Line congruence and transformation of projective surfaces

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Abstract

The aim of this article is to present and reformulate systematically what is known about surfaces in the projective 3-space, in view of transformations of surfaces, and to complement with some new results. Special emphasis will be laid on line congruences and Laplace transformations. A line congruence can be regarded as a transformation connecting one focal surface with the other focal surface. A Laplace transformation is regarded as a method of constructing a new surface from a given surface by relying on the asymptotic system the surface is endowed with. A principal object in this article is a class of projectively minimal surfaces. We clarify the procedure of getting new projectievely minimal surfaces from a given one, which was found by F. Marcus, as well as the procedure of Demoulin transformation of projective surfaces.

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Introduction

Transformations of surfaces have been a subject of strong interest to mathematicians for many years, from a geometric point of view as well as from an analytic point of view. G. Darboux opened his four-volume book "Leçons sur la théorie générale des surfaces," with a preface saying that his aim was to investigate new applications to geometry of the theory of partial differential equations. In his book, we can find several ways of extracting geometric conclusions from structures lying in systems of differential equations, which still stimulate us. Classical transformations such as the Darboux transformation and Bäcklund transformation are still even within the last ten years being found to be fundamental to knowing the structure of linear and/or nonlinear differental equations associated to geometric objects, and to constructing surfaces with special properties; this has lead to a renewal of interest in the theory of surfaces.

The aim of this article is to present and reformulate systematically what is known about surfaces in the projective 3-space, in view of transformations of surfaces, and to complement with some new results. Special emphasis will be laid on line congruences and Laplace transformations. A line congruence can be regarded as a transformation connecting one focal surface with the other focal surface. A Laplace transformation is regarded as a method of constructing a new surface from a given surface by relying on the asymptotic system the surface is endowed with.

A principal object in this article is a class of projectively minimal surfaces. By definition, a surface is projectively minimal if and only if the area functional, constructed in a projectively invariant way, attains an extremal value for this surface. The class of such surfaces includes the class of affine spheres that are fundamental in affine differential geometry and the class of Demoulin surfaces that are characterized by degeneracy of the envelope of Lie quadrics. In the last section of this article, we show that the class of projectively minimal surfaces fits nicely into the framework of line congruences and Laplace transformations which we prepare in the foregoing sections.

The contents are as follows: In Section 2, we give a formulation of projective surfaces and introduce several notions such as Wilczynski frame, Demoulin frame, projectively applicable surfaces and projectively minimal surfaces. This is a short introduction to the projective differential geometry of surfaces. Section 1 is a preparation for Section 2. In Section 3, we introduce the notion of line congruence and W-congruence, and in Section 6 we formulate a projective theory of line congruences.

In Section 4, we give a summary of Laplace transformations, mainly relying on the books by Darboux. First, in terms of line congruence, the notion of Laplace transformation is introduced and its fundamentals are given. Second, sequences of consecutive Laplace transformations are treated. Third, we turn our attention to the Euler-Darboux-Poisson equation, and we treat the hypergeometric systems by Appell as examples. A characterization of Demoulin surfaces by Godeaux sequences, a classical result, is then presented. Section 5 is a digression to affine spheres. The definition of affine spheres, the differential system describing affine spheres, and properties relative to the Laplace transformation will be given.

A line congruence can be regarded as a surface in the projective 5-space that is the moduli space of projective lines. It lies in a 4-dimensional quadratic hypersurface defined by the Plücker relation. A line congruence is said to belong to a linear complex if it lies, regarded as a surface, in a hyperplane. This is explained in Section 7. In Section 8, we treat Laplace transformations of line congruences.

A line congruence is described by a pair of two focal surfaces. In Section 9, the projective invariants of focal surfaces are explicitly given, and a characterization of line congruences for which both focal surfaces are quadrics, found by E. J. Wilczynksi, is proved.

There are a number of classical books on projective differential geometry. Among others, the books [FC1] and [FC2] by G. Fubini and E. Čeck, the book [L] by E. P. Lane, the three-volume book [Bol] by G. Bol, the book on nets [Tz1924] by G. Tzitzeica, and the book [W1906] by E. J. Wilczynski are fundamental. In his book [FC2], Fubini gave a method for constructing W-congruences, which will be reproduced here in Section 10. Section 11 treats Demoulin transformations by introducing Demoulin lines and Demoulin congruences. Explicit forms will be given by relying mainly on the works of [Fi1930] and [Su1957]. Section 12 treats again Demoulin transformations intrinsically by appealing to Demoulin coframes.

Section 13 gives a characteristic property of W-congruences between projectively minimal surfaces, and presents a transformation formula of projectively minimal surfaces originally due to F. Marcus [Mar1980].

In Appendix A, an example of a line congruence derived from one of Appell's systems is given; in Appendix B, we recall a fundamental formulation of line congruences in the Euclidean 3-space and show that W-congruence, or orginally Weingarten congruence, comes from the normal congruence associated with Weingarten surfaces. Appendix C gives a proof of a statement in Sect. 4.8.

Notation: $\mathbf{P}^n$ denotes the n-dimensional projective space with homogeneous coordinates $[z^0, z^1, \ldots, z^n]$. $\mathbf{P}_{n+1}$ denotes its dual projective space. The $\wedge$-product is frequently used in this article. For two vectors $z = [z^0, z^1, \ldots, z^n]$ and $w = [w^0, w^1, \ldots, w^n]$, the vector $z \wedge w = [z^i w^j - z^j w^i]$ $(0 \leq i \neq j \leq n)$ defines a vector in $\mathbf{P}^{\binom{n+1}{2}-1}$. The multi-wedge product $z \wedge w \wedge \cdots$ is similarly defined. The coefficeint number field is $\mathbf{R}$ or $\mathbf{C}$ throughout this article.

Remarks on references: The books mentioned above are used in several places without mention. The formulation in Section 1 is due to [W1906] and [Sa1999], and Section 2 is based on [Sa1988, Sa1999], [Fe2000a, Fe1999], [L]. Formulations in Section 4 are based essentially on [D], except for some additional matters in Sections 4.7-4.8. We refer the reader to [NS1994] for the affine differential geometry described in Section 5. A part of Section 7 and the contents of Sections 8 and 9 depend on [W1911]. The material of Section 10 comes from [FC1]; we refer also to [L]. Section 11 relies on [Fi1930]. Section 12 was written in part by using the work of [Su1957]. At the end of each section, more information on references will be given. In References, we list related books and papers that are not directly cited in the article.

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1 Ruled surfaces

A projective surface is an immersion of a two-dimensional manifold into a 3-dimensional projective space $\mathbf{P}^3$. It is given locally by a map $(x, y) \mapsto z(x, y) \in \mathbf{P}^3$. To study the immersion, it is fundamental to consider the ruled surface consisting of tangent lines to the family of curves, say, $x \mapsto z(x, y)$. In this section, we provide fundamental invariants of ruled surfaces and discuss a property of the Plücker embedding of a ruled surface. We start with recalling the fundamental invariants of space curves.

1.1 Space curves

It was G.H. Halphen who developed a projective theory of linear ordinary differential equations by introducing geometric invariants of curves defined by these equations. We give a sketch of his theory for space curves, without proofs.

Let us consider the linear differential equation

$$z^{(n+1)} + p_1 z^{(n)} + \cdots + p_n z' + p_{n+1} z = 0,$$

where $z(t)$ is an unknown function and p_1, …, p_{n+1} are scalar functions of the variable t. We choose arbitrarily a set of independent solutions, say, $z^1(t)$, …, $z^{n+1}(t)$. Then, the map

$$t \longmapsto [z^1(t), \ldots, z^{n+1}(t)]$$

defines a curve into the projective space $\mathbf{P}^n$. Any other set of solutions defines a curve which is a projective transformation of the original curve. Hence, we can say that the ordinary differential equation above defines a curve uniquely up to a projective transformation. Conversely, given a curve in the projective space $\mathbf{P}^n$, we can find an ordinary differential equation satisfied by each coordinate function of the associated mapping. However, the equation is not uniquely defined because the homogeneous coordinates are determined only up to scalar multiplication. We also have the ambiguity in the choice of parameter t, when we are starting with only the curve's image in $\mathbf{P}^n$. Hence, we can admit a change of variables such as

$$(z, t) \longmapsto (y = \lambda(t) z, s = f(t)) \tag{1.1}$$

in order to get geometrical information from the associated curve. This reasoning is essential for developing the projective theory of curves in relation to linear differential equations. Later, this was generalized by E. J. Wilczynsky to develop the projective differential geometry of submanifolds in relation to linear differential systems.

To illustrate the above reasoning, let us consider a space curve that is a mapping $t \mapsto [z(t)] \in \mathbf{P}^3$, where $z(t)$ is a vector in the homogeneous coordinate space of 4-dimension. At a point of the curve where $\det(z''', z'', z', z) \neq 0$, each component is a solution of an ordinary differential equation

$$z'''' + p_1 z''' + p_2 z'' + p_3 z' + p_4 z = 0, \tag{1.2}$$

where the p_j's are functions of t. By multiplying z by a scalar factor, the equation is changed to the form

$$z'''' + q_2 z'' + q_3 z' + q_4 z = 0. \tag{1.3}$$

Performing the transformation (1.1), the transformed equation has the form

$$\ddddot{y} + \{4\lambda'/(\lambda f') + 6 f''/(f')^2\} \dddot{y} + \cdots = 0,$$

where $\{\dot{}\}$ denotes the derivation with respect to s. Hence, by choosing λ so that $\lambda'/\lambda = -(3/2)f''/f'$, we get the equation into the form

$$\ddddot{y} + (q_2 - 10\{f;t\})/(f')^2 \ddot{y} + \cdots = 0,$$

where $\{f;t\} = (f''/f')'/2 - (f''/f')^2/4$ is the Schwarzian derivative of f with respect to t. We now solve the equation $q_2 = 10\{f;t\}$ to get a function f. Using this f, we finally have the equation in the form

$$\ddddot{y} + r_3 \dot{y} + r_4 y = 0.$$

Here note that the parameter s is not unique, but is determined uniquely up to a fractional linear transformation in view of the fractional invariance of the Schwarzian derivative. This means that the curve has a unique projective structure.

It is known that the two differential forms

$$\psi_3 = r_3 ds^3, \quad \psi_4 = (r_4 - \frac{1}{2}r_3')ds^4$$

are defined canonically with respect to the first equation (1.2), and are called fundamental invariants of the space curve. Relative to Equation (1.3), we see that $\psi_3 = (q_3 - q_2')dt^3$. When $r_3 \neq 0$, or equivalently when $q_3 \neq q_2'$, we can choose a special parameter t so that $\psi_3 = dt^3$, which is called the *projective length parameter* of the space curve.

Among several known geometrical interpretations of these invariants, we cite one for later use. At each point of the curve we associate the tangent line that can be regarded as the vector

$$\xi = y \wedge \dot{y},$$

in the space $\mathbf{P}^5 \equiv \overset{2}{\wedge} \mathbf{P}^3$. By successive differentiation, we get

$$L := \xi^{(5)} + r_3 \ddot{\xi} - 2(2r_4 - \dot{r_3}) \dot{\xi} - (2\dot{r_4} - \ddot{r_3})\xi = 2r_3 \dot{y} \wedge \ddot{y},$$

and

$$r_3 \dot{L} - \dot{r_3} L - r_3^2(\dddot{\xi} + r_3 \xi) = 0.$$

The last equation defines a 6th order differential equation relative to ξ. However, when $r_3 = 0$ this equation degenerates and ξ satisfies the equation $L = 0$, which is a 5th order equation. Namely, there exists a linear relation amongst the six components of the vector ξ.

When the Plücker image of the set of tangent lines in $\mathbf{P}^5$ is lying in a hyperplane, then the curve is said to belong to a *linear complex*. Hence, we can state the following.

Proposition 1.1 (G.H. Halphen [Ha1883, p.332]) *The space curve defined by* (1.3) *belongs to a linear complex if and only if* $q_3 = q_2'$.

1.2 Ruled surfaces

A surface in 3-dimensional projective space $\mathbf{P}^3$, simply called a projective surface, is an immersion denoted by

$$(x, y) \longrightarrow z(x, y) \in \mathbf{P}^3.$$

Before entering into the general treatment of projective surfaces in the next section, we recall a projective treatment of ruled surfaces.

We call a 1-parameter family of lines in 3-dimensional projective space $\mathbf{P}^3$ a *ruled surface*. In practice, it is given by a pair of curves $\{z_1(x), z_2(x)\}$ with curve parameter x: we associate a ruled surface with

$$(x, y) \longrightarrow z(x, y) = z_1(x) + yz_2(x).$$

Here we are regarding any point in $\mathbf{P}^3$ as a vector in the homogeneous coordinate space of 4-dimension. The curves z_1 and z_2 are called *generating curves*.

Example 1.2 *(1) Let $z_1(x) = [1, x, 0, 0]$ and $z_2(x) = [0, 0, 1, x]$ in the homogeneous coordinates. Then $z(x, y) = [1, x, y, xy]$, which denotes a quadric $z^1z^4 = z^2z^3$ relative to the homogeneous coordinates $[z^1, z^2, z^3, z^4]$.*
(2) For a given curve $z_1(x)$, define a second curve $z_2(x)$ by $z_2(x) = A + kz_1(x)$, where A is a fixed vector and k a scalar. Then the surface is nothing but a cone.
(3) Given a space curve $c(x)$, the pair $\{c(x), c'(x)\}$ defines a ruled surface consisting of tangent lines of the curve $c(x)$, which is called a tangent developable surface.

Definition 1.3 A ruled surface $\{z_1, z_2\}$ is said to be *developable* if

$$z_1 \wedge z_2 \wedge z_1' \wedge z_2' = 0.$$

A developable surface is locally a cone or a tangent developable surface. In the latter case, a generating space curve is called a *directrix curve*. A directrix curve is written as $c(x) = \alpha z_1(x) + \beta z_2(x)$ for certain scalars $\alpha(x)$ and $\beta(x)$, with the property that $c'(x) \equiv 0 \pmod{z_1, z_2}$.

In the following, we always assume that the ruled surface is *not* developable. With this assumption, we will have the following system of differential equations.

$$z_i'' = \sum_j p_i^j z_j' + \sum_j q_i^j z_j \qquad 1 \leq i, j \leq 2. \tag{1.4}$$

Conversely, a system of this form defines a ruled surface: we can see that this system has four independent solutions that define a pair of curves in $\mathbf{P}^3$, which in turn defines a ruled surface. The surface is defined uniquely up to a projective transformation.

However, the surface itself does not define the system uniquely, it still has the freedom of change of variables and generating curves. In fact, the change of variables $(x, z) \to (y, w)$ given by

$$w_i = \sum_j a_i^j(x) z_j, \quad \det(a_i^j) \neq 0 \tag{1.5}$$

keeps the surface unchanged. With this freedom of choice, let us try to simplify the expression of the system (1.4).

Definition 1.4 Let $z(x, y)$ be a surface. A curve on this surface defined by $x = x(t)$ and $y = y(t)$ is called an *asymptotic curve* if the four vectors z, z_x, z_y, and z_{tt} are linearly dependent:

$$z \wedge z_x \wedge z_y \wedge z_{tt} = 0.$$

For a ruled surface $z = z_1(x) + y z_2(x)$, we have

$$\begin{aligned} z \wedge z_x \wedge z_y &= z_1 \wedge z_1' \wedge z_2 + y\, z_1 \wedge z_2' \wedge z_2, \\ z_t &= (z_1' + y\, z_2')\dot{x} + z_2 \dot{y}, \\ z_{tt} &= (z_1' + y\, z_2')\ddot{x} + (z_1'' + y\, z_1'')(\dot{x})^2 + 2 z_2' \dot{x}\dot{y} + z_2 \ddot{y}, \end{aligned}$$

where $\{\dot{}\}$ denotes the derivation with respect to t. Hence

$$z \wedge z_x \wedge z_y \wedge z_{tt} = 2\dot{x}\,\dot{y}\, z_1 \wedge z_1' \wedge z_2 \wedge z_2' - \dot{x}^2 A,$$

where

$$A = z_1 \wedge z_2 \wedge z_1' \wedge z_1'' + y(z_1 \wedge z_2 \wedge z_2' \wedge z_1'' + z_1 \wedge z_2 \wedge z_1' \wedge z_2'') + y^2 z_1 \wedge z_2 \wedge z_2' \wedge z_2''.$$

Therefore, the asymptotic curves are determined by the equation

$$\dot{x}\{2\dot{y}\, z_1 \wedge z_1' \wedge z_2 \wedge z_2' - \dot{x}\, A\} = 0,$$

which has always two different solutions, i.e. two asymptotic curves pass through each point. One of these is a ruling line defined by $\dot{x} = 0$ and the other is given by a differential equation of Riccati type

$$2 z_1 \wedge z_1' \wedge z_2 \wedge z_2' dy - A\, dx = 0.$$

Now we reparametrize the surface assuming that both z_1 and z_2 are asymptotic curves. In this case we see that

$$z_1 \wedge z_2 \wedge z_1' \wedge z_1'' = z_1 \wedge z_2 \wedge z_2' \wedge z_2'' = 0,$$

which implies that $p_1^2 = p_2^1 = 0$ in the system (1.4). We next replace z_1 and z_2 by their scalar multiples λz_1 and μz_2. Then the coefficients p_1^1 and p_2^2 are changed by adding λ'/λ and μ'/μ respectively. So we can always find λ and μ so that $p_1^1 = p_2^2 = 0$. Hence we have proved the following:

Proposition 1.5 *A nondevelopable ruled surface is given by a system of differential equations*

$$\begin{aligned} z_1'' &= p\,z_1 + q\,z_2, \\ z_2'' &= r\,z_1 + s\,z_2. \end{aligned} \tag{1.6}$$

Example 1.6 *(1) Any nondegenerate quadric is a ruled surface given by the system*

$$z_1'' = z_2'' = 0.$$

(2) Cayley's cubic scroll is by definition

$$(z^2)^3 + z^1(z^1 z^4 + z^2 z^3) = 0,$$

relative to the coordinates $[z^1, z^2, z^3, z^4]$*. It is ruled by two generating curves* $z_1 = [1, -x, -x^2, 0]$ *and* $z_2 = [0, 0, 1, x]$*. Hence the system of equations is*

$$z_1'' = -2z_2 + 2xz_2', \qquad z_2'' = 0.$$

If we set

$$z_3 = z_1 - \frac{1}{2}x^2 z_2 + a\,z_2 = [1, -x, -\frac{3}{2}x^2 + a, -\frac{1}{2}x^3 + a\,x],$$

where a *is any constant, then the system is written in the form asserted in the proposition:*

$$z_3'' = -3z_2, \qquad z_2'' = 0.$$

1.3 Fundamental invariants of a ruled surface

Let us check the covariance of the system (1.6) relative to the transformation (1.5). Setting, for simplicity,

$$X = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \qquad Q = \begin{pmatrix} p & q \\ r & s \end{pmatrix},$$

the system is written as

$$X'' = QX.$$

Relative to the transformation $y = f(x)$ and $Y = AX$, where $A = (a_i^j)$, the variable Y satisfies

$$(f')^2\ddot{Y} = (2f'A'A^{-1} - f'')\dot{Y} + (A''A^{-1} + AQA^{-1} - 2A'A^{-1}A'A^{-1})Y,$$

where $\dot{\{\}} = d/dy$ and $\{'\} = d/dx$. Then $B = (f')^{-1/2}A$ should satisfy $B' = 0$ in order that the coefficient of $\dot{Y}$ vanishes. In this case, the system becomes

$$(f')^2\ddot{Y} = (2\{f;x\} + BQB^{-1})Y,$$

where $\{f;x\}$ is the Schwarzian derivative of f relative to the variable x. Since the trace of the coefficient matrix of Y is $\operatorname{tr} Q + 4\{f;x\}$, we can assume this coefficient vanishes by choosing f appropriately. Then any transformation preserving this condition is of the form

$$A = (f')^{1/2} B, \qquad \{f; x\} = 0;$$

i.e.,

$$f = \frac{\alpha x + \beta}{\gamma x + \delta}, \qquad A = (\gamma x + \delta)^{-1} B, \qquad \text{where } B \text{ is a constant matrix.} \tag{1.7}$$

The matrix-valued quadratic form Qdx^2 changes under this transformation as

$$Qdx^2 \longrightarrow B(Qdx^2)B^{-1}.$$

Definition 1.7 We call $Q\,dx^2$ the *fundamental invariant* of a ruled surface.

Note that the condition $Q = 0$ holds only if the ruled surface is a quadric; see Example 1.6. Summarizing the argument, we have seen the following theorem.

Theorem 1.8 (1) *Any ruled surface can be written as a solution of a system of the form*

$$X'' = Q\,X; \qquad X = {}^t(z_1, z_2)\,, \quad \mathrm{tr} Q = 0.$$

(2) *The transformation* (1.5) *preserving this normalization is given by* (1.7) *and the conjugate class of the matrix-valued quadratic form* $Q\,dx^2$ *is invariant; in particular,* $(\det Q)dx^4$ *is an absolute invariant.*
(3) *For two ruled surfaces given respectively by curves* $(z_1(x), z_2(x))$ *and curves* $(w_1(y), w_2(y))$*, let* $Q\,dx^2$ *and* $R\,dy^2$ *denote the fundamental invariants. If both are projectively equivalent, then there exists a diffeomorphism between the parameters,* $y = f(x)$*, and a non-singular constant matrix* B *such that*

$$f^*(R\,dy^2) = B(Q\,dx^2)B^{-1}.$$

Conversely, if there exists a mapping f *and a matrix* B *satisfying this identity, then the ruled surfaces* z *and* w *are projectively equivalent.*

Remark 1.9 Since the parameter is determined up to a fractional linear transformation, we can say that any ruled surface has a 1-dimensional projective structure.

1.4 Differential invariants of a ruled surface

Let I_x be a differential polynomial defined by using the matrix Q relative to the variable x. With I_y denoting the polynomial of the same form relative to y, we say I is an invariant of weight k if

$$I_y = I_x (f')^{-k}.$$

The following is a list of some examples of such differential invariant polynomials; we refer the reader to [W1906].

$$\begin{aligned}
\theta_4 &= -4 \det Q, \\
\theta_6 &= 9\det(Q') - 2(\det Q)'', \\
\theta_{10} &= -4\det Q \det(Q') + ((\det Q)')^2, \\
\theta_9 &= \det \begin{pmatrix} p & q & r \\ p' & q' & r' \\ p'' & q'' & r'' \end{pmatrix}.
\end{aligned}$$

The weights are 4, 6, 10, and 9, respectively.

As we have done for space curves, let us consider the Plücker coordinates of the line $\overline{z_1 z_2}$ joining the points z_1 and z_2 represented by the vector $\xi = z_1 \wedge z_2$. Its components are $\xi_{ij} = z_1^i z_2^j - z_1^j z_2^i$, where $z_k = [z_k^1, z_k^2, z_k^3, z_k^4]$. Then the point ξ lies on the quadratic hypersurface in $\mathbf{P}^5$ determined by the Plücker relation

$$\xi_{12}\xi_{34} - \xi_{13}\xi_{24} + \xi_{14}\xi_{23} = 0.$$

Furthermore, we have

Proposition 1.10 *The invariant θ_9 vanishes if and only if ξ lies in a hyperplane.*

Proof. Differentiating $\xi = z_1 \wedge z_2$, we get $\xi' = z_1' \wedge z_2 + z_1 \wedge z_2'$ and $\xi'' = 2z_1' \wedge z_2'$. Then, it is easy to see the following:

$$\begin{aligned}
\tfrac{1}{2}\xi''' + p\xi' &= 2pz_1 \wedge z_2' + qz_2 \wedge z_2' - rz_1 \wedge z_1', \\
\tfrac{1}{2}\xi'''' + p'\xi' + 2(p^2 + qr)\xi &= 2p'z_1 \wedge z_2' + q'z_2 \wedge z_2' - r'z_1 \wedge z_1', \\
\tfrac{1}{2}\xi^{(5)} + (p'' + 2(p^2 + qr))\xi' + 3(p^2 + qr)'\xi &= 2p''z_1 \wedge z_2' + q''z_2 \wedge z_2' - r''z_1 \wedge z_1'.
\end{aligned}$$

Hence, ξ satisfies a 5th order equation if and only if the right-hand sides of the above equations are linearly dependent, namely, if θ_9 vanishes.

Remark 1.11 Investigation of the fundamental invariants of curves in projective space was done by Laguerre and Forsyth. A modern treatment was given by A. Se-ashi [Se1987]. For more details regarding the contents of this section, we refer the reader to [W1906] and [Sa1999].

2 Projective theory of surfaces

This section aims at describing the theory of surfaces in the 3-dimensional projective space and introduces some typical classes of surfaces, such as Demoulin surfaces, projectively applicable surfaces and projectively minimal surfaces. To define invariants appropriately, we need to associate special frames to the surfaces, called normalized frames. The normalization process works also in higher dimensions, and we start with a description of invariants on hypersurfaces.

2.1 Projective invariants of hypersurfaces

Let $e_0 : M^n \to \mathbf{P}^{n+1}$ be an immersion of an n-dimensional manifold, which defines a hypersurface. We identify it with its (arbitrary) lift to $\mathbf{R}^{n+2} - \{0\}$ or $\mathbf{C}^{n+2} - \{0\}$. We choose linearly independent tangent vector fields to $e_0(M)$ denoted by $e_1, \ldots, e_n$, which are locally defined on M and linearly independent of e_0. Let e_{n+1} be a vector field, also defined locally, which are linearly independent of $e_0, e_1, \ldots, e_n$. The ordered set $e = (e_0, e_1, \ldots, e_n, e_{n+1})$ is called a (*projective*) *frame* along the immersion. It satisfies a differential equation

$$de = \omega e, \qquad \text{i.e.,} \qquad de_\alpha = \sum_\beta \omega_\alpha^\beta e_\beta;$$

thus defining an $(n+2) \times (n+2)$-matrix valued 1-form ω. Here and in the following, the index range of Greek letters is from 0 to $n+1$ and the index range of roman letters is from 1 to n. The form ω satisfies the integrability condition

$$d\omega = \omega \wedge \omega. \tag{2.1}$$

If we consider only frames e with the property $\det(e_0, e_1, \ldots, e_n, e_{n+1}) = 1$, then it also holds that

$$\sum_\alpha \omega_\alpha^\alpha = 0,$$

which we assume in the following. When necessary, we will denote $\omega(e)$ for the 1-form ω corresponding to a given frame e. Two frames, say e and $\tilde{e}$, are related by

$$\tilde{e} = ge,$$

where g is a mapping (locally) from M to the projective linear group PGL_{n+2}. Then, we see that

$$\omega(\tilde{e}) = dg\, g^{-1} + g\omega(e)g^{-1}. \tag{2.2}$$

We now further restrict the choice of e. We first look at the condition $\omega_0^{n+1} = 0$, which follows from the definition of e_{n+1}. By taking exterior derivation, we get

$$\sum_{i=1}^n \omega^i \wedge \omega_i^{n+1} = 0,$$

where $\omega^i = \omega_0^i$; by definition, these are independent basic forms. We can write $\omega_i^{n+1} = \sum_j h_{ij}\omega^j$ for a symmetric tensor h_{ij} and define a symmetric 2-form

$$\varphi_2 = \sum_{i,j} h_{ij}\omega^i\omega^j.$$

The conformal class of φ_2 can be seen to be unique. In order to see this, we need to know how the invariants transform under a change of the frame e to a new frame $\tilde{e} = ge$, where g is given by

$$g = \begin{pmatrix} \lambda & 0 & 0 \\ b & a & 0 \\ \mu & c & \nu \end{pmatrix}.$$

By the assumption made above, $\det g = 1$. Let us denote by adding tildes to the invariants relative to the frame $\tilde{e}$ and let A denote the inverse matrix of a. Then, a calculation using (2.2) shows

$$\tilde{\omega}^i = \lambda \sum_j A^i_j \omega^j \quad \text{and} \quad \tilde{\omega}^{n+1}_i = \nu^{-1} \sum_k a^k_i \omega^{n+1}_k,$$

from which we obtain

$$\tilde{h} = (\lambda\nu)^{-1} a\, h\, {}^t a, \tag{2.3}$$

where $h = (h_{ij})$. Hence, we see that $\tilde{\varphi_2} = \lambda\nu^{-1}\varphi_2$. We say that the immersion e_0 is *nondegenerate* when the matrix (h_{ij}) is nonsingular; we assume this property in the following. Then, the equation (2.3) implies that there exists a frame such that $|\det h_{ij}| = 1$. In particular, $|\lambda\nu| = 1$. The formula (2.2) shows that

$$\tilde{\omega}^0_0 + \tilde{\omega}^{n+1}_{n+1} = \omega^0_0 + \omega^{n+1}_{n+1} + \nu^{-1} \sum_i c^i \omega^{n+1}_i - \sum_{i,j} b_i A^i_j \omega^j$$

and we can assume $\omega^0_0 + \omega^{n+1}_{n+1} = 0$ and then, b and c are related by

$$b = \nu^{-1} a\, h\, {}^t c. \tag{2.4}$$

The exterior derivation of $\omega^0_0 + \omega^{n+1}_{n+1} = 0$ gives

$$\left(\sum_{j=1}^n h_{ij} \omega^j_{n+1} - \omega^0_i \right) \wedge \omega^i = 0.$$

Hence we can define ℓ_{ij} so that

$$\sum_{j=1}^n h_{ij} \omega^j_{n+1} - \omega^0_i = \sum_{j=1}^n \ell_{ij} \omega^j; \qquad \ell_{ij} = \ell_{ji}. \tag{2.5}$$

We set

$$\ell = \frac{1}{n} \sum_{i,j} h^{ij} \ell_{ij},$$

where (h^{ij}) is the inverse of the matrix $h = (h_{ij})$.

We will next treat the third-order information of the immersion. Define h_{ijk} by the equation

$$\sum_k h_{ijk} \omega^k = dh_{ij} - \sum_k h_{kj} \omega^k_i - \sum_k h_{ik} \omega^k_j. \tag{2.6}$$

It is seen that h_{ijk} is symmetric relative to all indices and satisfies the so-called *apolarity condition*:

$$\sum_{i,j} h^{ij} h_{ijk} = 0$$

for each k, which follows from the condition $|\det h| = 1$. We define a symmetric *cubic form* by

$$\varphi_3 = \sum_{i,j,k} h_{ijk}\omega^i\omega^j\omega^k$$

and a scalar called the *Fubini-Pick invariant* by

$$F = \sum_{i,j,k,p,q,r} h_{ijk}h_{pqr}h^{ip}h^{jq}h^{kr}.$$

The cubic form has the invariance

$$\lambda^2\nu\tilde{h}_{ijk} = h_{pqr}a_i^p a_j^q a_k^r;$$

namely,

$$\tilde{\varphi}_3 = \lambda\nu^{-1}\varphi_3.$$

Together with the invariance (2.3) we can conclude that $F\varphi_2$ is an absolutely-invariant 2-form, which is called the *projective metric form*. We lastly set

$$\omega_{n+1}^0 = -\sum_j \rho_j\omega^j.$$

Thus we have quantities $\{h_{ij}, h_{ijk}, \ell_{ij}, \rho_j\}$. These are canonically defined for a nondegenerate hypersurface and define invariants in the following sense. Continuing the computation of the form $\tilde{\omega}$, we can see the formula

$$\lambda^2\tilde{\ell}_{ij} = \sum_{p,q} a_i^p\ell_{pq}a_j^q + (2\mu - \nu^{-1}c\,h\,{}^tc)h_{ij} - \nu^{-1}\sum_{p,q,r} h_{pqr}c^p a_i^q a_j^r,$$

from which we get

$$\lambda\tilde{\ell} = \lambda^{-1}\ell + (2\mu - \nu^{-1}c\,h\,{}^tc).$$

Hence, we can find a frame so that $\ell = 0$.

Summarizing the arugument above, we have seen the following.

Proposition 2.1 (1) *There exists a frame e with the properties*

$$\det(e) = 1, \quad \omega_0^{n+1} = 0, \quad \omega_0^0 + \omega_{n+1}^{n+1} = 0, \quad |\det h_{ij}| = 1, \quad \text{and} \quad \ell = 0.$$

(2) *For such two frames, the connecting transformation g has the form*

$$\begin{pmatrix} \lambda & 0 & 0 \\ b & a & 0 \\ \mu & c & \nu \end{pmatrix} \qquad \begin{aligned} |\lambda\nu| &= 1 \\ b &= \nu^{-1}ah\,{}^tc \\ \mu &= (1/2)\nu^{-1}ch\,{}^tc, \end{aligned}$$

where a is an $n \times n$-matrix of $|\det a| = 1$.

The second property implies that to any nondegenerate hypersurface is associated canonically a conformal connection.

Further, the invariance is summarized in the following way.

Proposition 2.2 *Let* $\{\tilde{h}_{ij}, \tilde{h}_{ijk}, \tilde{\ell}_{ij}, \tilde{\rho}_j\}$ *be such invariants defined for the frame* ge, *where* g *is a matrix given in Proposition 2.1(2) with components* λ, ν, *and* $a = (a_i^j)$ *and* $c = (c^i)$. *Set*

$$\rho = \sum \rho_i \omega^i, \quad \mathcal{L} = \sum \ell_{ij}\omega^i\omega^j, \quad \mathcal{L}(c) = \sum \ell_{ij}\omega^i c^j/\nu,$$
$$H(c) = \sum h_{ijk}\omega^i\omega^j c^k/\nu, \quad H(c,c) = \sum h_{ijk}\omega^i c^j c^k/\nu^2.$$

Then the following transformation formulas hold:

$$\tilde{\varphi}_2 = \lambda\nu^{-1}\varphi_2, \quad \tilde{\varphi}_3 = \lambda\nu^{-1}\varphi_3,$$
$$\tilde{\mathcal{L}} = \mathcal{L} - H(c), \quad \lambda\nu^{-1}\tilde{\rho} = \rho + \mathcal{L}(c) - \frac{1}{2}H(c,c).$$

It can be proved that the set of invariants $\{h_{ij}, h_{ijk}, \ell_{ij}, \rho_j\}$ determines the immersion up to a projective transformation. In the case $n \geq 3$, the quantities ℓ_{ij} and ρ_j are given in terms of h_{ij}, h_{ijk} and their derivatives. But in the case $n = 2$, the situation is different. See Sect. 2.6.

The invariance in Proposition 2.1 means geometrically the following. Any point P in the space can be written as

$$P = p^0 e_0 + p^1 e_1 + \cdots + p^n e_n + p^{n+1} e_{n+1}$$

and thus we define the coordinates $p = (p^0, p^1, \ldots, p^n, p^{n+1})$ relative to the frame e: $P = pe$. We then set

$$H = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -h & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and define a quadratic hypersurface as the set $Q = \{P \mid X(P) = 0\}$, where $X(P) = pH\,{}^t p$. For the frame $\tilde{e}$, we see $P = \tilde{p}\tilde{e}$, where $p = \tilde{p}g$. It is easy to see $gH^t g = \pm\tilde{H}$ in view of the invariance (2.3) and (2.4). This implies that the quadratic hypersurface Q is well-defined independent of frames. It is called the *Lie quadratic hypersurface* and, by definition, tangent to the given hypersurface up to the second order.

We next see how the Lie quadratic hypersurface Q depends on the point of the given hypersurface. We fix a point P and consider it a function on the given hypersurface. Since it does not move, dP is proportional to P itself, which means that $dp = -p\omega - \kappa p$ for some 1-form κ. Hence, we can see that $dX = p\Omega^t p - 2\kappa pH\,{}^t p$, where

$$\Omega = \sum_k H_k \omega^k, \quad H_k = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -h_{ijk} & \ell_{ik} \\ 0 & \ell_{jk} & 2\rho_k \end{pmatrix}. \tag{2.7}$$

We now assume that $\varphi_3 = 0$. Then, the exterior derivation of (2.6) shows that $\ell_{ij} = 0$ by use of the integrability condition (2.1) and then the exterior derivation of (2.5) when $\ell_{ij} = 0$ shows that $\rho_i = 0$. Hence $dX(P) = 0$ for the point P on Q, which means that the quadratic hypersurface remains unchanged; namely, we have the following:

Theorem 2.3 *Assume that φ_2 is nondegenerate and $\varphi_3 = 0$. Then the hypersurface is projectively equivalent to a quadratic hypersurface.*

Given an immersion $e_0 : M^n \to \mathbf{P}^{n+1}$ and an associated projective frame $e = (e_0, e_1, \dots, e_n, e_{n+1})$, we define the dual frame $E = \{E^0, E^1, \dots, E^{n+1}\}$ by

$$E^\alpha = (-1)^\alpha e_0 \wedge \cdots \wedge \check{e}_\alpha \wedge \cdots \wedge e_{n+1}.$$

We have $\langle e_\alpha, E^\beta \rangle = \delta^\beta_\alpha$. The frame E satisfies the equation $dE = -E\omega$. The vector E^{n+1} can be identified with the tangent hyperplane of the given hypersurface and also can be regarded as an immersion into the dual projective space, called the *dual immersion*. A projective frame associated to this dual immersion is given by $\check{E} = (E^{n+1}, E^1, \dots, E^n, E^0)$, and if we set $d\check{E} = \Omega\check{E}$, then $\Omega^i = -\omega^{n+1}_i$, $\Omega^{n+1}_i = -\omega^i$, $\Omega^j_i = -\omega^j_i$, $\Omega^j_0 = -\omega^j_{n+1}$, $\omega^j_{n+1} = -\omega^0_j$, and $\Omega^0_{n+1} = -\omega^0_{n+1}$. When we denote the invariants of the dual immersion by adding asterisks, we can see that $h^*_{ij} = h^{ij}$, $h^*_{ijk} = \sum h^{ip}h^{jq}h^{kr}h_{pqr}$, $\ell^*_{ij} = -\sum h^{ip}h^{jq}\ell_{pq}$, and $\rho^*_i = \sum h^{ip}\rho_p$. Consequently $\varphi^*_2 = \varphi_2$ and $\varphi^*_3 = \varphi_3$.

Remark 2.4 *For the Lie quadratic hypersurface, we refer to [Bol, vol. 3, p. 438]. The case when $n = 2$ will be considered again in Sect. 11.2. For further details on the contents of this section, we refer to [Sa1988, Sa1999].*

2.2 Projective invariants of surfaces

Let us consider the case $n = 2$. We start with the system defined by an immersion

$$z : (x, y) \longrightarrow z(x, y) \in \mathbf{P}^3.$$

Assume that the vector z_{xy} considered as a mapping to $\mathbf{R}^4 - \{0\}$ or $\mathbf{C}^4 - \{0\}$ is linearly independent of z, z_x, and z_y. Then the system has the form

$$z_{xx} = \ell z_{xy} + a z_x + b z_y + pz, \quad z_{yy} = m z_{xy} + c z_x + d z_y + qz.$$

Relative to a frame $\{e_0 = z,\ e_1 = z_x,\ e_2 = z_y,\ e_3 = z_{xy}\}$, the coframe has the form

$$\omega = \begin{pmatrix} 0 & dx & dy & 0 \\ pdx & adx & bdx & \ell dx + dy \\ qdy & cdy & d\,dy & dx + mdy \\ * & * & * & * \end{pmatrix}.$$

Hence we see that

$$\varphi_2 = \ell dx^2 + 2dx\,dy + mdy^2.$$

In the following, we assume $1 - \ell m \neq 0$ so that φ_2 is nondegenerate. If the coordinates are chosen to be conjugate relative to φ_2 (see Sect. 4.1), then we get $\ell = m = 0$ and the system is reduced to

$$z_{xx} = a z_x + b z_y + pz, \quad z_{yy} = c z_x + d z_y + qz.$$

Here in the real case, we have assumed that φ_2 is indefinite. The associated coframe is

$$\omega = \begin{pmatrix} 0 & dx & dy & 0 \\ pdx & adx & bdx & dy \\ qdy & cdy & d\,dy & dx \\ \begin{matrix}(bq+p_y)dx \\ +(cp+q_x)dy\end{matrix} & \begin{matrix}(a_y+bc)dx \\ +(ac+c_x+q)dy\end{matrix} & \begin{matrix}(bd+b_y+p)dx \\ +(bc+d_x)dy\end{matrix} & adx+d\,dy \end{pmatrix}.$$

In particular, we have $d(e_0 \wedge e_1 \wedge e_2 \wedge e_3) = 2(adx + d\,dy)e_0 \wedge e_1 \wedge e_2 \wedge e_3$. Hence, $adx + d\,dy$ is an exact form and there exists a function θ such that $a = \theta_x$ and $d = \theta_y$. Then,

$$z_{xx} = \theta_x z_x + bz_y + pz, \quad z_{yy} = cz_x + \theta_y z_y + qz. \tag{2.8}$$

The integrability condition of the system is $d\omega = \omega \wedge \omega$, which consists of three equations:

$$\begin{aligned} L_y &= -2bc_x - cb_x, \\ M_x &= -2cb_y - bc_y, \\ bM_y + 2Mb_y + b_{yyy} &= cL_x + 2Lc_x + c_{xxx}, \end{aligned} \tag{2.9}$$

where L and M are traditional notations defined by

$$L = \theta_{xx} - \frac{1}{2}\theta_x^2 - b\theta_y - b_y - 2p, \quad M = \theta_{yy} - \frac{1}{2}\theta_y^2 - c\theta_x - c_x - 2q. \tag{2.10}$$

We can simplify the system (2.8) by replacing $e^{-\theta/2}z$ with w; then we get

$$w_{xx} = \overline{b}w_y + \overline{p}w, \quad w_{yy} = \overline{c}w_x + \overline{q}w,$$

where

$$\overline{b} = b, \quad \overline{c} = c, \quad \overline{p} = p - \frac{1}{2}\theta_{xx} + \frac{1}{4}\theta_x^2 + \frac{1}{2}b\theta_y, \quad \overline{q} = q - \frac{1}{2}\theta_{yy} + \frac{1}{4}\theta_y^2 + \frac{1}{2}c\theta_x. \tag{2.11}$$

The invariants L and M remain the same under this change of the unknown from z to w.

