Generalized complex structures in six dimensions

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1 Introduction

Mirror symmetry has revealed a rich structure in the complex and symplectic aspects of 3-dimensional Calabi-Yau manifolds. We attempt here to highlight some of the special features of six dimensions which underlie, at the classical level, these facts.

The first special feature in six dimensions was the basis of the description of Calabi-Yau moduli spaces in [H]. We repeat here the essentials. The starting point is the fact that if V is a 6-dimensional real vector space then $GL(V)$ has two open orbits in $\Lambda^3 V^*$. For ρ lying in one of them, say U, we have

$$\rho = \alpha + \bar{\alpha}$$

where α is a complex decomposable form and $\alpha \wedge \bar{\alpha} \neq 0$. We can then define

$$\phi(\rho) = i\alpha \wedge \bar{\alpha} \in \Lambda^6 V^*.$$

If M is a compact oriented 6-manifold, integrating $\phi(\rho)$ defines a functional on forms in U at each point, and critical points on a fixed cohomology class of closed forms corrrespond to the complex form α being closed.

This condition is equivalent to having a complex structure with trivial canonical bundle on M. A non-degeneracy condition (which follows from the $\partial\bar{\partial}$ lemma) then gives us the fact that an open set in $H^3(M, R)$ is a local moduli space for complex structures together with a holomorphic 3-form: the modulus is just the cohomology class of the real part of α.

One question that leads on from this is how to treat symplectic structures in a similar fashion. Non-degeneracy of $\omega \in \Lambda^2 V^*$ is an open condition, and the Liouville volume provides a functional similar to the one above. However if ω is closed this functional has constant value on a cohomology class in $H^2(M, R)$. If $\rho = \omega^2$ is closed this is not true and critical points on a fixed class in $H^4(M, R)$ give symplectic structures. Non-degeneracy follows from the strong Lefschetz property.

This situation becomes more visible if we extend the procedure and use another special fact about 6 dimensions concerning the group $Spin(6, 6)$.

2 $Spin(6,6)$ and its geometry

The two spin representations of $Spin(6,6)$ are real, 32-dimensional and have an invariant symplectic form $\langle\ ,\ \rangle$ on either of them. A concrete realization is to take $V \oplus V^*$ as the 12-dimensional space with the canonical pairing defining the inner product and then

$$S^{\pm} = \Lambda^{ev/odd} V^* \otimes (\Lambda^6 V)^{1/2}$$

The Lie algebra of $Spin(6,6)$ is built out of the endomorphisms of V and elements of $\Lambda^2 V^*$ and $\Lambda^2 V$ acting by external and interior products respectively. In S^+ it follows that the stabilizer in $Spin(6,6)$ of $\varphi_0 + \varphi_6$ is $SL(V)$ so that the orbit has dimension

$$\dim SO(6,6) - \dim SL(6) = 66 - 35 = 31 = 32 - 1$$

Thus $R^* \times Spin(6,6)$ has an open orbit in S^+ and also S^-.

We have a similar situation to the case of 3-forms. For ρ lying in one orbit U, we have

$$\rho = \alpha + \bar{\alpha}$$

where α is a complex pure spinor and $\langle \alpha, \bar{\alpha} \rangle \neq 0$. We can then define

$$\phi(\rho) = i\langle \alpha \wedge \bar{\alpha} \rangle \in R.$$

Equivalently, the function ϕ, which is homogeneous of degree 2, defines an invariant map from even or odd forms to $\Lambda^6 V^*$. For a 3-form this specializes to the function above. For the even form $1 - \omega^2/2$ it becomes the Liouville volume form.

If M is a compact oriented 6-manifold, integrating $\phi(\rho)$ defines a functional on even or odd forms and critical points on a fixed cohomology class of closed forms correspond to the complex form α being closed. Of course α depends on ρ in a highly nonlinear manner.

What is the geometry determined by α?

3 Generalized complex structures

A spinor φ is pure if the complex vectors x such that $x\varphi = 0$ (Clifford multiplication) form a maximal isotropic subspace. In our case, then, we get a maximal isotropic subbundle E of $(T \oplus T^*) \otimes C$. The condition $\langle \alpha, \bar{\alpha} \rangle \neq 0$ means that $E \cap \bar{E} = 0$ so E defines a complex structure J on $T \oplus T^*$. We also have a non-vanishing section of $\Lambda^6 E^*$ and J is compatible (in a pseudo-Kähler sense) with the inner product on $T \oplus T^*$. This is all linear algebra. There is an integrability condition however which is satisfied if $d\alpha = 0$. This involves the so-called *Courant bracket* [C].

