1 < C_2$, then repeated integrations by parts show, in view of (4.10), that the contribution of $I(T)$ is negligible. If $C_1E \le DT \le C_2E$, then by the second derivative test ([5, Lemma 2.2]) we have

$$I(T) \ll E^{3/4}D^{-5/4}. \tag{4.11}$$

The number of k's satisfying

$$||\,2\sqrt{k}(\sqrt{m}+\sqrt{n}) - 2\sqrt{mn}\,|| < \delta \quad (0 < \delta < \tfrac12) \tag{4.12}$$

for a fixed m, n is uniformly

$$\ll K\delta + K^{2/3} \tag{4.13}$$

by [22, Lemma 4]. This bound follows by applying standard exponential sum techniques. Namely $||x|| < \delta$ $(0 < \delta < 1/2)$ is equivalent to

$$[x+\delta] - [x-\delta] = \psi(x-\delta) - \psi(x+\delta) + 2\delta = 1$$

(the expression equals zero otherwise) with

$$\begin{aligned}\psi(x) = x - [x] - \tfrac12 &= -\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2\pi nx)}{n}\\ &= -\frac{1}{\pi}\sum_{n=1}^{N}\frac{\sin(2\pi nx)}{n} + O\left(\min\left(1,\,\frac{1}{N||x||}\right)\right) \quad (x \notin \mathbb{Z},\ N \ge 3).\end{aligned}$$

Thus the problem is quickly reduced to a problem involving the estimation of exponential sums. In our case $D \ll E/T \ll K^{3/2}/T$, hence (in view of (4.9)) in (4.12) we may take $\delta = CDK^{1/2} \ll K^2/T$ with a suitable $C > 0$. The contribution for which (4.7) and (4.10) hold is, on using (4.11) and (4.13),

$$\begin{aligned}R &\ll_\varepsilon T^{\varepsilon-1}G^{-1}E^{3/4}D^{-5/4}K(K^{3/2}D + K^{2/3})\\ &\ll_\varepsilon T^{\varepsilon}(T^{-3/4}K^{13/4}G^{-1} + T^{-1}G^{-1}(E/D)^{3/4}K^{5/3}D^{-1/2})\\ &\ll_\varepsilon T^{\varepsilon}(T^{-3/4}(TG^{-2})^{13/4}G^{-1} + T^{-1}G^{-1}T^{3/4}(TG^{-2})^{5/3}T^{1/4})\\ &\ll_\varepsilon T^{\frac52+\varepsilon}V^{-15} + T^{\frac53+\varepsilon}V^{-\frac{26}{3}}.\end{aligned} \tag{4.14}$$

If $D > \eta K^{-1/2}$, then (4.10) holds. From (4.8) it follows that there are $\ll DK^{1/2}$ possible values of ℓ, hence by trivial estimation, similar to the first bound in (4.14), we obtain a contribution which is $\ll_\varepsilon T^{5/2+\varepsilon}V^{-15}$.

If $D < T^{\varepsilon-1/2}$, then the bound in (4.13) and trivial estimation give

$$
\begin{aligned}
R &\ll_\varepsilon T^{\varepsilon-1}G^{-1}KT(K^{3/2}T^{-1/2}+K^{2/3}) \\
&\ll_\varepsilon T^{2+\varepsilon}V^{-12}+T^{\frac{5}{3}+\varepsilon}V^{-\frac{26}{3}}.
\end{aligned}
\tag{4.15}
$$

However, the term $K^{2/3}$ in (4.13) can be replaced, by applying e.g. the Bombieri-Iwaniec method (see e.g., M.N. Huxley [4]) for the estimation of the relevant exponential sum, by the sharper term $K^{7/11}$, yielding

$$
R \ll_\varepsilon T^{2+\varepsilon}V^{-12}+T^{\frac{18}{11}+\varepsilon}V^{-\frac{94}{11}}.
$$

Noting that $V \le T^{1/6}$ implies that $T^2V^{-12} \le T^{5/2}V^{-15}$, we obtain (under (4.2))

$$
R \ll_\varepsilon T^{\frac{5}{2}+\varepsilon}V^{-15}+T^{\frac{18}{11}+\varepsilon}V^{-\frac{94}{11}},
\tag{4.16}
$$

which is a little weaker than the twelfth moment estimate

$$
R \ll_\varepsilon T^{2+\varepsilon}V^{-12}.
\tag{4.17}
$$

Further improvements on the second term on the right-hand side of (4.16) are possible. Namely we used the bound in (4.13) assuming that m, n are fixed. However, summation over m, n in the relevant exponential sum leading to (4.13) produces in fact a three-dimensional exponential sum which can be treated e.g. by three-dimensional exponent pairs. In this way presumably one obtains $R \ll_\varepsilon T^{\frac{5}{2}+\varepsilon}V^{-15}$ if (4.2) holds, however if (4.2) does not hold, then this bound easily follows from (3.4). A further possibility for improvement is to note that the saddle point method, in case $C_1E \le DT \le C_2E$, furnishes in fact an asymptotic formula for $I(T)$, and not just the upper bound in (4.11). This will lead to the exponential sum

$$
\sum_{m,n,k,\ell} d(m)d(n)d(k)d(\ell)(mnk\ell)^{-1/4}\exp(2i\sqrt{DE}).
\tag{4.18}
$$

This sum is difficult to handle, since the variables in the exponential are not separated and, for fixed m, n, k, there will exist at most one ℓ. However, in view of $C_1E \le DT \le C_2E$, we expect heuristically that certain separation of variables in (4.18) is possible, which would lead to considerable cancellation. Presumably this will yield at least (4.17).

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Aleksandar Ivić, Katedra Matematike RGF-a Universiteta u Beogradu, Đušina 7, 11000 Beograd, Serbia (Yugoslavia).

Max-Planck-Institut für Mathematik Bonn, Vivatsgasse 7, D-53111 Bonn
E-mail address: `aivic@matf.bg.ac.yu, aivic@rgf.bg.ac.yu`