The *dual surface* is defined by the immersion $\xi = z \wedge z_x \wedge z_y$. It satisfies the system

$$\xi_{xx} = \theta_x \xi_x - b\xi_y + \overline{p}\xi, \quad \xi_{yy} = -c\xi_x + \theta_y \xi_y + \overline{q}\xi,$$

where

$$\overline{p} = p + b_y + b\theta_y, \qquad \overline{q} = q + c_x + c\theta_x.$$

2.3 Wilczynski frame of a surface

In this section, we treat the system

$$z_{xx} = bz_y + pz, \quad z_{yy} = cz_x + qz, \tag{2.12}$$

which we call the *canonical system* of the given immersion. Relative to this system, the integrability condition (2.9) simplifies to

$$\begin{gathered} p_y = bc_x + \tfrac{1}{2}b_x c - \tfrac{1}{2}b_{yy}, \quad q_x = cb_y + \tfrac{1}{2}bc_y - \tfrac{1}{2}c_{xx}, \\ b_{yyy} - bc_{xy} - 2bq_y - 2b_y c_x - 4qb_y \\ = c_{xxx} - cb_{xy} - 2cp_x - 2b_x c_y - 4pc_x. \end{gathered} \tag{2.13}$$

We now choose a frame defined by

$$e_0 = z, \quad e_1 = z_x, \quad e_2 = z_y, \quad e_3 = z_{xy} - \frac{1}{2}bcz. \tag{2.14}$$

Then the coframe ω for the canonical system (2.12) is

$$\omega = \begin{pmatrix} 0 & dx & dy & 0 \\ pdx + \frac{1}{2}bcdy & 0 & bdx & dy \\ qdy + \frac{1}{2}bcdx & cdy & 0 & dx \\ \begin{array}{c}(bq+p_y)dx + (cp+q_x)dy \\ -\frac{1}{2}d(bc)\end{array} & \begin{array}{c}(q+c_x)dy \\ +\frac{1}{2}bcdx\end{array} & \begin{array}{c}(p+b_y)dx \\ +\frac{1}{2}bcdy\end{array} & 0 \end{pmatrix}.$$

From this expression, we have

$$\begin{gathered} h_{11} = h_{22} = 0, \quad h_{12} = h_{21} = 1, \\ h_{111} = -2b, \quad h_{222} = -2c, \quad h_{112} = h_{122} = 0, \\ F = 8bc, \\ \ell_{11} = b_y, \quad \ell_{12} = \ell_{21} = 0, \quad \ell_{22} = c_x, \\ \rho_1 = \tfrac{1}{2}(bc)_x - bq - p_y, \quad \rho_2 = \tfrac{1}{2}(bc)_y - cp - q_x. \end{gathered} \tag{2.15}$$

In particular, the condition $b = c = 0$ is necessary and sufficient for the surface to be quadratic. The surface is ruled if and only if $F = 0$, i.e. $bc = 0$. In fact, assume $c = 0$. Then by (2.13), q is independent of x and we may assume $q = 0$ by multiplying z by some factor. Checking again (2.13), it is seen that the system has the form

$$z_{xx} = (\alpha y^2 + \beta y + \gamma)z_y + (-\alpha y + \delta)z, \qquad z_{yy} = 0, \tag{2.16}$$

where α, β, γ and δ are functions of x. This shows that z is linear relative to y and has the form $u(x) + yv(x)$, where u and v are solutions of the system $u_{xx} = \delta u + \gamma v$ and $v_{xx} = -\alpha u + (\beta + \delta)v$; thus z defines a ruled surface. The converse is shown similarly.

In some cases, it is useful to adopt another frame $\{z, z_1, z_2, \eta\}$ as far as $bc \neq 0$, called the *Wilczynski frame*, defined by

$$\begin{gathered} z = z, \quad z_1 = z_x - \frac{c_x}{2c}z, \quad z_2 = z_y - \frac{b_y}{2b}z, \\ \eta = z_{xy} - \frac{c_x}{2c}z_y - \frac{b_y}{2b}z_x + \left(\frac{b_y c_x}{4bc} - \frac{1}{2}bc\right)z. \end{gathered}$$

We introduce notations

$$\kappa_1 = \frac{bc - (\log b)_{xy}}{2}, \quad \kappa_2 = \frac{bc - (\log c)_{xy}}{2}, \tag{2.17}$$

and

$$P = p + \frac{b_y}{2} - \frac{c_{xx}}{2c} + \frac{c_x^2}{4c^2}, \quad Q = q + \frac{c_x}{2} - \frac{b_{yy}}{2b} + \frac{b_y^2}{4b^2}. \tag{2.18}$$

Then the system (2.12) can be written in the Pfaffian form

$$d\begin{pmatrix} z \\ z_1 \\ z_2 \\ \eta \end{pmatrix} = \left[\begin{pmatrix} \frac{c_x}{2c} & 1 & 0 & 0 \\ P & -\frac{c_x}{2c} & b & 0 \\ \kappa_1 & 0 & \frac{c_x}{2c} & 1 \\ bQ & \kappa_1 & P & -\frac{c_x}{2c} \end{pmatrix} dx + \begin{pmatrix} \frac{b_y}{2b} & 0 & 1 & 0 \\ \kappa_2 & \frac{b_y}{2b} & 0 & 1 \\ Q & c & -\frac{b_y}{2b} & 0 \\ cP & Q & \kappa_2 & -\frac{b_y}{2b} \end{pmatrix} dy\right] \begin{pmatrix} z \\ z_1 \\ z_2 \\ \eta \end{pmatrix}$$

For this expression of the coframe, the invariants are given as follows:

$$\begin{gathered} h_{11} = h_{22} = 0, \quad h_{12} = h_{21} = 1, \\ h_{111} = -2b, \quad h_{222} = -2c, \quad h_{112} = h_{122} = 0, \\ \ell_{11} = \ell_{12} = \ell_{21} = \ell_{22} = 0, \quad \rho_1 = -bQ, \quad \rho_2 = -cP. \end{gathered}$$

One merit of using the Wilczynski frame is that we can suppose all $\ell_{ij} = 0$. This is due to that the dimension is 2.

2.4 Demoulin frames of a surface

According to the invariance stated in Proposition 2.1, the expressions of invariants are dependent on the frame. In this section, we continue the procedure in Sect. 2.1 and try to find a frame so that $\rho_i = 0$.

To simplify notations, we assume that the surface is indefinite and choose a frame with the properties in Proposition 2.1(1) so that $h = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then, the apolarity condition and the condition $\ell = 0$ imply $h_{112} = h_{122} = 0$ and $\ell_{12} = 0$. Referring to the choice of frames in the previous subsection, we set $h_{111} = -2b$ and $h_{222} = -2c$. Then the transformation rule of Proposition 2.1 applied to the transformation g with $a = I_2$ and $\lambda = \nu = 1$ shows that the condition $\tilde{\rho}_i = 0$ is written as follows:

$$b(c^1)^2 + \ell_{11}c^1 + \rho_1 = 0, \qquad c(c^2)^2 + \ell_{22}c^2 + \rho_2 = 0. \tag{2.19}$$

Here (c^1, c^2) is a vector appearing in a change of the frame in Proposition 2.1. c^2 here is not $(c)^2$.

Proposition 2.5 *Assume that the surface is indefinite and non-ruled. Then, there is a frame so that $\rho_i = 0$. The number of such frames is at most four.*

Let us denote by Δ_1 and Δ_2 the discriminants of the equations of (2.19):

$$\Delta_1 = (\ell_{11})^2 - 4\rho_1 b \quad \text{and} \quad \Delta_2 = (\ell_{22})^2 - 4\rho_2 c,$$

respectively. We remark that, when Δ_1 and/or Δ_2 are negative or when h is definite, we need to make considerations in the complex number field.

Definition 2.6 We call such a frame a *Demoulin frame.*

Once a Demoulin frame is chosen, the remaining freedom of choice is very restricted. For simplicity consider the case $\lambda\nu = 1$. Then the only possible form of the frame change g is

$$\begin{pmatrix} \lambda & 0 & 0 & 0 \\ \lambda\alpha c^2 & \alpha & 0 & 0 \\ \lambda\alpha^{-1}c^1 & 0 & \alpha^{-1} & 0 \\ \lambda c^1 c^2 & c^1 & c^2 & \lambda^{-1} \end{pmatrix}$$

and the values c^1 and c^2 must satisfy the conditions

$$b(\lambda c^1)^2 + \ell_{11}\lambda c^1 = 0, \qquad c(\lambda c^2)^2 + \ell_{22}\lambda c^2 = 0.$$

For each solution, the new $\tilde{\ell}_{ij}$'s are given by

$$\tilde{\ell}_{11} = \ell_{11} + 2b\lambda c^1, \qquad \tilde{\ell}_{22} = \ell_{22} + 2c\lambda c^2,$$

and the last vector of the new frame is

$$\lambda^{-1}(e_3 + (\lambda c^1)e_1 + (\lambda c^2)e_2 + (\lambda^2 c^1 c^2)e_0).$$

Definition 2.7 When the vector e_3 of a Demoulin frame defines a surface, we call it a *Demoulin transform* of the original surface.

For a Demoulin frame, we set

$$\sum_j h_{ij}\omega_3^j = \sum_j q_{ij}\omega^j\,, \quad \omega_i^0 = \sum_j p_{ij}\omega^j. \tag{2.20}$$

(The letters p and q are reserved for denoting the coefficients of the system; the usage here for the matrices can be distingushed from the context.) The condition for e_3 to define a surface is that ω_3^1 and ω_3^2 are linearly independent, because $de_3 = \omega_3^3 e_3 + \omega_3^1 e_1 + \omega_3^2 e_2$. In other words,

$$\det q \neq 0\,, \quad q = (q_{ij}).$$

Under this condition, a set $\overline{e} = (e_3, e_1, e_2, e_0)$ in this order defines a projective frame of e_3, and the coframe $\overline{\omega}$ is

$$\overline{\omega} = \begin{pmatrix} \omega_3^3 & \omega_3^1 & \omega_3^2 & 0 \\ \omega_1^3 & \omega_1^1 & \omega_1^2 & \omega_1^0 \\ \omega_2^3 & \omega_2^1 & \omega_2^2 & \omega_2^0 \\ 0 & \omega^1 & \omega^2 & \omega_0^0 \end{pmatrix}$$

Therefore, the associated fundamental form $\overline{\varphi}_2$ is $\omega_3^1 \cdot \omega_1^0 + \omega_3^2 \cdot \omega_2^0$, which is nondegenerate when

$$\det p \neq 0\,, \quad p = (p_{ij}).$$

Since $\ell_{ij} = q_{ij} - p_{ij}$ satisfies the condition $\ell = \mathrm{tr}\ell_{ij} = 0$, we see that

$$p_{12} = q_{12}\,, \quad p_{21} = q_{21}, \tag{2.21}$$

from which we have

$$\overline{\varphi}_2 = p_{21}(p_{11}+q_{11})\omega^1\omega^1 + (2p_{12}p_{21}+p_{11}q_{22}+q_{11}p_{22})\omega^1\omega^2 + p_{12}(p_{22}+q_{22})\omega^2\omega^2. \tag{2.22}$$

Moreover, $\omega_3^0 = 0$ implies $\sum_i \omega_3^i \wedge \omega_i^0 = 0$. Hence

$$p_{11}q_{22} - p_{22}q_{11} = 0. \tag{2.23}$$

Let us continue our consideration of the system (2.12) by assuming $bc \neq 0$. By the expression in (2.15), a Demoulin frame is given by the change of the frame (2.14) by the tranformation

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ c^2 & 1 & 0 & 0 \\ c^1 & 0 & 1 & 0 \\ c^1c^2 & c^1 & c^2 & 1 \end{pmatrix},$$

where

$$c^1 = \frac{-b_y \pm \sqrt{\Delta_1}}{2b}, \quad c^2 = \frac{-c_x \pm \sqrt{\Delta_2}}{2c},$$

and

$$\Delta_1 = (b_y)^2 + 4b\,(bq + p_y - \frac{1}{2}(bc)_x), \quad \Delta_2 = (c_x)^2 + 4c\,(cp + q_x - \frac{1}{2}(bc)_y).$$

Hence, the Demoulin transforms w of the surface z are given by

$$w = (c^1c^2 - \frac{1}{2}b\,c)z + c^1 z_x + c^2 z_y + z_{xy}.$$

An explicit computation of Demoulin frames will be given in Sect. 11.6.

By the integrability (2.9), a simple computation shows

$$\Delta_1 = 4b^2 Q, \quad \Delta_2 = 4c^2 P. \tag{2.24}$$

By use of the Wilczynski frame $\{z, z_1.z_2, \eta\}$, the transform w is given by

$$w = \eta + \sigma z_1 + \tau z_2 + \sigma\tau z, \tag{2.25}$$

where $\sigma = \pm\sqrt{Q}$ and $\tau = \pm\sqrt{P}$.

Definition 2.8 We call a surface satisfying the conditions $bc \neq 0$ and $\det q \neq 0$ a *Demoulin surface* if it has only one Demoulin transform, equivalently if $P = Q = 0$. A surface with $P = 0$ or $Q = 0$ is called a *Godeaux-Rozet surface*.

The condition $P = Q = 0$ determines the coefficients p and q in terms of b and c by (2.18) and then the third equation of the integrability (2.13) turns out to be satisfied. The first and the second equations of the integtability (2.13) then give a system of nonlinear differential equations relative to b and c:

$$\begin{aligned} b^2 b_{xyy} - b b_x b_{yy} - b b_y b_{xy} + b_x {b_y}^2 - 2cb^3 b_y - b^4 c_y = 0, \\ c^2 c_{xxy} - c c_x c_{xx} - c c_x c_{xy} + {c_x}^2 c_y - 2bc^3 c_x - c^4 b_x = 0. \end{aligned} \tag{2.26}$$

Moreover, this system can be integrated as follows:

$$(\log b)_{xy} = bc + \frac{r(x)}{b}, \quad (\log c)_{xy} = bc + \frac{s(y)}{c} \tag{2.27}$$

by using indeterminate functions r of x and s of y. Hence, we have seen that a Demoulin surface is defined by the system

$$z_{xx} = bz_y + \left(\frac{c_{xx}}{2c} - \frac{c_x^2}{4c^2} - \frac{b_y}{2}\right) z, \quad z_{yy} = cz_x + \left(\frac{b_{yy}}{2b} - \frac{b_y^2}{4b^2} - \frac{c_x}{2}\right) z, \tag{2.28}$$

where b and c are satisfying the system (2.27).

Remark 2.9 Relative to a Demoulin frame, any Godeaux-Rozet surface for which $bc \neq 0$ is characterized by the condition $\ell_{11} = 0$ and $\ell_{22} \neq 0$, or, $\ell_{11} \neq 0$ and $\ell_{22} = 0$. Further, in view of (2.23), we must have $p_{11} = q_{11} = 0$ or $p_{22} = q_{22} = 0$, respectively.

2.5 Remarks on higher dimensional hypersurfaces

Let $\mathbf{A}^{n+1}$ be the $(n+1)$-dimensional affine space and $f_0 : M^n \to \mathbf{A}^{n+1}$ an immersion defining an affine hypersurface. We choose a set of vectors $\{f_1, \ldots, f_{n+1}\}$ along the immersion so that $f_1, \ldots, f_n$ are tangent to the hypersurface and $\det(f_1, \ldots, f_{n+1}) = 1$. Then we have a Pfaff equation $df_\alpha = \sum_\beta \tau_\alpha^\beta f_\beta$ $(0 \leq \alpha, \beta \leq n+1)$ and the connection form $\tau = (\tau_\alpha^\beta)$. The components of the form τ are

$$\begin{pmatrix} 0 & \tau^j & \tau_0^{n+1} \\ 0 & \tau_i^j & \tau_i^{n+1} \\ 0 & \tau_{n+1}^j & \tau_{n+1}^{n+1} \end{pmatrix}.$$

The integrability condition $d\tau = \tau \wedge \tau$ is satisfied. Similarly to the argument of Sect. 2.1, we have $\tau_0^{n+1} = 0$ and $\tau_i^{n+1} = \sum h_{ij} \tau^j$, where τ^j denotes τ_0^j. We may assume $|\det h| = 1$ and $\tau_{n+1}^{n+1} = 0$. Thus we get a quadratic form $\sum h_{ij} \tau^i \tau^j$. We define a tensor m_{ij} by $\sum h_{ij} \tau_{n+1}^j = \sum m_{ij} \tau^j$. It is called the affine mean curvature tensor, and $m = \frac{1}{n} \sum h^{ij} m_{ij}$ is called the affine mean curvature. When we regard the affine hypersurface as a projective hypersurface, the invariant h_{ij} remains the same, the invariant ℓ_{ij} is given by $\ell_{ij} = m_{ij} - m h_{ij}$, and the invariant form ω_{n+1}^0 equals $-\frac{1}{2} dm$. It suffices to normalize the frame by a transformation

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & I_n & 0 \\ -\frac{1}{2} m & 0 & 1 \end{pmatrix}.$$

When an affine hypersurface satisfies $m_{ij} - mh_{ij} = 0$, it is called an *affine hypersphere* and then m turns out to be constant. Namely, we have $\ell_{ij} = 0$ and $\rho_i = 0$ for affine hyperspheres. In the 2-dimensional case, this property is equivalent to the definition of a Demoulin surface. We introduce the following definition.

Definition 2.10 A projective hypersurface is called a *Demoulin hypersurface* if it has a frame so that $\ell_{ij} = 0$ and $\rho_i = 0$.

The property that $\ell_{ij} = 0$ and $\rho_i = 0$ for one frame is equivalent to the property that for any frame described in Proposition 2.1, there exists a vector a^k such that

$$\ell_{ij} = \sum h_{ijk}a^k \quad \text{and} \quad \rho_i = -\frac{1}{2}\sum h_{ijk}a^j a^k,$$

due to the invariance in Proposition 2.1.

We further consider whether there exists a frame so that $\rho_i = 0$ for a general hypersurface. To find such a frame is equivalent to solving the equation

$$\rho_i + \sum_j \ell_{ij}c^j - \frac{1}{2}\sum_{j,k} h_{ijk}c^j c^k = 0 \quad (1 \le i \le n) \tag{2.29}$$

relative to c^i; see Proposition 2.2. If the quadratic forms $H_i(c,c) = \sum h_{ijk}c^j c^k$ are in general position, then it is possible to solve the equations and the number of solutions is 2^n. However in our case the coefficients satisfy algebraic conditions

$$\sum_{i,j} h^{ij}\ell_{ij} = 0, \quad \sum_{i,j} h^{ij}h_{ijk} = 0$$

and also the conditions that the ℓ_{ij} and h_{ijk} are totally symmetric. A computer test done by M. Noro for the case $n = 3$ shows that the number of solutions is eight. Furthermore, when $n = 3$, the identities $\sum h_{ijk}c^j c^k = 0$ imply $c^1 = c^2 = c^3 = 0$, and especially the identities $\sum h_{ijk}c^k = 0$ imply $c^1 = c^2 = c^3 = 0$. This means that, for a Demoulin hypersurface, such a frame is unique. The author wonders if this also holds generically.

Remark 2.11 The equation (2.29) has the following geometrical meaning. Let us recall the definition of the Lie quadratic hypersurface Q defined by the equation $E = 0$ in Sect. 2.1. The set of hypersurfaces Q is stationary at a point P if and only if $dE = 0$ also at P. As we have seen, the condition $dE = 0$ is $pH_i{}^t p = 0$ for each i and this is written as

$$\rho_i(p^{n+1})^2 + \sum_j \ell_{ij}p^j p^{n+1} - \frac{1}{2}\sum_{j,k} h_{ijk}p^j p^k = 0 \quad (1 \le i \le n),$$

which is a homogeneous form of the equation (2.29). We refer to Sect. 11.3 and 11.4 for the detailed study of the 2-dimensional case.

2.6 Projectively applicable surfaces

Relative to the Wilczynski frames, the invariants $\kappa_1 = (bc - (\log b)_{xy})/2$ and $\kappa_2 = (bc - (\log c)_{xy})/2$ have played important roles. In this section, we examine their geometric meaning. For this purpose, we consider a space curve $z(x, y)$ by fixing the variable y. Set $w(x) = b^{-1/2}z$. Then a calculation shows

$$w_{xxxx} + q_2 w_{xx} + q_3 w_x + q_4 w = 0,$$

where

$$\begin{aligned} q_2 &= 2(\log b)_{xx} - ((\log b)_x)^2/2 - b_y - 2p, \\ q_3 &= 2(\log b)_{xxx} - (\log b)_x((\log b)_{xx} + b_y) - 2p_x - b^2 c. \end{aligned}$$

Hence the invariant form ψ_3 introduced in Sect. 1.1 is given by

$$\psi_3 = -b(bc - (\log b)_{xy})dx^3 = -2b\kappa_1 dx^3. \tag{2.30}$$

We give here a lemma without proof.

Lemma 2.12 *Set*

$$\begin{aligned} K^1 &= \log(b\kappa_1) = \log(b(bc - (\log b)_{xy})/2), \\ K^2 &= \log(c\kappa_2) = \log(c(bc - (\log c)_{xy})/2). \end{aligned} \tag{2.31}$$

Then

$$P_y = 2\kappa_2 K^2_x, \quad Q_x = 2\kappa_1 K^1_y.$$

In particular, for a Demoulin surface with $\kappa_1\kappa_2 \neq 0$,

$$(\log c\kappa_2)_x = 0, \quad (\log b\kappa_1)_y = 0.$$

For a Godeaux-Rozet surface, one of $(c(bc - (\log c)_{xy}))_x = 0$ and $(b(bc - (\log b)_{xy}))_y = 0$ will hold. When the former identity holds, we can set $c(bc - (\log c)_{xy}) = Y(y)$ and, when this does not vanish, we can assume $Y = 1$ by a simple argument on the coordinate dependence. On the other hand, this says that the parameter y is proportional to the projective length parameter. Hence, we have:

Proposition 2.13 *For any Demoulin surface, the coordinates (x, y) can be chosen so that x and y are projective length parameters of both coordinate curves.*

From (2.30), we can also conclude the following proposition.

Proposition 2.14 *Let z be a non-ruled surface. Then,* (1) *the space curve with parameter x (resp. parameter y) belongs to a linear complex if and only if $\kappa_1 = 0$ (resp. $\kappa_2 = 0$).* (2) *If both curves belong to linear complexes, then the system reduces to the case*

$$b = c, \quad (\log b)_{xy} = b^2. \tag{2.32}$$

Proof. Part (1) follows directly from Proposition 1.1. For the second part (2), we have

$$bc = (\log b)_{xy} = (\log c)_{xy}.$$

In particular, $(\log b/c)_{xy} = 0$. Hence, we can write $b/c = X(x)Y(y)$ for non-vanishing functions X and Y. Since, by coordinate change from (x, y) to $(\bar{x} = f(x), \bar{y} = g(y))$ the invariants b and c are changed to bg'/f'^2 and cf'/g'^2, we can assume $b = c$ and the conclusion follows.

We next treat the surface with the property (2.32). First, note that

$$L = -b_y - 2p, \quad M = -c_x - 2q.$$

Differentiating (2.32), we get the identity

$$((\log b)_{xx} + \frac{1}{2}(\log b)_x^2)_y = 3bb_x$$

and we see that $L_y = -3bb_x$, by (2.9). Hence, there exists a function $X_1(x)$ such that

$$L = -(\log b)_{xx} - \frac{1}{2}(\log b)_x^2 + X_1(x).$$

Similarly, we can set

$$M = -(\log b)_{yy} - \frac{1}{2}(\log b)_y^2 + Y_1(x).$$

Then, from the third formula of the integrability (2.9), it holds that

$$(b^2 Y_1)_y = (b^2 X_1)_x. \tag{2.33}$$

On the other hand, any solution of the equation $(\log b)_{xy} = b^2$ is written as

$$b = \frac{(X'Y')^{1/2}}{X + Y}, \quad \text{where} \quad X = X(x), \quad Y = Y(y).$$

Hence, integrating the form $b^2 Y_1 dx + b^2 X_1 dy$, which is closed by (2.33), we get the identity

$$-\frac{Y'Y_1}{X + Y} + Y_2 = -\frac{X'X_1}{X + Y} + X_2,$$

for appropriate functions $X_2(x)$ and $Y_2(y)$. By differentiation relative to x and y, we get

$$Y_2'X' = Y'X_2'.$$

This implies that there exist constants k_1, k_2, and k_3 so that

$$X_2 = k_1 X + k_2, \quad Y_2 = k_1 Y + k_3.$$

Then, we see that

$$X'X_1 - k_1 X^2 - (k_2 - k_3)X = Y'Y_1 - k_1 Y^2 - (k_3 - k_2)Y,$$

which must be constant, and let us call this constant c_3. By setting $k_2 - k_3 = c_2$ and $k_1 = c_1$, we finally get

$$X'X_1 = c_1X^2 + c_2X + c_3, \quad Y'Y_1 = c_1Y^2 - c_2Y + c_3.$$

Now we have explicit forms of p and q in terms of two functions X and Y and three constants:

$$\begin{aligned} 2p &= -b_y + (\log b)_{xx} + \frac{1}{2}(\log b)_x^2 - \frac{c_1X^2 + c_2X + c_3}{X'}, \\ 2q &= -b_x + (\log b)_{yy} + \frac{1}{2}(\log b)_y^2 - \frac{c_1Y^2 - c_2Y + c_3}{Y'}. \end{aligned} \tag{2.34}$$

This fact says that, for a surface with the property (2.32), the possible choices for p and q can include three arbitrary constants. Since this is a remarkable property, we recall the following definition.

Definition 2.15 Given two surfaces S_1 and S_2, a mapping $f : S_1 \to S_2$ is called a *projective deformation* if, for any point p in S_1, there exists a projective transformation T such that $T(p) = f(p)$ and that, for any curve C on S_1 passing through p, its image $T(C)$ has a second-order contact with the curve $f(C)$. A surface is said to be *projectively applicable* if it has a nontrivial projective deformation.

Proposition 2.16 *If $f : S_1 \to S_2$ is a projective deformation, then the asymptotic directions are preserved by f. Also, $b_1 = b_2$ and $c_1 = c_2$ when denoting the invariants for S_1 by b_1 and c_1 and for S_2 by b_2 and c_2, relative to the common asymptotic directions. These conditions are conversely sufficient for the two surfaces to be related by a projective deformation.*

Proof. From the second-order contactedness, it immediately follows that the asymptotic directions are preserved. Hence, both surfaces can be assumed to be given by systems like (2.12): Assume that S_1 is given by the system $(z_1)_{xx} = b_1(z_1)_y + p_1z_1$ and $(z_1)_{yy} = c_1(z_1)_x + q_1z_1$ and that S_2 is given by the system $(z_2)_{xx} = b_2(z_2)_y + p_2z_2$ and $(z_2)_{yy} = c_2(z_2)_x + q_2z_2$. Up to a projective transformation, we have $z_2 = \rho z_1$, $(z_2)_x = \rho_x z_1 + \rho z_{1x}$, $(z_2)_y = \rho_y z_1 + \rho z_1$, and $(z_2)_{xx} = \rho_{xx}z_1 + 2\rho_x(z_1)_x + \rho(z_1)_{xx}$ at the contact point. Hence, we can see that $(z_2)_{xx} = b_1(z_2)_y + 2\rho_x/\rho(z_2)_x + (p_1 + \rho_{xx}/\rho - b_1\rho_x/\rho - 2\rho_x^2/\rho^2)z_2$, which implies $b_1 = b_2$. Similarly, $c_1 = c_2$. The converse statement is seen by reversing the argument, since derivatives of ρ can take arbitrary values at the contact point.

Any (nondegenerate) ruled surface is projectively applicable, because the coefficient p includes an arbitrary function δ in the representation (2.16).

Next we assume $bc \neq 0$ and consider a surface given by the system (2.12). We want to find distinct surfaces with the same b and c. Let L_1, M_1 and L_2, M_2 be the quantities for two surfaces that are deformable to each other. By setting

$$\lambda = L_2 - L_1, \quad \mu = M_2 - M_1 \tag{2.35}$$

and checking the integrability, we get the conditions

$$\lambda_y = \mu_x = 0, \quad b\mu_y + 2\mu b_y = c\lambda_x + 2\lambda c_x. \tag{2.36}$$

Then, introducing ν by

$$\nu bc = b\mu_y + 2\mu b_y,$$

we get the compatibility condition

$$2\lambda(\log c)_{xy} = (\nu b)_y, \quad 2\mu(\log b)_{xy} = (\nu c)_x. \tag{2.37}$$

If the system of the linear differential equations (2.36) and (2.37) relative to λ, μ, and ν is solvable, then we have at most three independent solutions. We denote them by (λ_i, μ_i) for $i \leq 3$. General solutions are written as $\lambda = \sum c_i\lambda_i$ and $\mu = \sum c_i\mu_i$, where c_i are integration constants.

Proposition 2.17 *Assume $bc \neq 0$ and the number of independent solutions is three. Then the Gaussian curvature of the form $bcdxdy$ is constant.*

Proof. We compute ν_{xy} from (2.37) in two different orders $(\nu_x)_y$ and $(\nu_y)_x$ and get the identity

$$2c\lambda((\log c)_{xy}/bc)_x - 2b\mu((\log b)_{xy}/bc)_y = \nu(\log b/c)_{xy}.$$

Since we have assumed the existence of three independent solutions of the triple (λ, μ, ν), this equation should be trivial. Namely, we have

$$(\log b/c)_{xy} = 0, \quad ((\log c)_{xy}/bc)_x = 0, \quad ((\log b)_{xy}/bc)_y = 0. \tag{2.38}$$

The latter two equations show that $(\log bc)_{xy}/bc$ is constant, since $(\log b)_{xy} = (\log c)_{xy}$, namely, the Gaussian curvature of the metric form $bcdxdy$ is constant.

Remark 2.18 The first relation of (2.38) shows that we can assume $b = c$ by a suitable coordinate change. A surface with the condition $b = c$ was called an *isothermic-asymptotic surface* by Fubini.

Example 2.19 *The system*

$$z_{xx} = z_y + (kx + k_1)z, \quad z_{yy} = z_x + (kx + k_2),$$

where k, k_1, and k_2 are constants, is integrable. Any surface defined by this system is called a coincidence surface. *When these constants are all zero, the surface is equivalent to $XYZ = 1$ in affine space. By Proposition 2.16, any coincidence surface is projectively applicable to the surface $XYZ = 1$.*

Example 2.20 *The system called the Appell's system* (F_4) *is the following:*

$$
\begin{aligned}
z_{xx} &= \frac{2y}{1-x-y} z_{xy} + \frac{(a+b+1)x - c_1(1-y)}{x(1-x-y)} z_x \\
&\qquad + \frac{(a+b+1-c_2)y}{x(1-x-y)} z_y + \frac{ab}{x(1-x-y)} z, \\
z_{yy} &= \frac{2x}{1-x-y} z_{xy} + \frac{(a+b+1-c_1)x}{y(1-x-y)} z_x \\
&\qquad + \frac{(a+b+1)y - c_2(1-x)}{y(1-x-y)} z_y + \frac{ab}{y(1-x-y)} z,
\end{aligned}
$$

where a, b, c_1 *and* c_2 *are complex constants. An asymptotic coordinate system* (u,v) *is defined by the coordinate change* $(x,y) = (u(1-v), v(1-u))$. *Relative to the new unknown function* $w = (u(1-v))^{c_1}(v(1-u))^{1-c_2}(1-u-v)^{-e} z$ *where* $e = c_1 + c_2 - a - b - 1$, *the system is changed to*

$$
\begin{aligned}
z_{uu} &= -\frac{ev(1-v)}{u(1-u)(1-u-v)} z_v + pz, \\
z_{vv} &= -\frac{eu(1-u)}{v(1-v)(1-u-v)} z_u + qz,
\end{aligned}
$$

where

$$
\begin{aligned}
p &= p_0 + \frac{a_1}{u(1-u)} + \frac{a_2}{u^2} + \frac{a_3}{(1-u)^2}, \\
q &= q_0 + \frac{a_1}{v(1-v)} + \frac{a_3}{v^2} + \frac{a_2}{(1-v)^2},
\end{aligned}
$$

$$
\begin{aligned}
p_0 &= \frac{e}{2(1-u-v)^2} + \frac{e^2}{4}\left(\frac{2}{u(1-v)} + \frac{2}{v(1-u)} \right. \\
&\qquad \left. - \frac{2(1-2v)}{v(1-v)(1-u-v)} + \frac{3}{(1-u-v)^2}\right), \\
q_0 &= \frac{e}{2(1-u-v)^2} + \frac{e^2}{4}\left(\frac{2}{u(1-v)} + \frac{2}{v(1-u)} \right. \\
&\qquad \left. - \frac{2(1-2u)}{u(1-u)(1-u-v)} + \frac{3}{(1-u-v)^2}\right),
\end{aligned}
$$

and

$$
a_1 = (2ab - c_1c_2 + (a+b+1)e)/2, \ a_2 = c_1(-2+c_1)/4, \ a_3 = c_2(-2+c_2)/4.
$$

This shows that the system above defines projectively applicable surfaces with three parameters for each constant $e = c_1 + c_2 - a - b - 1$. *The Gaussian curvature referred to in Proposition 2.17 is equal to* $-2/e^2$.

Now we go back to the general situation. It is easy to see that λ and μ defined in (2.35) change to λ/f'^2 and μ/g'^2 by the coordinate change from (x, y) to $(\bar{x} = f(x), \bar{y} = g(y))$. Therefore, when $\lambda\mu \neq 0$, we can reduce the system to one where $\lambda = \mu = 1$ and, when $\lambda \neq 0$ and $\mu = 0$, to one where $\lambda = 1$ and $\mu = 0$. The corresponding conditions on b and c are either $c_x = b_y$ or $c_x = 0$.

Definition 2.21 The surface with $c_x = b_y$ is called an *R-surface* and the surface with $c_x = 0$ is called an *R_0-surface*.

For an R-surface, $\lambda = \mu$ can be any constant. Namely, the integrability condition holds for $p + k$ and $q + k$ for any constant k with the same cubic invariants b and c. For an R_0-surface, $\mu = 0$ and λ can be any constant. Hence, the integrability condition holds for $p + k$ and q with the same cubic invariants for any constant k.

2.7 Projectively minimal surfaces

In Sect. 2.2 we saw that the form $F\varphi_2$ is well-defined independent of the frames. When $F \neq 0$, $F\varphi_2$ defines an area functional and any critical surface relative to this functional is called a *projectively minimal surface*.

It is known that, relative to the system (2.12), the condition for projective minimality is written as

$$bM_y + 2Mb_y + b_{yyy} = 0, \quad cL_x + 2Lc_x + c_{xxx} = 0. \tag{2.39}$$

By the integrability (2.9), each of the above equations is equivalent to the other. In terms of the invariants P and Q, the equation is rewritten as

$$bQ_y + 2b_yQ = 0, \quad cP_x + 2c_xP = 0 \tag{2.40}$$

and, equivalently by (2.24),

$$(\Delta_1)_y = 0, \quad (\Delta_2)_x = 0. \tag{2.41}$$

We assume that the surface is of indefinite type and non-ruled, and let e be a Demoulin frame. Then the condition (2.41) is seen to be equivalent to

$$\det\, p = \det\, q, \tag{2.42}$$

where the matrices p and q are as defined in (2.20). For a proof we need explicit formulas of p and q in terms of the coefficients of the system, which will be given in (11.13). For the notations used in this formula, we refer to Sect. 11.3. Further, by the identities (2.21) and (2.23), the condition in (2.42) is equivalent to each one of

$$(1)\ \ell_{22}(p_{11} + q_{11}) = 0 \quad \text{and} \quad (2)\ \ell_{11}(p_{22} + q_{22}) = 0. \tag{2.43}$$

More generally, we can state the following proposition.

Proposition 2.22 *A non-degenerate surface is projectively minimal if and only if*

$$\sum_{i,j} \ell^{ij}(p_{ij} + q_{ij}) = 0, \quad \text{where } \ell^{ij} := \sum_{k,m} h^{ik} h^{jm} \ell_{km},$$

for a Demoulin frame.

The computation of the condition (2.39) was given in [Th1926]. The differential equation of projective minimality in general dimension was given in [Sa1987] in terms of affine invariants and by use of the formulation in [Sa1987] the above proposition was proved in [Sa1999]. Equivalence of (2.41) and (2.42) follow more directly from (11.8), (11.11), (11.12), and (11.13) by using an explicit form of a Demoulin frame. The next theorem states a fundamental property of projectively minimal surfaces relative to Demoulin transform:

Theorem 2.23 *Let S be a nondegenerate surface of indefinite type. Assume the conditions $bc \neq 0$, $\ell_{11}\ell_{22} \neq 0$ and $\det p \neq 0$ for a Demoulin frame. Then:* (1) *If S is projectively minimal, then the conformal structure of a Demoulin transform of S is the same as the conformal structure of S.* (2) *Conversely, if the conformal structure of a Demoulin transform is the same as the conformal structure of the original surface, then the original surface is projectively minimal or a surface with the property $p_{12}p_{21} = 0$.*

Proof. Since $\ell_{11}\ell_{22} \neq 0$ by assumption, the conditions in (2.43) show that $\overline{\varphi}_2$ is conformal to $\varphi_2 = \omega^1\omega^2$ in (2.22). The converse is also immediate.

Remark 2.24 A correspondence that preserves conformal structures in the theorem defines a W-congruence in the terminology explained in Sect. 3.2. The surface with $p_{12}p_{21} = 0$ is called a Q-surface in [Bol, vol. 2, p. 318].

Example 2.25 *By Definition 2.8 and* (2.40) *or by* (2.43)*, a Godeaux-Rozet surface is projectively minimal. In particular, a Demoulin surface is also projectively minimal.*

Remark 2.26 The books [FC1, FC2] and [L] as well as [Bol, AG] are good references to the theory of projective surfaces. We refer to [Sa1988, Sa1999] for further details regarding Sects. 2.1 to 2.5, and to [L] for Sect.2.2. A generalization of the subject in Sect. 2.1 is given in [Se1988, MSY1993, SYY1997]. The proof of Proposition 2.22 is given in [Sa1999]. The projective minimality in terms of affine invariants is given in [Sa1987]. We refer also to [Th1926, May1932]. The notations L and M were introduced in [FC2]. See [Sa2001] for further details on Example 2.20. For projective applicability and its further development, we refer to [FC2, Chap. 6] and [Ca1920]. J. Kanitani [Ka1922] classified projectively applicable surfaces with three parameters by determining the corresponding systems. Example 2.20 gives an explicit representation of such systems for the nonflat case. We refer also to Fubini [FC1, §69]. Demoulin surfaces were introduced by Demoulin [Dem1924]. For a general theory of R-surfaces and

R_0-surfaces, we refer to [FC2] and [Fe2000a]; the latter serves as a good introduction to the theory of projective surfaces in view of integrable systems. We refer also to [Fe1999].

3 Line congruences (1)

Line congruences and Laplace transformations are central notions treated in this article. In this section, we introduce elementary terminology on line congruences and explain what a W-conguence is, and in the next section we give fundamental properties of Laplace transformations.

3.1 Line congruences

We call a 2-parameter family of lines in 3-dimensional projective space $\mathbf{P}^3$ a *line congruence*. Given a surface in 3-dimensional Euclidean space, the family of normal lines to the surface is a typical example of a line congruence and is called the *normal congruence*. Its study was begun by Kummer [Ku1860]. We give a sketch of an elementary treatment of normal congruences in Appendix B.

Another method of constructing a line congruence is given as follows: Given a family of curves on a surface such that through any point passes a curve belonging to this family, the set of tangent lines to the curves form a family of lines parametrized by points of the surface. This congruence is called the *tangent congruence* associated to the family of curves.

A practical way of presenting a line congruence is to give a pair of surfaces $\{z(x,y), w(x,y)\}$ with surface parameters x and y. Then to these parameters we associate the line $\overline{zw}$ joining the two points $z(x,y)$ and $w(x,y)$, and thus obtain a line congruence. We denote it simply by $\{z,w\}$.

Let $t \mapsto (x(t), y(t))$ be a curve in the parameter space (x,y). Then we get a ruled surface $\{z(x(t),y(t)), w(x(t),y(t))\}$. The condition for this surface to be developable is

$$z \wedge w \wedge \frac{d}{dt}z \wedge \frac{d}{dt}w = 0,$$

which is equivalent to

$$P\left(\frac{dx}{dt}\right)^2 + 2Q\frac{dx}{dt}\frac{dy}{dt} + R\left(\frac{dy}{dt}\right)^2 = 0,$$

where

$$\begin{aligned} P &= z \wedge w \wedge z_x \wedge w_x, \\ 2Q &= z \wedge w \wedge z_x \wedge w_y + z \wedge w \wedge z_y \wedge w_x, \\ R &= z \wedge w \wedge z_y \wedge w_y. \end{aligned}$$

Hence, generally, there exist two directions on the parameter space so that, along the integral curves of these direction fields, the ruled surface is developable and its directrix curves for each family of directions comprise a surface; this surface is called the *focal surface* of the line congruence. We remark that the integral curves for normal congruence of a surface in Euclidean space are nothing but the curvature lines. We refer to Appendix B.

Let $\partial/\partial x$ be one of the degenerate directions, i.e., let us assume $P = 0$, and let z be the corresponding focal surface. We then have

$$z_x \equiv 0 \pmod{z, w}. \tag{3.1}$$

The tangent plane of z is generated by $\{z, w, z_y\}$. If $w_x \equiv 0 \pmod{z, w}$ throughout, then $(z \wedge w)_x = \lambda z \wedge w$; namely, the line $z \wedge w$ is stationary in the direction $\partial/\partial x$. To exclude this degenerate case, we assume now that $w_x \not\equiv 0 \pmod{z, w}$.

We next let $\partial/\partial y$ be the other degenerate direction, i.e., $R = 0$, and let w be the focal surface. We see that

$$w_y \equiv 0 \pmod{z, w} \tag{3.2}$$

and the tangent plane is generated by $\{z, w, w_x\}$. Here we need the condition $Q \neq 0$:

$$z \wedge w \wedge z_y \wedge w_x \neq 0, \tag{3.3}$$

which we assume in the following.

3.2 W-congruences

We write (3.1) and (3.2) more explicitly as

$$\begin{aligned} z_x &= mw + rz, \\ w_y &= sw + nz, \end{aligned}$$

where we assume $m \neq 0$ and $n \neq 0$ so that z and w really do depend on x and y, respectively. Further, we set

$$\begin{array}{rcll} z_{yy} & \equiv & pw_x & \pmod{z, w, z_y}, \\ w_{xx} & \equiv & qz_y & \pmod{z, w, w_x}. \end{array}$$

Then we have

$$\begin{array}{rcll} z_{xx} & \equiv & mw_x & \pmod{z, w}, \\ z_{xy} & \equiv & 0 & \pmod{z, w, z_y}, \\ w_{xy} & \equiv & 0 & \pmod{z, w, w_x}, \\ w_{yy} & \equiv & nz_y & \pmod{z, w}. \end{array}$$

We now compute the induced conformal structures on the surfaces. Set

$$\begin{aligned} E &= z \wedge z_x \wedge z_y \wedge z_{xx} = m^2 z \wedge w \wedge z_y \wedge w_x, \\ F &= z \wedge z_x \wedge z_y \wedge z_{xy} = 0, \\ G &= z \wedge z_x \wedge z_y \wedge z_{yy} = mpz \wedge w \wedge z_y \wedge w_x, \end{aligned}$$

then $Edx^2 + 2Fdxdy + Gdy^2$ defines the conformal structure on the surface z, which turns out to be

$$\varphi_z = mdx^2 + pdy^2. \tag{3.4}$$

Similarly, we get

$$\varphi_w = qdx^2 + ndy^2, \tag{3.5}$$

which defines the conformal structure on the surface w. For the surfaces to be nondegenerate, the condition

$$mnpq \neq 0 \tag{3.6}$$

is necessary and sufficient.

Definition 3.1 We say that the line congruence $\{z, w\}$ is a *W-congruence* (W is after Weingarten) if the conformal class of φ_z is equal to the conformal class of φ_w. Namely, if it holds that

$$W := mn - pq = 0. \tag{3.7}$$

Remark 3.2 If we say that two vectors X and Y are conjugate relative to a nondegenerate 2-form φ when $\varphi(X, Y) = 0$, then a W-congruence is a line congruence that preserves conjugate directions relative to φ_z and φ_w. We refer to Sect. 4.1 for an intrinsic definition of a conjugate system.

Given a line congruence $\{z, w\}$, we define a surface ξ lying in $\mathbf{P}^5$ by

$$\xi = z \wedge w,$$

which lies in the quadratic hypersurface of $\mathbf{P}^5$ defined by the Plücker relation.

Theorem 3.3 (G. Darboux) *A congruence is a W-congruence if and only if the second osculating space of the surface ξ is of codimension greater than* 1.