Let $X + \xi, Y + \eta$ be sections of $T \oplus \Lambda^p T^*$. The Courant bracket is defined by

$$[X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi + \frac{1}{2} d(i(X)\eta - i(Y)\xi)$$

If $d\alpha = 0$ then one can show that sections of E are closed under the Courant bracket for $p = 1$. This is the direct analogue of the Newlander-Nirenberg condition. We call such a structure a *generalized complex structure.*

Following the analogy with 3-forms we should have under suitable nondegeneracy conditions that the moduli spaces of these generalized complex structures are given by open sets in the even or odd cohomology spaces, but modulo what, one may ask? Diffeomorphisms certainly but there are also automorphisms of the Courant bracket. Unlike the Lie bracket of vector fields, there are gauge transformations of $T \oplus \Lambda^p T^*$ which preserve the bracket, namely

$$X + \xi \mapsto X + \xi + i(X)\beta$$

where β is a closed $p+1$-form. We consider structures modulo diffeomorphisms and *exact* 2-forms.

4 Example

If ω is symplectic then $1 - \omega^2/2$ is an even form lying in the open set U at each point. The pure spinor of which it is the real part is

$$\exp(i\omega)1 = 1 + i\omega - \frac{1}{2}\omega^2 - i\frac{1}{6}\omega^3$$

where we have exponentiated in $Spin(6,6)$ and applied it to the pure spinor $1 \in \Lambda^0$. In fact if φ is a closed pure spinor with the (necessarily) constant $\varphi_0 = c \neq 0$ then

$$\varphi = c\exp(B + i\omega)1$$

for a closed 2-form B. Thus a generic even generalized complex structure can be transformed to a symplectic structure by a (possibly non-exact) closed 2-form B acting as an automorphism of $T \oplus T^*$. This is essentially what physicists call the B-field.

Not all even generalized complex structures are like this, however. If $c = 0$ then an even pure spinor is of the form

$$\exp(\Omega)\gamma$$

where γ is a decomposable complex 2-form. So for example $M = S^2 \times K3$ where S^2 has a symplectic structure and the $K3$ surface has a complex structure together with a choice of holomorphic 2-form is an example.

5 Gerbes

The construction above seems to notice a difference between automorphisms of $T \oplus T^*$ defined by an exact form and those defined by simply a closed one. There is also a more subtle question of integrality lying beneath, which is only really explained by bringing in gerbes. To illustrate this, let's go one step down from the Courant bracket on $T \oplus T^*$ to that on $T \oplus 1$. Here the automorphisms are closed 1-forms.

The bracket here is

$$[X+f, Y+g] = [X,Y] + Xg - Yf$$

This has another interpretation –as the Lie bracket on S^1-invariant vector fields on $M \times S^1$. More generally, if P is a principal circle bundle then the bundle TP/S^1 is an extension (the Atiyah extension)

$$1 \to TP/S^1 \to T$$

whose sections are invariant vector fields on P. The Lie bracket defines a bracket on sections of the extension. A connection on P is a splitting of the sequence, or an isomorphism

$$TP/S^1 \cong T \oplus 1$$

so this gives a new bracket on sections of $T \oplus 1$ which turns out to be

$$[X+f, Y+g] + F(X,Y)$$

where F is the curvature of the connection. In this set-up an exact automorphism of $T \oplus 1$ is the 1-form $g^{-1}dg$ for a gauge transformation g connected to the identity. An arbitrary gauge transformation would be an automorphism which is a closed 1-form with integral periods. A flat connection on the other hand defines a non-exact automorphism.

Moving up to $T \oplus T^*$ we should say that a gauge transformation is a circle bundle with connection, and a non-exact automorphism is a flat gerbe. To do this we need to think of the hierarchy or mathematical objects which begins with maps to the circle, goes up to principal circle bundles and then next to gerbes. The B-field should then be interpreted as an automorphism defined by a flat gerbe.

6 Further possibilities?

The two special features about 6 dimensions we have used, facts about certain representations of $SL(6,R)$ and $Spin(6,6)$, form part of at least two lists. One concerns symplectic representations: any such representation on a vector space V defines a quadratic moment map μ taking values in the dual of the Lie algebra, or using the Killing form, the Lie algebra itself. The two groups above have the property that they act transitively on non-zero level sets of the invariant quartic (μ, μ). The second list is the classification of exceptional symplectic holonomy groups of Merkulov and Schwächhofer. Thirdly, in the work of Alekseevski and Cortes on special pseudo-Kähler manifolds the same list occurs, especially from the property that the orbit of a complex highest weight vector should be a Lagrangian cone. Our decomposable 3-forms and pure spinors are just such vectors.

One example at the end of the list is E_7 in its 56-dimensional representation. This can be realized by taking an 8-dimensional vector space V and considering

$$\Lambda^2 V^* \oplus \Lambda^6 V^*$$

We could use this to define an invariant functional on 2-forms and 6-forms on a compact 8-manifold, and consider the critical points as before, but what replaces complex structures or generalized complex structures I leave as an exercise.

References

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