Proof. The second osculating space of ξ is a space spanned by the derivatives of ξ up to second order. It is enough to show that there exists a linear relation among the six vectors ξ, ξ_x, ξ_y, ξ_{xx}, ξ_{xy}, and ξ_{yy} if and only if $W = 0$. By a computation, we see that

$$\begin{aligned}
\xi_x &= z \wedge w_x \qquad (\text{mod } \xi), \\
\xi_y &= z_y \wedge w \qquad (\text{mod } \xi), \\
\xi_{xx} &= mw \wedge w_x + qz \wedge z_y \qquad (\text{mod } \xi, \xi_x), \\
\xi_{xy} &= z_y \wedge w_x \qquad (\text{mod } \xi, \xi_x, \xi_y), \\
\xi_{yy} &= pw_x \wedge w + nz_y \wedge z \qquad (\text{mod } \xi, \xi_y).
\end{aligned}$$

Hence, for a linear relation such as

$$A\xi_{xx} + 2B\xi_{xy} + C\xi_{yy} \equiv 0 \quad (\text{mod } \xi, \xi_x, \xi_y)$$

to exist, it is enough to solve

$$\begin{aligned}(Aq - Cn)z \wedge z_y + 2Bz_y \wedge w_x + (Am - Cp)w \wedge w_x \\ \equiv 0 \pmod{z \wedge w, z \wedge w_x, z_y \wedge w}.\end{aligned}$$

Since $z \wedge w \wedge z_y \wedge w_x \neq 0$ by the assumption (3.3), this identity is equivalent to

$$Aq - Cn = 0, \qquad B = 0, \qquad Am - Cp = 0.$$

Therefore, $W = 0$ is necessary for nontrivial A and C to exist, and vice versa.

4 The Laplace transformation

This section treats the Laplace transformation of the differential equation

$$z_{xy} + az_x + bz_y + cz = 0,$$

which sends any solution of this equation to a solution of another differential equation of the same form. We first interpret it in terms of tangent congruence when z is regarded as an immersion of a surface, and then give fundamental notions relative to the Laplace transformation following Darboux [D].

4.1 Laplace invariants

Let S be a surface in $\mathbf{P}^3$ defined by an immersion $z(x, y)$ with parameter $\{x, y\}$ and the tangent planes $T_{(x,y)}$.

Definition 4.1 The coordinate system $\{x, y\}$ is called a *conjugate system* if the limit of the line $T_{(x,y)} \cap T_{(x+dx,y)}$, as $dx \to 0$, tends to a line that is parallel to the vector z_y, or equivalently, as seen below, if the limit of the line $T_{(x,y)} \cap T_{(x,y+dy)}$, as $dy \to 0$, tends to a line that is parallel to the vector z_x.

We can interpret this definition in terms of z as follows. Let Z denote the coordinate vector of $\mathbf{P}^3$. Then, for each (x, y), there exists a vector $A(x, y)$ in the dual space such that the tangent plane is written as

$$T_{(x,y)} \ : \quad \langle A(x, y), Z \rangle = 0,$$

where $\langle \cdot, \cdot \rangle$ denotes the dual pairing, and z satisfies

$$\langle A, z \rangle = \langle A, z_x \rangle = \langle A, z_y \rangle = 0.$$

Since

$$T_{(x+dx,y)} \ : \quad \langle A(x, y) + A_x(x, y)dx, Z \rangle = 0,$$

the limit of the intersection of these planes is defined by the equation

$$\langle A, Z \rangle = \langle A_x, Z \rangle = 0.$$

Hence the condition of the coordinates being conjugate is

$$\langle A, z_y\rangle = \langle A_x, z_y\rangle = 0.$$

Therefore, we also have

$$0 = \langle A_x, z_y\rangle = \langle A, z_y\rangle_x - \langle A, z_{xy}\rangle = -\langle A, z_{xy}\rangle.$$

Hence, we see that the four vectors z, z_x, z_y, and z_{xy} are linearly dependent. Also, because the surface is regular, the first three are linearly independent. This implies an equation of the form

$$(E) \qquad z_{xy} + az_x + bz_y + cz = 0$$

holds. Conversely, this differential equation characterizes conjugate systems.

However, the derivation of the equation (E) is not unique. This is because the system of homogeneous coordinates is determined only up to a scalar multiple and, in the definition above, only the directions of the coordinates are considered. Namely, we have the following freedom:

1) we can choose $w = \lambda^{-1}z$ in place of z, where λ is a scalar function,
2) we may choose coordinates (s, t), for which $s = s(x)$ and $t = t(y)$.

Though exchange of x and y is also possible, for the sake of simplicity we do not consider it. Relative to the freedom above the equation (E) is changed as folows:

$$\begin{aligned} 1) \qquad & w_{xy} + (a + (\log\lambda)_y)w_x + (b + (\log\lambda)_x)w_y \\ & \quad + (c + a(\log\lambda)_x + b(\log\lambda)_y + \lambda_{xy}/\lambda)w = 0, \\ 2) \qquad & z_{st} + a\frac{dy}{dt}z_s + b\frac{dx}{ds}z_t + c\frac{dx}{ds}\frac{dy}{dt}z = 0. \end{aligned}$$

Now let us set

$$h = ab + a_x - c, \qquad k = ab + b_y - c. \tag{4.1}$$

Then, if we denote the h and k for the new equations by h' and k', we have the identities

$$h = h', \quad k = k'$$

for 1) and

$$hdxdy = h'dsdt, \quad kdxdy = k'dsdt$$

for 2). Hence, we have

Proposition 4.2 *The 2-forms $hdxdy$ and $kdxdy$ are invariantly defined for the conjugate system.*

Definition 4.3 We call these two 2-forms (h and k for short) the *Laplace invariants.*

Remark 4.4 (1) According to 1), the quantity $ab - c$ is changed into $ab - c - (\log\lambda)_{xy}$. Hence, we can always reduce the equation to the form $ab = c$ by choosing λ appropriately. In this case, the invariants are $h = a_x$ and $k = b_y$.
(2) If $h = k$, then $a_x = b_y$. Hence

$$a + (\log\lambda)_y = 0, \qquad b + (\log\lambda)_x = 0$$

is solvable. This implies that the equation (E) can be reduced to

$$z_{xy} + cz = 0.$$

(3) If we choose λ so that $a + (\log\lambda)_y = 0$, then the new c, which is equal to $c + a(\log\lambda)_x + b(\log\lambda)_y + \lambda_{xy}/\lambda$, turns out to be h itself. Hence, if $h = 0$, then the equation is reduced to a simpler equation of the form

$$z_{xy} + bz_y = 0. \tag{4.2}$$

Furthermore, if $h = 0$, then the equation (E) is solvable as follows. Since the equation (E) is written as $(z_y + az)_x = -b(z_y + az)$ and the equation $z^1_x = -bz^1$ is solved by $z^1 = Y(y)e^{\int bdx}$ where $Y(y)$ is a space curve, z has the form

$$z = e^{-\int ady}\left\{X(x) + \int^y Y(t)e^{\int ady - \int bdx}dt\right\},$$

where X is an arbitrary space curve.

4.2 The Laplace transformation

We define the Laplace transformation in terms of tangent congruence. Let z be a surface with a conjugate system of coordinates $\{x, y\}$. We choose a point w on each tangent line in the direction y; it is written as

$$w = z_y + \lambda z$$

with parameter λ. We assume that the point w draws a surface and that the line $\overline{zw}$ is again tangent to the surface w; this implies that, relative to the congruence $\{z, w\}$, the two surfaces z and w are focal surfaces. Since

$$w_x = (\lambda - a)z_x - bz_y + (\lambda_x - c)z,$$

this assumption is satisfied if

$$\lambda = a.$$

Thus we can introduce the following definition.

Definition 4.5 The surface $z^1 = z_y + az$ is called the *first Laplace transform* of z and, similarly, the surface $z^{-1} = z_x + bz$ is called the *minus-first Laplace transform.*

We give some remarks on the definition. From (E) we see that

$$z_x^1 = -bz^1 + hz,$$

where h is one of the Laplace invariants, and

$$z_{xy}^1 + a_1 z_x^1 + b_1 z_y^1 + c_1 z^1 = 0,$$

where

$$\begin{aligned} a_1 &= a - (\log h)_y, \\ b_1 &= b, \\ c_1 &= c - a_x + b_y - b(\log h)_y. \end{aligned} \tag{4.3}$$

Hence,

$$(z^1)^{-1} = hz$$

and, if $h \neq 0$, the surface $(z^1)^{-1}$ is the same as the surface z. Similarly, if $k \neq 0$, the surface $(z^{-1})^1$ is the same as z.

Let us next examine the case when z^1 is degenerate. This is the case where

$$Az_x^1 + Bz_y^1 + Cz^1 = 0$$

for certain scalar functions A, B, and C. If B is identically zero, then we have $A(-bz^1 + hz) + Cz^1 = 0$, which holds only if $h = 0$. If B is not vanishing, then $(z^1)_y \equiv 0 \pmod{z^1, (z^1)_x}$; hence, $(z^1)_y \equiv 0 \pmod{z, z_y}$. Since $z_y^1 = z_{yy} + (az)_y$, we have $z_{yy} \equiv 0 \pmod{z, z_y}$. This means that z is ruled. If we further have $h \neq 0$, then z^1 is a curve and the ruling lines are tangent to this curve, hence the surface z is developable.

Proposition 4.6 *The transform z^1 is degenerate if and only if $h = 0$ or if the surface z is developable.*

Remark 4.7 The osculating plane of z^1 along the x-curve is $z^1 \wedge z_x^1 \wedge z_{xx}^1$. Since

$$\begin{aligned} z_x^1 &= -bz_y + (a_x - c)z, \\ z_{xx}^1 &= hz_x + (b^2 - b_x)z_y + (bc + a_{xx} - c_x)z, \end{aligned}$$

we have

$$z^1 \wedge z_x^1 \wedge z_{xx}^1 = h^2 z \wedge z_x \wedge z_y,$$

i.e., the osculating plane coincides with the tangent plane of z, unless $h = 0$.

Example 4.8 *Consider the equation*

$$(x-y)\frac{\partial^2 z}{\partial x \partial y} + n\frac{\partial z}{\partial x} - m\frac{\partial z}{\partial y} = 0,$$

where m and n are constants. A surface satisfying this equation is, for example, given by

$$z = [(x-a)^m(y-a)^n, (x-b)^m(y-b)^n, (x-c)^m(y-c)^n, (x-d)^m(y-d)^n]$$

for arbitrary constants a, b, c and d. Then we see that

$$z^1 = z_y + \frac{n}{x-y}z = \frac{n}{x-y} \qquad \big[(x-a)^{m+1}(y-a)^{n-1}, (x-b)^{m+1}(y-b)^{n-1},$$
$$(x-c)^{m+1}(y-c)^{n-1}, (x-d)^{m+1}(y-d)^{n-1}\big].$$

Remark 4.9 S. S. Chern [Ch1944, Ch1947] first considered the Laplace transformation when the dimension n is greater than 2. A systematic new treatment was given in [KT1996].

4.3 Recursive relations of Laplace invariants

It is interesting to know how the Laplace invariants vary with Laplace transformations. We denote by z^i the successive transforms, namely $z^{i\pm1} = (z^i)^{\pm1}$, and we denote the Laplace invariants of z^i by h_i and k_i. The sequence $\{z^i\}$, when well-defined, is called the *Laplace sequence.*

Proposition 4.10 *The Laplace invariants satisfy the following relations.*

$$\begin{array}{ll}
(1) & h_{i+1} = 2h_i - k_i - \dfrac{\partial^2 \log h_i}{\partial x \partial y}, \\
 & k_{i+1} = h_i, \\
 & \quad k_i = 2k_{i+1} - h_{i+1} - \dfrac{\partial^2 \log k_{i+1}}{\partial x \partial y}; \\
(2) & h_{i+1} + h_{i-1} = 2h_i - \dfrac{\partial^2 \log h_i}{\partial x \partial y}, \\
 & k_{i+1} + k_{i-1} = 2k_i - \dfrac{\partial^2 \log k_i}{\partial x \partial y}; \\
(3) & h_{i+1} = h_i + h - k - (\log(hh_1 \cdots h_i))_{xy}, \\
 & k_{i+1} = k_i + h - k - (\log(k_1 \cdots k_i))_{xy}.
\end{array}$$

Proof. It is enough to check the identities when $i = 0$ for (1). Set $H = \log h$. Then we have, by (4.1) and (4.3),

$$\begin{array}{rcl}
h_1 & = & (a - H_y)_x - (c - a_x + b_y - bH_y) + b(a - H_y) \\
 & = & 2(ab + a_x - c) - (ab + b_y - c) - H_{xy}, \\
 & = & 2h - k - H_{xy}, \\
k_1 & = & b_y - (c - a_x + b_y - bH_y) + b(a - H_y) \\
 & = & ab + a_x - c, \\
 & = & h,
\end{array}$$

which yield (1). The identities (2) and (3) follow from (1).

We write the equation of z^i as

$$(E_i) \qquad (z^i)_{xy} + a_i(z^i)_x + b_i(z^i)_y + c_i z = 0.$$

The coefficients a_i, b_i, and c_i are determined recursively by the formula (4.3). Namely, we see that

$$\begin{aligned} a_i &= a_{i-1} - (\log h_{i-1})_y, \\ b_i &= b, \\ c_i &= c_{i-1} - (a_{i-1})_x + b_y - b(\log h_{i-1})_y, \end{aligned}$$

in the case $i > 0$. A similar formula holds for the case $i < 0$.

4.4 Periodic Laplace sequences

We say that a Laplace sequence $\{z^i\}$ is *periodic* of period n if the surface z^n coincides with the starting surface z. In this case, we have $h_n = h$ and $k_n = k$. We first give two examples.

Example 4.11 *Assume $(E) = (E_1)$, i.e., $h_1 = h$ and $k_1 = k$. Then, we get $k = h$ and $(\log h)_{xy} = 0$. Hence $h = X(x)Y(y)$. By a change of coordinates, we can assume $h = 1$, as long as $h \neq 0$. This means that the equation is nothing but*

$$z_{xy} = z.$$

Example 4.12 *Assume $(E) = (E_2)$, i.e., the transformation is doubly periodic, and $hk \neq 0$. We have $h_2 = h$ and $k_2 = k$. From Proposition 4.10 (1) above,*

$$2k - 2h = \frac{\partial^2 \log h}{\partial x \partial y}, \qquad 2h - 2k = \frac{\partial^2 \log k}{\partial x \partial y}.$$

This implies $(\log hk)_{xy} = 0$; hence, $hk = X(x)Y(y)$ and we can find coordinates such that $hk = 1$. In this case, by setting $h = e^\theta$, we have

$$\frac{\partial^2 \theta}{\partial x \partial y} = 4 \sinh \theta.$$

Conversely, for any solution θ of this equation, let us define the equation (E), where the coefficients a, b, and c are defined by solving the equations

$$a_x = h = e^\theta, \qquad b_y = k = e^{-\theta}, \qquad c = ab;$$

then we get a surface and its Laplace sequence that is doubly periodic.

We next treat the general periodic case. By assumption,

$$z^n = \mu z$$

for a nonvanishing function μ. Assume $h_i \neq 0$ in the following. By Proposition 4.10 (3), it holds that

$$(\log h h_1 \cdots h_{n-1})_{xy} = 0.$$

Hence, $hh_1 \cdots h_{n-1} = X(x)Y(y)$ for certain functions X and Y. By a coordinate change, we can assume

$$hh_1 \cdots h_{n-1} = 1;$$

see Proposition 4.2. From (4.3),

$$\begin{gathered} b_n = b, \\ a_n = a_{n-1} - (\log h_{n-1})_{xy} = \cdots = a - (\log hh_1 \cdots h_{n-1})_{xy} = a. \end{gathered}$$

From $h_n = h$, we see that $(a_n)_x + a_n b_n - c_n = a_x + ab - c$; hence, we also get $c_n = c$. Therefore, z^n and z satisfy the same equation and μ is constant. We can see moreover that

$$\begin{aligned} z^{n+i} &= \mu z^i \quad \text{for} \quad i > 0, \\ z^{n-i} &= \frac{\mu}{h_{n-1}h_{n-2}\cdots h_{n-i}} z^{-i} \quad \text{for} \quad 0 < i < n. \end{aligned}$$

We restrict our further consideration to only the case $h = k$.

Proposition 4.13 *Assume that the sequence is n-periodic and $h = k$. Choose coordinates so that $hh_1 \cdots h_{n-1} = 1$. Then*

$$\begin{array}{lll} (1) \quad h_{2p-i} = h_{i-1} & (i = 1, 2, \ldots, p) & \textit{when } n = 2p, \\ (2) \quad h_{2p+1-i} = h_{i-1} & (i = 1, 2, \ldots, p) & \textit{when } n = 2p+1. \end{array}$$

Proof. For the case where $h = k$, Proposition 4.10 (3) implies

$$h_i = h_{i-1} - (\log hh_1 \cdots h_{i-1})_{xy}.$$

When $i = 2p - 1$, we have

$$h_{2p-1} = h_{2p} + (\log hh_1 \cdots h_{2p-1})_{xy} = h_{2p} = h.$$

Assume that (1) holds for $i = 1, 2, \ldots, j$. Then

$$h_{j-1} = h_{j-2} - (\log hh_1 \cdots h_{j-2})_{xy} = h_{j-2} - (\log h_{2p-1}h_{2p-2} \cdots h_{2p-j+1})_{xy}.$$

Combining this with the identity above for $i = 2p - j$, we get the result for (1). The case (2) can be similarly shown.

Example 4.14 *Consider the case where $p = 1$ and $n = 3$. Since $hh_1h_2 = 1$ and $h_2 = h$, we have $h_1 = 1/h^2$. Hence, h satisfies the equation*

$$(\log h)_{xy} = h - 1/h^2. \tag{4.4}$$

The associated differential equation is

$$z_{xy} = hz$$

and the Laplace sequence satisfies

$$z^3 = mz, \quad z^2 = (m/h)z^{-1}, \quad z^1 = mhz^{-2}.$$

On the other hand, by definition,

$$z^1 = z_y, \quad z^2 = z_{yy} - (\log h)_y z_y, \quad z^{-1} = z_x, \quad z^{-2} = z_{xx} - (\log h)_x z_x.$$

Therefore, z satisfies a system of differential equations

$$z_{xx} = \frac{h_x}{h} z_x + \frac{1}{mh} z_y, \quad z_{xy} = hz, \quad z_{yy} = \frac{m}{h} z_x + \frac{h_y}{h} z_y. \tag{4.5}$$

Since the number of independent solutions of this system is three, the surface z lies in a plane in the projective space.

Example 4.15 *We next consider the case where $p = 2$ and $n = 4$. Here $hh_1h_2h_3 = 1$, $h_2 = h_1$, and $h_3 = h$. We may assume $h_1 = 1/h$. Then h must satisfy*

$$(\log h)_{xy} = h - \frac{1}{h}, \tag{4.6}$$

and z is a solution of the system of differential equations

$$z_{xy} = hz, \quad z^4 = mz, \quad z^3 = \frac{m}{h} z^{-1}, \quad z^2 = mz^{-2}, \quad z^1 = mhz^{-3}.$$

It is straightforward to see that this system is the same as

$$z_{xy} = hz, \quad mz_{xx} - z_{yy} = m\frac{h_x}{h} z_x - \frac{h_y}{h} z_y. \tag{4.7}$$

Conversely, for any h satisfying (4.6), *four independent solutions of* (4.7) *define a surface with the required periodicity and with equal Laplace invariants.*

The simplest example of h is $h = 1$. The system is $z_{xy} = z$ and $z_{xx} - z_{yy} = 0$; the associated mapping is projectively equivalent to $[e^{x+y}, e^{-x-y}, \cos(x - y), \sin(x - y)]$ in $\mathbf{P}^3$ with the homogeneous coordinates $[X, Y, Z, U]$. The surface is nothing but the quadric $XY = Z^2 + U^2$. Differentiating relative to y successively, we get $z^1 = z_y$, $z^2 = z_{yy}$, $z^3 = z_{yyy}$, and $z^4 = z$. z^1 is the quadric $-XY = Z^2 + U^2$.

Remark 4.16 A Laplace sequence $\{z^i\}$ is said to be *self-projective* if there exists a projective transformation T and an integer such that $Tz^i = z^{m+i}$ for any i. This is a more general notion than periodicity.

4.5 Terminating Laplace sequences

Historically, a special interest was paid to Laplace sequences that terminate in a finite number of steps, say, $h_i = 0$ but $h_{i-1} \neq 0$. We exhibit some of the treatment of this case. As in Remark 4.4 (3), the equation (E_i) is written as

$$\frac{\partial}{\partial x}\left(\frac{\partial z^i}{\partial y} + a_i z^i\right) + b_i \left(\frac{\partial z^i}{\partial y} + a_i z^i\right) = 0.$$

The general solution has the form

$$z^i = e^{-\int a_i dy} \left\{ X(x) + \int^y Y(t) e^{\int a_i dt - \int b_i dx} dt \right\},$$

where X and Y are arbitrary functions, so long as the integral is valid. Then the solution of the original equation is given by tracing through the Laplace transformations in reverse order.

We assume $h_i = 0$, where $i > 0$, for simplicity. Then by (4.2) the equation (E_i) is equivalent to

$$z_{xy} + b z_y = 0.$$

We define a function α by $b = -(\log \alpha)_x$ so that

$$(E_i) \qquad z_{xy} - (\log \alpha)_x z_y = 0.$$

Then $h_{i-1} = k_i = -(\log \alpha)_{xy}$ by Proposition 4.10 (1). Now

$$h_{i-2} = 2h_{i-1} - \frac{\partial^2 \log h_{i-1}}{\partial x \partial y} = -\frac{\partial^2}{\partial x \partial y} \log (\alpha \alpha_{xy} - \alpha_x \alpha_y).$$

Following Darboux, we introduce the notation

$$D_x(\alpha_1, \ldots, \alpha_{m+1}) = \det \begin{pmatrix} \alpha_1 & \partial \alpha_1/\partial x & \cdots & \partial^m \alpha_1/\partial x^m \\ \vdots & \vdots & \ldots & \vdots \\ \alpha_{m+1} & \partial \alpha_{m+1}/\partial x & \cdots & \partial^m \alpha_{m+1}/\partial x^m \end{pmatrix}.$$

Analogously, we introduce D_y. Then we set

$$H_m = D_x \left(\alpha, \frac{\partial \alpha}{\partial y}, \ldots, \frac{\partial^m \alpha}{\partial y^m} \right) = D_y \left(\alpha, \frac{\partial \alpha}{\partial x}, \ldots, \frac{\partial^m \alpha}{\partial x^m} \right).$$

Lemma 4.17 *For $m \geq 0$, we have*

$$h_{i-m} = -\frac{\partial^2 \log H_{m-1}}{\partial x \partial y}.$$

Proof. By a formula for the expansion of determinants, we have

$$H_{m-1} H_{m+1} = H_m \frac{\partial^2 H_m}{\partial x \partial y} - \frac{\partial H_m}{\partial x} \frac{\partial H_m}{\partial y}.$$

The righthand side is equal to $H_m^2 (\log H_m)_{xy}$. Hence, by derivation of both sides, we have

$$(\log H_{m-1})_{xy} + (\log H_{m+1})_{xy} = 2(\log H_m)_{xy} + \left(\log(\log H_m)_{xy}\right)_{xy}.$$

This identity is the same as the recurrence formula satisfied by h_i, up to a sign:

$$h_{m-1} + h_{m+1} = 2h_m - (\log h_m)_{xy}.$$

Taking into account the expressions of h_{i-1} and h_{i-2} given above, we have the conclusion.

This lemma shows that the invariants h and k of the equation (E_{i-m}) are $-(\log H_{m-1})_{xy}$ and $-(\log H_m)_{xy}$, respectively. The associated equation is

$$z_{xy} - (\log H_{m-1})_y z_x - (\log H_m)_x z_y + (\log H_{m-1})_y (\log H_m)_x z = 0.$$

Here we assume $H_{-1} = 0$. For any solution θ of (E_i), we define the function

$$\theta_m = D_x\left(\theta, \alpha, \frac{\partial\alpha}{\partial y}, \ldots, \frac{\partial^{m-1}\alpha}{\partial y^{m-1}}\right), \quad m \geq 1.$$

Then we can assert:

Proposition 4.18 *θ_m is a solution of (E_{i-m}).*

We refer to [D, no 379; II, pp. 138-139] for the proof. If the Laplace sequence terminates on both sides, then we must have $(\log H_{m-1})_{xy} = 0$ for certain m. This implies $H_m = D_x(\alpha, \partial\alpha/\partial y, \ldots, \partial^m\alpha/\partial y^m) = 0$.

Hence the α must satisfy an ordinary differential equation of order m with coefficients being functions of y alone. This property can be traced back to the starting equation (E_i) and we can get a general description of solutions of (E_i) in this case. We refer again to [D, no 382–386] for a detailed treatment.

4.6 The Euler-Poisson-Darboux equation

The Euler-Poisson-Darboux equation is by definition a differential equation of the form

$$z_{xy} - \frac{n}{x-y} z_x - \frac{m}{y-x} z_y - \frac{p}{(x-y)^2} z = 0,$$

where n, m, and p are constants. It is a special case of (E).

When we set $z = (x-y)^\alpha w$, w satisfies an equation of the same form with constants replaced by

$$n \to n + \alpha, \qquad m \to m + \alpha, \qquad p \to p + \alpha^2 + \alpha(m + n - 1).$$

Hence, for an appropriate α, we can reduce the equation to

$$(E(\beta, \beta')) \qquad z_{xy} - \frac{\beta'}{x-y} z_x - \frac{\beta}{y-x} z_y = 0.$$

First we note that by setting $z = (x-y)^{1-\beta-\beta'} w$ the equation for w remains the same, but with coefficients replaced by

$$\beta \to 1 - \beta \qquad \text{and} \qquad \beta' \to 1 - \beta'.$$

This is an involutive transformation from $E(\beta, \beta')$ to $E(1-\beta', 1-\beta)$.

We next remark that the equation has special solutions, by use of Gauss hypergeometric functions. Let us set $y/x = t$ and assume z has the form $z = x^\lambda \varphi(t)$. Then φ satisfies the Gauss hypergeometric equation

$$t(1-t)\varphi'' + \{(1-\beta-\lambda) - (1+\beta'-\lambda)t\}\varphi' + \lambda\beta'\varphi = 0.$$

In particular, this implies that $E(\beta, \beta')$ has independent solutions

$$x^\lambda F(-\lambda, \beta', 1-\beta-\lambda; y/x) \quad \text{and} \quad x^{-\beta} y^{\beta+\lambda} F(\beta, \beta+\beta'+\lambda, 1+\beta+\lambda; y/x),$$

where λ is an arbitrary constant and F denotes the Gauss hypergeometric funtion. Furthermore, when λ is a positive integer, the first solution is a homogeneous polynomial of degree λ.

The second remark we make is that the equation $E(\beta, \beta')$ has special solutions of the form $z = X(x)Y(y)$. Relative to this z, the equation is written as

$$(x-y)X_x Y_y - \beta' X_x Y + \beta X Y_y = 0.$$

Hence $x + \beta X/X_x = y + \beta' Y/Y_y$ must be a constant a; we get

$$X = (x-a)^{-\beta} \quad \text{and} \quad Y = (y-a)^{-\beta'}$$

up to multiplicative constants. Namely, $(x-a)^{-\beta}(y-a)^{-\beta'}$ is a solution for any constant a. By the involution stated above,

$$(y-x)^{1-\beta-\beta'}(x-a)^{\beta'-1}(y-a)^{\beta-1}$$

is also a solution.

The third remark we make is on the symmetry of the equation. We easily see that the Laplace invariants are

$$h = \frac{\beta'(1-\beta)}{(x-y)^2} \quad \text{and} \quad k = \frac{\beta(1-\beta')}{(x-y)^2}.$$

Let $(x, y) \to (s, t)$ be a coordinate transformation of the form

$$x = \frac{as+b}{cs+d} \quad \text{and} \quad y = \frac{at+b}{ct+d}.$$

Since it holds that

$$x - y = -\frac{ad-bc}{(cs+d)(ct+d)}(s-t), \quad dx = \frac{ad-bc}{(cs+d)^2}ds, \quad dy = \frac{ad-bc}{(ct+d)^2}dt,$$

and because $hdxdy = h'dsdt$ and $kdxdy = k'dsdt$, where h' and k' are the invariants for the new equation relative to (s, t), we see that

$$h' = \frac{\beta'(1-\beta)}{(s-t)^2} \quad \text{and} \quad k' = \frac{\beta(1-\beta')}{(s-t)^2}.$$

This implies that, if we denote by $S(\beta, \beta')$ the space of solutions, then the space $S(\beta, \beta')$ has an SL_2-action. More precisely, we have

Proposition 4.19 (Appell) *For any solution $\varphi(x,y)$ of the equation $E(\beta,\beta')$, the function*

$$(ax+b)^{-\beta}(ay+b)^{-\beta'}\varphi\left(\frac{cx+d}{ax+b},\frac{cy+d}{ay+b}\right)$$

is also a solution of $E(\beta,\beta')$, where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is any element of SL_2. The associated infinitesimal action is given by

$$X=\partial_x+\partial_y, \qquad Y=x\partial_x+y\partial_y+\frac{1}{2}(\beta+\beta'),$$
$$Z=x^2\partial_x+y^2\partial_y+\beta x+\beta' y.$$

The correspondence with the matrix elements is given by $X\leftrightarrow\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $Y\leftrightarrow\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and $Z\leftrightarrow\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. They satisfy the expected relations: $[X,Z]=2Y$, $[Y,X]=-X$, and $[Y,Z]=Z$.

So far, we have two kinds of symmetries of the family $\{E(\beta,\beta')\}$. We next give two other kinds of symmetries. The first one is given by the Laplace transformation. Starting with the equation $E(\beta,\beta')$, let E_i be the ith Laplace transform with invariants h_i and k_i. By use of Proposition 4.10, we easily see that

$$h_i=\frac{(i+\beta')(i-\beta+1)}{(x-y)^2} \quad \text{and} \quad k_i=\frac{(i+\beta'-1)(i-\beta)}{(x-y)^2};$$

this means that $E_i=E(\beta-i,\beta'+i)$.

The second symmetry is given by the operation that associates to each solution z of $E(\beta,\beta')$ the derivative $\partial z/\partial x$. Differentiating the equation $E(\beta,\beta')$, it is direct to see that $\partial z/\partial x$ belongs to $S(\beta+1,\beta')$. Similarly, $\partial z/\partial y$ belongs to $S(\beta,\beta'+1)$.

Proposition 4.20 *Assume $\beta\neq 0$. Then the operator $\partial/\partial x$ is surjective from $S(\beta,\beta')$ to $S(\beta+1,\beta')$.*

Proof. Given $z^1\in S(\beta+1,\beta')$, it is necessary to solve $\partial z/\partial x=z^1$. If z exists in $S(\beta,\beta')$, then $(x-y)\partial z^1/\partial y-\beta' z^1+\beta z_y=0$. Hence, we must have

$$dz=z^1dx+\left(\frac{\beta'}{\beta}z^1-\frac{x-y}{\beta}z^1_y\right)dy.$$

The integrability of this Pfaff equation is nothing but the condition $z^1\in S(\beta+1,\beta')$, as can be seen by taking exterior differentiation.

Remark 4.21 The general solution of the exceptional equation $E(0,\beta')$ is seen to be of the form $\int^x X(t)(y-t)^{-\beta'}dt+Y(y)$ in case $\beta'\neq 0$, while E(0,0) is the trivial equation $z_{xy}=0$. As for the equation $E(1,\beta')$, the first invariant vanishes. Considering the involution between $E(1,\beta')$ and $E(1-\beta',0)$, we get a general solution of the form

$$(x-y)^{-\beta'}\left\{X(x)+\int^y Y(t)(x-t)^{\beta'-1}dt\right\}.$$

When both β and β' are integers, the equation $E(1,1)$ is important. For any solution z of this case, we set $w=(x-y)z$. Then it is straightforward to see that $w_{xy}=0$. Hence, the general solution of $E(1,1)$ is of the form $(X(x)-Y(y))/(x-y)$.

Now we consider a more general case: the case where $0<\mathrm{Re}\beta<1$ and $0<\mathrm{Re}\beta'<1$. The formula called Poisson-Appell treats this case and asserts that the general solution has the form

$$\int^y \varphi(u)(u-x)^{-\beta}(y-u)^{-\beta'}ds+(x-y)^{1-\beta-\beta'}\int_x^y \psi(u)(u-x)^{\beta'-1}(y-u)^{\beta-1}ds,$$

in the case $\beta+\beta'\neq 1$ and

$$\int_0^1 \varphi(x+(y-x)t)t^{-\beta}(1-t)^{\beta-1}dt+\int_0^1 \psi(x+(y-x)t)t^{-\beta}(1-t)^{\beta-1}\log(t(1-t)(y-x))dt,$$

in the case $\beta+\beta'=1$, where φ and ψ are arbitrary functions, as long as the integral can be defined. The proof is given by using the connection formula of the Gauss hypergeometric functions. We refer to [D, no 362] for an elegant theory on the Euler-Poisson-Darboux equation.

Remark 4.22 We refer [D] for Sect. 4.1-4.6.

4.7 The Échell of hypergeometric functions

We now make a short digression and discuss Appell's hypergeometric system, which gives an example of the Euler-Poisson-Darboux equation.

Appell's system denoted by (F_1) is a system defined by

$$(F_1)\qquad \begin{cases}\theta(\theta+\theta'+\gamma-1)z-x(\theta+\theta'+\alpha)(\theta+\beta)z=0,\\ \theta'(\theta+\theta'+\gamma-1)z-y(\theta+\theta'+\alpha)(\theta+\beta')z=0,\end{cases}$$

where $\theta=x\partial_x$ and $\theta'=y\partial_y$ and α, β, β' and γ are complex parameters. Appell's hypergeometric function denoted by F_1 is a solution that is holomorphic around the origin:

$$F_1(\alpha,\beta,\beta',\gamma;x,y)=\sum_{m,n\geq 0}\frac{(\alpha,m+n)(\beta,m)(\beta',n)}{(\gamma,m+n)(1,m)(1,n)}x^m y^n;$$

we use here the notation $(a,m)=a(a+1)\cdots(a+m-1)$. By a straightforward computation, we see that any solution z of (F_1) satisfies the Euler-Poisson-Darboux equation $E(\beta,\beta')$. Hence, the first Laplace transform is

$$w=z_y-\frac{\beta'}{x-y}z,$$

and w itself satisfies the system (F_1) with different parameters. Relative to the function F_1, the transformation can be interpreted by the identity

$$[\beta' - (x-y)\partial_y]F_1(\alpha,\beta,\beta',\gamma;x,y) = \beta' F_1(\alpha,\beta-1,\beta'+1,\gamma;x,y), \tag{4.8}$$

called the *contiguity relation* of Appell's system. In particular, we can prove that the solution space of (F_1) is invariant under Laplace transformations.

Let us describe some details. For the pair $A = (\beta,\beta')$ we introduce the notation $A+1 = (\beta-1,\beta'+1)$, in view of the translation in (4.8) of parameters. We define

$$D(A) = (x-y)\partial_x + \beta, \qquad U(A) = (x-y)\partial_y - \beta',$$
$$L(A) = \partial_x\partial_y - \frac{\beta'}{x-y}\partial_x + \frac{\beta}{x-y}\partial_y;$$

Then we see that

$$D(A+1)U(A) = \beta'(1-\beta) + (x-y)^2 L(A),$$
$$U(A-1)D(A) = \beta(1-\beta') + (x-y)^2 L(A).$$

Hence, if we denote $F_1(\alpha,\beta,\beta',\gamma;x,y)$ by $F(A)$, then

$$D(A)F(A) = \beta F(A-1), \qquad U(A)F(A) = -\beta' F(A+1).$$

These identities show that the system is invariant under Laplace transformations.

Using this invariance, K. Okamoto [O1988] gave a solution of the Toda equation as follows: To make the formula simpler, let us define

$$f_n = 1/(n+1-\beta), \quad g_n = 1/(n-1+\beta'),$$
$$X = \partial_x, \quad Y = \partial_y,$$
$$L_n = XY - \frac{\beta'+n}{x-y}X + \frac{\beta-n}{x-y}Y,$$

and set

$$B_n = g_n((x-y)X + \beta - n) = g_n D(A+n),$$
$$H_n = f_n((x-y)Y - \beta' - n) = f_n U(A+n).$$

Then, introducing the series of functions $\{\phi_n\}$ by

$$\phi_0 = F(A), \qquad \phi_n = H_n\phi_0,$$

we see that

$$H_n\phi_n = \phi_{n+1}, \qquad B_n\phi_n = \phi_{n-1}.$$

In particular, defining

$$\psi_n = (x-y)^{(\beta-n)(\beta'+n)}\phi_n,$$

we finally have:

Proposition 4.23

$$f_n g_n XY \log \psi_n = -\frac{\psi_{n+1}\psi_{n-1}}{\psi_n^2}.$$

A similar formula of this kind also appears for the second Appell's system defined by

$$(F_2) \qquad D_1 z = 0 \quad \text{and} \quad D_2 z = 0,$$

where

$$\begin{aligned} D_1 &= x(1-x)\partial_{xx} - xy\partial_{xy} + \{\gamma - (\alpha+\beta+1)x\}\partial_x - \beta y \partial_y - \alpha\beta, \\ D_2 &= y(1-y)\partial_{yy} - xy\partial_{xy} + \{\gamma' - (\alpha+\beta'+1)y\}\partial_y - \beta' x \partial_x - \alpha\beta'. \end{aligned}$$

The following series is a solution that is holomorphic around the origin.

$$F_2(\alpha,\beta,\beta',\gamma,\gamma';x,y) = \sum_{m,n\geq 0} \frac{(\alpha,m+n)(\beta,m)(\beta',n)}{(\gamma,m)(\gamma',n)(1,m)(1,n)} x^m y^n.$$

The system has four independent solutions and any set of independent solutions defines a surface in $\mathbf{P}^3$. The surface is uniquely defined up to a projective transformation and it has a unique conformal structure, a nondegenerate quadratic form $y/(1-x)dx^2 + 2dxdy + x/(1-y)dy^2$. Relative to this form, one set of conjugate directions is given by $x\partial_x$ and $(x-1)\partial_x + y\partial_y$; see Appendix A. In view of this fact, we define two operators by

$$\begin{aligned} H_n &= x\partial_x + (\beta+n), \\ B_n &= \frac{1}{(\beta+n-1)(\beta-\gamma+n)} \left(x((x-1)\partial_x + y\partial_y) + \alpha x + (\beta+n) - \gamma\right). \end{aligned}$$

Then

$$B_{n+1}H_n - 1 = a_n L(\beta+n), \qquad H_{n+1}B_n - 1 = b_n L(\beta+n),$$

where

$$a_n = \frac{\alpha x + \beta + n - \gamma}{(\beta+n-1)(\beta+n-\gamma)}, \qquad b_n = \frac{x}{(\beta+n-1)(\beta+n-\gamma)}.$$

The corresponding contiguity relation is

$$H_n F_2(\alpha,\beta+n,\beta',\gamma,\gamma';x,y) = (\beta+n)F_2(\alpha,\beta+n+1,\beta',\gamma,\gamma';x,y).$$

If we set

$$\chi_n = c_n (x-1)^n \Gamma(\beta+n) F_2(\alpha,\beta+n,\beta',\gamma,\gamma';x,y),$$

where c_n is a certain constant depending on n, then we have

$$\partial_x((x-1)\partial_x + y\partial_y)\log\chi_n = \frac{\chi_{n+1}\chi_{n-1}}{{\chi_n}^2}.$$

4.8 Godeaux sequences

L. Godeaux gave a method for studying projective surfaces through their Plücker images in $\mathbf{P}^5$. His method relies on the consideration of the Laplace sequence assocated with the Plücker image, called the Godeaux sequence. In this section, we show a charaterization of Demoulin surfaces by Godeaux sequence and a characterization of the Plücker image of surfaces in view of distingushed choices of frames of $\mathbf{P}^5$.

Given a surface by the system (2.12), we define two vectors in $\mathbf{P}^5$ by

$$U = z \wedge z_x, \quad V = z \wedge z_y. \tag{4.9}$$

It is easy to see that

$$U_x = bV, \quad V_y = cU,$$

and that U and V satisfy the Laplace equations:

$$U_{xy} = (\log b)_y U_x + bcU, \quad V_{xy} = (\log c)_x V_y + bcV.$$

By following the procedure given in Sect. 4.2, the first Laplace transform of U relative to the coordinate y is given by

$$U^1 = U_y - (\log b)_y U.$$

It satisfies

$$U^1_x = h_1 U, \quad U^1_{xy} = h_1 U^1 + (\log bh_1)_y U^1_x, \quad h_1 = 2\kappa_1,$$

where $\kappa_1 = (bc - (\log b)_{xy})/2$ was defined in (2.17). Continuing this process successively, we can define

$$U^{n+1} = U^n_y - (\log bh_1 \cdots h_n)_y U^n,$$

where h_n is defined by

$$h_n = h_{n-1} - (\log bh_1 \cdots h_{n-1})_{xy}$$

and U^n satisfies

$$U^n_{xy} = h_n U^n + (\log bh_1 \cdots h_n)_y U^n_x.$$

Similarly for V, we have a recursive definition

$$\begin{aligned} V^{n+1} &= V^n_x - (\log ck_1 \cdots k_n)_x V^n, \quad k_1 = 2\kappa_2, \\ k_n &= k_{n-1} - (\log ck_1 \cdots k_{n-1})_{xy}, \\ V^n_{xy} &= k_n V^n + (\log ck_1 \cdots k_n)_x V^n_y, \end{aligned}$$

where $\kappa_2 = (bc - (\log c)_{xy})/2$. The sequence $\{\ldots, U^n, \ldots, U^1, U, V, V^1, \ldots, V^n, \ldots\}$ is called the *Godeaux sequence*. This sequence has a special property for a Demoulin surface, which we now describe.

By Lemma 2.12 for a Demoulin surface, we see that

$$h_4 = k_1,\ h_3 = bc,\ h_2 = h_1; \quad k_2 = k_1,\ k_3 = bc,\ k_4 = h_1.$$

(Recall that we have set $h_1 = 2\kappa_1$ and $k_1 = 2\kappa_2$.) This tells us that some periodicity occurs in the sequence.

In order to look into the details, we introduce four vectors

$$M_1 = z \wedge z_{xy}, \quad M_2 = z_x \wedge z_y, \quad M_3 = z_x \wedge z_{xy}, \quad M_4 = z_y \wedge z_{xy},$$

following Godeaux [G1934]. Then, by defining a frame $T = {}^t(U, V, M_1, M_2, M_3, M_4)$ in $\mathbf{P}^6$, we see that

$$T_x = \begin{pmatrix} 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ bc & b_y + p & 0 & 0 & 1 & 0 \\ 0 & p & 0 & 0 & 1 & 0 \\ -bq - p_y & 0 & p & b_y + p & 0 & b \\ 0 & -bq - p_y & 0 & -bc & 0 & 0 \end{pmatrix} T,$$

$$T_y = \begin{pmatrix} 0 & 0 & 1 & -1 & 0 & 0 \\ c & 0 & 0 & 0 & 0 & 0 \\ c_x + q & bc & 0 & 0 & 0 & 1 \\ -q & 0 & 0 & 0 & 0 & -1 \\ -cp - q_x & 0 & 0 & bc & 0 & 0 \\ 0 & -cp - q_x & q & -c_x - q & c & 0 \end{pmatrix} T.$$

From these relations, we can express U^1, U^2, V^1, and V^2 in terms of the six vectors above.

$$\begin{aligned} U^1 &= -(\log b)_y U + M_1 - M_2, \\ U^2 &= \lambda U + bcV - ((\log b)_y + K_{1y})(M_1 - M_2) + 2M_4, \\ V^1 &= -(\log c)_x V + M_1 + M_2, \\ V^2 &= bcU + \mu V - ((\log c)_x + K_{2x})(M_1 + M_2) + 2M_3, \end{aligned}$$

where

$$\lambda = 2Q + \frac{b_y^2}{2b^2} + (\log b)_y K_{1y}, \quad \mu = 2P + \frac{c_x^2}{2c^2} + (\log c)_x K_{2x},$$

and where P and Q are given in (2.18) and K_1 and K_2 are given in (2.31). Since U, V, M_1, M_2, M_3, and M_4 are independent, as we assumed that the surface is nondegenerate, there are no linear relations among U, V, U^1, U^2, V^1, and V^2, as can be seen from the above expressions. Now a slightly long computation gives the formulas:

$$\begin{aligned} U^3 &= -((\log bh_2)_y + 2K_{1y})U^2 + (4Q - K_{1yy} - (\log b)_y K_{1y} - (K_{1y})^2)U^1 \\ &\quad + (4(\log b)_y Q + 2Q_y)U - 4cPV + cK_{2x}V^1 + cV^2, \\ V^3 &= -((\log ck_2)_x + 2K_{2x})V^2 + (4P - K_{2xx} - (\log c)_x K_{2x} - (K_{2x})^2)V^1 \\ &\quad + (4(\log c)_x P + 2P_x)V - 4bQU + bK_{1y}U^1 + bU^2. \end{aligned}$$

Theorem 4.24 *Assume that $bc \neq 0$ and $\kappa_1\kappa_2 \neq 0$. Then the Godeaux sequence is periodic with period six if and only if the surface is Demoulin.*

Proof. If the surface is Demoulin, we saw that $P = Q = 0$. Lemma 2.12 says that the expressions above reduce to $U^3 = cV^2$ and $V^3 = bU^2$, which implies the periodicity. The converse statement also follows from the expressions above.

In general, any consective sequence of six vectors forms a frame in $\mathbf{P}^5$ as we have seen. We modify the frame T by introducing the vectors

$$\begin{aligned} N_1 &= z \wedge z_{xy} - z_x \wedge z_y = M_1 - M_2 = U_y, \\ N_2 &= z \wedge z_{xy} + z_x \wedge z_y = M_1 + M_2 = V_x, \\ N_3 &= 2z_y \wedge z_{xy} + bcz \wedge z_y = 2M_4 + bcV, \\ N_4 &= 2z_x \wedge z_{xy} + bcz \wedge z_x = 2M_3 + bcU. \end{aligned} \tag{4.10}$$

The set of vectors $\{U, V, N_1, \ldots, N_4\}$ are chosen to be orthonormal in the following sense. Given two vectors u and v in $\overset{2}{\wedge}\mathbf{R}^4$, the product $u \wedge v$ is a vector lying in $\overset{4}{\wedge}\mathbf{R}^4$ which we identity with the scalar field $\mathbf{R}$ and we get a scalar value $(u, v) = u \wedge v$. We thus have an inner product on the space $\overset{2}{\wedge}\mathbf{R}^4$. By the identification $z \wedge z_x \wedge z_y \wedge z_{xy} = -1/2$, we can see that

$$(U, N_3) = -1, \quad (V, N_4) = 1, \quad (N_1, N_1) = 1, \quad (N_2, N_2) = -1 \tag{4.11}$$

and the remaining products vanish. In particular, they are linearly independent and we thus get a new frame $\mathcal{T} = {}^t(U, V, N_1, N_2, N_3, N_4)$ in the space $\mathbf{P}^5$. We remark that the signature of the inner product is $(3, 3)$.

The frame $\mathcal{T}$ satisfies a Pfaff system

$$d\mathcal{T} = \omega\mathcal{T},$$

where

$$\omega = \begin{pmatrix} 0 & bdx & dy & 0 & 0 & 0 \\ cdy & 0 & 0 & dx & 0 & 0 \\ bcdx + \kappa dy & b_y dx & 0 & 0 & dy & 0 \\ c_x dy & bcdy + \delta dx & 0 & 0 & 0 & dx \\ 0 & \mu dx + \nu dy & bcdx + \kappa dy & -c_x dy & 0 & cdy \\ \mu dx + \nu dy & 0 & -b_y dx & bcdy + \delta dx & bdx & 0 \end{pmatrix} \tag{4.12}$$

and

$$\kappa = 2q + c_x, \quad \delta = 2p + b_y, \quad \mu = b_{yy} - b\kappa, \quad \nu = c_{xx} - c\delta. \tag{4.13}$$

The integrability condition $d\omega = \omega \wedge \omega$ is

$$\kappa_x = (bc)_y + cb_y, \quad \delta_y = (bc)_x + bc_x, \quad \nu_x - \mu_y = \delta c_x - \kappa b_y, \tag{4.14}$$

which is the same as the integrability condition of (2.13).

Let us make a simple consideration on the frame $\mathcal{T}$. By definition we have

$$U_x = bV \text{ and } V_y = cU, \quad N_1 = U_y \text{ and } N_2 = V_x.$$

While the line $\overline{UV}$ is lying in the quadric $Q_4 = \{u \in \mathbf{P}^5 | (u,u) = 0\}$, the points N_1 and N_2 are lying outside the quadric. We next have

$$dN_1 \equiv dyN_3, \quad dN_2 \equiv dxN_4 \qquad (\text{mod } U, V).$$

Hence, both $(N_1)_x$ and $(N_2)_y$ lie on the line $\overline{UV}$ and

$$(N_1)_y \equiv N_3, \qquad (N_2)_x \equiv N_4 \qquad (\text{mod } U, V).$$

We further see that

$$(N_3)_x = \mu V + bcN_1 \equiv bcN_1, \quad (N_4)_y = \nu U + bcN_2 \equiv bcN_2 \quad (\text{mod } U, V).$$

We conversely start with a projective frame $t = {}^t(t_1, \ldots, t_6)$ of $\mathbf{P}^5$ depending on two parameters (x, y) such that it satisfies the orthonormality such as $(t_i, t_j) = h_{ij}$, where $h = (h_{ij})$ is a nondegenerate constant matrix of signature $(3, 3)$. Let us denote by $so(h)$ the Lie algebra of the orthogonal group relative to h, which is isomorphic to $so(3, 3; \mathbf{R})$. Then the equation of motion satisfied by t is written as

$$dt = \omega t, \quad dt_i = \sum_j \omega_i^j t_j,$$

where ω is a 1-form with values in $so(h)$. For a given t to be a frame associated to a projective surface, we impose the conditions that

$$dt_1 \equiv 0 \qquad (\text{mod } t_1, t_2, t_3), \quad dt_2 \equiv 0 \qquad (\text{mod } t_1, t_2, t_4) \tag{4.15}$$

and that

$$\omega_1^3 =: \omega^2 \quad \text{and} \quad \omega_2^4 =: \omega^1 \quad \text{are linearly independent.} \tag{4.16}$$

In order that the line $\overline{t_1 t_2}$ lies in the quadric Q_4 and the frame satisfies the property (4.11), we further impose the condition that

$$h = \begin{pmatrix} & & -J \\ & J & \\ -J & & \end{pmatrix}, \qquad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

With these conditions, the frame t can be normalized. We say a change of frame: $t \to gt$ is allowable if $gh\,{}^tg = h$. Then we have the following theorem.

Theorem 4.25 *There is an allowable change of frame so that the form ω has the form given in (*4.12).

Proof is given by a detailed examination of the integrability, which will be given in Appendix C, where we also give a remark on the analogous presentation relative to Lie sphere geometry of Euclidean surfaces.

Remark 4.26 For the present section, see [G1934] and [Fe2000a].

5 Affine spheres and the Laplace transformation

In previous sections we defined several notions necessary in the following study of surfaces. In this section, apart from the projective treatment of surfaces, we examime the Laplace transformation of surfaces in the affine setting. To do this, we first recall the historical notion of affine spheres, called S-surfaces by Tzitzeica, and then explain a transformation of affine spheres called the Tzitzeica transformation. We fix one affine chart of projective space $\mathbf{P}^3$, which we denote by $\mathbf{A}^3$.

5.1 Affine surfaces

We consider an immersed surface given by a mapping $(x, y) \mapsto z(x, y) \in \mathbf{A}^3$. For a mapping to define a surface, the three vectors z, z_x, and z_y must be independent, and then we have a system of the form

$$\begin{aligned} z_{xx} &= az_x + bz_y + cz, \\ z_{xy} &= a'z_x + b'z_y + c'z, \\ z_{yy} &= a''z_x + b''z_y + c''z. \end{aligned} \tag{5.1}$$

Each coordinate of z satisfies this system. Conversely, given a system as above, any set of three independent solutions defines an immersion, thus defining a surface. Two sets of independent solutions differ only by an affine transformation leaving the origin fixed; hence the system defines a surface uniquely up to an affine motion preserving the origin. The conformal structure of the surface is well-defined and is given by

$$cdx^2 + 2c'dxdy + c''dy^2. \tag{5.2}$$

(See Sect. 5.4.) We assume that this symmetric 2-form is nondegenerate. When the coefficient field is the real number field, we assume further that the form is indefinite, because we need a separate treatment for the definite case. Then the coordinates (x, y) are asymptotic relative to this form if $c = c'' = 0$, which in the following we fix once and for all.

Proposition 5.1 *The equation $z_{xy} = a'z_x + b'z_y + c'z$ has equal Laplace invariants.*

Proof. We compute the integrability conditions of the system (5.1) by using the identities:

$$(z_{xx})_y = (z_{xy})_x \quad \text{and} \quad (z_{xy})_y = (z_{yy})_x.$$

The former implies

$$\begin{gathered} a_y + a''b = a'_x + a'b' + c', \\ b_y + ab' + bb'' = b'_x + a'b + b'^2, \\ ac' = c'_x + b'c', \end{gathered} \tag{5.3}$$

and the latter implies

$$\begin{aligned} b''_x + a''b &= b'_y + a'b' + c', \\ a'_y + a'^2 + a''b' &= a''_x + aa'' + b''a', \\ b''c' &= c'_y + a'c'. \end{aligned} \tag{5.4}$$

Hence, we get $b' = a - (\log c')_x$ and $a' = b'' - (\log c')_y$. In particular, we have $b'_y = a_y - (\log c')_{xy}$ and $a'_x = b''_x - (\log c')_{xy}$. Hence $a'_x - b'_y = b''_x - a_y$. On the other hand, the first equations of each of the two sets of equations above imply that $a'_x - b'_y = a_y - b''_x$. Therefore $a'_x = b'_y$, which shows that the Laplace invariants h and k are equal.

By this proposition, we can set

$$a' = \lambda_y/\lambda \quad \text{and} \quad b' = \lambda_x/\lambda \tag{5.5}$$

for some function λ.

5.2 Affine spheres

We next fix a Euclidean structure on the affine space $\mathbf{A}^3$. Let $Edx^2 + 2Fdxdy + Gdy^2$ be the usual induced metric and set $\Delta = (EG - F^2)^{1/2}$. The induced second fundamental form is

$$\frac{1}{\Delta}\{|z_{xx}\ z_x\ z_y|\,dx^2 + 2\,|z_{xy}\ z_x\ z_y|\,dxdy + |z_{yy}\ z_x\ z_y|\,dy^2\},$$

which is equal to $(2c'/\Delta)|z\ z_x\ z_y|dxdy$ in the notation of the previous section. Let $d = d(x,y)$ be the distance from the origin to the tangent plane at $z(x,y)$. It is given by

$$d = \frac{|z\ z_x\ z_y|}{\Delta}.$$

Then, denoting the Gauss curvature by K, we get the formula

$$\frac{K}{d^4} = -\frac{c'^2}{|z\ z_x\ z_y|^2}.$$

We define a scalar function μ by $\mu^4 = |d^4/K|$. Then $\mu^2 c' = \pm\,|z\ z_x\ z_y|$, whose logarithmic derivative gives

$$2\frac{\mu_x}{\mu} + \frac{c'_x}{c'} = a + b' \quad \text{and} \quad 2\frac{\mu_y}{\mu} + \frac{c'_y}{c'} = b'' + a'.$$

The identities $a = b' + (\log c')_x$ and $b'' = a' + (\log c')_y$, which follow from (5.3) and (5.4), then imply

$$a' = \mu_y/\mu \quad \text{and} \quad b' = \mu_x/\mu,$$

which together with (5.5) imply that $\mu = m\lambda$ for some constant m. We have seen that

$$\frac{K}{d^4} = -\frac{1}{m^4\lambda^4}.$$

Corollary 5.2 *The quantity K/d^4 is uniquely determined from the system, up to a constant factor.*

Definition 5.3 The surface z is called an *affine sphere* if K/d^4 is constant.

This notion coincides with the definition of an affine sphere in affine differential geometry. We refer to [NS1994].

The discussion above shows that an affine sphere is defined by a system of the form

$$z_{xx} = az_x + bz_y, \quad z_{xy} = hz, \quad z_{yy} = a''z_x + b''z_y,$$

where $h = c'$ because μ is constant. We know $h \neq 0$ by nondegeneracy. The integrability condition is the following:

$$\begin{aligned} &h_x = ah, \quad a_y + a''b = h, \quad b_y + bb'' = 0, \\ &h_y = b''h, \quad a''_x + aa'' = 0, \quad b''_x + a''b = h. \end{aligned}$$

Suppose that $b = 0$. Then the surface is ruled with x as the line coordinate. When $a'' = 0$, y is the line coordinate. To exclude these cases, we assume $a''b \neq 0$. Then the integrability condition is written as

$$a = \frac{h_x}{h}, \quad b'' = \frac{h_y}{h}, \quad a'' = \frac{Y(y)}{h}, \quad b = \frac{X(x)}{h}$$

and

$$(\log h)_{xy} = h - \frac{X(x)Y(y)}{h^2},$$

where X is a function of x and Y is a function of y. By suitably choosing coordinates, we can further assume X and Y are constants so that $XY = 1$. Namely, we have seen the following:

Proposition 5.4 *Any affine sphere is given, up to an affine motion, by a system of the form*

$$z_{xx} = \frac{h_x}{h}z_x + \frac{m}{h}z_y, \quad z_{xy} = hz, \quad z_{yy} = \frac{1}{mh}z_x + \frac{h_y}{h}z_y, \tag{5.6}$$

where h is a solution of

$$(\log h)_{xy} = h - \frac{1}{h^2} \tag{5.7}$$

and m is a constant.

Remark 5.5 We have already met the system above in Example 4.13 (see (4.4) and (4.5)).

Example 5.6 *When $h = 1$, the system is $z_{xx} = z_y$, $z_{xy} = z$, and $z_{yy} = z_x$. For any cubic root ω, $e^{\omega x + \omega^2 y}$ is a solution. The associated surface is $XYZ = 1$ in the complex affine space with coordinates (X, Y, Z). In the real case, we also have a representation of the form $X(Y^2 + Z^2) = 1$.*

5.3 Affine polar surfaces

Let $\mathbf{A}_3$ denote the dual space of $\mathbf{A}^3$ and $\langle\ ,\ \rangle$ the dual pairing. Given a surface z in $\mathbf{A}^3$, we define its polar surface $\zeta : (x,y) \mapsto \zeta(x,y) \in \mathbf{A}_3$ by requiring

$$\langle\zeta,\ z\rangle = 1, \quad \langle\zeta,\ z_x\rangle = 0, \quad \langle\zeta,\ z_y\rangle = 0.$$

Lemma 5.7 *Assume the surface z is given by the system* (5.1) *with $c = c'' = 0$. Then the surface ζ is determined by the system*

$$\begin{aligned}
\zeta_{xx} &= (a - 2b')\zeta_x - b\zeta_y, \\
\zeta_{xy} &= -a'\zeta_x - b'\zeta_y + c'\zeta, \\
\zeta_{yy} &= -a''\zeta_x + (b'' - 2a')\zeta_y.
\end{aligned} \tag{5.8}$$

Proof. Set

$$\zeta_{xx} = p\zeta_x + q\zeta_y + r\zeta, \quad \zeta_{xy} = p'\zeta_x + q'\zeta_y + r'\zeta, \quad \zeta_{yy} = p''\zeta_x + q''\zeta_y + r''\zeta.$$

By differentiation of the parings above, we can get the following. From $\langle\zeta,\ z\rangle = 1$,

$$(1)\ \langle\zeta_x,\ z\rangle = 0,\ \langle\zeta_y,\ z\rangle = 0.$$

From $\langle\zeta,\ z_x\rangle = 0$ and $\langle\zeta,\ z_y\rangle = 0$,

$$(2)\ \langle\zeta_x,\ z_x\rangle = 0, \quad (3)\ \langle\zeta_y,\ z_x\rangle = \langle\zeta_x,\ z_y\rangle = -c', \quad (4)\ \langle\zeta_y,\ z_y\rangle = 0.$$

Differentiating (1) again, we see that $\langle\zeta_{xx},\ z\rangle = 0, \quad \langle\zeta_{xy},\ z\rangle = c', \quad \langle\zeta_{yy},\ z\rangle = 0$; namely, $r = 0$, $r' = c'$, and $r'' = 0$. Differentiating (2), we get $\langle\zeta_{xx},\ z_x\rangle + \langle\zeta_x,\ z_{xx}\rangle = 0$ and $\langle\zeta_{xy},\ z_x\rangle + \langle\zeta_z,\ z_{xy}\rangle = 0$; both imply $q = -b$ and $q' = -b'$. Similarly, from (4), we get $p' = -a'$ and $p'' = -a''$. Differentiating (3), we get $\langle\zeta_{xx},\ z_y\rangle + \langle\zeta_x,\ z_{xy}\rangle = c'_x$, which implies $p = -b' + (\log c')_x$. Since $(\log c')_x = a - b'$ by the integrability, we see that $p = a - 2b'$. Similarly, we have $q'' = b'' - 2a'$.

Corollary 5.8 *The asymptotic coordinates for the surface z are also asymptotic for the polar surface ζ. When z is an affine sphere, with parameter m, the polar surface ζ is also an affine sphere, with parameter $-m$.*

5.4 From an affine surface to a projective surface

Let z^1, z^2, and z^3 be independent solutions of the system (5.1). The surface $z(x,y) = (z^1(x,y), z^2(x,y), x^3(x,y))$ defines a surface in projective space by the mapping

$$(x,y) \longrightarrow w(x,y) = [\lambda z^1, \lambda z^2, \lambda z^3, \lambda] =: [w^1, w^2, w^3, w^4],$$

for an arbitrary nonvanishing scalar function λ. The system that has solutions w^i is determined by the following

$$\begin{vmatrix} w & w_x & w_y & w_{xy} & w_{xx} \\ w^1 & w^1_x & w^1_y & w^1_{xy} & w^1_{xx} \\ w^2 & w^2_x & w^2_y & w^2_{xy} & w^2_{xx} \\ w^3 & w^3_x & w^3_y & w^3_{xy} & w^3_{xx} \\ w^4 & w^4_x & w^4_y & w^4_{xy} & w^4_{xx} \end{vmatrix} = 0, \qquad \begin{vmatrix} w & w_x & w_y & w_{xy} & w_{yy} \\ w^1 & w^1_x & w^1_y & w^1_{xy} & w^1_{yy} \\ w^2 & w^2_x & w^2_y & w^2_{xy} & w^2_{yy} \\ w^3 & w^3_x & w^3_y & w^3_{xy} & w^3_{yy} \\ w^4 & w^4_x & w^4_y & w^4_{xy} & w^4_{yy} \end{vmatrix} = 0,$$

where w is an indeterminate. Since the projective class of the surface is independent of the choice of λ, we here assume for simplicity that $\lambda = 1$. Then, the system is written down as follows.

$$\begin{aligned} c' w_{xx} &= c w_{xy} + (c'a - ca')w_x + (c'b - cb')w_y, \\ c' w_{yy} &= c'' w_{xy} + (c'b'' - c''b')w_y + (c'a'' - c''a')w_x. \end{aligned}$$

Hence, the conformal structure of the surface is $cdx^2 + 2c'dxdy + c''dy^2$, as we have already claimed.

If $c = c'' = 0$, then the system reduces to

$$w_{xx} = aw_x + bw_y, \qquad w_{yy} = a''w_x + b''w_y.$$

In particular, for the system of affine spheres (5.6), the associated projective system is

$$z_{xx} = \frac{h_x}{h} z_x + \frac{m}{h} z_y, \qquad z_{yy} = \frac{1}{mh} z_x + \frac{h_y}{h} z_y,$$

with the additional condition (5.7): $(\log h)_{xy} = h - 1/h^2$.

5.5 Laplace transforms of affine spheres

Given an affine sphere z by the system (5.6), we define its first Laplace transform

$$w^1 = \lambda z_y$$

in the affine space, where λ is a nonvanishing scalar function determined later. Assume here $m = 1$ for simplicity. By computation,

$$w^1_x = (\log \lambda)_x w^1 + \lambda h z, \quad w^1_y = (\log \lambda h)_y w^1 + \frac{\lambda}{h} z.$$

Hence, z and z_x can be recovered by

$$z = \frac{1}{\lambda h}(w^1_x - (\log \lambda)_x w^1), \qquad z_x = \frac{h}{\lambda}(w^1_y - (\log \lambda h)_y w^1).$$

By differentiation, we get

$$\begin{aligned} w^1_{xx} &= (\log \lambda^2 h)_x w^1_x + h^2 w^1_y + \big((\log\lambda)_{xx} - (\log \lambda h)_x (\log \lambda)_x - h^2 (\log \lambda h)_y\big)\, w^1, \\ w^1_{xy} &= (\log \lambda)_x w^1_y + (\log \lambda h)_y w^1_x + ((\log \lambda)_{xy} - (\log \lambda h)_y (\log\lambda)_x + h)\, w^1, \\ w^1_{yy} &= \frac{1}{h} w^1_x + (\log \lambda^2)_y w^1_y + \left((\log \lambda h)_{yy} - (\log(\lambda/h))_y (\log \lambda h)_y - \frac{1}{h}(\log \lambda)_x\right) w^1. \end{aligned}$$

From the second equation, the Laplace invariants of the surface w, which we denote by h_1 and k_1, are

$$h_1 = 1/h^2, \qquad k_1 = h.$$

By definition, the second Laplace transform w^2 is given by

$$w^2 = \mu(w^1_y - (\log \lambda h)_y w^1)$$

for a certain scalar μ, and we see that this is equal to $(\lambda\mu/h)z_x$. By computation, we get

$$\begin{aligned} w^2_x &= (\log \lambda\mu)_x w^2 + (\mu/h^2)w^1, \\ w^2_y &= ((\log u)_y + (\log \lambda/h)_y + h(\log \lambda)x(\log \lambda\mu)x)\, w^2 + (\mu/h)w^1_x + (1/h)(\log \lambda)_x w^2_x. \end{aligned}$$

Hence, we can write w^1, w^1_x, and w^1_y in terms of w^2 as

$$\begin{aligned} w^1 &= \frac{h^2}{\mu}(w^2_x - (\log \lambda\mu)_x w^2), \\ w^1_y &= \frac{h^2}{\mu}(\log \lambda h)_y w^2_x + \frac{1}{\mu}(1 - h^2(\log \lambda h)_y(\log \lambda\mu)_x)w^2, \\ w^1_x &= \frac{h}{\mu}\left(w^2_y - \frac{1}{h}(\log \lambda)_x w^2_x - ((\log \lambda\mu/h)_y + h(\log \lambda)_x(\log \lambda\mu)_x)w^2\right). \end{aligned}$$

From these relations we have

$$\begin{aligned} w^2_{xy} = {} & (\log \lambda\mu/h)_y w^2_x + (\log \lambda\mu)_x w^2_y \\ & + \left((\log \lambda\mu)_{xy} + 1/h^2 - (\log \lambda\mu)_x(\log \lambda\mu/h)_y\right) w^2. \end{aligned}$$

This implies that the third Laplace transform is given by

$$w^3 = \nu(w^2_y - (\log \lambda\mu/h)_y w^2)$$

for a certain scalar ν, and the Laplace invariants h_2 and k_2 are

$$h_2 = h, \qquad k_2 = \frac{1}{h^2}.$$

Now it is easy to see that

$$w^3 = \lambda\mu\nu z.$$

Proposition 5.9 *Assume $\lambda\mu\nu = 1$. Then the affine sphere z is 3-periodic relative to Laplace transformation.*

5.6 Tzitzeica transforms of affine spheres

Tzitzeica found a transformation formula that sends a given affine sphere to a new affine sphere. We reproduce the formula in our setting.

We start by recalling the Moutard transformation of the equation

$$z_{xy} = hz, \tag{5.9}$$

which is a part of the system (5.6). For notational simplicity, we introduce the operator

$$M(z) = \frac{z_{xy}}{z}.$$

Then, the equation (5.9) is simply written as $M(z) = h$. Let R be an arbitrary scalar solution of (5.9) and define a 1-form ω associated to any solution z by

$$\omega = (Rz_x - R_x z)dx - (Rz_y - R_y z)dy.$$

Then computation shows that ω is closed. This means that there is a function u such that $\omega = du$, namely,

$$u_x = Rz_x - R_x z, \qquad u_y = -(Rz_y - R_y z).$$

Furthermore, for $w = u/R$, we see that

$$w_{xy} = M\left(\frac{1}{R}\right) w, \tag{5.10}$$

and

$$(Rw)_x = R^2\left(\frac{z}{R}\right)_x, \quad (Rw)_y = -R^2\left(\frac{z}{R}\right)_y.$$

Hence, w is defined by the integral

$$w = \frac{1}{R}\int R^2\left\{\left(\frac{z}{R}\right)_x dx - \left(\frac{z}{R}\right)_y dy\right\}.$$

The transformation of any solution z of (5.9) to a solution w of (5.10) is called a *Moutard transformation*.

Let now z be any solution of the system (5.6) with parameter $m = a$ and choose a solution R of the same system with parameter $m = b$. Assuming $a \neq b$, we define a new function w by

$$w = z - \frac{2a(\log R)_x}{(a-b)h} z_y + \frac{2b(\log R)_y}{(a-b)h} z_x. \tag{5.11}$$

Then computation shows that

$$(Rw)_x = \frac{a+b}{a-b}(Rz_x - R_x z), \quad (Rw)_y = -\frac{a+b}{a-b}(Rz_y - R_y z),$$

which implies that w is a Moutard transformation of z, up to a constant multiple. Further, we can see that w satisfies the system (5.6) with the same constant m and with $h - 2(\log R)_{xy}$ as the new h. Therefore we have the following theorem due to Tzitzeica.

Theorem 5.10 *Let z be an affine sphere and define a new surface w by* (5.11). *Then,* (1) *The surface w is again an affine sphere and the associated system is determined by the potential $h - 2(\log R)_{xy}$ for h and the same parameter $m = a$ as for z.* (2) *The congruence made by lines joining $z(x, y)$ and $w(x, y)$ is a W-congruence.*

Part (2) follows from the fact that the conformal structure of both surfaces is conformal to $dxdy$. We remark that the potential $\overline{h} = h - 2(\log R)_{xy}$ satisfies the equation $(\log \overline{h})_{xy} = \overline{h} - 1/\overline{h}^2$.

Remark 5.11 A reference for affine differential geometry is [NS1994]. Fundamental references for Tzitzeica's work are [Tz1907, Tz1924]. We refer to Section 13 for a generalization of Theorem 5.10, and to [RS2002] for the Tzitzeica transformation. We refer also to [BS1999]. [Dem1920] is a good reference for the Moutard transformation.

6 Line congruences (2)

In Section 3 we defined the notion of line congruence and explained a geometrical characterization of W-congruence. This section aims at continuing the consideration in a slightly different manner, using the moving frame method and discussing the relation with a differential system.

6.1 The Weingarten invariant W

Definition 6.1 A projective frame $e = \{e_1, e_2, e_3, e_4\}$ in $\mathbf{P}^3$ defined along an immersed surface M^2 is said to belong to a line congruence if the connection form ω, defined by $de_i = \sum_j \omega_i^j e_j$, satisfies the condition

$$\omega_1^4 = \omega_2^3 = 0, \quad \omega_1^3 \text{ and } \omega_2^4 \text{ are linearly independent.} \tag{6.1}$$

We hereafter use the notation $\omega^1 = \omega_1^3$ and $\omega^2 = \omega_2^4$.

A geometric interpretation is the following: The vector e_1 is an immersion of M and defines a surface S_1, and e_2 defines a second surface S_2; the set of lines joining two points e_1 and e_2 is a line congruence in the sense defined in Section 3. The Fig. 1 shows a line congruence between two paraboloids. The tangent lines to the parameter curves of one of two surfaces are tangent to the other surface.

Since we can see $e_1 \wedge e_2 \wedge de_1 \wedge de_2 = \omega^1 \cdot \omega^2 \; e_1 \wedge e_2 \wedge e_3 \wedge e_4$ by definition (remark that the wedge product is made relative to vectors and the product of differential forms is a symmetric product), the ruled surface given as the union of lines through the integral curve of the equation $\omega^1 = 0$ is developable. The same is true for $\omega^2 = 0$. Hence, the vectors e_1 and e_2 describe focal surfaces of the congruence.

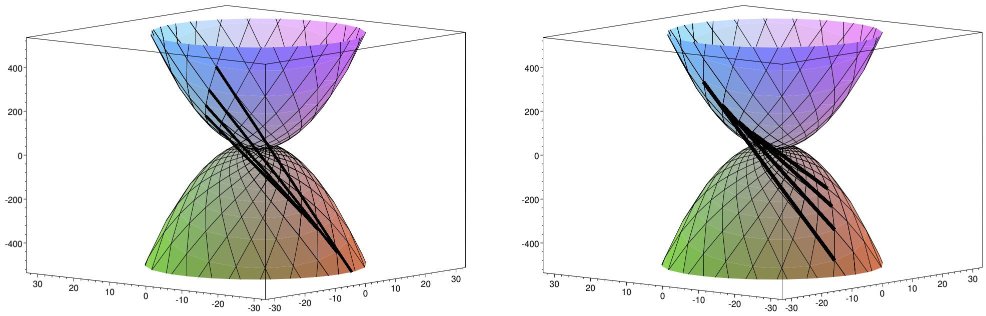

Figure 1: Line congruence between two paraboloids

Two frames belonging to the same line congruence are related by a matrix of the form

$$g = \begin{pmatrix} \lambda & \beta & 0 & 0 \\ \gamma & \mu & 0 & 0 \\ p & q & \rho & 0 \\ r & s & 0 & \sigma \end{pmatrix},$$

which acts on the frame from the left. Such matrices form a group, say G, and the set of such frames define a bundle over M with G the fiber group.

Let us see what the first condition of (6.1) implies. From $d\omega_1^4 = 0$, we have $\omega_1^2 \wedge \omega_2^4 + \omega_1^3 \wedge \omega_3^4 = 0$. Hence, if we set

$$\omega_1^2 = h_{11}\omega^1 + h_{12}\omega^2, \quad \omega_3^4 = h_{31}\omega^1 + h_{32}\omega^2,$$

then $h_{11} + h_{32} = 0$. From $d\omega_2^3 = 0$, we have $\omega_2^1 \wedge \omega_1^3 + \omega_2^4 \wedge \omega_4^3 = 0$ and we can set

$$\omega_2^1 = h_{21}\omega^1 + h_{22}\omega^2, \quad \omega_4^3 = h_{41}\omega^1 + h_{42}\omega^2,$$

so that $h_{22} + h_{41} = 0$.

Lemma 6.2 *There always exists a frame satisfying $h_{11} = h_{32} = h_{22} = h_{41} = 0$. For such a frame, we have*

$$\omega_1^2 = h_{12}\omega^2, \quad \omega_3^4 = h_{31}\omega^1, \quad \omega_2^1 = h_{21}\omega^1, \quad \omega_4^3 = h_{42}\omega^2. \tag{6.2}$$

Proof. We define a new frame $\overline{e} = \{\overline{e}_1, \overline{e}_2, \overline{e}_3, \overline{e}_4\}$ by

$$\begin{aligned} \overline{e}_1 &= e_1, \\ \overline{e}_2 &= e_2, \\ \overline{e}_3 &= pe_1 + qe_2 + e_3, \\ \overline{e}_4 &= re_1 + se_2 + e_4. \end{aligned}$$

Then it is not difficult to see that

$$\begin{aligned}
d\overline{e}_1 &= (\omega_1^1 - p\omega_1^3)\overline{e}_1 + (\omega_1^2 - q\omega_1^3)\overline{e}_2 + \omega_1^3\overline{e}_3,\\
d\overline{e}_2 &= (\omega_2^1 - r\omega_2^4)\overline{e}_1 + (\omega_2^2 - s\omega_2^4)\overline{e}_2 + \omega_2^4\overline{e}_4,\\
d\overline{e}_3 &\equiv (\omega_3^4 + q\omega_2^4)\overline{e}_4 \quad (\mathrm{mod}\ \overline{e}_1, \overline{e}_2, \overline{e}_3),\\
d\overline{e}_4 &\equiv (\omega_4^3 + r\omega_1^3)\overline{e}_3 \quad (\mathrm{mod}\ \overline{e}_1, \overline{e}_2, \overline{e}_4).
\end{aligned}$$

This implies that the connection form $\overline{\omega}$ corresponding to $\overline{e}$ has an expression of the form

$$\begin{aligned}
\overline{\omega}_1^2 &= \omega_1^2 - q\omega^1 = (h_{11} - q)\omega^1 + h_{12}\omega^2,\\
\overline{\omega}_2^1 &= \omega_2^1 - r\omega^2 = h_{21}\omega^1 + (h_{22} - r)\omega^2,\\
\overline{\omega}_3^4 &= \omega_3^4 + q\omega^2 = h_{31}\omega^1 + (h_{32} + q)\omega^2,\\
\overline{\omega}_4^3 &= \omega_4^3 + r\omega^1 = (h_{41} + r)\omega^1 + h_{42}\omega^2.
\end{aligned}$$

Hence the result follows, because $h_{11} + h_{32} = 0$ and $h_{22} + h_{41} = 0$.

We now assume that the frame satisfies (6.2). Along the surface S_1, the projective frame in the previous section is given by $\{e_1, e_2, e_3, e_4\}$; hence the induced conformal structure, which we now denote by ϕ_1, is given by

$$\phi_1 = \omega_1^2 \cdot \omega_2^4 + \omega_1^3 \cdot \omega_3^4 = h_{31}\omega^1\omega^1 + h_{12}\omega^2\omega^2.$$

Similarly, for the surface S_2, we get the conformal structure

$$\phi_2 = h_{21}\omega^1\omega^1 + h_{42}\omega^2\omega^2.$$

Definition 6.3 We call $W = h_{12}h_{21} - h_{31}h_{42}$ the *Weingarten invariant* of the line congruence. This coincides with the definition of W given in (3.7); see Sect. 6.3.

6.2 Covariance of frames

We now consider how frames satisfying the condition (6.2) can vary. Let e and $\tilde{e}$ be such frames and let g be the connecting matrix such that $\tilde{e} = ge$:

$$g = \begin{pmatrix} A & 0 \\ P & \Lambda \end{pmatrix}; \quad A = \begin{pmatrix} \lambda & \beta \\ \gamma & \mu \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \rho & 0 \\ 0 & \sigma \end{pmatrix}, \quad P = \begin{pmatrix} p & q \\ r & s \end{pmatrix}.$$

Then the connection forms are ω above and $\tilde{\omega} = dg \cdot g^{-1} + g\omega g^{-1}$. We denote components of ω by

$$\omega = \begin{pmatrix} \omega_1 & \omega_2 \\ \omega_3 & \omega_4 \end{pmatrix},$$

where ω_1, ..., ω_4 are 2×2-matrix-valued 1-forms. The $\tilde{\omega}$ is also decomposed similarly. Computation shows that

$$dg \cdot g^{-1} = \begin{pmatrix} dA \cdot A^{-1} & 0 \\ dP \cdot A^{-1} - d\Lambda \cdot \Lambda^{-1} P A^{-1} & d\Lambda \cdot \Lambda^{-1} \end{pmatrix},$$

$$g\omega g^{-1} = \begin{pmatrix} A\omega_1 A^{-1} - A\omega_2\Lambda^{-1}PA^{-1} & A\omega_2\Lambda^{-1} \\ P\omega_1 A^{-1} + \Lambda\omega_3 A^{-1} - P\omega_2\Lambda^{-1}pA^{-1} & P\omega_2\Lambda^{-1} \\ -\Lambda\omega_4\Lambda^{-1}PA^{-1} & +\Lambda\omega_4\Lambda^{-1} \end{pmatrix}.$$

Thus we see that

$$\tilde{\omega}_2 = A\omega_2\Lambda^{-1} = \begin{pmatrix} \lambda & \beta \\ \gamma & \mu \end{pmatrix} \begin{pmatrix} \rho^{-1}\omega_1^3 & 0 \\ 0 & \sigma^{-1}\omega_2^4 \end{pmatrix}.$$

In order that this has a form $\begin{pmatrix} \tilde{\omega}_1^3 & 0 \\ 0 & \tilde{\omega}_2^4 \end{pmatrix}$, it is necessary that $\beta = \gamma = 0$. In this case,

$$\begin{cases} \tilde{e}_1 &= \lambda e_1 \\ \tilde{e}_2 &= \mu e_2 \end{cases} \qquad \text{where} \quad A = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}.$$

We then have

$$\tilde{\omega}_1^3 = \lambda\rho^{-1}\omega_1^3, \qquad \tilde{\omega}_2^4 = \mu\sigma^{-1}\omega_2^4.$$

For the component $\tilde{\omega}_1$, we get

$$\begin{aligned} \tilde{\omega}_1 &= A\omega_1 A^{-1} - A\omega_2\Lambda^{-1}pA^{-1} + dA \cdot A^{-1} \\ &= \begin{pmatrix} \omega_1^1 & \mu^{-1}\lambda\omega_1^2 \\ \lambda^{-1}\mu\omega_2^1 & \omega_2^2 \end{pmatrix} - \begin{pmatrix} \tilde{\omega}_1^3 & 0 \\ 0 & \tilde{\omega}_2^4 \end{pmatrix} \begin{pmatrix} \lambda^{-1}p & \mu^{-1}q \\ \lambda^{-1}r & \mu^{-1}s \end{pmatrix} + \begin{pmatrix} d\log\lambda & 0 \\ 0 & d\log\mu \end{pmatrix}. \end{aligned}$$

Hence

$$\tilde{\omega}_1^2 = \lambda\mu^{-1}(\omega_1^2 - \rho^{-1}q\omega_1^3), \qquad \tilde{\omega}_2^1 = \lambda^{-1}\mu(\omega_2^1 - \sigma^{-1}r\omega_1^3).$$

Similarly,

$$\begin{aligned} \tilde{\omega}_4 &= P\omega_2\Lambda^{-1} + \Lambda\omega_4\Lambda^{-1} + d\Lambda \cdot \Lambda^{-1} \\ &= \begin{pmatrix} p\omega_1^3 & q\omega_2^4 \\ r\omega_1^3 & s\omega_2^4 \end{pmatrix} \begin{pmatrix} \rho^{-1} & \\ & \sigma^{-1} \end{pmatrix} + \begin{pmatrix} \rho\omega_3^3 & \rho\omega_3^4 \\ \sigma\omega_4^3 & \sigma\omega_4^4 \end{pmatrix} \begin{pmatrix} \rho^{-1} & \\ & \sigma^{-1} \end{pmatrix} + \begin{pmatrix} d\log\rho & 0 \\ 0 & d\log\sigma \end{pmatrix}. \end{aligned}$$

Hence

$$\begin{aligned} &\tilde{\omega}_3^3 = \rho^{-1}(p\omega_1^3 + \rho\omega_3^3) + d\log\rho, \quad \tilde{\omega}_3^4 = \sigma^{-1}(q\omega_2^4 + \rho\omega_3^4), \\ &\tilde{\omega}_4^3 = \rho^{-1}(r\omega_1^3 + \sigma\omega_4^3), \quad \tilde{\omega}_4^4 = \sigma^{-1}(s\omega_2^4 + \sigma\omega_4^4) + d\log\sigma. \end{aligned}$$

We define $\tilde{h}_{ij}$ and $\tilde{\phi}_i$ for $\tilde{\omega}$ similarly as was done for ω. Then the formulas above show the following:

$$\begin{aligned} \tilde{h}_{11} &= \rho\mu^{-1}h_{11} - \mu^{-1}q, & \tilde{h}_{21} &= \lambda^{-2}\rho\mu h_{21}, \\ \tilde{h}_{12} &= \sigma\lambda\mu^{-2}h_{12}, & \tilde{h}_{22} &= \sigma\lambda^{-1}h_{22} - \lambda^{-1}r, \\ \tilde{h}_{31} &= \lambda^{-1}\sigma^{-1}\rho^2 h_{31}, & \tilde{h}_{41} &= \lambda^{-1}\sigma h_{41} + \lambda^{-1}r, \\ \tilde{h}_{32} &= \mu^{-1}\rho h_{32} + \mu^{-1}q, & \tilde{h}_{42} &= \rho^{-1}\mu^{-1}\sigma^2 h_{42}, \end{aligned}$$

and

$$\tilde{\phi}_1 = \lambda\sigma^{-1}\phi_1, \qquad \tilde{\phi}_2 = \rho^{-1}\mu\phi_2.$$

Summarizing the above considerations, we have the following covariance relation between normalized frames.

Lemma 6.4 *The change of frames keeping the condition $h_{11} = h_{32} = h_{22} = h_{41} = 0$ has the form*

$$g = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ p & 0 & \rho & 0 \\ 0 & s & 0 & \sigma \end{pmatrix}.$$

The connection forms satisfy the following covariance relations:

$$\begin{aligned} \tilde{\omega}_1^1 &= \omega_1^1 - \rho^{-1} p \omega^1 + d \log \lambda, & \tilde{\omega}_3^3 &= \omega_3^3 + \rho^{-1} p \omega^1 + d \log \rho, \\ \tilde{\omega}_1^2 &= \mu^{-1} \lambda \omega_1^2, & \tilde{\omega}_3^4 &= \sigma^{-1} \rho \omega_3^4, \\ \tilde{\omega}_2^1 &= \lambda^{-1} \mu \omega_2^1, & \tilde{\omega}_4^3 &= \rho^{-1} \sigma \omega_4^3, \\ \tilde{\omega}_2^2 &= \omega_2^2 - \sigma^{-1} s \omega^2 + d \log \mu, & \tilde{\omega}_4^4 &= \omega_4^4 + \sigma^{-1} s \omega^2 + d \log \sigma. \end{aligned} \tag{6.3}$$

6.3 Systems of differential equations of a line congruence

We continue the considerations of Sect. 3.2. We have seen that the congruence $\{z, w\}$ with parameter (x, y) is written by the system of equations:

$$\begin{aligned} z_y &= mw, & z_{xx} &= az + bw + cz_x + dw_y, \\ w_x &= nz, & w_{yy} &= a'z + b'w + c'z_x + d'w_y. \end{aligned} \tag{6.4}$$

So we get

$$\begin{aligned} z_{xy} &= mnz + m_x w, & w_{xy} &= n_y z + mnw, \\ z_{yy} &= m_y w + m w_y, & w_{xx} &= n_x z + n z_x. \end{aligned} \tag{6.5}$$

If we define a projective frame as follows:

$$e_1 = z, \qquad e_2 = w, \qquad e_3 = z_x, \qquad e_4 = w_y,$$

then the associated connection form is

$$\omega = \begin{pmatrix} 0 & mdy & dx & 0 \\ ndx & 0 & 0 & dy \\ adx + mndy & bdx + m_x dy & cdx & ddx \\ a'dy + n_y dx & b'dy + mndx & c'dy & d'dy \end{pmatrix}. \tag{6.6}$$

In particular, we see that

$$\omega^1 = dx, \qquad \omega^2 = dy, \tag{6.7}$$

$$\begin{aligned} h_{11} &= 0, & h_{12} &= m, & h_{31} &= d, & h_{32} &= 0, \\ h_{21} &= n, & h_{22} &= 0, & h_{41} &= 0, & h_{42} &= c', \end{aligned} \tag{6.8}$$

and

$$W = mn - c'd.$$

The last expression coincides with the definition of W given in Sect. 3.2. The invariant quadratic forms are

$$\phi_1 = d(dx)^2 + m(dy)^2, \quad \phi_2 = n(dx)^2 + c'(dy)^2.$$

Compare (3.4) and (3.5) with (6.7) and (6.8). In the following, we assume $mnc'd \neq 0$ (refer to (3.6)) so that both focal surfaces are nondegenerate; we say that such a line congruence is *nondegenerate*.

The integrability condition of the system is nothing but $d\omega = \omega \wedge \omega$. A calculation shows that

$$\omega \wedge \omega = \begin{pmatrix} 0 & m_x & 0 & 0 \\ -n_y & 0 & 0 & 0 \\ (mn)_x - a_y & m_{xx} - b_y & -c_y & -d_y \\ a'_x - n_{yy} & b'_x - (mn)_y & c'_x & d'_x \end{pmatrix} dx \wedge dy.$$

Hence, the integrability condition of the system (6.4) consists of the next eight equations.

$$\begin{aligned} (mn)_x + nm_x - a_y - cmn - a'd = 0, \\ m_{xx} - b_y - am - cm_x - b'd = 0, \\ c_y - mn + c'd = 0, \\ d_y + b + dd' = 0, \\ (mn)_y + mn_y - b'_x - bc' - mnd' = 0, \\ n_{yy} - a'_x - b'n - d'n_y - ac' = 0, \\ d'_x - mn + c'd = 0. \\ c'_x + a' + cc' = 0, \end{aligned} \tag{6.9}$$

From the third and the seventh equations, we get $c_y = d'_x$. This implies that there is a nonvanishing function f so that

$$c = -f_x/f, \qquad d' = -f_y/f. \tag{6.10}$$

In particular,

$$W = mn - c'd = -(\log f)_{xy}. \tag{6.11}$$

If we set

$$\Delta = z \wedge w \wedge z_x \wedge w_y,$$

then a simple calculation shows

$$\Delta_x = c\Delta, \qquad \Delta_y = d'\Delta.$$

Namely, we have $\Delta = 1/f$ from (6.10), up to a constant multiple.

The system (6.4) is not uniquely determined by the given congruence. It has some freedom of choice. One is a change of parameters

$$(1) \qquad (x, y) \longmapsto (\overline{x} = X(x), \overline{y} = Y(y)),$$

and the other is a change of coordinates in $\mathbf{P}^3$

$$(2) \qquad (z, w) \longmapsto (\overline{z} = z/\lambda(x), \overline{w} = w/\mu(y)).$$

We look for dependence of coefficients separately for the two cases. By the change (1), the system is written as follows:

$$
\begin{aligned}
z_{\overline{y}} &= \frac{m}{Y_y} w, \quad & z_{\overline{x}\overline{x}} &= \frac{1}{X_x^2}(az + bw + (cX_x - X_{xx})z_{\overline{x}} + dY_y w_{\overline{y}}), \\
w_{\overline{x}} &= \frac{n}{X_x} z, \quad & w_{\overline{y}\overline{y}} &= \frac{1}{Y_y^2}(a'z + b'w + c'X_x z_{\overline{x}} + (d'Y_y - Y_{yy})w_{\overline{y}}).
\end{aligned}
$$

In particular, the Weingarten invariant $\overline{W}$ of the new system is

$$\overline{W} = \frac{1}{X_x Y_y} W.$$

Under the change (2), the new system relative to $(\overline{z}, \overline{w})$ is

$$
\begin{aligned}
\overline{z}_y &= \frac{\mu}{\lambda} m \overline{w}, \quad \overline{w}_x = \frac{\lambda}{\mu} n \overline{z}, \\
\overline{z}_{xx} &= \left(a + \frac{\lambda_x}{\lambda} c - \frac{\lambda_{xx}}{\lambda}\right) \overline{z} + \left(\frac{\mu}{\lambda} b + \frac{\mu_y}{\lambda} d\right) \overline{w} + \left(c - \frac{2\lambda_x}{\lambda}\right) \overline{z}_x + \frac{\mu}{\lambda} d \overline{w}_y, \\
\overline{w}_{yy} &= \left(\frac{\lambda}{\mu} a' + \frac{\lambda_x}{\mu} c'\right) \overline{z} + \left(b' + \frac{\mu_y}{\mu} d' - \frac{\mu_{yy}}{\mu}\right) \overline{w} + \frac{\lambda}{\mu} c' \overline{z}_x + \left(d' - \frac{2\mu_y}{\mu}\right) \overline{w}_y.
\end{aligned}
$$

It is easy to see that the Weingarten invariant is unchanged. Since $c_y = W$ and $d'_x = -W$, we can solve $c - 2\lambda_x/\lambda = 0$ and $d' - 2\mu_y/\mu = 0$ in the case $W = 0$:

Corollary 6.5 *If $W = 0$, then the system can be reduced to the case $c = d' = 0$.*

By composing the above two kinds of changes, we get

Lemma 6.6 *By the change of variables*

$$(x, y; z, w) \longmapsto (\overline{x} = X(x), \overline{y} = Y(y); \overline{z} = z/\lambda(x), \overline{w} = w/\mu(y)),$$

the coefficients m, n, c' and d are changed as follows:

$$m \longmapsto \frac{\mu}{\lambda Y_y} m, \quad n \longmapsto \frac{\lambda}{\mu X_x} n, \quad c' \longmapsto \frac{\lambda X_x}{\mu Y_y^2} c', \quad d \longmapsto \frac{\mu Y_y}{\lambda X_x^2} d.$$

6.4 The dual of a line congruence

Let $\{z, w\}$ be a line congruence. We denote by T_z (resp. T_w) the tangent space of the surface z (resp. w). Each line of the congruence is represented by the intersection $T_z \cap T_w$. The dual of the plane T_z, which we identify with the vector $z \wedge z_x \wedge z_y$, describes a surface in the dual projective space. Similarly, we have the surface given by $w \wedge w_x \wedge w_y$. Thus we get a congruence consisting of lines connecting two surfaces that are points in the dual projective space. We call this congruence the dual of a given congruence, or more briefly the *dual congruence*. We now give the system of the dual congruence.

Since $z \wedge z_x \wedge z_y = mz \wedge z_x \wedge w$ and $w \wedge w_x \wedge w_y = nw \wedge z \wedge w_y$, and since we have assumed $m \neq 0$ and $n \neq 0$, the dual congruence is well-defined. Then the dual image is described by two vectors

$$U = \frac{1}{n\Delta} w \wedge w_x \wedge w_y = -\frac{1}{\Delta} z \wedge w \wedge w_y,$$
$$V = \frac{1}{m\Delta} z \wedge z_x \wedge z_y = -\frac{1}{\Delta} z \wedge w \wedge z_x.$$

Proposition 6.7 *The system of the dual line congruence $\{U, V\}$ is the following:*

$$\begin{array}{rclrcl} U_y & = & c'V, & U_{xx} & = & (a - c_x)U + (nd' - n_y)V - cU_x + nV_y, \\ V_x & = & dU, & V_{yy} & = & (cm - m_x)U + (b' - d'_y)V + mU_x - d'V_y. \end{array}$$

The Weingarten invariant W^d of the dual congruence is

$$W^d = c'd - mn = -W.$$

Corollary 6.8 *If the dual congruence is projectively equivalent to the original congruence, then the Weingarten invariant vanishes.*

In this case, because of Corollary 6.5, the system can be reduced to

$$\begin{array}{rclrcl} U_y & = & c'V, & U_{xx} & = & aU - n_yV + nV_y, \\ V_x & = & dU, & V_{yy} & = & -m_xU + b'V + mU_x. \end{array} \tag{6.12}$$

7 Linear complexes

Each line of a line congruence can be regarded a point of $\mathbf{P}^5$ through the Plücker embedding, and thus any line congruence can be regarded as a surface in $\mathbf{P}^5$ contained in the quadratic hypersurface determined by the Plücker relation. In this section, we consider such a line congruence in the case that the image surface is contained in a hyperplane.

7.1 Linear complexes

We say that a 3-dimensional family of lines is a *linear complex* when its Plücker image into $\mathbf{P}^5$ lies in a hyperplane.

If $y = [y^1, y^2, y^3, y^4]$ and $z = [z^1, z^2, z^3, z^4]$ are two points on a line, the Plücker image of the line is the vector $y \wedge z$. The homogeneous coordinates of $y \wedge z$ are $[\tau_{12}, \tau_{13}, \tau_{14}, \tau_{23}, \tau_{42}, \tau_{34}]$, where $\tau_{ij} = y^i z^j - y^j z^i$. When they satisfy a linear relation

$$a_{34}\tau_{12} + a_{42}\tau_{13} + a_{23}\tau_{14} + a_{14}\tau_{23} + a_{13}\tau_{42} + a_{12}\tau_{34} = 0,$$

we have an associated vector $\xi = [\xi_1, \xi_2, \xi_3, \xi_4]$ given by

$$\xi = \begin{pmatrix} 0 & -a_{34} & -a_{42} & -a_{23} \\ a_{34} & 0 & -a_{14} & a_{13} \\ a_{42} & a_{14} & 0 & -a_{12} \\ a_{23} & -a_{13} & a_{12} & 0 \end{pmatrix} \begin{pmatrix} y^1 \\ y^2 \\ y^3 \\ y^4 \end{pmatrix}$$

where ξ is also regarded as a column vector. Then, we see that the linear relation is written as $\sum \xi_i z^i = 0$. Hence, ξ can be regarded as the vector defining a hyperplane, on which lies the point z. By definition, the point y itself lies on the hyperplane. This means that, when given a linear complex, for any point y, we can associate a hyperplane through y so that the lines on this hyperplane through y altogether form the linear complex. The correspondence of y to ξ is a linear transformation of the space $\mathbf{P}^3$ to its dual space. Conversely, such a transformation of the above form defines a linear complex.

7.2 The Plücker image of a line congruence

The Plücker image of a line congruence $\{z, w\}$ is identified with

$$\xi = z \wedge w.$$

The mapping $(x, y) \mapsto \xi(x, y)$ defines generally a surface in $\mathbf{P}^5$. Since we know any surface in $\mathbf{P}^5$ can be described by a system of third-order differential equations, we compute the associated system for ξ here.

For simplicity, we set

$$\eta = z \wedge z_x \quad \text{and} \quad \zeta = w \wedge w_y.$$

By using (6.4), we see that

$$\xi_x = z_x \wedge w, \quad \xi_y = z \wedge w_y,$$

and, by (6.4) and (6.5),

$$\begin{aligned} \xi_{xx} &= a\xi + c\xi_x - n\eta - d\zeta, \\ \xi_{xy} &= mn\xi + z_x \wedge w_y, \\ \xi_{yy} &= b'\xi + d'\xi_y + c'\eta + m\zeta. \end{aligned} \tag{7.1}$$

Then differentiating once more, we get third-order relations:

$$\begin{aligned} \xi_{xxx} &= c\xi_{xx} + (a + c_x)\xi_x - 2dn\xi_y + (a_x - bn + dn_y)\xi - (cn + n_x)\eta - d_x\zeta, \\ \xi_{xxy} &= c\xi_{xy} + 2mn\xi_x + a\xi_y + ((mn)_x - cmn)\xi - n_y\eta + b\zeta, \\ \xi_{xyy} &= d'\xi_{xy} + b'\xi_x + 2mn\xi_y + ((mn)_y - mnd')\xi - a'\eta + m_x\zeta, \\ \xi_{yyy} &= d'\xi_{yy} - 2c'm\xi_x + (b' + d'_y)\xi_y + (b'_y - a'm + c'm_x)\xi + c'_y\eta + (m_y + md')\zeta. \end{aligned} \tag{7.2}$$

From the first and third relations of (7.1), we have

$$\begin{aligned}
W\eta &= -m(\xi_{xx} - c\xi_x - a\xi) - d(\xi_{yy} - d'\xi_y - b'\xi),\\
W\zeta &= c'(\xi_{xx} - c\xi_x - a\xi) + n(\xi_{yy} - d'\xi_y - b'\xi),
\end{aligned}$$

where $W = mn - c'd$. Hence, if the Weingarten invariant does not vanish, the equations in (7.2) can be written in closed forms depending only on ξ and thus the system associated with ξ is derived. For later use, we list the system up to second-order terms.

$$\begin{aligned}
\xi_{xxx} &= \left(c + \frac{cmn + mn_x - c'd_x}{W}\right)\xi_{xx} + \frac{(cn + n_x)d - nd_x}{W}\xi_{yy} \pmod{\xi_x, \xi_y, \xi},\\
\xi_{xxy} &= \frac{mn_y + bc'}{W}\xi_{xx} + c\xi_{xy} + \frac{dn_y + bn}{W}\xi_{yy} \pmod{\xi_x, \xi_y, \xi},\\
\xi_{xyy} &= \frac{a'm + c'm_x}{W}\xi_{xx} + d'\xi_{xy} + \frac{a'd + m_x n}{W}\xi_{yy} \pmod{\xi_x, \xi_y, \xi},\\
\xi_{yyy} &= \frac{(m_y + md')c' - c'_y m}{W}\xi_{xx} + \left(d' + \frac{(m_y + md')n - c'_y d}{W}\right)\xi_{yy} \pmod{\xi_x, \xi_y, \xi}.
\end{aligned}$$

If the Weingarten invariant vanishes, ξ must satisfy a second-order differential equation

$$m(\xi_{xx} - c\xi_x - a\xi) + d(\xi_{yy} - d'\xi_y - b'\xi) = 0,$$

which shows Theorem 3.3 again.

Given a line congruence $\{z, w\}$, we next construct the following sets of lines:

$$\begin{aligned}
A_1(x,y) &= \text{the set of lines joining } z(x,y)\\
&\qquad\qquad \text{and any point on the line } \overline{w(x,y)w_y(x,y)},\\
L_1 &= \cup_{x,y} A_1(x,y),\\
A_2(x,y) &= \text{the set of lines joining } w(x,y)\\
&\qquad\qquad \text{and any point on the line } \overline{z(x,y)z_x(x,y)},\\
L_2 &= \cup_{x,y} A_2(x,y).
\end{aligned}$$

For a line congruence $\{z, w\}$, the set of vectors $\{z, w, z_x, w_y\}$ defines a projective frame. We associate to it a moving frame in $\mathbf{P}^5$ by defining

$$\begin{aligned}
\xi_{12} &= z \wedge w, & \xi_{13} &= z \wedge z_x, & \xi_{14} &= z \wedge w_y,\\
\xi_{23} &= w \wedge z_x, & \xi_{24} &= w \wedge w_y, & \xi_{34} &= z_x \wedge w_y.
\end{aligned}$$

The Plücker coordinates p_{ij} of any point of $p \in \mathbf{P}^5$ relative to the frame $\{\xi_{ij}\}$ are defined by setting

$$p = p_{12}\xi_{12} + p_{13}\xi_{13} + p_{14}\xi_{14} + p_{23}\xi_{23} + p_{24}\xi_{24} + p_{34}\xi_{34}. \tag{7.3}$$

The Plučker image of a line in L_1 is given by

$$\eta = z \wedge (w + tw_y),$$

where t is a parameter. We regard η as a point in $\mathbf{P}^5$ with three parameters x, y, and t. By a computation we have

$$\begin{aligned} \eta_x &= z_x \wedge w + tz \wedge w_y + tmnz \wedge w, \\ \eta_y &= tmw \wedge w_y + (1 + td')z \wedge w_y + tb'z \wedge w + tc'z \wedge z_x, \\ \eta_t &= z \wedge w. \end{aligned}$$

Hence the coordinates of these vectors are written as follows:

	p_{12}	p_{13}	p_{14}	p_{23}	p_{24}	p_{34}
η	1	0	t	0	0	0
η_x	tmn	0	0	-1	0	t
η_y	tb'	tc'	$1 + td'$	0	tm	0
η_t	0	0	1	0	0	0

From this table of coordinates, we can see that the hyperplane

$$mp_{13} - c'p_{24} = 0$$

is the unique hyperplane tangent to the set L_1 and including lines in $A_1(x, y)$ at each fixed value (x, y). Similarly, for the set L_2, we get the hyperplane

$$dp_{13} - np_{24} = 0.$$

These two hyperplanes coincide if and only if $W = 0$.

7.3 Line congruences belonging to a linear complex

We next want to consider when the hyperplanes obtained in Sect. 7.2 do not depend on (x, y); namely, when the line congruence belongs to a linear complex.

Let us set

$$\zeta = mp_{13} - c'p_{24},$$

considered as a vector-valued function of (x, y). The condition we need is

$$\zeta_x \equiv 0 \quad \text{and} \quad \zeta_y \equiv 0 \pmod{\zeta}.$$

By differentiating (7.3), we see that $\sum_{i<j}(dp_{ij}\xi_{ij} + p_{ij}d\xi_{ij}) = 0$. If we write $d(\xi_{ij}) = (Mdx + Ndy)(\xi_{ij})$, where M and N are 6×6-matrices, we have

$$p_{ijx} = -\sum_{k\ell} p_{k\ell} M_{ij}^{k\ell} \quad \text{and} \quad p_{ijy} = -\sum_{k\ell} p_{k\ell} N_{ij}^{k\ell}.$$

Since

$$\xi_{12x} = -\xi_{23}, \quad \xi_{13x} = b\xi_{12} + c\xi_{13} + d\xi_{14},$$
$$\xi_{14x} = mn\xi_{12}, \quad \xi_{23x} = -a\xi_{12} + n\xi_{13} + c\xi_{23} + d\xi_{24},$$
$$\xi_{24x} = -n_y\xi_{12} + n\xi_{14},$$
$$\xi_{34x} = -n_y\xi_{13} + a\xi_{14} - mn\xi_{23} + b\xi_{24} + c\xi_{34},$$

we have

$$M = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ b & c & d & 0 & 0 & 0 \\ mn & 0 & 0 & 0 & 0 & 1 \\ -a & n & 0 & c & d & 0 \\ -n_y & 0 & n & 0 & 0 & 0 \\ 0 & -n_y & a & -mn & b & c \end{pmatrix}.$$

Namely, we get

$$p_{13x} = -cp_{13} - np_{23} + n_yp_{34}, \quad p_{24x} = -dp_{23} - bp_{34}.$$

Therefore, we get

$$\zeta_x = (m_x - cm)p_{13} - (mn - c'd)p_{23} - c'_xp_{24} + (mn_y + bc')p_{34},$$

and the condition that $\zeta_x \equiv 0 \pmod{\zeta}$ is equal to

$$W = mn - c'd = 0, \quad mn_y + bc' = 0, \quad \begin{vmatrix} m_x - cm & c'_x \\ m & c' \end{vmatrix} = 0. \tag{7.4}$$

By a similar computation, we have the following:

$$\xi_{12y} = \xi_{14}, \quad \xi_{13y} = m_x\xi_{12} + m\xi_{23},$$
$$\xi_{14y} = b'\xi_{12} + c'\xi_{13} + d'\xi_{14} + m\xi_{24}, \quad \xi_{23y} = -mn\xi_{12} - \xi 34,$$
$$\xi_{24y} = -a'\xi_{12} + c'\xi_{23} + d'\xi_{24},$$
$$\xi_{34y} = -a'\xi_{13} + mn\xi_{14} - b'\xi_{23} + m_x\xi_{24} + d'\xi_{34},$$

and

$$N = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ m_x & 0 & 0 & m & 0 & 0 \\ b' & c' & d' & 0 & m & 0 \\ -mn & 0 & 0 & 0 & 0 & -1 \\ -a' & 0 & 0 & c' & d' & 0 \\ 0 & -a' & mn & -b' & m_x & d' \end{pmatrix}.$$

Hence

$$p_{13y} = -c'p_{14} + a'p_{34}, \quad p_{24y} = mp_{14} - d'p_{24} - m_xp_{34}.$$

Then we see

$$\zeta_y = m_yp_{13} - (c'_y - c'd')p_{24} + (ma' + c'm_x)p_{34},$$

and the condition that $\zeta_y \equiv 0 \pmod{\zeta}$ is equivalent to

$$ma' + c'm_x = 0, \qquad \begin{vmatrix} m_y & c'_y - c'd' \\ m & c' \end{vmatrix} = 0. \tag{7.5}$$

Since $W = 0$ by (7.4), we need not examine the set L_2. Namely, the condition that a line congruence belongs to a linear complex is (7.4) and (7.5).

Theorem 7.1 *A nondegenerate line congruence belongs to a linear complex if and only if the condition*

$$c = d' = 0, \quad c' = m, \quad d = n, \quad b = -n_y, \quad a' = -m_x \tag{7.6}$$

is satisfied. The system (6.4) *then reduces to*

$$\begin{aligned} z_y &= mw, & z_{xx} &= az - n_y w + nw_y, \\ w_x &= nw, & w_{yy} &= -m_x z + b'w + mz_x. \end{aligned}$$

Proof. Since $W = 0$, we may assume that $c = 0$ and $d' = 0$, by Corollary 6.5. Then the two determinants in (7.4) and (7.5) are

$$\begin{vmatrix} m_x & c'_x \\ m & c' \end{vmatrix} = \begin{vmatrix} m_y & c'_y \\ m & c' \end{vmatrix} = 0,$$

from which c'/m is seen to be a constant. By a frame change in Lemma 6.6, the quantity c'/m can be multiplied by $\lambda^2 X_x/\mu^2 Y_y$. Hence we may assume

$$m = c'.$$

Note that by this change the condition $c = d' = 0$, i.e., the condition $\omega_3^3 = \omega_4^4 = 0$ in view of (6.6), is preserved, by the formula (6.3) applied to the case where $p = s = 0$. Thus we have completed the proof.

Remark 7.2 Under the condition (7.6), the dual congruence is

$$\begin{aligned} U_y &= mV, & U_{xx} &= aU + bV + nV_y, \\ V_x &= nU, & V_{yy} &= a'U + b'V + mU_x. \end{aligned}$$

Hence it is autoreciprocal; we refer to (6.12).

Remark 7.3 The argument in this section relies on [W1911]. We refer to S.S. Chern [Ch1936] for some higher order invariants by which the family of quadratic complexes associated to a line congruence is described.

8 Laplace transforms of a line congruence

Let $\{z, w\}$ be a line congruence normalized as in (6.4). The point w lies on the tangent line of the surface z in the direction of the parameter y. This means that the line congruence is one of the tangent congruences of z associated with parameter curves. Hence, the other line congruence determined by the parameter x should be canonically associated to the given one. This section deals with this correspondence.

8.1 Laplace transforms of a line congruence

Suppose we are given a line congruence $\{z, w\}$, which we denote by Γ_0. Let us consider the tangent congruence $\{z_x, z\}$, for which The surface z is by definition one of the focal surfaces. We want to find the other focal surface, which should have the form

$$z_1 = z_x + \alpha z.$$

The α is determined by supposing

$$z_{1y} \equiv 0 \pmod{z, z_1}.$$

Since $z_{1y} = (\alpha_y + mn)z + (\alpha m + m_x)w$ by (6.4), we must have $\alpha = -m_x/m$. In this case, we call the congruence $\Gamma_1 = \{z_1, z\}$ the *Laplace transform* of Γ_0.

We now compute the system associated with Γ_1. Set $w_1 = z/m$. Then

$$\begin{aligned} z_{1y} &= (mn - (\log m)_{xy})mw_1, \\ z_{1x} &= (a - (\log mn)_{xx})z + bw + (c - \frac{m_x}{m})z_x + dw_y, \\ w_{1x} &= \frac{1}{m}z_1, \quad w_{1y} = -\frac{m_y}{m^2}z + w. \end{aligned}$$

From these relations, we can compute the coefficients of the system describing the congruence $\{z_1, w_1\}$. Denoting the coefficients by attaching the subscript 1, we get the following:

$$\begin{aligned} m_1 &= m(mn - (\log m)_{xy}), \\ n_1 &= 1/m, \\ a_1 &= a - (\log fm^2)_{xx} + (\log d/m)_x(\log fm)_x, \\ b_1 &= m(a_x - ad_x/d + bn + dn_y - (\log m)_{xxx}) \\ &\quad +(\log m)_y(b_x - bd_x/d + mnd) \\ &\quad +m_x(a - (\log fm^2)_{xx} + (\log fm)_x(\log d/m)_x) \\ &\quad -m(\log m)_{xx}(\log fm/d)_x, \\ c_1 &= -(\log fm/d)_x, \\ d_1 &= b_x - bd_x/d + dmn, \\ a'_1 &= (\log fm)_x/d, \\ b'_1 &= -(\log m)_{yy} \\ &\quad -\{am - m(\log m)_{xx} + b(\log m)_y - m_x(\log fm)_x\}/d, \\ c'_1 &= 1/d, \\ d'_1 &= -b/d - (\log m)_y. \end{aligned} \tag{8.1}$$

where f was defined in (6.10).

The transform in the inverse direction is $\Gamma_{-1} = \{z_{-1}, w_{-1}\}$ given by

$$z_{-1} = \frac{w}{n}, \qquad w_{-1} = w_y - \frac{n_y}{n}w.$$

Theorem 8.1 ([Dem1911, W1915]) *If the first transform of a W-congruence is a W-congruence, then the same is true for all of its Laplace transforms.*

Proof. Since $W = -(\log f)_{xy}$ by (6.11) and $W_1 = m_1 n_1 - c_1' d_1$ equals to $-(\log m)_{xy} + (\log d/f)_{xy}$ by (8.1) and by the identity $b = -d_y - dd'$ in (6.9), we get

$$W_1 - W = (\log d)_{xy} - (\log m)_{xy}.$$

Further, we get

$$\frac{m_1}{d_1} = \frac{m(mn - (\log m)_{xy})}{d(mn - (\log d/f)_{xy})}.$$

Then, $W_1 = W = 0$ implies that $(\log f)_{xy} = 0$ and $(\log d)_{xy} = (\log m)_{xy}$, which imply the identity $m_1/d_1 = m/d$. This means that we can continue the process while preserving the identity.

Remark 8.2 We can define the sequence of Laplace transforms successively. B. Su [Su1935] and H. Hu [Hu1993] studied and showed interesting results for the case where the sequence is four-times periodic.

8.2 Laplace transforms of a linear complex

The property of belonging to a linear complex reflects a kind of degeneration of line congruences. Wilczynski considered the case where both Γ_0 and Γ_1 belong to linear complexes and derived the sinh-Gordon equation describing such congruences. We will reproduce his computation in this section. The number field in this section is the real field.

The condition that Γ_0 belongs to a linear complex was given in (7.6):

$$c = d' = 0, \quad m = c', \quad n = d, \quad b = -n_y, \quad a' = -m_x.$$

In this case, the invariants of Γ_1 are

$$m_1 = m(mn - (\log m)_{xy}), \quad n_1 = 1/m, \quad c_1' = 1/n, \quad d_1 = n(mn - (\log n)_{xy})$$

and the Weingarten invariant is $W_1 = (\log(n/m))_{xy}$. Since W_1 vanishes when Γ_1 belongs to a linear complex, we can write

$$m/n = \alpha(x)\beta(y)$$

for certain nonvanishing functions α and β. (Recall that we are assuming $m \neq 0$ and $n \neq 0$.) On the other hand, a change of variables from $(x, y; z, w)$ to $(\overline{x}, \overline{y}, \overline{z}, \overline{w})$ that preserves the condition (7.6) is subject to the condition

$$\lambda^2 X_x = \mu^2 Y_y = \text{const},$$

and m/n is multiplied by $\mu^2 X_x / \lambda^2 Y_y$. Hence, we can assume

$$m = \pm n.$$

We have thus reduced the system to a simpler form with coefficients

$$c = d' = 0, \quad m = n = \varphi, \quad c' = d = \epsilon\varphi, \quad a' = -\varphi_x, \quad b = -\varphi_y,$$

with $\epsilon = \pm 1$. Then the coefficients of Γ_1 are computed as follows:

$$\begin{gathered} m_1 = \varphi(\epsilon\varphi^2 - (\log\varphi)_{xy}), \quad n_1 = 1/\varphi, \quad a_1 = a - 2(\log\varphi)_{xx}, \\ b_1 = \varphi a_x - \varphi(\log\varphi)_{xxx} - 2\varphi_x(\log\varphi)_{xx} - \epsilon\varphi_y(\log\varphi)_{xy} + \varphi^2\varphi_y, \\ c_1 = 0, \quad d_1 = \varphi(\varphi^2 - \epsilon(\log\varphi)_{xy}), \quad a_1' = \epsilon\varphi_x/\varphi^2, \\ b_1' = -\varphi_{yy}/\varphi + 2\varphi_y^2/\varphi^2 + \epsilon\varphi_{xx}/\varphi - \epsilon a, \quad c_1' = \epsilon/\varphi, \quad d_1' = 0. \end{gathered}$$

The conditions given in (7.4) and (7.5) so that Γ_1 is a linear complex are seen to be

$$a = \varphi_{xx}/\varphi + Y(y), \quad m_1/c_1' = \epsilon\varphi^2(\epsilon\varphi^2 - (\log\varphi)_{xy}) =: k,$$

where Y is a function of y and k is a constant. Next we check the integrability condition (8.1). It shows that

$$Y' = 0, \quad \varphi(a + \epsilon b') = \varphi_{xx} + \epsilon\varphi_{yy}, \quad b_x' = 4\epsilon\varphi\varphi_y.$$

Hence $Y = \ell$ is a constant and we have

$$(\log\varphi)_{xy} = \epsilon(\varphi^2 - k\varphi^{-2}).$$

The coefficients are

$$\begin{gathered} m = \varphi, \quad n = \epsilon\varphi, \\ a = \frac{\varphi_{xx}}{\varphi} + \ell, \qquad b = -\epsilon\varphi_y, \qquad c = 0, \qquad d = \epsilon\varphi, \\ a' = -\varphi_x, \qquad b' = \frac{\varphi_{yy}}{\varphi} - \epsilon\ell, \qquad c' = \varphi, \qquad d' = 0. \end{gathered}$$

Now it is easy to see that a simple change of variables (x, y) shows that we can assume $k = \pm 1$. The result is summarized in the following theorem.

Theorem 8.3 (E.J. Wilczynski [W1911]) *The congruences Γ_0 and Γ_1 both belong to linear complexes if and only if there exists a nonvanishing function φ satisfying the equation*

$$\frac{\partial^2 \log\varphi}{\partial x \partial y} = \epsilon(\varphi^2 - k\varphi^{-2}),$$

and the line congruence is given by the system

$$\begin{aligned} z_y &= \varphi w, \quad z_{xx} = (\varphi_{xx}/\varphi + \ell)z - \epsilon\varphi_y w + \epsilon\varphi w_y, \\ w_x &= \epsilon\varphi z, \quad w_{yy} = -\varphi_x z + (\varphi_{yy}/\varphi - \epsilon\ell)w + \varphi z_x, \end{aligned}$$

where ℓ is any constant, $\epsilon = \pm 1$ and $k = \pm 1$.

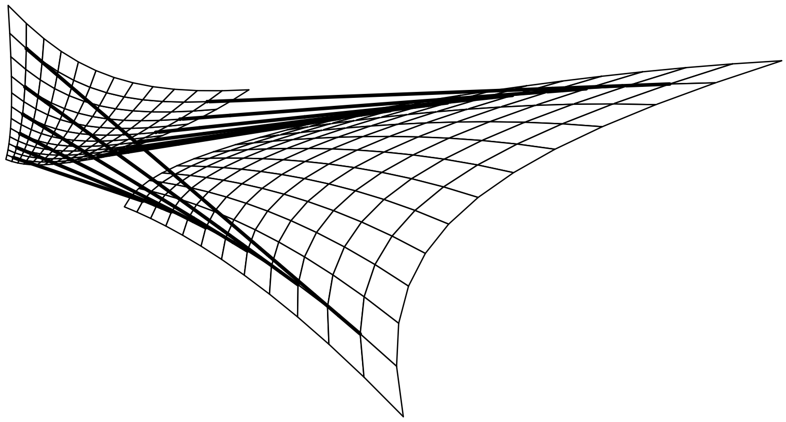

Figure 2: Line congruence between two hyperboloids 1

We call the function φ the *potential* for the pair $\{\Gamma_0, \Gamma_1\}$.

Corollary 8.4 *Assume both Γ_0 and Γ_1 satisfy the condition in the theorem. Let Γ_2 be the (positive) Laplace transform of Γ_1. Then Γ_2 is projectively equivalent to Γ_0. In other words, Γ_0 is doubly periodic.*

Proof. By a direct calculation we can see that the potential of the pair $\{\Gamma_1, \Gamma_2\}$ is equal to $1/\varphi$.

Corollary 8.5 *The congruence Γ_0 is projectively equivalent to Γ_1 if and only if $\varphi = \pm 1$.*

Example 8.6 *The focal surfaces when $\varphi = \pm 1$ are hyperboloids.*

The lines in Fig. 2 are the tangent lines to x-curves of the right surface and are tangent to the left surface along a curve parametrized by x; and the lines in Fig. 3 are the tangent lines to y-curves of the left surface and are tangent to the right surface along a curve parametrized by y. The figures above were drawn by W. Rossman.

9 Invariants of focal surfaces

A line congruence gives a correspondence between two focal surfaces. In this section, we compute such a line congruence when both of the focal surfaces are quadrics. This was first done by Wilczynski and we will follow a part of his description of such congruences. To do this, we need to compute invariants of focal surfaces in terms of invariants of line congruences. We continue to work on the real field.

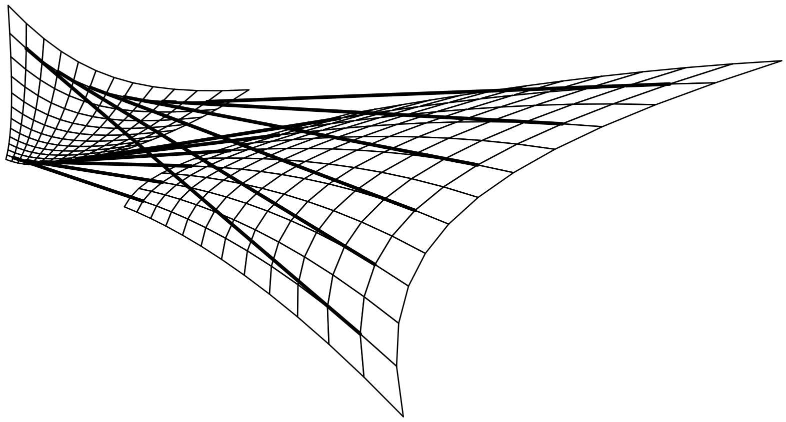

Figure 3: Line congruence between two hyperboloids 2

9.1 Invariants of focal surfaces

Let a projective frame $\{e_1, e_2, e_3, e_4\}$ be a line congruence, as was explained in Section 6. We regard the frame $\{e_1, e_2, e_3, e_4\}$ as a frame associated with the first focal surface e_1. Then by the process introduced in Section 2, we can compute the invariants. Let ω denote the associated coframe; see (6.6):

$$\omega = \begin{pmatrix} 0 & mdy & dx & 0 \\ ndx & 0 & 0 & dy \\ adx + mndy & bdx + m_x dy & cdx & ddx \\ a'dy + n_y dx & b'dy + mndx & c'dy & d'dy \end{pmatrix}.$$

Since it is necessary to normalize the coframe, as required in Proposition 2.1, we consider a change of frame of the form

$$\begin{pmatrix} \overline{e}_1 \\ \overline{e}_2 \\ \overline{e}_3 \\ \overline{e}_4 \end{pmatrix} = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ gr & g & 0 & 0 \\ hs & 0 & h & 0 \\ 0 & \nu p & \nu q & \nu \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix},$$

for which we assume $\lambda g h \nu > 0$. The new coframe $\overline{\omega}$ relative to this frame should satisfy

$$\overline{\omega}^1_1 = 0, \quad \overline{\omega}^2_2 + \overline{\omega}^3_3 = 0, \quad \overline{\omega}^4_4 = 0.$$

Since we get

$$\begin{aligned} d\overline{e}_1 &= (d\log\lambda - rmdy - sdx)\overline{e}_1 + \frac{\lambda m}{g} dy\overline{e}_2 + \frac{\lambda}{h} dx\overline{e}_3, \\ d\overline{e}_2 &= (d(gr) + gndx)e_1 + +(d\log g + (rm - p)dy)(\overline{e}_2 - gre_1) \\ &\qquad + \frac{g}{h}(rdx - qdy)(\overline{e}_3 - hse_1) + \frac{g}{\nu} dy\overline{e}_4, \end{aligned}$$

$$\begin{aligned}
d\overline{e}_3 &= \{d(hs)+h(adx+mndy)\}e_1 \\
&\qquad +\frac{h}{g}\{(b-dp)dx+(m_x+sm)dy\}(\overline{e}_2-gre_1) \\
&\qquad +\{d\log h+(c+s-qd)dx\}(\overline{e}_3-hse_1)+\frac{hd}{\nu}dx\overline{e}_4, \\
d\overline{e}_4 &\equiv \{(\nu qddx+(\nu d'+\nu p)dy+d\nu\}e_4 \pmod{e_1,e_2,e_3},
\end{aligned}$$

the condition we need is the following:

$$\begin{gathered}
d\log\lambda - sdx - rmdy = 0, \\
d\log\nu + qddx + (d'+p)dy = 0, \\
d\log(gh) + (c+s-qd)dx + (rm-p)dy = 0.
\end{gathered} \tag{9.1}$$

By assuming

$$g=\lambda m, \quad h=\lambda,$$

we see that

$$\overline{\omega}^1 := \overline{\omega}^2_1 = dy, \quad \overline{\omega}^2 := \overline{\omega}^3_1 = dx.$$

Then we have

$$\overline{\omega}^4_2 = \frac{\lambda m}{\nu}dy, \qquad \overline{\omega}^4_3 = \frac{\lambda d}{\nu}dx$$

and the fundamental tensor of the new frame is

$$\overline{h} = \frac{\lambda}{\nu}\begin{pmatrix} m & 0 \\ 0 & d \end{pmatrix}.$$

Hence the condition $|\det\overline{h}|=1$ is satisfied when

$$\lambda^2|md| = \nu^2.$$

In particular, the condition (9.1) gives the identity

$$d\log(\lambda^3\nu m) + cdx + d'dy = 0.$$

On the other hand, we know that we can set

$$c = -f_x/f, \quad d' = -f_y/f, \tag{9.2}$$

by (6.10). Then we may assume $\lambda^3 m\nu = f > 0$. In the following, we set $\epsilon_1 = 1$ when $\bar{h}$ is positive definite and $\epsilon_1 = -1$ when it is indefinite. To be more precise, assume $m > 0$ and ϵ_1 denotes the sign of d. And furthermore, let us drop overlines from the notation. Then, from $\epsilon_1\lambda^2 md = \nu^2$ and $\lambda^3 m\nu = f$, we have

$$\begin{gathered}
\lambda = f^{1/4}(\epsilon_1 m^3 d)^{-1/8}, \quad g=\lambda m, \quad h=\lambda, \quad \nu = f^{1/4}(\epsilon_1 md^3)^{1/8}, \\
\omega^4_2 = |m/d|^{1/2}dy, \quad \omega^4_3 = \epsilon_1|d/m|^{1/2}dx.
\end{gathered}$$

From (9.1.1), we have

$$
\begin{aligned}
r &= \frac{1}{m}(\log\lambda)_y = \frac{1}{8m}\left(\frac{2f_y}{f}-\frac{d_y}{d}-\frac{3m_y}{m}\right),\\
s &= (\log\lambda)_x = \frac{1}{8}\left(\frac{2f_x}{f}-\frac{d_x}{d}-\frac{3m_x}{m}\right)
\end{aligned}
$$

and, from (9.1.2),

$$
p = \frac{1}{8}\left(\frac{6f_y}{f}-\frac{3d_y}{d}-\frac{m_y}{m}\right), \quad q = -\frac{1}{8d}\left(\frac{2f_x}{f}+\frac{3d_x}{d}+\frac{m_x}{m}\right). \tag{9.3}
$$

Thus we have a normalized coframe as follows:

$$
\begin{aligned}
\omega_2^1 &= m(dr - r d\log\lambda + n dx + (pr+qs)dy),\\
\omega_3^1 &= ds + (a - rb + rdp - cs - s^2 + qds)dx + (mn - rm_x - rsm)dy,\\
\omega_2^2 &= d\log\lambda m + (rm - p)dy,\\
\omega_2^3 &= m(rdx - qdy),\\
m\omega_3^2 &= (b - pd)dx + (m_x + sm)dy,\\
\omega_3^3 &= d\log\lambda + (c + s - qd)dx,\\
\lambda m\omega_4^2 &= d(\nu p) + \nu(mn + bq)dx + \nu(b' + qm_x)dy,\\
\lambda\omega_4^3 &= d(\nu q) + \nu cq dx + \nu c' dy,\\
\lambda\omega_4^1 &= -r\{d(\nu p) + \nu(bq + mn)dx + \nu(b' + qm_x)dy\}\\
&\quad -s\{d(\nu q) + c\nu q dx + \nu c' dy\}\\
&\quad +\nu\{(n_y + aq + pn)dx + (a' + qmn)dy\}.
\end{aligned}
$$

By setting

$$\mu = |m/d|^{1/2},$$

the fundamental tensor is

$$h_{11} = \mu, \quad h_{12} = h_{21} = 0, \quad h_{22} = \epsilon_1/\mu;$$

namely, the fundamental form φ_2 of the surface e_1 is given by

$$\varphi_2 = \mu dy^2 + \epsilon_1\mu^{-1}dx^2. \tag{9.4}$$

We next compute the cubic tensor h_{ijk}, which is by definition

$$\sum_k h_{ijk}\omega^k = dh_{ij} - \sum_k h_{ik}\omega_{j+1}^{k+1} - \sum_k h_{kj}\omega_{i+1}^{k+1} \quad \text{for } 1 \le i,j,k \le 2.$$

(Note that the indexing of ω differs by 1 from that used in Section 2.) In fact, we have

$$h_{111}dy + h_{112}dx = dh_{11} - 2h_{11}\omega_2^2$$

$$= \quad d\mu - 2\mu(d\log\lambda m + (rm-p)dy)$$

and

$$\begin{aligned} h_{221}dy + h_{222}dx &= dh_{22} - 2h_{22}\omega_3^3 \\ &= d(\epsilon_1/\mu) - (2\epsilon_1/\mu)(d\log\lambda + (c+s-qd)dx). \end{aligned}$$

Hence computation shows that

$$h_{111} = \mu\beta, \quad h_{112} = -\mu\gamma, \quad h_{122} = -\epsilon_1\mu^{-1}\beta, \quad h_{222} = \epsilon_1\mu^{-1}\gamma,$$

where

$$\beta = \frac{1}{4}\left(\log\frac{f^2}{d^3m}\right)_y, \qquad \gamma = \frac{1}{4}\left(\log f^2dm^3\right)_x. \tag{9.5}$$

This means that the cubic form φ_3 of the surface e_1 is given by

$$\varphi_3 = \mu\beta dy^3 - \mu\gamma dxdy^2 - \epsilon_1\mu^{-1}\beta dx^2dy + \epsilon_1\mu^{-1}\gamma dx^3. \tag{9.6}$$

For the second focal surface e_2, the fundamental form ψ_2 and the cubic form ψ_3 of the surface e_2 are given by

$$\psi_2 = \mu_1 dx^2 + \epsilon_2\mu_1^{-1}dy^2, \tag{9.7}$$

$$\psi_3 = \mu_1\beta_1 dx^3 - \mu_1\gamma_1 dydx^2 - \epsilon_2\mu_1^{-1}\beta_1 dy^2dx + \epsilon_2\mu_1^{-1}\gamma_1 dy^3, \tag{9.8}$$

where

$$\begin{gathered} \epsilon_2 = \mathrm{sign}(nc'), \quad \mu_1 = |n/c'|^{1/2}, \\ \beta_1 = \frac{1}{4}\left(\log\frac{f^2}{nc'^3}\right)_x, \quad \gamma_1 = \frac{1}{4}\left(\log f^2n^3c'\right)_y. \end{gathered} \tag{9.9}$$

9.2 Line congruences both of whose focal surfaces are quadrics

We next describe line congruences whose focal surfaces are quadrics. Since the signatures of φ_2 and ψ_2 are important in the following argument, we assume $m > 0$ and $n > 0$, which is possible by Lemma 6.6, and set $\epsilon_1 = \mathrm{sign}(d)$, $\epsilon_2 = \mathrm{sign}(c')$, and $\epsilon = \epsilon_1\epsilon_2$. If $\epsilon = 1$, both of φ_2 and ψ_2 have the same signature and, if $\epsilon = -1$, the signature of φ_2 is opposite to the signature of ψ_2. By Theorem 2.3, and by referring to (9.5) and (9.9), both focal surfaces are quadrics if and only if

$$\begin{aligned} \left(\log\frac{f^2}{d^3m}\right)_y &= 0, \quad \left(\log f^2dm^3\right)_x = 0, \\ \left(\log\frac{f^2}{nc'^3}\right)_x &= 0, \quad \left(\log f^2n^3c'\right)_y = 0. \end{aligned} \tag{9.10}$$

From these relations, we see that

$$(\log md)_{xy} = (\log nc')_{xy} = 0$$

and

$$(\log mn^3c'd^3)_y = 0, \quad (\log m^3nc'^3d)_x = 0.$$

Then, for some positive-valued scalar functions $X_1(x)$, $X_2(x)$, $X_3(x)$, $Y_1(y)$, $Y_2(y)$, $Y_3(y)$, we may set

$$md = \epsilon_1 X_1^2 Y_1^2, \quad nc' = \epsilon_2 X_2^2 Y_2^2,$$

and

$$mn^3c'd^3 = \epsilon X_3(x), \quad m^3nc'^3d = \epsilon Y_3(y).$$

Hence, we have

$$X_3 = k^2(X_1X_2)^8, \quad Y_3 = \frac{1}{k^2}(Y_1Y_2)^8,$$

for some positive constant k. We next define φ by

$$m = Y_1^3 Y_2 \varphi.$$

Then we have

$$n = kX_1X_2^3\varphi, \quad d = \frac{\epsilon_1 X_1^2}{Y_1Y_2\varphi}, \quad c' = \frac{\epsilon_2 Y_2^2}{kX_1X_2\varphi}.$$

Now, in view of Lemma 6.6, by a change of variables from $(x, y; z, w)$ to $(\overline{x} = X(x), \overline{y} = Y(y); \overline{z} = z/\lambda(x), \overline{w} = w/\mu(y))$, the ratios m/n and d/c' become $\mu^2 X_x Y_1^3 Y_2/(k\lambda^2 Y_y X_1 X_2^3)$ and $\epsilon k \mu^2 Y_y^3 X_1^3 X_2/(\lambda^2 X_x^3 Y_1 Y_2^3)$ respectively. This shows that we may assume $m = n$ and $c' = \epsilon d$ by solving the equations $X_x/\lambda^2 = kX_1X_2^3$, $\lambda^2 X_x^3 = kX_1^3X_2$, $Y_y/\mu^2 = Y_1^3Y_2$ and $\mu^2 Y_y^3 = Y_1Y_x^3$. Then we can suppose that X_1, X_2, Y_1 and Y_2 are constants. Namely, we have reduced to the case $m = n = \varphi$ and $\epsilon_2 c' = \epsilon_1 d = k_1/\varphi$ for some constant k_1. Applying Lemma 6.6 once more, we see that we may assume $k_1 = 1$:

$$m = n = \varphi, \quad d = \epsilon_1/\varphi, \quad c' = \epsilon_2/\varphi.$$

In this case, (9.10) shows that $f\varphi$ is constant, and from (9.2) we see that

$$c = \varphi_x/\varphi, \quad d' = \varphi_y/\varphi.$$

Then a check of the integrability (6.9) implies that

$$a' = 0, \quad b = 0, \quad a_y = 2\varphi\varphi_x, \quad b'_x = 2\varphi\varphi_y,$$

$$(\log\varphi)_{xx} = a + \frac{\epsilon_1 b'}{\varphi^2}, \quad (\log\varphi)_{yy} = b' + \frac{\epsilon_2 a}{\varphi^2}$$

and

$$(\log\varphi)_{xy} = \varphi^2 - \frac{\epsilon}{\varphi^2}.$$

Then we have the following theorem due to Wilczynski.

Theorem 9.1 *Assume both focal surfaces are quadrics. Then there exists a nonvanishing function φ such that*

$$m = n = \varphi, \quad d = \frac{\epsilon_1}{\varphi}, \quad c' = \frac{\epsilon_2}{\varphi}.$$

Unless $\varphi^2 = 1$ and $\epsilon = 1$, it must satisfy the following system of equations:

$$\begin{aligned}(\log\varphi)_{xy} &= \varphi^2 - \epsilon\varphi^{-2},\\ 2\varphi\varphi_x &= ((\varphi^2(\log\varphi)_{xx} - \epsilon_1(\log\varphi)_{yy})/(\varphi^2 - \epsilon\varphi^{-2}))_y,\\ 2\varphi\varphi_y &= ((\varphi^2(\log\varphi)_{yy} - \epsilon_2(\log\varphi)_{xx})/(\varphi^2 - \epsilon\varphi^{-2}))_x.\end{aligned}$$

The coefficients of the congruence are given by

$$\begin{aligned}a &= (\varphi^2(\log\varphi)_{xx} - \epsilon_1(\log\varphi)_{yy})/(\varphi^2 - \epsilon\varphi^{-2}),\\ b' &= (\varphi^2(\log\varphi)_{yy} - \epsilon_2(\log\varphi)_{xx})/(\varphi^2 - \epsilon\varphi^{-2}),\\ b &= 0, \quad c = \varphi_x/\varphi, \quad a' = 0, \quad d' = \varphi_y/\varphi.\end{aligned}$$

Remark 9.2 For the line congruence above to be a W-congruence, the condition that $\epsilon = 1$ and $\varphi^2 = 1$ is necessary. When $\varphi = 1$ and $\epsilon = 1$, a and b' are seen to be constants and must satisfy $a + b' = 0$. The system is then written as

$$z_y = w, \quad w_x = z, \quad z_{xx} = az + w_y, \quad w_{yy} = -aw + z_x,$$

where a is a constant and this case is thus included in Theorem 8.3. We refer to Sect. 10.2. Getting solutions of the above system in Theorem 9.1 is an open problem.

10 Construction of W-congruences

G. Fubini and E. Čech [FC1] gave a method for constructing W-congruences with a given surface as one of the focal surfaces. Since this method is fundamental for discussing W-congruences, we give a summary of the related computations. One application is the construction of all W-congruences whose focal surfaces are quadrics. Sect. 10.3 treats the composition formula of two W-congruences, starting from a given surface.

10.1 A description of W-congruences

Let the surface z be given by the system

$$z_{xx} = \theta_x z_x + b z_y + pz, \quad z_{yy} = c z_x + \theta_y z_y + qz.$$

We consider a family of curves on the surface infinitesimally written as $Ady - Bdx = 0$, where A and B are nonvanishing functions on the surface. Let

$$w = \mu z + 2(Az_x + Bz_y) \tag{10.1}$$

be any point on the tangent line of curves of the given family, where μ is a scalar parameter on the line. We are going to determine μ so that the point w is the focal point. We first check the condition that the point z lies on the tangent plane at w of the second surface, *i.e.*, the condition $z \wedge w \wedge w_x \wedge w_y = 0$. Since

$$\begin{aligned} w_x &= (\mu + 2A_x + 2A\theta_x)z_x + 2(B_x + bA)z_y + 2Bz_{xy} + (\mu_x + 2pA)z, \\ w_y &= 2(A_y + cB)z_x + (\mu + 2B_y + 2B\theta_y)z_y + 2Az_{xy} + (\mu_y + 2qB)z, \end{aligned}$$

the condition is seen to be the identity

$$\mu = -A_x - B_y - A\theta_x - B\theta_y + \frac{B(A_y + cB)}{A} + \frac{A(B_x + bA)}{B}. \tag{10.2}$$

Furthermore, a computation shows that

$$\begin{aligned} w_x &= \frac{1}{B}(B_x + bA)w + Ez - \lambda z_x + 2Bz_{xy}, \\ w_y &= \frac{1}{A}(A_x + cB)w + Fz + \lambda z_y + 2Az_{xy}, \end{aligned}$$

where

$$\begin{aligned} \lambda = -A_x + B_y - \theta_x A + \theta_y B + \frac{A}{B}(B_x + bA) - \frac{B}{A}(A_y + cB), \\ E = \mu_x + 2pA - \mu\frac{B_x + bA}{B}, \\ F = \mu_y + 2qB - \mu\frac{A_y + cB}{A}. \end{aligned}$$

In particular,

$$\begin{aligned} Aw_x - Bw_y &= (A\mu_x - B\mu_y + 2A^2p - 2B^2q)z \\ &\quad +(A\mu + 2AA_x - 2BA_y - 2cB^2 + 2A^2\theta_x)z_x \\ &\quad -(B\mu + 2BB_y - 2AB_x - 2bA^2 + 2B^2\theta_y)z_y. \end{aligned}$$

Hence, the identity (10.2) implies that the vector $Aw_x - Bw_y$ is pointing in the direction of the lines of the congruence. In fact, if we set

$$N = \lambda\mu + 2AE - 2BF, \tag{10.3}$$

then we get a relation which is the reverse of (10.1):

$$Nz = -\nu w + 2(Aw_x - Bw_y),$$

where

$$\nu = -B_y - \theta_y B + A_x + \theta_x A + \frac{A}{B}(B_x + bA) - \frac{B}{A}(A_y + cB).$$

We assume $N \neq 0$ in the following so that w is nondegenerate.

Proposition 10.1 *The line congruence joining z and w is a W-congruence if and only if*

$$\left(\frac{A_y + cB}{A}\right)_x = \left(\frac{B_x + bA}{B}\right)_y. \tag{10.4}$$

Proof. If the congruence is a W-congruence, then the x-curves are asymptotic curves of the surface w, namely it holds that $w \wedge w_x \wedge w_y \wedge w_{xx} = 0$. By a computation, this is seen to be equivalent to (10.4). The symmetry of (10.4) implies that the y-curves are also asymptotic.

We remark that the conformal structure of w is given by the 2-form

$$N(4\alpha(Bdx + Ady)^2 - 2Ndxdy),$$

where

$$\alpha = \left(\frac{A_y + cB}{A}\right)_x - \left(\frac{B_x + bA}{B}\right)_y.$$

We next compute the system defining the surface w, assuming that the congruence is a W-congruence. The condition (10.4) shows the existence of a function φ such that

$$\frac{A_y + cB}{A} = \frac{\varphi_y}{\varphi}, \quad \frac{B_x + bA}{B} = \frac{\varphi_x}{\varphi}.$$

Replacing A/φ and B/φ with A and B, the condition (10.4) can be reduced to

$$A_y + cB = 0, \qquad B_x + bA = 0. \tag{10.5}$$

In this case

$$\begin{aligned} \mu &= -A_x - B_y - \theta_x A - \theta_y B, \\ \lambda &= -A_x + B_y - \theta_x A + \theta_y B, \end{aligned}$$

and

$$Nz = \lambda w + 2Aw_x - 2Bw_y. \tag{10.6}$$

From the identities

$$\begin{aligned} w_x &= (\mu_x + 2Ap)z - \lambda z_x + 2Bz_{xy}, \\ w_y &= (\mu_y + 2Bq)z + \lambda z_y + 2Az_{xy} \end{aligned} \tag{10.7}$$

we know that the system of differential equations for the surface w is

$$\begin{aligned} w_{xx} &= \overline{\theta}_x w_x + \overline{b} w_y + \overline{p} w, \\ w_{yy} &= \overline{c} w_x + \overline{\theta}_y w_y + \overline{q} w, \end{aligned} \tag{10.8}$$

where

$$\begin{aligned} \overline{\theta}_x = \theta_x + (\log N)_x, &\quad \overline{\theta}_y = \theta_y + (\log N)_y, \\ \overline{b} = -b - \frac{B}{A}(\log N)_x, &\quad \overline{c} = -c - \frac{A}{B}(\log N)_y, \\ \overline{p} = p + b_y + b\theta_y &+ \frac{\lambda}{2A}(\log N)_x, \\ \overline{q} = q + c_x + c\theta_x &- \frac{\lambda}{2B}(\log N)_y. \end{aligned} \tag{10.9}$$

Remark 10.2 By using Lemma 10.4 proven later, we have a remarkable identity:

$$\overline{b}\overline{c} = bc - (\log N)_{xy}. \tag{10.10}$$

For later use, we set

$$\mathcal{L} = AA_{xx} - \frac{1}{2}A_x^2 + A^2 L, \quad \mathcal{M} = BB_{yy} - \frac{1}{2}B_y^2 + B^2 M, \tag{10.11}$$

then

$$N = 2(\mathcal{M} - \mathcal{L}). \tag{10.12}$$

Let us make a remark on the case when N is a constant. In this case, the system satisfied by w is nothing but the dual of the system satisfied by z. Hence, by recalling the reasoning in Sect. 7.1, we see that the line congruence $\{z, w\}$ belongs to a linear complex.

10.2 W-congruences whose focal surfaces are quadrics

Let us consider the special case where the surface z is a quadric and the congruence is a W-congruence. Since $b = c = 0$ (and hence we may assume $p = q = 0$ and θ is constant), the condition (10.4) implies that $A = X$ is a function of x and $B = Y$ is a function of y. Then we have

$$\begin{aligned} \mu &= -X' - Y', \quad \lambda = -X' + Y', \\ N &= X'^2 - Y'^2 - 2XX'' + 2YY'', \\ \overline{b} &= 2YX'''/N, \quad \overline{c} = 2XY'''/N. \end{aligned}$$

Here $\{'\}$ means derivation relative to the respective variable. For $\overline{b} = \overline{c} = 0$, it is necessary and sufficient that $X'''(x) = Y'''(y) = 0$ because A and B are assumed to be not zero.

Theorem 10.3 *When X and Y are polynomials of degree at most two, the surface w is also a quadric.*

Assume that X and Y are polynomials of degree at most two; in particular, N is constant. Then the system for w is

$$w_{xx} = \theta_x w_x + pw, \qquad w_{yy} = \theta_y w_y + qw,$$

which means that the surface w satisfies the same system as for z; geometrically, w is a projective transformation of z. An explicit correspondence is given below:

Assume, for simplicity, that $p = q = 0$ and θ is constant; the surface z is parametrized as $z = [1, x, y, xy]$ in homogeneous coordinates. Set

$$X = p_1 x^2 + 2p_2 x + p_3 \quad \text{and} \quad Y = q_1 y^2 + 2q_2 y + q_3.$$

Then the surface w is given by

$$\begin{aligned} w &= [-p_2 - q_2 - p_1 x - q_1 y, p_3 + (p_2 - q_2)x - q_1 xy, \\ &\qquad q_3 + (q_2 - p_2)y - p_1 xy, q_3 x + p_3 y + (p_2 + q_2)xy]. \end{aligned}$$

Namely,

$$w = gz; \qquad g = \begin{pmatrix} -p_2 - q_2 & -p_1 & -q_1 & 0 \\ p_3 & p_2 - q_2 & 0 & -q_1 \\ q_3 & 0 & q_2 - p_2 & -p_1 \\ 0 & q_3 & p_3 & p_2 + q_2 \end{pmatrix},$$

where $\det g = p_2^2 - q_2^2 - p_1 p_3 + q_1 q_3$. To simplify the representation of the congruence, we introduce a new parametrization of the surface by defining new coordinates (ξ, η) by

$$\partial_\xi = X\partial_x + Y\partial_y, \qquad \partial_\eta = (X\partial_x - Y\partial_y)/\det(g).$$

We set

$$\overline{z} = \rho z, \qquad \overline{w} = \rho w, \qquad \text{where} \quad \rho = (XY)^{-1/2}.$$

Then it is easy to see that the congruence is written as

$$\overline{w} = \partial_\xi \overline{z}, \qquad \overline{z} = \partial_\eta \overline{w}.$$

Thus the parametrization by (ξ, η) defines a net on the surface associated to the congruence. In this way, we get all W-congruences whose focal surfaces are quadrics.

We add a few remarks. When one and only one of X and Y is a polynomial of degree at most two, then the surface w is ruled. Thus, we get W-congruences joining a quadratic surface and a ruled surface.

The second remark is on the case where w is not ruled: $\overline{bc} \neq 0$. Then

$$(\log \overline{b})_{xy} = -(\log N)_{xy} = N_x N_y / N = \overline{bc}.$$

Similarly, $(\log \overline{c})_{xy} = \overline{bc}$. Hence, by Proposition 2.12, both parameter curves on w belong to respective linear complexes. Conversely, given a surface both of whose parameter curves belong to linear complexes, namely when the condition (2.32) is holding, we can construct a W-congruence joining the surface and a quadratic surface. We refer to [FC2, §47] for details.

10.3 Composition of W-congruences

Given two W-congruences w_1 and w_2, each given by

$$w_i = \mu_i z + 2(A_i z_x + B_i z_y), \quad i = 1, 2,$$

Fubini constructed a third surface z that is joined with w_1 and w_2 by respective W-congruences. Here, we cite his construction.

Let λ_i, μ_i and N_i be the corresponding quantities and define

$$\begin{array}{ll} A = c^1 A_1 + c^2 A_2, & B = c^1 B_1 + c^2 B_2, \\ \lambda = c^1 \lambda_1 + c^2 \lambda_2, & \mu = c^1 \mu_1 + c^2 \mu_2, \end{array}$$

where c^1 and c^2 are constants, and define N by the formula (10.3). Since N is quadratic in A and B, it is possible to set

$$N = (c^1)^2 N_1 + 2c^1 c^2 N_{12} + (c^2)^2 N_2,$$

by appropriately defining N_{12}. In fact, we have

$$N_{12} = \frac{1}{2}(\lambda_1\mu_2 + \lambda_2\mu_1) + 4(A_1A_2p - B_1B_2q) + (A_1\mu_{2x} + A_2\mu_{1x} - B_1\mu_{2y} - B_2\mu_{1y}).$$

Further, a computation shows that

$$\frac{N_x}{A} = \mu_{xx} - \theta_x\mu_x + b\mu_y - \mu(b_y + b\theta_y) + 4Bbq + 4A_xp + 2Ap_x + 2Bp_y,$$

$$-\frac{N_y}{A} = \mu_{yy} - \theta_v\mu_v + c\mu_x - \mu(c_x + c\theta_x) + 4Acp + 4B_yq + 2Bq_y + 2Aq_x,$$

and note that the right-hand sides are linear in A, B and μ. Hence, we have

$$\frac{N_x}{2A} = c^1\frac{N_{1x}}{2A_1} + c^2\frac{N_{2x}}{2A_2}.$$

From this follows

$$(N_{12})_x = \frac{A_2}{2A_1}N_{1x} + \frac{A_1}{2A_2}N_{2x}.$$

Similarly,

$$(N_{12})_y = \frac{B_2}{2B_1}N_{1y} + \frac{B_1}{2B_2}N_{2y}.$$

Lemma 10.4 *N satisfies the equation*

$$N_{xy} + \frac{cB}{A}N_x + \frac{bA}{B}N_y = 0. \qquad (10.13)$$

The proof will be given in Sect. 13.1 (see Lemma 13.1). We now look for a scalar function f and g so that

$$f_x = \frac{A_2}{A_1}N_{1x}, \quad f_y = \frac{B_2}{B_1}N_{1y}; \quad g_x = \frac{A_1}{A_2}N_{2x}, \quad g_y = \frac{B_1}{B_2}N_{2y}.$$

From (10.5) and (10.13),

$$\begin{aligned} (f_x)_y &= \frac{A_{2y}}{A_1}N_{1x} - \frac{A_2A_{1y}}{A_1^2}N_{1x} + \frac{A_2}{A_1}N_{1xy} \\ &= -\frac{cB_2}{A_1}N_{1x} - \frac{bA_2}{B_1}N_{1y}, \end{aligned}$$

and the same holds for $(f_y)_x$. Hence, it is possible to find f by integration. Similarly, we can find g. By definition, $f + g - N_{12}$ is constant, and we may assume it is equal to 0. Now we define two new surfaces

$$\begin{aligned} Z_{12} &= fz - \lambda_2 w_1 + 2(-A_2 w_{1x} + B_2 w_{1y}), \\ Z_{21} &= gz - \lambda_1 w_2 + 2(-A_1 w_{2x} + B_1 w_{2y}). \end{aligned}$$

Inserting the derivations of w_1 and w_2 given in (10.7), we get

$$\begin{aligned} Z_{12} &= (f - \lambda_2\mu_1 - 2A_2\mu_{1x} - 4A_1A_2p + 2B_2\mu_{1y} + 4B_1B_2q)z \\ &\quad +2(\lambda_1A_2 - \lambda_2A_1)z_x + 2(\lambda_1B_2 - \lambda_2B_1)z_y + 4(B_2A_1 - A_2B_1)z_{xy}, \\ Z_{21} &= (g - \lambda_1\mu_2 - 2A_1\mu_{2x} - 4A_1A_2p + 2B_1\mu_{2y} + 4B_1B_2q)z \\ &\quad +2(\lambda_2A_1 - \lambda_1A_2)z_x + 2(\lambda_2B_1 - \lambda_1B_2)z_y + 4(B_1A_2 - A_1B_2)z_{xy}. \end{aligned}$$

Therefore,

$$Z_{12} + Z_{21} = (f + g - 2N_{12})z = 0.$$

Now we write Z for Z_{12} and we show that Z is one of the required surfaces. In fact, by setting

$$\overline{\mu} = -\lambda_2 + \frac{\lambda_1 a}{N_1}, \quad \overline{A} = -A_2 + \frac{A_1 a}{N_1}, \quad \overline{B} = B_2 - \frac{B_1 a}{N_1},$$

where a is a constant of integration relative to f, we have

$$Z = \overline{\mu} w_1 + 2(\overline{A} w_{1x} + \overline{B} w_{1y}); \tag{10.14}$$

see (10.6). If we denote by $\overline{\theta}$, b_1 and c_1 the invariants θ, b and c for w_1, then we see by a simple calculation that

$$\overline{A}_y = -c_1\overline{B}, \quad \overline{B}_x = -b_1\overline{A}$$

and

$$\overline{A}_x + \overline{\theta}_x\overline{A} + \overline{B}_y + \overline{\theta}_y\overline{B} = \lambda_2 - \lambda_1 a/N_1.$$

Hence, we have proved the following theorem.

Theorem 10.5 *Let $\{w_1, z\}$ and $\{w_2, z\}$ be two W-congruences. Define a third surface Z by* (10.14). *Then, both $\{Z, w_1\}$ and $\{Z, w_2\}$ are W-congruences. The constants c^1 and c^2 are arbitrary and the choice of f and g includes a constant of integration.*

11 Lie quadrics and Demoulin transforms

In Sect. 2.4 we have defined the Demoulin frame associated with a surface and the Demoulin transform of the surface. In this section we revisit these notions, following the development by S. Finikow and O. Mayer. We first recall how to attach a quadric called the Lie quadric to each point of the surface, and then prove that the envelope of Lie quadrics generally consists of four surfaces, each being a Demoulin transform. We next define Demoulin congruences joining Demoulin transforms and prove that the Demoulin congruence is a W-congruence if and only if the original surface is projectively minimal. In the last section, we compute explicitly the invariants of the Demoulin transform.

11.1 Osculating quadrics

Let $z = z_1(x) + yz_2(x)$ be a ruled surface given by the normalized system (1.6). We now define an osculating quadric to this ruled surface.

Given a set of skew three lines on the ruled surface, we generally have a quadric including these lines. If these lines tend to one limit line, then the quadric tends to a limit position, again a quadric. This is called an *osculating quadric*, which might be degenerate. Assume that the limit quadric Q is defined by the equation ${}^t zAz = 0$. We denote ${}^t zAz = \langle z, z\rangle$ for simplicity. The limit line is in Q if and only if

$$\langle z_1, z_1\rangle = \langle z_1, z_2\rangle = \langle z_2, z_2\rangle = 0.$$

The limit process implies that the first derivatives of the limiting lines are also in Q:

$$\langle z_1, z_1'\rangle = \langle z_1, z_2'\rangle + \langle z_1', z_2\rangle = \langle z_2, z_2'\rangle = 0.$$

Differentiating a second time implies

$$\langle z_1', z_1'\rangle = \langle z_1', z_2'\rangle = \langle z_2', z_2'\rangle = 0,$$

in view of (1.6). If we write a general point z in $\mathbf{P}^3$ as $w = p^0 z_1 + p^1 z_2 + p^2 z_1' + p^3 z_2'$, then

$$\langle w, w\rangle = 2(p^0p^3 - p^1p^2)\langle z_1, z_2'\rangle.$$

Hence the osculating quadric is defined by the equation $p^0p^3 - p^1p^2 = 0$. A part of the condition $\langle z_1, z_1\rangle = \langle z_1, z_1'\rangle = \langle z_1', z_1'\rangle = 0$ implies that the asymptotic tangent belongs to the osculating quadric. Since this property does not depend on y, all asymptotic lines through the limit ruling line give a ruling of the osculating quadric.

11.2 Lie quadrics

Let $z(x, y)$ be a nondegenerate surface:

$$\begin{aligned} z_{xx} &= \theta_x z_x + bz_y + pz, \\ z_{yy} &= cz_x + \theta_y z_y + qz. \end{aligned} \tag{11.1}$$

We consider one of the associated ruled surfaces

$$w(x, s) = z_y(x, y_0) + sz(x, y_0),$$

which consists of tangent lines to y-curves parametrized by x. The osculating quadric to this ruled surface is determined by the asymptotic directions, as was remarked in the previous section.

Let $s = s(x)$ be such an asymptotic curve. Since

$$w_x = z_{xy} + s'z + sz_x,$$
$$w_{xx} = z_{xxy} + s''z + 2s'z_x + sz_{xx},$$
$$z_{xxy} = \theta_x z_{xy} + (bc + \theta_{xx})z_x + (b\theta_y + b_y + p)z_y + (bq + p_y)z,$$

and then

$$w_{xx} \equiv (bc + 2s' + \theta_{xy})z_x \qquad (\text{mod } z, z_y, w_x)$$

in order that $w_{xx} \equiv 0 \pmod{()z, z_y, w_x}$, it is necessary that

$$bc + 2s' + \theta_{xy} = 0.$$

Hence we have

$$w_x = z_{xy} + sz_x - (1/2)(bc + \theta_{xy})z.$$

The asymptotic direction is written as

$$w_x + tw = z_{xy} + sz_x + tz_y + (st - (1/2)(bc + \theta_{xy}))z,$$

where t is a line parameter. If we write any point in $\mathbf{P}^3$ as $p^0 z + p^1 z_x + p^2 z_y + p^3 z_{xy}$, then the quadric consisting of the asymptotic directions above is

$$p^0 p^3 - p^1 p^2 = -(1/2)(bc + \theta_{xy})(p^3)^2. \tag{11.2}$$

(If the original surface is ruled, then this coincides with the osculating quadric. In fact, $bc = 0$ for a ruled surface, and θ is constant for a ruled surface written in the form (1.6).) This quadric is called a *Lie quadric*, which is the same as the Lie quadratic hypersurface given in Sect. 2.2 in view of the frame (2.14).

11.3 Demoulin Transforms

To each point of the surface is associated a Lie quadric. The envelope of Lie quadrics generally consists of four surfaces. We shall see that these four surfaces are nothing but the Demoulin transforms of the original surface, defined in Sect. 2.4.

We assume that $bc \neq 0$ so that the surface is not ruled. Any Lie quadric is of the form

$$w = z_{xy} + sz_x + tz_y + (st - \alpha)z, \quad \alpha = \frac{1}{2}(bc + \theta_{xy}).$$

In order that w belongs to the envelope surface, its tangent vector, say w_x, belongs to the tangent plane of the Lie quadric. By a direct computation, we have

$$\begin{aligned} w_x \;=\; & (t + \theta_x)w + (\alpha + s_x)(z_x + tz) \\ & \quad + (b\theta_y + b_y + p + t_x + bs - t^2 - t\theta_x)(z_y + sz) + Bz, \end{aligned}$$

where

$$B = (bq + p_y - \alpha_x + \alpha\theta_x) - (b_y + b\theta_y)s - bs^2.$$

Since the tangent plane of the Lie quadric is spanned by the vectors w, $w_s = z_x + tz$, and $w_t = z_y + sz$, the required condition for the vector w_x is $B = 0$; namely,

$$bs^2 + (b_y + b\theta_y)s - (bq + p_y - \alpha_x + \alpha\theta_x) = 0. \tag{11.3}$$

Similarly, for w_y we have

$$\begin{aligned} w_y &= (s+\theta_y)w + (\alpha + t_y)(z_y + sz) \\ &\quad + (c_x + c\theta_x + q + s_y + ct - s^2 - s\theta_y)(z_x + tz) + Cz, \end{aligned}$$

where

$$C = (cp + q_x - \alpha_y + \alpha\theta_y) - (c_x + c\theta_x)t - ct^2.$$

The required condition for the vector w_y is $C = 0$; namely,

$$ct^2 + (c_x + c\theta_x)t - (cp + q_x - \alpha_y + \alpha\theta_y) = 0. \tag{11.4}$$

Hence the envelope surfaces are given by solving two equations (11.3) and (11.4); generally, we have four solutions.

We denote the discriminants of both equations (11.3) and (11.4) by Δ_1 and Δ_2, respectively. They are given as follows, in view of integrability:

$$\Delta_1 = b_y^2 - 2bb_{yy} - 2b^2M, \qquad \Delta_2 = c_x^2 - 2cc_{xx} - 2c^2L. \tag{11.5}$$

Referring to (2.10), we can see that these discriminants are the same as those defined by (2.19). Since we have not assumed that θ is constant, we need to replace p and q in (2.10) by $p - \theta_{xx}/2 + \theta_x^2/4 + b\theta_x/2$ and $q - \theta_{yy}/2 + \theta_y^2/4 + c\theta_y/2$.

Note that

$$\Delta_{1y} = -2b(b_{yyy} + 2Mb_y + bM_y), \quad \Delta_{2x} = -2c(c_{xxx} + 2Lc_x + cL_x), \tag{11.6}$$

hence the third integrability condition of (2.9) implies

$$c\Delta_{1y} = b\Delta_{2x},$$

and conversely, this identity is equivalent to the third integrability condition, provided $bc \neq 0$.

To keep the notations P and Q defined in (2.18), in accordance with the case where θ is not necessarily constant, we need to change the formulas to

$$P = p + \frac{b_y}{2} - \frac{c_{xx}}{2c} + \frac{c_x^2}{4c^2} - \frac{\theta_{xx}}{2} + \frac{1}{4}\theta_x^2 + \frac{b\theta_y}{2}$$

and

$$Q = q + \frac{c_x}{2} - \frac{b_{yy}}{2b} + \frac{b_y^2}{4b^2} - \frac{\theta_{yy}}{2} + \frac{1}{4}\theta_y^2 + \frac{c\theta_x}{2}. \tag{11.7}$$

We refer to (2.11). For the benefit of later use, we introduce the notations σ and τ by

$$\sigma^2 = \frac{\Delta_1}{4b^2} = Q, \qquad \tau^2 = \frac{\Delta_2}{4c^2} = P; \tag{11.8}$$

we refer to (2.24). Then

$$s = -\frac{b_y}{2b} - \frac{1}{2}\theta_y \pm \sigma, \qquad t = -\frac{c_x}{2c} - \frac{1}{2}\theta_x \pm \tau \tag{11.9}$$

and the Demoulin transform w is written as

$$\begin{aligned} w &= z_{xy} + s z_x + t z_y + (st - \frac{1}{2}bc)z \\ &= z_{xy} - \left(\frac{b_y}{2b} + \frac{1}{2}\theta_y\right) z_x - \left(\frac{c_x}{2c} + \frac{1}{2}\theta_y\right) z_y \\ &\quad + \left((\frac{b_y}{2b} + \frac{1}{2}\theta_y)(\frac{c_x}{2c} + \frac{1}{2}\theta_x) - \frac{1}{2}(bc + \theta_{xy}) \right) z \\ &\quad + \sigma\left(z_x - (\frac{c_x}{2c} + \frac{1}{2}\theta_x) z \right) + \tau\left(z_y - (\frac{b_y}{2b} + \frac{1}{2}\theta_x) z \right) + \sigma\tau z. \end{aligned}$$

We remark that this reduces to (2.25) when θ is constant. In terms of σ and τ, the formula in Lemma 2.12 can be written as follows:

$$b(\sigma^2)_x = (b\kappa_1)_y, \qquad c(\tau^2)_y = (c\kappa_2)_x, \tag{11.10}$$

where κ_1 and κ_2 are defined in (2.17).

11.4 Demoulin Lines

Here we look at the Demoulin transforms from a different point of view, by repeating the computation given in Sect. 2.1 in part. For simplicity, we assume that θ is constant in this section.

We recall that the dual coordinate system (p^0, p^1, p^2, p^3) was defined by

$$P = p^0 z + p^1 z_x + p^2 z_y + p^3 z_{xy}.$$

Relative to this coordinate system, the Lie quadric is defined by the equation $E = 0$, where

$$E := p^0 p^3 - p^1 p^2 + \alpha (p^3)^2, \qquad \alpha = \frac{1}{2}bc.$$

We look for the characteristic points of the family of quadrics parametrized by x. They are defined by the equations $E = 0$ and $E_x = 0$. To compute E_x, we need variation formulas of the homogeneous coordinates p^i when the point P remains fixed.

Since we have

$$\begin{aligned} P_x &= (p^0_x + p^1 p + p^3(p_y + bq))z + (p^0 + p^1_x + p^3 bc) z_x \\ &\quad + (bp^1 + p^2_x + p^3(b_y + p)) z_y + (p^2 + p^3_x) z_{xy}, \\ P_y &= (p^0_y + p^2 q + p^3(q_x + cp))z + (cp^2 + p^1_y + p^3(c_x + q)) z_x \\ &\quad + (p^0 + p^2_y + p^3 bc) z_x + (p^1 + p^3_y) z_{xy}, \end{aligned}$$

if we assume the point P remains fixed, then the coordinates p^i vary by the following rules:

$$\begin{aligned}
dp^0 &= -(p^1p + p^3(p_y + bq))dx - (p^2q + p^3(q_x + cp))dy, \\
dp^1 &= -(p^0 + p^3bc)dx - (cp^2 + p^3(c_x + q))dy, \\
dp^2 &= -(bp^1 + p^3(b_y + p))dx - (p^0 + p^3bc)dy, \\
dp^3 &= -p^2dx - p^1dy.
\end{aligned}$$

Now it is easy to see that

$$E_x = b(p^1)^2 + b_yp^1p^3 - (b_y + bq - \alpha_x)(p^3)^2,$$

and that the equation $E_x = 0$ is nothing but the homogeneous form of the equation (11.3). Any point satisfying the equation lies on the lines defined by the equations

$$f := p^1 - sp^3 = 0, \qquad g := p^0 - sp^2 + \alpha p^3,$$

where s is one of the solutions of (11.3). Similarly, for the family parametrized by y, we get the lines

$$p^2 - tp^3 = 0, \qquad p^0 - tp^1 + \alpha p^3 = 0,$$

where t is one of the solutions of (11.4). Thus we get four distinguished lines on the Lie quadric, which were called the *Demoulin lines*.

Let us consider the line congruence consisting of Demoulin lines for each fixed choice of s or t, which we call the *Demoulin congruence*.

We want to find the focal points of the Demoulin congruence. At such a point, we have a direction along which $df = 0$ and $dg = 0$ hold. By computation, we see that

$$df = -(s_x + \alpha)p^3dx - (cp^2 + (c_x + q + s_y - s^2)p^3)dy$$

and

$$dg = -(s_x + \alpha)p^2dx + ((s^2 - q - s_y)p^2 + (\alpha_y - q_x - cp)p^3)dy.$$

Hence the direction is determined so that both df and dg are proportional, i.e., when the following identity holds:

$$(s_x + \alpha)\{c(p^2)^2 + c_xp^2p^3 + (\alpha_y - q_x - cp)(p^3)^2\} = 0.$$

Since the focal surface is not determined if $s_x+\alpha = 0$, we assume in the following that $s_x + \alpha \neq 0$. Then the condition above is the same as the condition $E_y = 0$. Thus, we have seen that the Demoulin congruence is stationary at the points where the line in one family meets the line in the other family.

11.5 Demoulin congruences

We compute the asymptotic directions of the Demoulin transforms. The induced conformal form ω on the surface is determined by the identity

$$w \wedge w_x \wedge w_y \wedge (w_{xx}dx^2 + 2w_{xy}dxdy + w_{yy}dy^2) = \omega z \wedge z_x \wedge z_y \wedge z_{xy}.$$

If we write $\omega = Adx^2 + 2Bdxdy + Cdy^2$, then a computation using the integrability conditions (2.9) gives the formula

$$\begin{aligned} A &= \rho(s_x+\alpha)\left(2\frac{(ct)_x}{c} + \frac{c_{xx}}{c}\right), \qquad C = \rho(t_y+\alpha)\left(2\frac{(bs)_y}{b} + \frac{b_{yy}}{b}\right), \\ B &= \rho(s_x+\alpha)(t_y+\alpha) - \rho(bs + \frac{1}{2}b_y)(ct + \frac{1}{2}c_x) \\ &\quad + \rho\left(s_y + \frac{b_y}{b}s + \frac{b_{yy}}{2b}\right)\left(t_x + \frac{c_x}{c}t + \frac{c_{xx}}{2c}\right), \end{aligned}$$

where

$$\rho = (b_y + bs + t_x + p - t^2)(c_x + s_y + cqt - s^2) - (s_x+\alpha)(t_y+\alpha).$$

If $\rho = 0$, then the Demoulin transform does not make a surface. Hence we assume $\rho \neq 0$ in the following. By definitions (11.8) and (11.9), we have

$$s_x + \alpha = \kappa_1 + \sigma_x, \qquad t_y + \alpha = \kappa_2 + \tau_y,$$

$$bs + \frac{1}{2}b_y = b\sigma, \qquad ct + \frac{1}{2}c_x = c\tau.$$

Hence, we can conclude that the conformal structure of the Demoulin transform is defined by the form

$$\omega = \left((\kappa_1 + \sigma_x)dx + \frac{(b\sigma)_y}{b}dy\right)\left(\frac{(c\tau)_x}{c}dx + (\kappa_2 + \tau_y)dy\right) - bc\sigma\tau dxdy.$$

We now define the *nondegeneracy* for a Demoulin transform by

$$\rho\sigma\tau(\kappa_1 + \sigma_x)(\kappa_2 + \tau_y) \neq 0.$$

Since

$$(\sigma^2)_x = (\log b)_y\kappa_1 + (\kappa_1)_y, \qquad (\tau^2)_y = (\log c)_x\kappa_2 + (\kappa_2)_x$$

by (11.10), we must have $\kappa_1\kappa_2 \neq 0$ under the assumption of nondegeneracy.

Now we can state the following theorem:

Theorem 11.1 (S. Finikov [Fi1930]) *Assume that the Demoulin transform is nondegenerate. Then the Demoulin congruence is a W-congruence if and only if the original surface is projectively minimal.*

Proof. For a fixed s, consider a Demoulin congruence joining two Demoulin transforms corresponding to two values of t. In terms of σ, one corresponds to the value σ and the other corresponds to $-\sigma$. Since the asymptotic directions for each transform are defined by the equation $\omega = 0$, in order that the congruence be a W-congruence, it is necessary and sufficient that the equation $\omega = 0$ for σ and the equation $\omega = 0$ for $-\sigma$ define the same direction. Then, taking into account the nondegeneracy defined above, we can see that

$$(b\sigma)_y = 0 \quad \text{and} \quad (c\tau)_x = 0.$$

To complete the proof, it is enough to see that this condition is equivalent to (2.40) for projective minimality.

11.6 An explicit form of Demoulin frames

This section aims at getting a detailed form of a Demoulin transform.

We recall the notation of a nondegenerate surface $z(x, y)$:

$$z_{xx} = \theta_x z_x + b z_y + p z, \quad z_{yy} = c z_x + \theta_y z_y + q z.$$

Differentiating these equations once, we get

$$\begin{aligned} z_{xxy} &= \theta_x z_{xy} + (bc + \theta_{xy}) z_x + (b\theta_y + b_y + p) z_y + (bq + p_y) z, \\ z_{xyy} &= \theta_y z_{xy} + (c_x + c\theta_x + q) z_x + (bc + \theta_{xy}) z_y + (cp + q_x) z. \end{aligned}$$

We introduce the notations

$$\lambda = b\sigma + \frac{(c\tau)_x}{c}, \quad \lambda' = c\tau + \frac{(b\sigma)_y}{b}, \quad \mu = \alpha + s_x, \quad \mu' = \alpha + t_y, \tag{11.11}$$

and we remark that the following identities hold:

$$\sigma(b\sigma)_y = \tau(c\tau)_x, \quad b\sigma\lambda' = c\tau\lambda. \tag{11.12}$$

We define vectors Y, Z, and X by

$$Y = tz + z_x, \quad Z = sz + z_y, \quad X = z_{xy} + s z_x + t z_y + (st - \alpha) z.$$

Then, from what we have shown, we get the following Pfaff equation:

$$d \begin{pmatrix} z \\ Y \\ Z \\ X \end{pmatrix} = \omega \begin{pmatrix} z \\ Y \\ Z \\ X \end{pmatrix},$$

where

$$\omega = \begin{pmatrix} -(tdx + sdy) & dx & dy & 0 \\ (\lambda - 2b\sigma)dx + \mu' dy & (\theta_x + t)dx - sdy & bdx & dy \\ \mu dx + (\lambda' - 2c\tau)dy & cdy & -tdx + (\theta_y + s)dy & dx \\ 0 & \mu dx + \lambda' dy & \lambda dx + \mu' dy & \begin{matrix}(\theta_x + t)dx \\ +(\theta_y + s)dy\end{matrix} \end{pmatrix}$$

The integrability condition $d\omega = \omega \wedge \omega$ can be checked by using the identities given above and the additional identities

$$\mu'_x - \lambda_y = \mu'(2t + \theta_x) - b\lambda', \qquad \mu_y - \lambda'_x = \mu(2s + \theta_y) - c\lambda.$$

Provided that $R := \lambda\lambda' - \mu\mu' \neq 0$, the vector X defines a surface. Considering the Pfaff equation above, it is easy to see that the conformal structure on X is given by the 2-form

$$b\mu(c\tau)_x dx^2 + bc(\lambda\lambda' + \mu\mu' - b\sigma\lambda' - c\tau\lambda)dxdy + c(b\sigma)_y dy^2.$$

The above coframe ω satisfies the required condition for the frame $\{z, Y, Z, X\}$ to be a Demoulin frame defined in Sect. 2.4, up to scalar multiplication. The invariants are seen to be as follows:

$$\omega^1 = dx, \quad \omega^2 = dy, \quad h_{111} = -2b, \quad h_{112} = h_{122} = 0, \quad h_{222} = -2c,$$

$$(p_{ij}) = \begin{pmatrix} \lambda - 2b\sigma & \mu' \\ \mu & \lambda' - 2c\tau \end{pmatrix}, \quad (q_{ij}) = \begin{pmatrix} \lambda & \mu' \\ \mu & \lambda' \end{pmatrix},$$
$$(L_{ij}) = \begin{pmatrix} 2b\sigma & 0 \\ 0 & 2c\tau \end{pmatrix}. \tag{11.13}$$

Now we assume that the surface z is projectively minimal, i.e., $(b\sigma)_y = 0$ and $(c\tau)_x = 0$. In this case, we simply get

$$\lambda = b\sigma, \qquad \lambda' = c\tau,$$

and, provided that $R \neq 0$, the induced conformal structure on the surface X is the same as that on z. The system of equations defining X is given as follows:

$$X_{xx} = \overline{\theta}_x X_x + \overline{b} X_y + \overline{p} X, \quad X_{yy} = \overline{c} X_x + \overline{\theta}_y X_y + \overline{q} X.$$

To simplify notations, we set

$$\nu = \theta_x + t, \quad \nu' = \theta_y + s.$$

Then, we can see that

$$\begin{gathered} \overline{\theta}_x = \theta_x + \frac{R_x}{R}, \quad \overline{\theta}_y = \theta_y + \frac{R_y}{R}, \\ \overline{b} = \frac{\mu}{\mu'} b + \frac{R\lambda_x - R_x\lambda}{R\mu'}, \quad \overline{c} = \frac{\mu'}{\mu} c + \frac{R\lambda'_y - R_y\lambda'}{R\mu}, \\ \overline{p} = \nu_x + \nu^2 + \lambda - \nu\overline{\theta}_x - \nu'\overline{b}, \\ \overline{q} = \nu'_y + \nu'^2 + \lambda' - \nu'\overline{\theta}_y - \nu\overline{c}. \end{gathered} \tag{11.14}$$

We also have expressions

$$\begin{aligned} \overline{b} &= \frac{1}{R}(\lambda\mu(\nu + t) + \lambda\mu_x - \lambda_x\mu - b\mu^2), \\ \overline{c} &= \frac{1}{R}(\lambda'\mu'(\nu' + s) + \lambda'\mu'_y - \lambda'_y\mu' - c\mu'^2). \end{aligned}$$

Let us normalize the frame ${}^t(X, Y, Z, z)$. We know

$$d\begin{pmatrix} X \\ Y \\ Z \\ z \end{pmatrix} = \begin{pmatrix} \rho & \mu dx + \lambda' dy & \lambda dx + \mu' dy & 0 \\ dy & \nu dx - s dy & b dx & -\lambda dx + \mu' dy \\ dx & c dy & -t dx + \nu' dy & \mu dx - \lambda' dy \\ 0 & dx & dy & -t dx - s dy \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ z \end{pmatrix},$$

where $\rho = (\theta_x + t)dx + (\theta_y + s)dy$. We define new vectors

$$\overline{z} = -Rz, \quad \overline{Y} = \mu Y + \lambda Z, \quad \overline{Z} = \lambda' Y + \mu' Z.$$

Relative to the frame $f = {}^t(X, \overline{Y}, \overline{Z}, \overline{z})$, we get $df = \Omega f$, where

$$\Omega = \begin{pmatrix} \rho & dx & dy & 0 \\ \lambda dx + \mu dy & \delta dx + \nu' dy & \overline{b} dx & dy \\ \mu' dx + \lambda' dy & \overline{c} dy & \nu dx + \delta' dy & dx \\ 0 & \mu' dx - \lambda' dy & -\lambda dx + \mu dy & \begin{matrix} (R_x/R - t)dx \\ +(R_y/R - s)dy \end{matrix} \end{pmatrix},$$

and where

$$\begin{aligned} \delta &= (-\mu' \mu_x + \lambda' \lambda_x - \mu\mu'\nu - \lambda\lambda' t + \mu\lambda' b)/R, \\ \delta' &= (-\mu\mu'_y + \lambda\lambda'_y - \mu\mu'\nu - \lambda\lambda' s + \mu'\lambda c)/R, \\ \overline{b} &= (\lambda\mu_x - \mu\lambda_x + \lambda\mu\nu - \mu^2 b + \lambda\mu t)/R, \\ \overline{c} &= (\lambda'\mu'_y - \mu'\lambda'_y + \lambda'\mu'\nu' - \mu'^2 c + \lambda'\mu' s)/R. \end{aligned}$$

The frame f is the Demoulin frame of the Demoulin transform with Ω as its coframe when the original surface is projectively minimal.

Remark 11.2 The contents in this section were originally given by [Fi1930] and [May1932]. [L] is also helpful.

12 An intrinsic description of Demoulin transforms of projectively minimal surfaces

In the previous section we have seen that projectively minimal surfaces enjoy a special feature relative to Demoulin transforms. Referring to the intrinsic formulation of projectively minimal surface in Sect. 2.7, we compute normalized frames of Demoulin transforms and then get the formula of the second Demoulin transforms. Relying on explicit forms of the second Demoulin transforms, we reprove the result by O. Mayer and B. Su that a projectively minimal surface generally yields nine second Demoulin transforms. As an example, we explicitly give the second Demoulin transforms among the coincidence surfaces.

12.1 The normalized frame of a Demoulin transform

Let us recall the notation defined in Sect. 2.4: e denotes a Demoulin frame and ω its coframe. Let p and q denote the matrices given in (2.20) and set $P = \det p$.

We assume now that the surface is projectively minimal and that $P \neq 0$ so that every Demoulin transform defines a surface. Then by Proposition 2.22 we have the following expressions:

$$
\begin{gathered}
(p_{ij}) = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}, \quad (q_{ij}) = \begin{pmatrix} -p_{11} & p_{12} \\ p_{21} & -p_{22} \end{pmatrix}, \\
(\ell_{ij}) = \begin{pmatrix} -2p_{11} & 0 \\ 0 & -2p_{22} \end{pmatrix}.
\end{gathered}
\tag{12.1}
$$

The frame $\tilde{e} = (e_3, e_1, e_2, e_0)$ is associated with the Demoulin transform e_3, and the coframe defined by $d\tilde{e} = \tilde{\omega}\tilde{e}$ has the form

$$
\tilde{\omega} = \begin{pmatrix} \omega_3^3 & p_{21}\omega^1 - p_{22}\omega^2 & -p_{11}\omega^1 + p_{12}\omega^2 & 0 \\ \omega^2 & \omega_1^1 & b\omega^1 & p_{11}\omega^1 + p_{12}\omega^2 \\ \omega^1 & c\omega^2 & \omega_2^2 & p_{21}\omega^1 + p_{22}\omega^2 \\ 0 & \omega^1 & \omega^2 & \omega_0^0 \end{pmatrix}.
$$

This is not normalized, and in order to get a normalized frame, we need a slight change of the frame as follows:

$$
\begin{gathered}
\overline{e}_0 = \delta e_3, \quad \overline{e}_1 = \delta(p_{21}e_1 - p_{11}e_2), \\
\overline{e}_2 = \delta(p_{12}e_2 - p_{22}e_1), \quad \overline{e}_3 = -\delta P e_0,
\end{gathered}
\tag{12.2}
$$

where $\delta = 1/\sqrt{P}$. Then, the normalized coframe defined by $d\overline{e} = \overline{\omega e}$ has the form

$$
\overline{\omega} = \begin{pmatrix} d\log\delta + \omega_3^3 & \omega^1 & \omega^2 & 0 \\ p_{21}\omega^2 - p_{11}\omega^1 & \overline{\omega}_1^1 & \overline{\omega}_1^2 & \omega^2 \\ p_{12}\omega^1 - p_{22}\omega^2 & \overline{\omega}_1^2 & \overline{\omega}_2^2 & \omega^1 \\ 0 & p_{12}\omega^1 + p_{22}\omega^2 & p_{11}\omega^1 + p_{21}\omega^2 & \omega_0^0 - d\log\delta \end{pmatrix},
\tag{12.3}
$$

where

$$
\begin{aligned}
P\overline{\omega}_1^1 &= -dP/2 - p_{12}dp_{21} + p_{22}dp_{11} - p_{12}p_{21}\omega_1^1 \\
&\qquad + p_{11}p_{12}\omega_2^1 - p_{22}p_{21}\omega_1^2 + p_{11}p_{22}\omega_2^2, \\
P\overline{\omega}_1^2 &= -p_{11}dp_{21} + p_{21}dp_{11} - p_{11}p_{21}\omega_1^1 \\
&\qquad + (p_{11})^2\omega_2^1 - (p_{21})^2\omega_1^2 + p_{11}p_{21}\omega_2^2, \\
P\overline{\omega}_2^1 &= p_{12}dp_{22} - p_{22}dp_{12} + p_{12}p_{22}\omega_1^1 \\
&\qquad - (p_{12})^2\omega_2^1 + (p_{22})^2\omega_1^2 - p_{12}p_{22}\omega_2^2, \\
P\overline{\omega}_2^2 &= -dP/2 + p_{11}dp_{22} - p_{21}dp_{12} + p_{11}p_{22}\omega_1^1 \\
&\qquad - p_{11}p_{12}\omega_2^1 + p_{21}p_{22}\omega_1^2 - p_{12}p_{21}\omega_2^2.
\end{aligned}
\tag{12.4}
$$

We note that the conditions $\overline{\omega}_0^0+\overline{\omega}_3^3=0$, $\overline{\omega}_1^1+\overline{\omega}_2^2=0$, and $\overline{\omega}_3^0=0$ are satisfied as required for normalization. The associated invariants $\overline{h}$, $\overline{p}$, $\overline{q}$, and $\overline{\ell}$ are

$$\overline{h}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \overline{p}=\begin{pmatrix}-p_{11} & p_{21}\\ p_{12} & -p_{22}\end{pmatrix},$$
$$\overline{q}=\begin{pmatrix}p_{11} & p_{21}\\ p_{12} & p_{22}\end{pmatrix},\quad \overline{\ell}=\begin{pmatrix}2p_{11} & 0\\ 0 & 2p_{22}\end{pmatrix}.$$

These formulas say that, for the Demoulin transform e_3, p_{11} and p_{22} are changed into their negatives and p_{12} and p_{21} are interchanged. In particular for Godeaux-Rozet surfaces, we have the following corollary in view of Remark 2.9.

Corollary 12.1 *Any Demoulin transform of a Godeaux-Rozet surface is also a Godeaux-Rozet surface.*

The cubic invariants $\overline{b}$ and $\overline{c}$ are defined by the formulas $\overline{h}_{111}=-2\overline{b}$ and $\overline{h}_{222}=-2\overline{c}$, and we see that

$$\overline{\omega}_1^2=\overline{b}\omega^1,\qquad \overline{\omega}_2^1=\overline{c}\omega^2.$$

We now calculate the cubic invariants explicitly.

We define covariant derivatives of p_{ij} by

$$dp_{ij}-\sum_k p_{ik}\omega_j^k-\sum_k p_{kj}\omega_i^k+2p_{ij}\omega_0^0=\sum_k p_{ij,k}\omega^k. \tag{12.5}$$

Lemma 12.2 $p_{11,2}=p_{12,1}=0$ *and* $p_{21,2}=p_{22,1}=0$.

Proof. The differentiation of $\omega_i^0=p_{ij}\omega^j$ implies $d\omega_i^0-dp_{ij}\wedge\omega^j-p_{ij}d\omega^j=0$, and the left-hand side is equal to $\omega^j\wedge(dp_{ij}-p_{ik}\omega_j^k-p_{kj}\omega_i^k+2p_{ij}\omega_0^0)$. Hence, we have $p_{ij,k}=p_{ik,j}$.

Next we differentiate $\omega_1^0=\sum p_{1j}\omega^j$ and get

$$dp_{11}\wedge\omega^1+dp_{12}\wedge\omega^2=2p_{11}(\omega_1^1-\omega_0^0)\wedge\omega^1-2p_{12}\omega_0^0\wedge\omega^2+bp_{22}\omega^1\wedge\omega^2.$$

The differentiation of $\omega_3^2=\sum q_{1j}\omega^j$ gives

$$dq_{11}\wedge\omega^1+dq_{12}\wedge\omega^2=2q_{11}(\omega_1^1-\omega_0^0)\wedge\omega^1-2q_{12}\omega_0^0\wedge\omega^2-bq_{22}\omega^1\wedge\omega^2.$$

Since $q_{11}=-p_{11}$, $q_{12}=p_{12}$, and $q_{22}=-p_{22}$, the latter identity means

$$-dp_{11}\wedge\omega^1+dp_{12}\wedge\omega^2=-2p_{11}(\omega_1^1-\omega_0^0)\wedge\omega^1-2p_{12}\omega_0^0\wedge\omega^2+bp_{22}\omega^1\wedge\omega^2.$$

Hence, we have

$$\begin{aligned}dp_{11}\wedge\omega^1&=2p_{11}(\omega_1^1-\omega_0^0)\wedge\omega^1,\\ dp_{12}\wedge\omega^2&=bp_{22}\omega^1\wedge\omega^2-2p_{12}\omega_0^0\wedge\omega^2.\end{aligned} \tag{12.6}$$

By a similar computation for the identities $\omega_2^0 = p_{2j}\omega^j$ and $\omega_3^1 = q_{2j}\omega^j$, we also have

$$\begin{aligned} dp_{22} \wedge \omega^2 &= -2p_{22}(\omega_1^1 + \omega_0^0) \wedge \omega^2, \\ dp_{21} \wedge \omega^1 &= -cp_{11}\omega^1 \wedge \omega^2 - 2p_{21}\omega_0^0 \wedge \omega^1. \end{aligned} \tag{12.7}$$

We now insert $dp_{11} = p_{11,1}\omega^1 + p_{11,2}\omega^2 + 2p_{11}\omega_1^1 + (p_{12} + p_{21})\omega_1^2 - 2p_{11}\omega_0^0$, which is one of the formulas in (12.5), into the first equation of (12.6) to get $p_{11,2} = 0$ by using the property that $\omega_1^2 = b\omega^2$ contains no term with ω^1. The other claimed identities are similarly shown.

Now it is easy to compute $P\overline{\omega}_1^2$ defined in (12.4) by use of the dp_{21} and dp_{11} given in (12.5), and we get

$$P\overline{\omega}_1^2 = (p_{21}p_{11,1} - p_{11}p_{21,1})\omega^1 - P\omega_1^2.$$

Hence, we have

Lemma 12.3 *Let $P = \det(p)$ and let $\overline{b}$ and $\overline{c}$ denote the cubic invariants of the Demoulin transfrom. Then*

$$\begin{aligned} P\overline{b} &= -Pb + p_{21}p_{11,1} - p_{11}p_{21,1}, \\ P\overline{c} &= -Pc + p_{12}p_{22,2} - p_{22}p_{12,2}. \end{aligned}$$

We assume in the following that $L_{11}L_{22} \neq 0$ so that we have four distinct Demoulin transforms. Let us recall the treatment in Sect. 2.4 again. Once we get a Demoulin frame e, the other Demoulin frames are represented by ge, where g is the transformation of the form

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \rho^2 & 1 & 0 & 0 \\ \rho^1 & 0 & 1 & 0 \\ \rho^1\rho^2 & \rho^1 & \rho^2 & 1 \end{pmatrix};$$

the values ρ^1 and ρ^2 are determined by the conditions

$$b(\rho^1)^2 + \ell_{11}\rho^1 = 0 \qquad \text{and} \qquad c(\rho^2)^2 + \ell_{22}\rho^2 = 0.$$

We set

$$t^1 = -\ell_{11}/b, \qquad t^2 = -\ell_{22}/c.$$

Then, ρ^1 takes the value 0 or t^1 and ρ^2 takes the value 0 or t^2.

We denote one of the new frames by $\tilde{e} = \{\tilde{e}_0, \tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$, where

$$\tilde{e}_0 = e_0, \quad \tilde{e}_1 = \rho^2 e_0 + e_1, \quad \tilde{e}_2 = \rho^1 e_0 + e_2, \quad \tilde{e}_3 = \rho^1\rho^2 e_0 + \rho^1 e_1 + \rho^2 e_2 + e_3,$$

and then the new $\tilde{\ell}_{ij}$'s are given by

$$\tilde{\ell}_{11} = \ell_{11} + 2b\rho^1, \qquad \tilde{\ell}_{22} = \ell_{22} + 2c\rho^2.$$

The coframe $\tilde{\omega}$, $d\tilde{e} = \tilde{\omega}\tilde{e}$, is

$$\tilde{\omega} = \begin{pmatrix} \omega_0^0 - \rho^2\omega^1 - \rho^1\omega^2 & \omega^1 & \omega^2 & 0 \\ \tilde{\omega}_1^0 & \omega_1^1 + \rho^2\omega^1 - \rho^1\omega^2 & \omega_1^2 & \omega^2 \\ \tilde{\omega}_2^0 & \omega_2^1 & \omega_2^2 + \rho^1\omega^2 - \rho^2\omega^1 & \omega^1 \\ 0 & \tilde{\omega}_3^1 & \tilde{\omega}_3^2 & \omega_3^3 + \rho^2\omega^1 + \rho^1\omega^2 \end{pmatrix},$$

where

$$\begin{aligned} \tilde{\omega}_1^0 &= \omega_1^0 - \rho^1\omega_1^2 + d\rho^2 - (\rho^2)^2\omega_1^1 - \rho^1\omega_1^2 + \rho^2\omega_0^0, \\ \tilde{\omega}_2^0 &= \omega_2^0 - \rho^2\omega_2^1 + d\rho^1 - (\rho^1)^2\omega^2 - \rho^1\omega_2^2 + \rho^1\omega_0^0, \\ \tilde{\omega}_3^1 &= \omega_3^1 + \rho^2\omega_2^1 + d\rho^1 - (\rho^1)^2\omega^2 + \rho^1\omega_1^1 - \rho^1\omega_3^3, \\ \tilde{\omega}_3^2 &= \omega_3^2 + \rho^1\omega_1^2 + d\rho^2 - (\rho^2)^2\omega_1 + \rho^2\omega_2^2 - \rho^2\omega_3^3. \end{aligned}$$

We remark that, for any choice of frame, the cubic invariants b and c remain the same.

We now assume that the surface is projectively minimal.

Lemma 12.4 *The derivations of $t^1 = -\ell_{11}/b$ and $t^2 = -\ell_{22}/c$ have the following expressions:*

$$\begin{aligned} dt^1 + t^1(\omega_0^0 + \omega_1^1) - (t^1)^2\omega^2 = t^1_{,1}\omega^1, \\ dt^2 + t^2(\omega_0^0 + \omega_2^2) - (t^2)^2\omega^1 = t^2_{,2}\omega^2, \end{aligned}$$

where $t^1_{,1}$ and $t^2_{,2}$ are thus defined.

Proof. The exterior derivative of $\omega_3^2 - \omega_1^0 = \ell_{11}\omega^1$ yields

$$(d\ell_{11} + 2\ell_{11}(\omega_0^0 - \omega_1^1)) \wedge \omega^1 = 0,$$

and the exterior derivative of $\omega_1^2 = b\omega^1$ yields

$$db \wedge \omega^1 = -b(\omega_0^0 - 3\omega_1^1) \wedge \omega^1 - \ell_{11}\omega^1 \wedge \omega^2.$$

Hence, we have

$$(bd\ell_{11} - \ell_{11}db + b\ell_{11}(\omega_0^0 + \omega_1^1) + (\ell_{11})^2\omega^2) \wedge \omega^1 = 0,$$

from which, by dividing both sides by b^2, we get the first assertion. The second assertion is similarly proved.

By this lemma, the new coframe simplifies to

$$\begin{aligned} \tilde{\omega}_1^0 &= \omega_1^0 - \rho^1\omega_1^2 + t^2_{,2}\omega^2, & \tilde{\omega}_2^0 &= \omega_2^0 - \rho^2\omega_2^1 + t^1_{,1}\omega^1, \\ \tilde{\omega}_3^1 &= \omega_3^1 + \rho^2\omega_2^1 + t^1_{,1}\omega^1, & \tilde{\omega}_3^2 &= \omega_3^2 + \rho^1\omega_1^2 + t^2_{,2}\omega^2. \end{aligned} \tag{12.8}$$

Here we understand that $t^1_{,1} = 0$ when $\rho^1 = 0$ and $t^2_{,2} = 0$ when $\rho^2 = 0$.

Summing up the discussion above, we get the following lemma.

Proposition 12.5 *Assume that the original surface is a non-ruled indefinite projectively minimal surface and that $\ell_{11}\ell_{22} \neq 0$. Let $e = \{e_0, e_1, e_2, e_3\}$ be one of the Demoulin frames. Then the four Demoulin frames are given as follows.*

type	ρ^1	ρ^2	frame	invariant p
1	0	0	e_0, e_1, e_2, e_3	$\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$
2	t^1	0	$e_0, e_1, e_2 + t^1 e_0, e_3 + t^1 e_1$	$\begin{pmatrix} -p_{11} & p_{12} \\ p'_{21} & p_{22} \end{pmatrix}$
3	0	t^2	$e_0, e_1 + t^2 e_0, e_2, e_3 + t^2 e_2$	$\begin{pmatrix} p_{11} & p'_{12} \\ p_{21} & -p_{22} \end{pmatrix}$
4	t^1	t^2	$e_0, e_1 + t^2 e_0, e_2 + t^1 e_0, e_3 + t^1 e_1 + t^2 e_2 + t^1 t^2 e_0$	$\begin{pmatrix} -p_{11} & p'_{12} \\ p'_{21} & -p_{22} \end{pmatrix}$,

where

$$p'_{21} = p_{21} + t^1_{,1}, \qquad p'_{12} = p_{12} + t^2_{,2}.$$

The following table gives the normalized frames and invariants for each of these Demoulin transforms; let Ti $(1 \leq i \leq 4)$ denote the transform of type i given in Lemma 12.5. Let $e^i = \{e^i_0, e^i_1, e^i_2, e^i_3\}$ denote the normalized frame for the transform Ti. The matrix in the last column of the table below represents the invariants of the transformed surface, whose determinant is denoted by P^i.

$T1$	$e^1_0 = e_3$ $e^1_1 = p_{21}e_1 - p_{11}e_2$ $e^1_2 = -p_{22}e_1 + p_{12}e_2$ $e^1_3 = -P^1 e_0$	$\begin{pmatrix} -p_{11} & p_{21} \\ p_{12} & -p_{22} \end{pmatrix}$
$T2$	$e^2_0 = e_3 + t^1 e_1$ $e^2_1 = p'_{21}e_1 + p_{11}(e_2 + t^1 e_0)$ $e^2_2 = p_{12}(e_2 + t^1 e_0) - p_{22}e_1$ $e^2_3 = -P^2 e_0$	$\begin{pmatrix} p_{11} & p'_{21} \\ p_{12} & -p_{22} \end{pmatrix}$
$T3$	$e^3_0 = e_3 + t^2 e_1$ $e^3_1 = p_{21}(e_1 + t^2 e_0) - p_{11}e_2$ $e^3_2 = p'_{12}e_2 + p_{22}(e_1 + t^2 e_0)$ $e^3_3 = -P^3 e_0$	$\begin{pmatrix} -p_{11} & p_{21} \\ p'_{12} & p_{22} \end{pmatrix}$
$T4$	$e^4_0 = e_3 + t^1 e_1 + t^2 e_2 + t^1 t^2 e_0$ $e^4_1 = p'_{21}(e_1 + t^2 e_0) + p_{11}(e_2 + t^1 e_0)$ $e^4_2 = p'_{12}(e_2 + t^1 e_0) + p_{22}(e_1 + t^2 e_0)$ $e^4_3 = -P^4 e_0$	$\begin{pmatrix} p_{11} & p'_{21} \\ p'_{12} & p_{22} \end{pmatrix}$

The above computation is valid even when $\ell_{11}\ell_{22} = 0$ provided $bc \neq 0$. When both of ℓ_{11} and ℓ_{22} vanish, namely, when the surface is a Demoulin surface, we get only one surface. When $\ell_{11} = 0$ and $\ell_{22} \neq 0$, namely, when the surface is a Godeaux-Rozet surface, we need to set $t^1 = 0$ in the above computation and we get $t^1_{,1} = 0$. Then, we see that $p'_{21} = p_{21}$. Since $p_{11} = 0$ in this case, we get $T1 = T3$ and $T2 = T4$. Similarly when $\ell_{11} \neq 0$ and $\ell_{22} = 0$, we get $T1 = T3$ and $T2 = T4$.

12.2 Second Demoulin transforms

The repeated use of the above procedure gives the second transforms, which are listed below. Let us denote by b^i and c^i the respective cubic invariants for Ti. and we use the label Tij for denoting the second transforms of Ti, where the number j is given by following the same rule as used for labelling Ti. We remark that for these transforms to really define surfaces, we need to assume $P^i \neq 0$, $b^i \neq 0$, and $c^i \neq 0$.

$T11$	$-P^1e_0$
$T12$	$-b^1P^1e_0 - 2p_{11}(p_{21}e_1 - p_{11}e_2)$
$T13$	$-c^1P^1e_0 - 2p_{22}(-p_{22}e_1 + p_{12}e_2)$
$T14$	$-b^1c^1P^1e_0 - 2c^1p_{11}(p_{21}e_1 - p_{11}e_2) - 2b^1p_{22}(p_{12}e_2 - p_{22}e_1)$ $+4p_{11}p_{22}e_3$
$T21$	$-P^2e_0$
$T22$	$-b^2P^2e_0 + 2p_{11}(p'_{21}e_1 + p_{11}(e_2 + t^1e_0))$
$T23$	$-c^2P^2e_0 - 2p_{22}(-p_{22}e_1 + p_{12}(e_2 + t^1e_0))$
$T24$	$-b^2c^2P^2e_0 + 2c^2p_{11}(p'_{21}e_1 + p_{11}(e_2 + t^1e_0))$ $-2b^2p_{22}(p_{12}(e_2 + t^1e_0) - p_{22}e_1) - 4p_{11}p_{22}(e_3 + t^1e_1)$
$T31$	$-P^3e_0$
$T32$	$-b^3P^3e_0 - 2p_{11}(p_{21}(e_1 + t^2e_0) - p_{11}e_2)$
$T33$	$-c^3P^3e_0 + 2p_{22}(p'_{12}e_2 + p_{22}(e_1 + t^2e_0))$
$T34$	$-b^3c^3P^3e_0 - 2c^3p_{11}(p_{21}(e_1 + t^2e_0) - p_{11}e_2)$ $+2b^3p_{22}(p'_{12}e_2 + p_{22}(e_1 + t^2e_0)) - 4p_{11}p_{22}(e_3 + t^2e_2)$
$T41$	$-P^4e_0$
$T42$	$-b^4P^4e_0 + 2p_{11}(p'_{21}(e_1 + t^2e_0) + p_{11}(e_2 + t^1e_0))$
$T43$	$-c^4P^4e_0 + 2p_{22}(p'_{12}(e_2 + t^1e_0) + p_{22}(e_1 + t^2e_0))$
$T44$	$-b^4c^4P^4e_0 - 2c^4p_{11}(p'_{21}(e_1 + t^2e_0) - p_{11}(e_2 + t^1e_0))$ $+2b^4p_{22}(p'_{12}(e_2 + t^1e_0) + p_{22}(e_1 + t^2e_0))$ $-4p_{11}p_{22}(e_3 + t^1e_1 + t^2e_2 + t^1t^2e_0)$

As is seen above, each $Ti1$ $(1 \leq i \leq 4)$ coincides with the original surface. Further,

Proposition 12.6 *The following coincidences of surfaces hold:* $T13 = T23$, $T12 = T32$, $T22 = T42$, *and* $T33 = T43$.

Proof. We prove the coincidence $T13 = T23$. By definition,

$$T13 \quad = \quad -c^1P^1e_0 - 2p_{22}(-p_{22}e_1 + p_{12}e_2),$$

$$
\begin{aligned}
T23 &= -c^2P^2e_0 - 2p_{22}(-p_{22}e_1 + p_{12}(e_2 + t^1e_0)) \\
&= -(c^2P^2 + 2p_{22}p_{12}t^1)e_0 - 2p_{22}(-p_{22}e_1 + p_{12}e_2),
\end{aligned}
$$

Thus, it is enough to see that

$$
P^1c^1 = P^2c^2 + 2p_{22}p_{12}t^1 = P^2c^2 + 4p_{11}p_{22}p_{12}/b, \tag{12.9}
$$

where $P^2 = -p_{11}p_{22} - p_{12}p'_{21}$ is the determinant of the invariant matrix for $T23$:

$$
\begin{pmatrix} \tilde{p}_{11} & \tilde{p}_{12} \\ \tilde{p}_{21} & \tilde{p}_{22} \end{pmatrix} = \begin{pmatrix} p_{11} & p'_{21} \\ p_{12} & -p_{22} \end{pmatrix}.
$$

By Lemma 12.3,

$$
P^1c^1 = -P^1c + p_{12}p_{22,2} - p_{22}p_{12,2}
$$

for the transform $T13$ and

$$
P^2c^2 = -P^2c + \tilde{p}_{12}\tilde{p}_{22,2} - \tilde{p}_{22}\tilde{p}_{12,2}
$$

for the transform $T23$. The coframe for $T23$ is, by (12.10),

$$
\tilde{\omega} = \begin{pmatrix} \omega_0^0 - t^1\omega^2 & \omega^1 & \omega^2 & 0 \\ \tilde{\omega}_1^0 & \omega_1^1 - t^1\omega^2 & \omega_1^2 & \omega^2 \\ \tilde{\omega}_2^0 & \omega_2^1 & \omega_2^2 + t^1\omega^2 & \omega^1 \\ 0 & \tilde{\omega}_3^1 & \tilde{\omega}_3^2 & \omega_3^3 + t^1\omega^2 \end{pmatrix}.
$$

By using this form,

$$
\begin{aligned}
\sum \tilde{p}_{22,k}\omega^k &= d\tilde{p}_{22} - \sum(\tilde{p}_{2k} + \tilde{p}_{k2})\tilde{\omega}_2^k + 2\tilde{p}_{22}\tilde{\omega}_0^0 \\
&= dp_{22} - (p_{21} + p_{12} + t^1_{,1})\omega_2^1 - 2p_{22}(\omega_2^2 + c^1\omega^2) + 2p_{22}(\omega_0^0 - c^1\omega^2) \\
&= \sum p_{22,k}\omega^k - (ct^1_{,1} + 4t^1p_{22})\omega^2
\end{aligned}
$$

and

$$
\begin{aligned}
\sum \tilde{p}_{12,k}\omega^k &= d\tilde{p}_{12} - \tilde{p}_{11}\tilde{\omega}_2^1 - \tilde{p}_{12}(\tilde{\omega}_2^2 + \tilde{\omega}_1^1) - \tilde{p}_{22}\tilde{\omega}_1^2 + 2\tilde{p}_{12}\tilde{\omega}_0^0 \\
&= dp_{12} + p_{11}\omega_2^1 - p_{12}(\omega_2^2 + \omega_1^1) - p_{22}\omega_1^2 + 2p_{12}(\omega_0^0 - c^1\omega^2) \\
&= \sum p_{12,k}\omega^k + (2p_{11}c - 2p_{12}t^1)\omega^2.
\end{aligned}
$$

Here, we used $\omega_1^2 = b\omega^2$, $\omega_2^1 = c\omega^1$, and $\omega_1^1 + \omega_2^2 = 0$. Hence,

$$
\tilde{p}_{12}\tilde{p}_{22,2} - \tilde{p}_{22}\tilde{p}_{12,2} = p_{12}p_{22,2} - p_{22}p_{12,2} - ct^1_{,1}p_{12} - 2cp_{11}p_{22} - 2t^1p_{12}p_{22}.
$$

Now it is easy to show (12.9) by using $t^1 = 2p_{11}/b$.

The arrangement of transforms can be illustrated as in the following diagram.

$$
\boxed{
\begin{array}{ccccc}
 & & T32 = T12 & & \\
T34 \ \bullet & & \circ & & \bullet \ T14 \\
 & T3 \ * & & * \ T1 & \\
T33 = T43 \ \circ & & \odot & & \circ \ T13 = T23 \\
 & T4 \ * & & * \ T2 & \\
T44 \ \bullet & & \circ & & \bullet \ T24 \\
 & & T42 = T22 & &
\end{array}
}
$$

where $\odot$ is the original surface, $*$ denotes the first Demoulin transforms, and $\circ$ and $\bullet$ denote the second Demoulin transforms.

We next consider the line congruence joining the original projectively minimal surface and four of the second transforms. We will show:

Proposition 12.7 *The line congruence joining the original projectively minimal surface and each one of the surfaces $T13$, $T12$, $T33$, and $T22$ is a W-congruence, and these four surfaces are focal surfaces of each congruence.*

Proof. We give a proof for $T13$. Let us recall that the transform $T1$ has a normalized frame

$$e_0^1 = e_3, \quad e_1^1 = p_{21}e_1 - p_{11}e_2, \quad e_2^1 = -p_{22}e_1 + p_{12}e_2, \quad e_3^1 = -P^1 e_0 \tag{12.10}$$

up to a scalar factor. The invariant matrix, which we denote by $\overline{p}$, is

$$\overline{p} = \begin{pmatrix} \overline{p}_{11} & \overline{p}_{12} \\ \overline{p}_{21} & \overline{p}_{22} \end{pmatrix} = \begin{pmatrix} -p_{11} & p_{21} \\ p_{12} & -p_{22} \end{pmatrix}.$$

Let $\overline{t}^2 = -2p_{22}/c^1$ be the value of t^2 for the transform $T1$. Then, the normalized Demoulin frame for $T1$ that is needed define $T13$ is given by

$$\tilde{e}_0 = e_0^1, \quad \tilde{e}_1 = e_1^1 + \overline{t}^2 e_0^1, \quad \tilde{e}_2 = e_2^1, \quad \tilde{e}_3 = e_3^1 + \overline{t}^2 e_2^1. \tag{12.11}$$

The corresponding invariant matrix is

$$\tilde{p} = \begin{pmatrix} \tilde{p}_{11} & \tilde{p}_{12} \\ \tilde{p}_{21} & \tilde{p}_{22} \end{pmatrix} = \begin{pmatrix} \overline{p}_{11} & \overline{p}'_{12} \\ \overline{p}_{21} & -\overline{p}_{22} \end{pmatrix},$$

where $\overline{p}'_{12} = \overline{p}_{12} + \overline{t}^2_{,2}$ and $\overline{t}^2_{,2}$ is defined by

$$d\overline{t}^2 + \overline{t}^2(\overline{\omega}_0^0 - \overline{\omega}_1^1) - (\overline{t}^2)^2\omega^1 = \overline{t}^2_{,2}\omega^2$$

relative to the connection form $\overline{\omega}$ in (12.3). Now we see that the normalized Demoulin frame of $T13$ is given by

$$f_0 = \tilde{e}_3, \quad f_1 = \tilde{p}_{21}\tilde{e}_1 - \tilde{p}_{11}\tilde{e}_2, \quad f_2 = -\tilde{p}_{22}\tilde{e}_1 + \tilde{p}_{12}\tilde{e}_2, \quad f_3 = -\tilde{P}\tilde{e}_0,$$

up to a multiplicative factor, where $\tilde{P}$ denotes the determinant of the matrix $\tilde{p}$. Then, we see that

$$\begin{aligned} f_1 &= p_{12}(p_{21}e_1 - p_{11}e_2 + \overline{t}^2 e_3) - (-p_{11})(p_{12}e_2 - p_{22}e_1), \\ f_2 &= (p_{21} + \overline{t}^2_{,2})(p_{12}e_2 - p_{22}e_1) - p_{22}(p_{21}e_1 - p_{11}e_2 + \overline{t}^2 e_3). \end{aligned}$$

Thus, we get

$$p_{22}f_1 + p_{12}f_2 = (p_{11}p_{22} + p_{12}p_{21} + p_{12}\overline{t}^2_{,2})(-p_{22}e_1 + p_{12}e_2).$$

This means that the line joining the original surface and $T13$ is tangent to each surface. The congruence obtained is a W-congruence because each Demoulin transform preserves the conformal structure. Thus we have Proposition 12.7.

Now let us set aside the long procedure above for a moment and give a summary. Given a (indefinite) projectively minimal non-ruled surface z with a Demoulin frame e and coframe ω, we defined the Demoulin transform. We denoted by

$$p = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$$

the invariant of the surface and assumed $\det(p) \neq 0$ so that the transform really defines a surface, and we assumed $p_{11}p_{22} \neq 0$ so that we get four distinct transforms. Then, each transform is projectively minimal. We now define a new surface w by

$$w = \frac{1}{2}\kappa z + p_{21}e_1 - p_{11}e_2 \quad \text{or} \quad w = \frac{1}{2}\kappa' z - p_{22}e_1 + p_{12}e_2, \tag{12.12}$$

where κ and κ' are certain scalar functions determined by composing (12.10) and (12.11). Then the transform

$$z \longmapsto w$$

defines a W-congruence whose focal surfaces are both projectively minimal. By using the notation in Sect. 11.6, the new surface is

$$w = \frac{1}{2}\kappa z + \mu z_x + \lambda z_y \quad \text{or} \quad w = \frac{1}{2}\kappa' z + \lambda' z_x + \mu' z_y,$$

respectively.

In Sect. 11.4 we defined a Demoulin congruence, and in Sect. 11.5 we proved that the Demoulin congruence is a W-congruence when the surface is projectively minimal. In the formulation of this section, this corresponds to the following fact: Let us consider the line congruence joining two surfaces $T1 = e_3$ and $T2 = e_3 + t^1 e_1$. The tangent plane of $T1$ is spanned by the vectors $e_1^1 = p_{21}e_1 - p_{11}e_2$ and $e_2^1 = -p_{22}e_1 + p_{12}e_2$, and the tangent plane of $T2$ is spanned by $e_1^2 = p'_{21}e_1 + p_{11}(e_2 + t^1 e_0)$ and $e_2^2 = p_{12}(e_2 + t^1 e_0) - p_{22}e_1$. Hence, $-P^1 e_1$ is included in the tangent plane of $T1$ and $-P^2 e_1$ is included in the tangent plane of $T3$. Since the line joining $T1$ and $T2$ is in the direction of the vector e_1, both surfaces $T1$ and $T2$ are focal surfaces of the congruence.

12.3 Coincidence surfaces

The surface defined by the system

$$z_{xx} = z_y + (kx + k_1)z, \quad z_{yy} = z_x + (ky + k_2)z$$

where k, k_1, and k_2 are constants, is called a coincidence surface. We refer to Example 2.19. It is easy to check that this surface z is projectively minimal when $k = 0$, and it is a Demoulin surface only when $k = k_1 = k_2 = 0$. We assume $k = 0$. Then the function $\exp(\mu x + \nu y)$ is a solution if $\mu^2 = \nu + k_1$ and $\nu^2 = \mu + k_2$. According to the multiplicity of solutions (μ, ν), the surface has different expressions: $Z = X^\ell Y^m$ when all solutions are distinct, $Z = n \log X - 1/n \log Y$ when two solutions coincide, and $Z = \exp(Y - X^2)$ when three solutions coincide. Here (X, Y, Z) denote the affine coordinates and ℓ, m, and n are constants.

In the case $k = 0$, the system can be written as

$$z_{xx} = z_y + t^2 z, \quad z_{yy} = z_x + s^2 z.$$

The invariants are $L = -2t^2$, $M = -2s^2$, $\sigma^2 = s^2$, $\tau^2 = t^2$, $\lambda = \sigma$, $\lambda' = \tau$, and $\mu = \mu' = 1/2$ in the notations in Sect. 11.6. In the following $\sigma = \pm s$ and $\tau = \pm t$. The Demoulin transform is

$$w = z_{xy} + \sigma z_x + \tau z_y + (\sigma\tau - 1/2)z),$$

and the normalized frame is

$$e = {}^t(e_0, e_1, e_2, e_3) = {}^t(z, z_x + \tau z, z_y + \sigma z, w).$$

Then e satisfies the Pfaff equation:

$$de = \begin{pmatrix} -\tau dx - \sigma dy & dx & dy & 0 \\ -\sigma dx + \frac{1}{2}dy & \tau dx - \sigma dy & dx & dy \\ \frac{1}{2}dx - \tau dy & dy & -\tau dx + \sigma dy & dx \\ 0 & \frac{1}{2}dx + \tau dy & \sigma dx + \frac{1}{2}dy & \tau dx + \sigma dy \end{pmatrix} e.$$

With the notation above, it is easy to compute the system satisfied by w, which turns out to be the same system satisfied by z; namely, Demoulin transforms do not yield any new surfaces, and, rather, the transforms give line congruences among the coincidence surfaces. The second Demoulin transform u is given by

$$u = w_{xy} + \sigma' w_x + \tau' w_y + (\sigma'\tau' - 1/2)w,$$

where $\sigma' = \pm s$ and $\tau' = \pm t$. A computation shows that

$$\begin{aligned} u \;=\; & (\sigma + \sigma')(\tau + \tau')z_{xy} \\ & +\{\sigma(\sigma + \sigma')(\tau + \tau') + \tau(\tau + \tau') + (\sigma + \sigma')/2\}z_x \\ & +\{\tau(\sigma + \sigma')(\tau + \tau') + \sigma(\sigma + \sigma') + (\tau + \tau')/2\}z_y \\ & +\{\sigma\tau(\sigma + \sigma')(\tau + \tau') + \sigma^2(\sigma + \sigma') + \tau^2(\tau + \tau') - (\sigma\tau + \sigma'\tau')/2 + 1/4\}z, \end{aligned}$$

and we get the following list:

$T1$	$z_{xy} + sz_x + tz_y + (st - 1/2)z]$	$T2$	$z_{xy} - sz_x + tz_y + (-st - 1/2)z$
$T3$	$z_{xy} + sz_x - tz_y + (-st - 1/2)z$	$T4$	$z_{xy} - sz_x - tz_y + (st - 1/2)z$
$T12$	$2t^2 z_x + tz_y + (2t^3 + 1/4)z$	$T13$	$sz_x + 2s^2 z_y + (2s^3 + 1/4)z$
$T24$	$-sz_x + 2s^2 z_y + (-2s^3 + 1/4)z$	$T34$	$2t^2 z_x - tz_y + (-2t^3 + 1/4)z$
$T14$	$S(\sigma = t, \tau = s)$	$T24$	$S(\sigma = t, \tau = -s)$
$T34$	$S(\sigma = -t, \tau = s)$	$T44$	$S(\sigma = -t, \tau = -s)$,

where $S(\sigma, \tau)$ denotes the surface given by the vector

$$\begin{gathered}4\sigma\tau z_{xy} + (4\sigma^2\tau + 2\tau^2 + \sigma)z_x + (4\sigma\tau^2 + 2\sigma^2 + \tau)z_y \\ +(4\sigma^2\tau^2 + 2\sigma^3 + 2\tau^3 - \sigma\tau + 1/4)z.\end{gathered}$$

Remark 12.8 *The contents for Sects. 12.1 and 12.2 are based on [May1932] and [Su1936, Su1957] with modification relying only on the moving frame. We refer to [Mar1979] for Sect. 12.3.*

13 Transformations of projectively minimal surfaces

In Sect. 5.6, we showed the formula for Tzitzeica transformation of affine spheres. This section gives its generalization, a formula that transforms a given projectively minimal surface to a new projectively minimal surface. The formula is originally due to R. Marcus [Mar1980].

13.1 W-congruences of projectively minimal surfaces

In Section 10, we exhibited how to construct W-congruences, where one of the focal surfaces is given by the system

$$z_{xx} = \theta_x z_x + b z_y + pz, \quad z_{yy} = c z_x + \theta_y z_y + qz,$$

and the other focal surface is defined by

$$w = \mu z + 2(A z_x + B z_y).$$

In this section, we restrict our attention to projectively minimal surfaces.

We recall that we have assumed (10.5):

$$A_y + cB = 0, \quad B_x + bA = 0 \tag{13.1}$$

so that the congruence is a W-congruence, and we defined μ by

$$\mu = -A_x - A\theta_x - B_y - B\theta_y.$$

We have also defined $\mathcal{L}$ and $\mathcal{M}$ in (10.11):

$$\mathcal{L} = AA_{xx} - \frac{1}{2}A_x^2 + A^2L, \quad \mathcal{M} = BB_{yy} - \frac{1}{2}B_y^2 + B^2M,$$

and saw that

$$N = 2(\mathcal{M} - \mathcal{L}).$$

We assume here that $N \neq 0$.

Lemma 13.1 *Assume the surface is projectively miminal. Then, $\mathcal{L}$, $\mathcal{M}$, and N satisfy the same equation*

$$X_{xy} + \frac{cB}{A}X_x + \frac{bA}{B}X_y = 0.$$

Proof. Making use of the identity (13.1) and one of the integrability conditions $L_y = -2bc_x - cb_x$, we see that

$$\mathcal{L}_{xy} + \frac{cB}{A}\mathcal{L}_x + \frac{bA}{B}\mathcal{L}_y = -AB(c_{xxx} + cL_x + 2c_xL).$$

The condition of projective minimality implies that the right-hand side vanishes, and we get the result for $\mathcal{L}$. The case for $\mathcal{M}$ is similarly done.

The invariants for the surface w were given in (10.9). By the expressions of these invariants, we can compute L and M for w, which we denote by $\overline{L}$ and $\overline{M}$:

$$\overline{L} = L + \frac{N_{xx}}{N} - \frac{3}{2}\left(\frac{N_x}{N}\right)^2 + \frac{A_x}{A}\frac{N_x}{N},$$

$$\overline{M} = M + \frac{N_{yy}}{N} - \frac{3}{2}\left(\frac{N_y}{N}\right)^2 + \frac{B_y}{B}\frac{N_y}{N}.$$

We also recall Δ_1 and Δ_2 given in (11.5):

$$\Delta_1 = b_y^2 - 2bb_y - 2b^2M, \quad \Delta_2 = c_x^2 - 2cc_x - 2c^2L.$$

Then we have the following lemma on such invariants for w by direct computation:

Lemma 13.2 *Assume the surface z is projectively miminal. Then*

$$\overline{\Delta}_1 = \Delta_1 + \frac{4N_x}{A^2N^2}(\mathcal{M}\mathcal{L}_x - \mathcal{L}\mathcal{M}_x), \quad \overline{\Delta}_2 = \Delta_2 + \frac{4N_y}{B^2N^2}(\mathcal{M}\mathcal{L}_y - \mathcal{L}\mathcal{M}_y).$$

Provided that $bc \neq 0$, the conditions $\Delta_{1y} = 0$ and $\Delta_{2x} = 0$ are equivalent, and both imply the surface is projectively minimal; see (11.6). Hence we have the following proposition.

Proposition 13.3 *Assume that the surface z is projectively minimal and that*

$$\mathcal{M} = k\mathcal{L} \tag{13.2}$$

for a constant $k \neq 1$. Then the surface w is also projectively minimal.

Proof. From $\mathcal{M} = k\mathcal{L}$, we see that $\overline{\Delta}_1 = \Delta_1$. Hence $\overline{\Delta}_1$ is independent of y, which implies the result. $k \neq 1$ is necessary for $N \neq 0$.

Moreover, since $\Delta_1 = \Delta_2 = 0$ means that the surface z is a Demoulin surface, and since either one of the conditions $\Delta_1 = 0$ and $\Delta_2 = 0$ implies that the surface is a Godeaux-Rozet surface, we have the following proposition.

Proposition 13.4 *Assume that the surface z is projectively minimal and that*

$$\mathcal{M} = k\mathcal{L}$$

for some constant $k \neq 1$. If z is a Godeaux-Rozet surface, then so is w. If z is a Demoulin surface, then so is w.

13.2 Transformations of projectively minimal surfaces

We now apply the propositions above by finding A and B satisfying (13.1) and (13.2). Let us consider the surface given by

$$z_{xx} = bz_y + (\pi - \frac{1}{2}b_y)z, \quad z_{yy} = cz_x + (\rho - \frac{1}{2}c_x)z \tag{13.3}$$

so that $L = -2\pi$ and $M = -2\rho$ take simple forms. We assume z is projectively minimal:

$$bM_y + 2Mb_y + b_{yyy} = 0, \quad cL_x + 2Lc_x + c_{xxx} = 0.$$

For a parameter $\alpha(\neq 1)$, we define a new system

$$\overline{z}_{xx} = \alpha b\overline{z}_y + (\pi - \frac{\alpha}{2}b_y)\overline{z}, \quad \overline{z}_{yy} = \frac{1}{\alpha}c\overline{z}_x + (\rho - \frac{1}{2\alpha}c_x)\overline{z}. \tag{13.4}$$

Since $L = -2\pi$ and $M = -2\rho$ also for this system, L and M are unchanged and the product bc is also unchanged. Hence, the system for $\overline{z}$ is integrable.

We now let φ and ψ be two independent solutions of (13.4) and define

$$A = -\alpha(\psi\varphi_y - \varphi\psi_y), \quad B = \psi\varphi_x - \varphi\psi_x. \tag{13.5}$$

It is easy to see that the condition (13.2) is satisfied; in fact

$$\begin{aligned} A_y &= -\alpha(\psi\varphi_{yy} - \varphi\psi_{yy}) \\ &= -\alpha\left(\psi(\frac{1}{\alpha}c\varphi_x + (\rho - \frac{1}{2}c_x)\varphi) - \varphi(\frac{1}{\alpha}c\psi_x + (\rho - \frac{1}{2}c_x)\psi)\right) \\ &= -cB. \end{aligned}$$

Furthermore, by computation, we see that

$$\begin{aligned} \frac{2}{\alpha^2}\mathcal{L} &= 2bc(\psi\varphi_y - \varphi\psi_y)(\psi\varphi_x - \varphi\psi_x) - (\varphi_x\psi_y - \psi_x\varphi_y)^2 - (\varphi\psi_{xy} - \psi\varphi_{xy})^2 \\ &\quad +(-4\psi\varphi_x\varphi_y + 2\varphi(\varphi_x\psi_y + \psi_x\varphi_y))\psi_{xy} \\ &\quad +(-4\varphi\psi_x\psi_y + 2\psi(\varphi_x\psi_y + \psi_x\varphi_y))\varphi_{xy}. \end{aligned}$$

The computation of $2\mathcal{M}$ yields the same right-hand side. Namely, we have $\mathcal{L} = \alpha^2\mathcal{M}$.

Theorem 13.5 *Assume the surface* (13.3) *is projectively minimal. For any constant* $\alpha(\neq \pm 1)$ *and independent solutions* φ *and* ψ *of the system* (13.4), *define* A *and* B *by* (13.5), *and define* μ *by*

$$\mu = (\alpha+1)(\psi_x\varphi_y - \varphi_x\psi_y) + (\alpha-1)(\psi\varphi_{xy} - \varphi\psi_{xy}).$$

Then the surface

$$w = \mu z + 2(Az_x + Bz_y)$$

defines a projectively minimal surface. If z *is a Demoulin (resp. Godeaux-Rozet) surface, then so is* w.

Let $\varphi_1, \ldots, \varphi_4$ be independent solutions of the system (13.4). Then φ and ψ are linear combinations of these solutions, say, $\varphi = \sum_i k^i\varphi_i$ and $\psi = \sum_i m^i\varphi_i$. For $i < j$, we define the coefficients

$$A_{ij} = -\alpha(\varphi_j\varphi_{iy} - \varphi_i\varphi_{jy}), \quad B_{ij} = \varphi_j\varphi_{ix} - \varphi_i\varphi_{jx},$$
$$\mu_{ij} = (\alpha+1)(\varphi_{jx}\varphi_{iy} - \varphi_{ix}\varphi_{jy}) + (\alpha-1)(\varphi_j\varphi_{ixy} - \varphi_i\varphi_{jxy}),$$

and the surface

$$w_{ij} = \mu_{ij}z + 2(A_{ij}z_x + B_{ij}z_y).$$

Then, the surface w associated with the pair φ and ψ is a combination of w_{ij}:

$$w = \sum_{i<j} c^{ij}w_{ij},$$

where $c^{ij} = k^i m_j - k^j m^i$. The coefficients c^{ij} can be regarded as the wedge product of the vectors (k^i) and (m^j) and they satisfy the Plücker relation $c^{12}c^{34} - c^{13}c^{24} + c^{14}c^{23} = 0$. We remark that the system satisfied by the surface w can be computed by using the formulas (10.8) and (10.9).

We next modify the transformation formula in Theorem 13.5 to show that the Tzitzeica transformation is derived from the above transformation. We start with the projectively minimal surface z given by the system of a general form

$$z_{xx} = \theta_x z_x + bz_y + pz, \quad z_{yy} = cz_x + \theta_y z_y + qz$$

and then we introduce another system by

$$\overline{z}_{xx} = \overline{\theta}_x\overline{z}_x + \alpha b\overline{z}_y + \overline{pz}, \quad \overline{z}_{yy} = \frac{c}{\alpha}\overline{z}_x + \overline{\theta}_y\overline{z}_y + \overline{qz},$$

so that $\overline{L} = L$ and $\overline{M} = M$. Since both systems can be normalized to the forms in (13.3) and (13.4) by multiplying z and $\overline{z}$ by some factors, existence of such a system is trivial; however, finding explicit representations of $\overline{\theta}$, $\overline{p}$, and $\overline{q}$ is not simple. Not withstanding this ambiguity, we define A and B by

$$A = -\alpha\exp(-\overline{\theta})(\psi\varphi_y - \varphi\psi_y), \quad B = \exp(-\overline{\theta})(\psi\varphi_x - \varphi\psi_x).$$

Furthermore, we can modifiy the systems above so that $\overline{p} = 0$ and $\overline{q} = 0$:

$$z_{xx} = \theta_x z_x + b z_y, \quad z_{yy} = c z_x + \theta_y z_y,$$

and

$$\overline{z}_{xx} = \overline{\theta}_x \overline{z}_x + \alpha b \overline{z}_y, \quad \overline{z}_{yy} = \frac{c}{\alpha}\overline{z}_x + \overline{\theta}_y \overline{z}_y,$$

so that $\psi = 1$ is a trivial solution. Then, we get

$$A = -\alpha \exp(-\overline{\theta})\varphi_y, \quad B = \exp(-\overline{\theta})\varphi_x.$$

We assume further that $\overline{\theta} = \theta$. Then, $b\theta_y + b_y = 0$ and $c\theta_x + c_x = 0$ are necessary to conclude that $\overline{L} = L$ and $\overline{M} = M$. Since $(\log(b/c))_{xy} = 0$ in this case, we can assume that $c = b$. Now the situation is reduced to the case where the original system has the form

$$z_{xx} = -\frac{b_x}{b} z_x + b z_y, \quad z_{yy} = b z_x - \frac{b_y}{b} z_y.$$

The integrability condition is

$$(\log b)_{xy} = b^2 + \frac{k}{b},$$

where k is any constant. If we set $b = 1/h$, then these equations are written as

$$z_{xx} = \frac{h_x}{h} z_x + \frac{1}{h} z_y, \quad z_{yy} = \frac{1}{h} z_x + \frac{h_y}{h} z_y.$$

The integrability condition is

$$(\log h)_{xy} = -kh - \frac{1}{h^2}.$$

This is the case we have considered in Sect. 5.6 for Tzitzeica transformation of affine spheres. In fact, setting $k = -1$, we assume that z satisfies also

$$z_{xy} = hz,$$

so that z defines an affine sphere. Let φ be a nontrivial solution of

$$\overline{z}_{xx} = \frac{h_x}{h}\overline{z}_x + \frac{\alpha}{h}\overline{z}_y, \quad \overline{z}_{yy} = \frac{1}{\alpha h}\overline{z}_x + \frac{h_y}{h}\overline{z}_y,$$

and assume it satisfies also

$$\varphi_{xy} = h\varphi.$$

Then we get the transformed surface

$$w = \frac{1}{2}(\alpha - 1)\varphi z - \frac{\alpha \varphi_y}{h} z_x + \frac{\varphi_x}{h} z_y,$$

which is a new affine sphere. This gives nothing but the Tzitzeica transformation stated in Theorem 5.10.

Remark 13.6 In [Fe1999], E. V. Ferapontov and W. K. Schief gave the transformation formula of 5.10 relative to Demoulin surfaces by appealing to the system of nonlinear equations (2.27) and using the Moutard transformation. Their formulation was generalized in [RS2002] to projectively minimal surfaces by C. Rogers and W. K. Schief, resulting the same formula.

A Line congruences derived from Appell's system (F_2)

Appell's system is defined in Sect. 4.7:

$$D_1 z = 0 \qquad \text{and} \qquad D_2 z = 0,$$

where

$$\begin{aligned} D_1 &= \quad x(1-x)\partial_{xx} - xy\partial_{xy} + \{\gamma - (\alpha+\beta+1)x\}\partial_x - \beta y \partial_y - \alpha\beta, \\ D_2 &= y(1-y)\partial_{yy} - xy\partial_{xy} + \{\gamma' - (\alpha+\beta'+1)y\}\partial_y - \beta' x \partial_x - \alpha\beta'. \end{aligned}$$

Appell's fucntion $F_2 = F_2(\alpha, \beta, \beta', \gamma, \gamma'; x, y)$ is given also in Sect. 4.7. When certain number of parameters are increased or decreased by 1, the set of parameters is said to be contiguous to the original set of parameters and any function F_2 with (a set of) contiguous parameters, say, $F_2(\alpha \pm 1, \beta, \beta', \gamma.\gamma'; x, y)$, $F_2(\alpha \pm 1, \beta \pm 1, \ldots; x, y)$, $\cdots$, is called a contiguous function of F_2. Since the dimension of the solution space of the system is four, contiguous functions are written by using F_2, $(F_2)_x$, $(F_2)_y$, and $(F_2)_{xy}$ and formulas representing such relations are called *contiguity relations*. However, some contiguous functions are written by using only the first three not including second order derivative. For example, the operator H_0 increases β by $+1$ and the operator B_0 decreases β by -1 and both give contiguous functions; H_n and B_n are defined in Sect. 4.7. Geometrically speaking, in such a case, any point of the surface defined by the system of contiguous parameters (to be called a contiguous surface) is lying on the tangent plane at the point of the same coordinates of the original surface. We refer to [Sa1991] for the list of the basic contiguous relations of Appell's system corresponding to Appell's function usually denoted by F_3; note that the system is equivalent to (F_2). From the list, we can read that the pair of H_0 and B_0 is essentially unique in the sense that the contiguous surface is lying on the tangent space of the original surface as well as the original surface is lying on the tangent space of the contiguous surface. This means that these two surfaces are focal surfaces of the line congruence joining them. In the following, we describe explicitly this congruence in terms of Laplace transformation.

Since the invariant quadratic form is conformal to $(y/(1-x))dx^2 + 2dxdy + (x/(1-y))dy^2$, one choice of conjugate directional fields is given by $(x-1)\partial_x$ and $(x-1)\partial_x + y\partial_y$. We adopt here the notation

$$X = (x-1)\partial_x, \qquad Y = (x-1)\partial_x + y\partial_y. \tag{A.1}$$

They are commutative: $[X, Y] = 0$.

In terms of X and Y, the operators D_1 and D_2 are written as follows:

$$\begin{aligned} D_1 &= -\frac{1}{x-1}(xXY + ((\beta-\gamma)+\alpha x)X + \beta(x-1)Y + \alpha\beta(x-1)), \\ D_2 &= -\frac{y-1}{y}YY + \left(\frac{x}{x-1} - \frac{y-1}{y}\right)XX + AY + BX + C, \end{aligned} \tag{A.2}$$

where

$$A = -2\beta\frac{(x-1)(y-1)}{xy} + \beta - \beta' - \alpha + \frac{\gamma' - 1}{y},$$
$$B = \frac{2(\beta-\gamma)(1-y)}{xy} + \frac{2\alpha+1-\gamma'}{y} + \frac{\alpha+\beta-\beta'-\gamma}{x-1},$$
$$C = -\frac{2\alpha\beta(x-1)(y-1)}{xy} + \alpha(\beta-\beta').$$

The first equation is

$$XYz + \left(\frac{\beta-\gamma}{x} + \alpha\right) Xz + \beta\frac{x-1}{x}Yz + \alpha\beta\frac{x-1}{x}z = 0. \tag{A.3}$$

Hence the Laplace transforms are

$$z^1 = Yz + \left(\frac{\beta-\gamma}{x} + \alpha\right) z, \quad z^{-1} = Xz + \beta\frac{x-1}{x}z, \tag{A.4}$$

by definition. Note that z^1 is equal to $H_0 z$ and z^{-1} to $B_0 z$ up to scalar multiple. The Laplace invariants are

$$h = (\beta-1)(\beta-\gamma)\frac{x-1}{x^2}, \qquad k = \beta(\beta-\gamma+1)\frac{x-1}{x^2} - \alpha\beta\frac{x-1}{x}.$$

The Laplace equation for z^1 is, by computation,

$$XYz^1 + \left(\alpha+1+\frac{\beta-\gamma-2}{x}\right) Xz^1 + \beta\frac{x-1}{x}Yz^1 + \left(\beta(\alpha+1)\frac{x-1}{x} - \gamma\frac{x-1}{x^2}\right) z^1 = 0.$$

We next define a projective frame to show how to compute the invariants of the line congruence $\{z, z^{-1}\}$ by

$$\begin{aligned}
e_1 &= x^\beta z, \\
e_2 &= x^{\gamma-\beta-1}(x-1)^{\alpha+\beta-\gamma}(xXz + \beta(x-1)z), \\
e_3 &= x^\beta\left(Yz + \frac{\beta(x-1)}{x}z\right), \\
e_4 &= Xe_2,
\end{aligned}$$

and introduce coordinates (u, v) by requiring $X = \partial_u$ and $Y = \partial_v$. Then for any function f we have $df = (Xf)du + (Yf)dv$. By using the expressions (A.2), we can prove the following:

$$\omega^1 = \omega_1^3 = dv, \qquad \omega^2 = \omega_2^4 = du,$$
$$\omega_1^2 = x^{2\beta-\gamma}(x-1)^{\gamma-\alpha\beta}du,$$
$$\omega_2^1 = \beta(\beta-\gamma+1)x^{\gamma-2\beta-2}(x-1)^{\alpha+\beta-\gamma+1}dv,$$
$$\omega_3^4 = \frac{x+y-1}{y-1}x^{2\beta-\gamma}(x-1)^{\gamma-\alpha-\beta-1}dv,$$
$$\omega_4^3 = \beta(\beta-\gamma+2)\frac{y-1}{x+y-1}x^{\gamma-2\beta-2}(x-1)^{\alpha+\beta-\gamma+2}du.$$

Therefore, by the definition of the invariant W in Sect. 4, we see that

$$W = h_{12}h_{21} - h_{31}h_{42} = -\beta\frac{x-1}{x^2},$$

which means the line congruence $\{z, z^{-1}\}$ is a W-congruence only when $\beta = 0$.

B Line congruences in $\mathbf{E}^3$

Let us give a brief summary of the theory of line congruences in the 3-dimensional Euclidean space $\mathbf{E}^3$, first formulated by Kummer [Ku1860], and then explain how the notion of W-congruence was introduced.

1. Each line in the congruence parametrized by (x, y) is described by the pair consisting of a base point $z(x, y)$ on the line and the directional unit vector $\phi(x, y)$. The line is expressed as $z(x, y) + t\phi(x, y)$ with the line parameter t. On the other hand, we can regard $z(x, y)$ as a surface and $\phi(x, y)$ as a spherical surface. The surface z is called a reference surface. If we choose another reference surface $z'(x, y)$, we have a relation such as $z'(x, y) = z(x, y) + \ell\phi(x, y)$. The fundamental invariants related to this pair are the metric form of the spherical surface

$$\mathrm{I} = \langle d\phi, d\phi\rangle = Edx^2 + 2F dx\, dy + Gdy^2, \tag{B.1}$$

and the 2-form

$$\mathrm{I\!I} = -\langle dz, de\rangle = pdx^2 + (q + q')dx\, dy + rdy^2, \tag{B.2}$$

where

$$E = \langle \phi_x, \phi_x\rangle, F = \langle \phi_x, \phi_y\rangle, G = \langle \phi_y, \phi_y\rangle,$$
$$p = \langle \phi_x, z_x\rangle, q = \langle \phi_x, z_y\rangle, q' = \langle \phi_y, z_x\rangle, r = \langle \phi_y, z_y\rangle.$$

The form $\mathrm{I\!I}$ for the reference surface z', denoted by $\mathrm{I\!I}'$, is

$$\mathrm{I\!I}' = \mathrm{I\!I} + \ell\mathrm{I}.$$

If we write I as $\sum g_{ij}dx^i dx^j$, where $x^1 = x$ and $x^2 = y$, then the Gauss equation for the map ϕ is

$$\phi_{ij} = \sum_k \Gamma^k_{ij}\phi_k - g_{ij}z, \tag{B.3}$$

where Γ^k_{ij}'s are the Christoffel symbols relative to $\{g_{ij}\}$.
2. The line congruence is called a *normal congruence* if there exists a surface z' such that ϕ coincides with the unit normal of the surface z'. The condition is $\langle dz', \phi\rangle = 0$, which implies $\langle dz, \phi\rangle + dt = 0$. Then, by exterior differentiation, we must have $\langle z_x, \phi_y\rangle = \langle z_y, \phi_x\rangle$, namely, $q = q'$. Conversely, if $q = q'$, we can find such a z'.

3. The focal points on each line are $Z(x,y) = z(x,y) + t\phi(x,y)$ for which dZ is parallel to ϕ. This implies that $\mathrm{II} + t\mathrm{I} = 0$. In particular, the value t is determined by the quadratic equation

$$\begin{vmatrix} p + tE & q + tF \\ q' + tF & r + tG \end{vmatrix} = 0.$$

For each t, the direction $dx : dy$ determines the degenerate ruled surface, in which lies a focal point.

4. The fundamental equations of the congruence $\{z, \phi\}$ are the Gauss equation (B.3) and the system

$$\begin{aligned} z_x &= a\phi_x + b\phi_y + m\phi, \\ z_y &= c\phi_x + d\phi_y + n\phi. \end{aligned} \tag{B.4}$$

The coefficients are determined by the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} p & q' \\ q & r \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}, \quad m = \langle \phi, z_x \rangle, \quad n = \langle \phi, z_y \rangle.$$

The integrability condition of the last system is computed by developing the equation $(z_x)_y = (z_y)_x$ into a linear combination of ϕ, ϕ_x, and ϕ_y, by using (B.3). The resulting equations are

$$\begin{aligned} &a_y + a\Gamma^1_{12} + b\Gamma^1_{22} - c_x - c\Gamma^1_{11} - d\Gamma^1_{12} - n = 0, \\ &b_y + a\Gamma^2_{12} + b\Gamma^2_{22} - d_x - c\Gamma^2_{11} - d\Gamma^2_{12} + m = 0, \\ &m_y - n_x - aF - bG + cE + dF = 0. \end{aligned}$$

By making use of the formula expressing the Christoffel symbol Γ^k_{ij} in terms of E, F and G, we can see that the system above is equivalent to the system

$$\left\{ \begin{array}{c} p_y - q_x - \Gamma^1_{12}p + \Gamma^1_{11}q - \Gamma^2_{12}q' + \Gamma^2_{11}r + Fm - En = 0, \\ q'_y - r_x - \Gamma^1_{22}p + \Gamma^1_{12}q - \Gamma^2_{22}q' + \Gamma^2_{12}r + Gm - Fn = 0, \\ m_y - n_x + q - q' = 0. \end{array} \right. \tag{B.5}$$

We summarize the above argument as follows:

Theorem B.1 *The line congruence $\{\phi, z\}$ described above is determined by the systems* (B.3) *and* (B.4) *satisfying the integrability condition* (B.5)*. Conversely, given a spherical surface ϕ and the coefficients p, q, q', r, m, and n satisfying* (B.5)*, any solution of* (B.4) *gives a reference surface z, and thus the line congruence.*

Example B.2 *Let us consider the normal congruence for a surface in the Euclidean space. The surface is z and ϕ is its Gauss mapping. The shape operator A is given by $\nabla_X \phi = -AX$. The directional field $dx : dy$ for determining the focal points is given by $-AX \equiv 0 \pmod{X}$. The integral curves are nothing but the curvature lines.*

5. A surface in $\mathbf{E}^3$ is called a Weingarten surface if there exists a functional relation between the two principal curvatures. We here recall how normal congruence is related to the W-congruence.

Assume (x, y) are the curvature coordinates so that the first fundamental form of the surface is $edx^2 + gdy^2$ (to avoid confusion with the first fundamental form I for the surface ϕ, we do not use the usual notation E and G) and the second fundamental form is nothing but $\mathbb{I}$ above with the condition $q = q' = 0$.

The principal curvatures k_1 and k_2 are given by $k_1 = p/e$, $k_2 = r/g$.

The Codazzi identity is written as

$$(k_2)_x + \frac{1}{2}(k_2 - k_1)\frac{\partial \log g}{\partial x} = 0, \quad (k_1)_y + \frac{1}{2}(k_1 - k_2)\frac{\partial \log e}{\partial y} = 0.$$

We assume $k_1 k_2 \neq 0$ and set $R_i = 1/k_i$, which are the principal radii. Then the coordinates have the property that

$$z_x + R_1\phi_x = 0, \qquad z_y + R_2\phi_y = 0.$$

For the surface z, we define two mappings by

$$w_i = z + R_i\phi \qquad i = 1, 2. \tag{B.6}$$

They define in general two surfaces called the central surfaces. (We do not treat the case where these mappings are degenerate.)

Let $w = w_1$ for the moment. The relation above shows

$$w_x = R_{1x}\phi, \qquad w_y = (1 - R_1/R_2)z_y + R_{1y}\phi,$$

which imply that the unit normal to w is $\phi_1 = z_x/\sqrt{e}$ up to sign. Then it is easily seen, by using the Coddazi identity, that the coefficients of the second fundamental form of w, denoted by p_1, q_1, and r_1, are given by

$$p_1 = -\sqrt{e}(\log R_1)_x, \quad q_1 = 0, \quad r_1 = -\frac{gR_1}{\sqrt{e}R_2}(\log R_2)_y.$$

Thus, the second fundamental form of the first surface is conformal to

$$eR_2^2 R_{1x}dx^2 - gR_1^2 R_{2x}dy^2.$$

Similarly, for the second surface we have

$$eR_2^2 R_{1y}dx^2 - gR_1^2 R_{2y}dy^2.$$

These computations show the following theorem.

Theorem B.3 *Assume the central surfaces are nondegenerate. Then the normal congruence between them is a W-congruence if and only if the principal radii satisfy the relation*

$$\begin{vmatrix} R_{1x} & R_{2x} \\ R_{1y} & R_{2y} \end{vmatrix} = 0. \tag{B.7}$$

The original surface satisfying this relation is classically called a Weingarten surface, and this is the origin of the naming of W-congruence.

For a general theory of normal congruence, we refer to e.g. [D, no 447–455] and [E1909, Chap. XII].

C Plücker image of projective surfaces into $\mathbf{P}^5$

We here give a proof of Theorem 4.25.

From the asummption $(t,t) = h$, the connection form ω satisfies $\omega h + h\,{}^t\omega = 0$. Together with the condition $\omega_1^4 = \omega_1^5 = \omega_1^6 = 0$ and $\omega_2^3 = \omega_2^5 = \omega_2^6 = 0$ from (4.14), the form ω can be written as

$$\omega = \begin{pmatrix} \omega_1^1 & \omega_1^2 & \omega^2 & 0 & 0 & 0 \\ \omega_2^1 & \omega_2^2 & 0 & \omega^1 & 0 & 0 \\ \omega_3^1 & \omega_3^2 & 0 & \sigma & \omega^2 & 0 \\ \omega_4^1 & \omega_4^2 & \sigma & 0 & 0 & \omega^1 \\ 0 & \tau & \omega_3^1 & -\omega_4^1 & -\omega_1^1 & \omega_2^1 \\ \tau & 0 & -\omega_3^2 & \omega_4^2 & \omega_1^2 & -\omega_2^2 \end{pmatrix},$$

where we set $\sigma = \omega_3^4$ and $\tau = \omega_5^2$. It satisfies the following integrability conditions.

$$\begin{array}{rcccccccc} d\omega_1^1 & = & & & \omega_1^2 \wedge \omega_2^1 & + & \omega^2 \wedge \omega_3^1, & & \\ d\omega_1^2 & = & \omega_1^1 \wedge \omega_1^2 & + & \omega_1^2 \wedge \omega_2^2 & + & \omega^2 \wedge \omega_3^2, & & \\ d\omega_2^1 & = & \omega_2^1 \wedge \omega_1^1 & + & \omega_2^2 \wedge \omega_2^1 & + & \omega^1 \wedge \omega_4^1, & & \\ d\omega_2^2 & = & & & \omega_2^1 \wedge \omega_1^2 & + & \omega^1 \wedge \omega_4^2, & & \\ d\omega^1 & = & \omega_2^2 \wedge \omega^1, & & & & & & \\ d\omega^2 & = & \omega_1^1 \wedge \omega^2, & & & & & & \\ d\omega_3^1 & = & \omega_3^1 \wedge \omega_1^1 & + & \omega_3^2 \wedge \omega_2^1 & + & \sigma \wedge \omega_4^1, & & \\ d\omega_3^2 & = & \omega_3^1 \wedge \omega_1^2 & + & \omega_3^2 \wedge \omega_2^2 & + & \sigma \wedge \omega_4^2 & + & \omega^2 \wedge \tau, \\ d\omega_4^1 & = & \omega_4^1 \wedge \omega_1^1 & + & \omega_4^2 \wedge \omega_2^1 & + & \sigma \wedge \omega_3^1 & + & \omega^1 \wedge \tau, \\ d\omega_4^2 & = & \omega_4^1 \wedge \omega_1^2 & + & \omega_4^2 \wedge \omega_2^2 & + & \sigma \wedge \omega_3^2, & & \\ d\sigma & = & \omega_3^2 \wedge \omega^1 & - & \omega^2 \wedge \omega_4^1, & & & & \\ d\tau & = & \tau \wedge \omega_1^1 & + & \tau \wedge \omega_2^2 & - & \omega_4^1 \wedge \omega_4^2 & + & \omega_3^1 \wedge \omega_3^2, \\ 0 & = & \omega_1^2 \wedge \omega^1 & - & \sigma \wedge \omega^2, & & & & \\ 0 & = & \omega_2^1 \wedge \omega^2 & - & \sigma \wedge \omega^1. & & & & \end{array} \tag{C.1}$$

To check the action of the transformation g, we devide ω and g into 2×2-matrix components:

$$\omega = \begin{pmatrix} \Omega_1^1 & \Omega_1^2 & 0 \\ \Omega_2^1 & \Omega_2^2 & \Omega_2^3 \\ \Omega_3^1 & \Omega_3^2 & \Omega_3^4 \end{pmatrix}, \quad g = \begin{pmatrix} G_{11} & 0 & 0 \\ G_{21} & G_{22} & 0 \\ G_{31} & G_{32} & G_{33} \end{pmatrix}.$$

Since $g\,h\,{}^t g = h$, we have

$$\begin{gathered} G_{33} = J({}^tG_{11})^{-1}J, \quad G_{22}J^tG_{22} = J, \quad G_{22}J^tG_{32} = G_{21}J^tG_{33}, \\ G_{32}J^tG_{32} = G_{33}J^tG_{31} + G_{31}J^tG_{33}, \end{gathered} \tag{C.2}$$

By a transformation g, ω changes to $\tilde{\omega} = dg \cdot g^{-1} + g\omega g^{-1}$ that is written as

$$\tilde{\omega} = (\tilde{\omega}_i^j) = \begin{pmatrix} \tilde{\Omega}_1^1 & \tilde{\Omega}_1^2 & 0 \\ \tilde{\Omega}_2^1 & \tilde{\Omega}_2^2 & \tilde{\Omega}_2^3 \\ \tilde{\Omega}_3^1 & \tilde{\Omega}_3^2 & \tilde{\Omega}_3^4 \end{pmatrix},$$

where

$$
\begin{array}{rcl}
\tilde\Omega^1_1 &=& dG_{11}G_{11}^{-1} + G_{11}(\Omega^1_1 - \Omega^2_1 G_{22}^{-1}G_{21})G_{11}^{-1},\\
\tilde\Omega^2_1 &=& G_{11}\Omega^2_1 G_{22}^{-1},\\
\tilde\Omega^2_2 &=& dG_{22}G_{22}^{-1} + G_{22}(G_{22}^{-1}G_{21}\Omega^2_1 + \Omega^2_2 - \Omega^3_2 G_{33}^{-1}G_{32})G_{22}^{-1},\\
\tilde\Omega^3_2 &=& G_{22}\Omega^3_2 G_{33}^{-1},\\
\tilde\Omega^1_2 &=& dG_{21} - (dG_{22} + G_{21}\Omega^2_1 + G_{22}\Omega^2_2)G_{22}^{-1}G_{21}G_{11}^{-1}\\
&& +(G_{21}\Omega^1_1 + G_{22}\Omega^1_2)G_{11}^{-1} + G_{22}\Omega_{23}G_{33}^{-1}(-G_{31} + G_{32}G_{22}^{-1}G_{21})G_{11}^{-1},\\
\tilde\Omega^1_3 &=& dG_{31}G_{11}^{-1} - dG_{32}G_{22}^{-1}G_{21}G_{11}^{-1}\\
&& +(dG_{33} + G_{32}\Omega^3_2 + G_{33}\Omega_{33})G_{33}^{-1}(-G_{31} + G_{32}G_{22}^{-1}G_{21})G_{11}^{-1}\\
&& +(G_{31}\Omega^1_1 + G_{32}\Omega^1_2 + G_{33}\Omega^1_3)G_{11}^{-1}\\
&& -(G_{31}\Omega^2_1 + G_{32}\Omega_{22} + G_{33}\Omega^2_3)G_{22}^{-1}G_{21}G_{11}^{-1}.
\end{array}
\tag{C.3}
$$

Now let us take

$$
g = \begin{pmatrix} I & & 0 \quad 0 \\ G_{21} & I & 0 \\ \bullet & G_{32} & I \end{pmatrix}
$$

and set $G_{21} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$. Then $G_{32} = J^t G_{21} J = \begin{pmatrix} p & -r \\ -q & s \end{pmatrix}$ and

$$
\tilde\Omega^2_2 = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix} + \begin{pmatrix} 0 & q\omega^1 + r\omega^2 \\ q\omega^1 + r\omega^2 & 0 \end{pmatrix}.
$$

Hence, we can assume $\tilde\Omega^2_2 = 0$. In order to keep the form $\Omega^2_1 = \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega^1 \end{pmatrix}$, *i.e.*, in order to have $\tilde\omega^4_1 = \tilde\omega^3_2 = 0$, it is necessary to assume either (1): both G_{11} and G_{22} are diagonal or (2): both G_{11} and G_{22} are anti-diagonal. However, by the second condition of (C.2), the case (2) does not occur. Hence we can see that G_{11} is diagonal: $G_{11} = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$, $G_{22} = I_2$, and $G_{33} = G_{11}^{-1}$; followingly, $\tilde\Omega^2_1 = \begin{pmatrix} a\omega^2 & 0 \\ 0 & d\,\omega^1 \end{pmatrix} = \tilde\Omega^3_2$. Now insert $\Omega^2_2 = \tilde\Omega^2_2 = 0$ in the second identity of (C.3) and then we can see that G_{21} must have the form $\begin{pmatrix} p & 0 \\ 0 & s \end{pmatrix}$ and $G_{32} = \begin{pmatrix} p/a & 0 \\ 0 & s/d \end{pmatrix}$. By setting $G_{31} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$, we check the 4-th condition of (C.2). Then we see that $e = p^2/(2d)$, $h = s^2/(2d)$, $g/a = f/d =: \lambda$. Now we have seen that $\tilde\Omega^1_1$ can take the form:

$$
\tilde\Omega^1_1 = \begin{pmatrix} \omega^1_1 + d\log a - p\omega^2 & \frac{a}{d}\omega^2_1 \\ \frac{d}{a}\omega^1_2 & \omega^2_2 + d\log d - s\omega^1 \end{pmatrix}.
$$

By the 5th formula of (C.3), we can see

$$
\tilde\Omega^1_2 = \begin{pmatrix} \frac{p}{a}\omega^1_1 + \frac{1}{a}(dp + \omega^1_3) - e\omega^2 & \frac{p}{d}\omega^2_1 + \frac{1}{d}\omega^2_3 - a\lambda\omega^2 \\ \frac{s}{a}\omega^1_2 + \frac{1}{a}\omega^1_4 - d\lambda\omega^1 & \frac{s}{d}\omega^2_2 + \frac{1}{d}(ds + \omega^1_4) - h\omega^1 \end{pmatrix}.
$$

It remains to see what is $\tilde{\Omega}^1_3$. A computation shows:

$$\tilde{\Omega}^1_3 = \begin{pmatrix} 0 & \tilde{\tau} \\ \tilde{\tau} & 0 \end{pmatrix}, \qquad \begin{array}{rcl} \tilde{\tau} & = & \frac{1}{ad}\tau + d\lambda + \lambda(d(\log d) + \omega^1_1 + \omega^2_2 - s\omega^1 - p\omega^2) \\ & & + \frac{p^2}{2ad}\omega^2_1 + \frac{s^2}{2ad}\omega^1_2 + \frac{p}{ad}\omega^2_3 + \frac{s}{ad}\omega^1_4. \end{array}$$

Thus we have reduced the form of ω. Next, we examin the integrability. By $d\omega^1 = \omega^2_2 \wedge \omega^1$ and $d\omega^2 = \omega^1_1 \wedge \omega^2$, we can find coordinates (x, y) such that $\omega^1 = dx$ and $\omega^2 = dy$. Then G_{11} must be a constant matrix and we may assume $G_{11} = I_2$. In this case, we must have $\omega^2_2 \wedge \omega^1 = 0$ and $\omega^1_1 \wedge \omega^2 = 0$ and, by the formulas $\tilde{\omega}^1_1 = \omega^1_1 - p\omega^2$ and $\tilde{\omega}^2_2 = \omega^2_2 - s\omega^1$, we can assume $\omega^1_1 = \omega^2_2 = 0$. Then $G_{21} = 0$. Since we saw $\sigma = 0$, it holds $\omega^2_1 \wedge \omega^1 = \omega^1_2 \wedge \omega^2 = 0$. This means that we can set $\omega^2_1 = b\omega^1$ and $\omega^1_2 = c\omega^2$. The first identity of (C.1) becomes $d\omega^1_1 = bc\omega^1 \wedge \omega^2 + \omega^2 \wedge \omega^1_3$ while $d\omega^1_1 = 0$. This implies that there exists a scalar κ so that $\omega^1_3 = bc\omega^1 + \kappa\omega^2$. Similarly, from the fourth identity of (C.1), $\omega^2_4 = bc\omega^2 + \delta\omega^1$ for another scalar δ. From the second and the third identities, we may set $\omega^2_3 = b_y\omega^1 + \beta\omega^2$ and $\omega^1_4 = c_x\omega^2 + \gamma\omega^1$. Further, $\sigma = 0$ implies $\omega^2_3 \wedge \omega^1 = \omega^2 \wedge \omega^1_4$. Hence, $\beta = \gamma$. Here we look at the formula of $\tilde{\Omega}^1_2$, which says that we can make $\beta = \gamma = 0$ by choosing λ appropriately. Now we have a final form of ω that is given in (4.12) by setting $\tau = \mu\omega^1 + \nu\omega^2$. The conditions relative to $d\omega^2_3$ and $d\omega^1_4$ show that $\mu = b_{yy} - \kappa b$ and $\nu = c_{xx} - \delta c$. The conditions relative to $d\omega^1_3$, $d\omega^2_4$, and $d\tau$ give the three integrability conditions in (4.14). We thus complete the proof.

Remark C.1 The consideration above works also for the case when the signature is $(4, 2)$, which occurs in the Lie sphere geometry of surfaces in Euclidean space. Let $t = {}^t(t_1, \cdots, t_6)$ be a frame of $\mathbf{P}^5$ endowed with the inner product defined by

$$(u, v) = -u^0v^0 + u^1v^1 + u^2v^2 + u^3v^3 + u^4v^4 - u^5v^5$$

for two vectors $u = [u^0, \cdots, u^5]$ and $v = [v^0, \cdots, v^5]$. We assume t satisfies the orthonormality $(t, t) = h_{\mathcal{L}}$, where

$$h_{\mathcal{L}} = \begin{pmatrix} 0 & 0 & -I_2 \\ 0 & I_2 & 0 \\ -I_2 & 0 & 0 \end{pmatrix}.$$

Let $Q_{\mathcal{L}} = \{u \in \mathbf{P}^5; (u, u) = 0\}$ be a quadratic hypersurface called the Lie hyperquadric of signature $(4, 2)$. We assume that the line $\overline{t_1t_2}$ lies in $Q_{\mathcal{L}}$ and the conditions (4.15) and (4.16). Then, similarly as in the case of signature $(3, 3)$, we can find a transformation g with $gh_{\mathcal{L}}{}^tg = h_{\mathcal{L}}$ such that the connection form relative to the frame gt reduces to

$$\omega = \begin{pmatrix} 0 & bdx & dy & 0 & 0 & 0 \\ cdy & 0 & 0 & dx & 0 & 0 \\ bcdx+\kappa dy & b_y dx & 0 & 0 & dy & 0 \\ c_x dy & bcdy+\delta dx & 0 & 0 & 0 & dx \\ 0 & \mu dx+\nu dy & bcdx+\kappa dy & c_x dy & 0 & -cdy \\ -\mu dx-\nu dy & 0 & b_y dx & bcdy+\delta dx & -bdx & 0 \end{pmatrix} \tag{C.4}$$

and, in this case, we have identities

$$\mu = b_{yy} - b\kappa, \qquad \nu = -c_{xx} + c\delta \tag{C.5}$$

and the integrability condition is

$$\kappa_x = (bc)_y + cb_y, \quad \delta_y = (bc)_x + bc_x, \quad \nu_x - \mu_y = -\delta c_x - \kappa b_y. \tag{C.6}$$

The formulas (C.4)-(C.6) are different from the formulas (4.12)-(4.14) in Sect. 4.8 only in $\pm$-signs of certain variables. It is in [Fe2000b] and [Fe2002] that the dual correspondence is shown to be possible. To the case where the system (4.12) describes the image of a projectively minimal surface corresponds the case where the system (C.4) desribes a Lie minimal surface. A unified approach to both minimal surfaces was given in [BHJ2002].

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