Einstein-Hermitian metrics on non-compact Kähler manifolds

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In his paper [D], Donaldson showed that there is a natural correspondence between the moduli space of anti-self-dual connections with finite action on the complex Euclidean plane $\mathbf{C}^2$ and the moduli space of holomorphic vector bundles on the complex projective plane $\mathbf{P}^2$ whose restrictions to the complex line at infinity are trivial. The purpose of the paper is to generalize the result for certain class of Kähler manifolds.

Let $\bar{X}$ be an n-dimensional ($n \geq 2$) compact Kähler manifold and D a smooth divisor which has positive normal line bundle. We denote the complement of D by X and put a cone-like Kähler metric ω on X. We fix a point o in X and denote the distance from o by r. Then our main result is

Theorem 1. *There is a natural correspondence between the moduli space of Einstein-Hermitian holomorphic vector bundles on (X, ω) which satisfy the curvature decay condition*

$$|F| = O(r^{-2-\epsilon}), \qquad \text{with } \epsilon > 0,$$

and have trivial holonomy at infinity, and the moduli space of holomorphic vector bundles on $\bar{X}$ whose restrictions to D are $U(r)$-flat.

Corollary 2. *If (X, ω) is asymptotically locally Euclidean,* ALE *in short, then in the Theorem 1 we can replace the curvature decay condition by*

$$\int_X |F|^n < \infty.$$

And in this case it is equivalent to

$$|F| = O(r^{-(n+2-\epsilon)}), \qquad \text{for any } \epsilon > 0.$$

Corollary 3. *Let X be a non-singular compact Kähler surface, C a non-singular curve with positive self intersection $C^2 > 0$ and E a holomorphic* *vector bundle on X. If the restriction $E|_C$ of E is poly-stable with vanishing first Chern class $c_1(E|_C) = 0 \in H^2(C, \mathbf{R})$ then we have*

$$2c_2(E) - c_1(E)^2 \geq 0.$$

Moreover the equality holds if and only if E is flat.

Remark.

(i) In Corollary 3, if $E|_C$ is poly-stable but may have non-vanishing first Chern class, then considering $E \otimes E^*$ one can get the following inequality.

$$2rc_2(E) - (r-1)c_1(E)^2 \geq 0, \qquad r = \text{rank} E.$$

One can also show that the equality holds if and only if E is projectively flat.

(ii) Let X is a compact normal surface, C a smooth ample divisor and E a holomorphic vector bundle on X whose restriction to C is poly-stable with first Chern class zero. If we take a resolution, we can apply Corollary 3.

Here we remark that Theorem 1 can be considered as a sort of removable singularity theorem of holomorphic vector bundles across divisors. For a removable singularity theorem across subvarieties of higher co-dimension, the readers are referred to [B] and [B-S].

The author would like to express his thanks to Professors Mabuchi and Nakajima for helpful discussions. He also would like to acknowledge his gratitude to Max-Planck-Institut für Mathematik for hospitality. This work was done during his stay in Bonn.

1. Existence of Einstein-Hermitian metrics

Let $(\bar{X}, \omega_0)$ be a compact n-dimensional ($n \geq 2$) Kähler manifold and D a smooth divisor which has positive normal line bundle. We denote the Poincaré dual of D and its restriction to D by the same notation $[D]$. Set $X = \bar{X} \backslash D$. By assumption, there exists a Hermitian metric $\|\cdot\|$ on the line bundle L_D defined by D such that its curvature form θ is positive definite on a neighborhood of D. Pick a holomorphic section σ of L_D on $\bar{X}$ whose zero divisor is D. Put $t = \log\|\sigma\|^{-2}$. Then $\theta = \sqrt{-1}\partial\bar{\partial}\log\|\sigma\|^{-2} = \sqrt{-1}\partial\bar{\partial}t$. Fix an arbitrary positive number $a > 0$ and a sufficiently large positive constant C. We define a Kähler metric ω on X by

$$\begin{aligned}\omega &= \sqrt{-1}\partial\bar{\partial}\frac{1}{a}\exp(at) + C\omega_0 \\ &= \exp(at)\,\theta + a\exp(at)\sqrt{-1}\partial t \wedge \bar{\partial} t + C\omega_0.\end{aligned}$$

Here we identify a Kähler metric and its Kähler form. Then it is easy to see that ω is a $C^{k,\alpha}$-cone-like metric for any positive integer k, real number $0 < \alpha < 1$ and some positive number $\tau > 0$.

Definition. A complete Riemannian manifold (M, g) is said to be $C^{k,\alpha}$-cone-like of order $\tau > 0$ if there exists a compact subset K of M, a compact Riemannian manifold (N, h), a compact subset K' of the cone CN over N and a diffeomorphism $\phi : CN\backslash K' \longrightarrow M\backslash K$ such that it holds that up to $C^{k,\alpha}$-order

$$\phi^* g = dr^2 + r^2 h + O(r^{-\tau}).$$

Then in particular (M, g) is asymptotically flat in the following sense ([BK]).

Definition. A complete Riemannian metric g on a manifold M is said to be of $C^{k,\alpha}$-asymptotically flat geometry if for each point $p \in M$ with distance r from a fixed point o in M, there exists a harmonic coordinates system $x = (x^1, x^2, \ldots, x^m)$ centered at p which satisfies the following conditions:

(i) The coordinate x runs over a unit ball $\mathbf{B}^m$ in $\mathbf{R}^m$.

(ii) If we write $g = \sum g_{ij}(x)dx^i dx^j$, then the matrix $(r^2+1)^{-1}(g_{ij})$ is bounded from below by a constant positive matrix independent of p.

(ii) The $C^{k,\alpha}$-norms of $(r^2+1)^{-1}g_{ij}$, as functions in x, are uniformly bounded.

On such a manifold we can define the Banach space $C^{k,\alpha}_\delta$ of weighted $C^{k,\alpha}$-bounded functions: We may assume that $(r^2+1)^{-1}(g_{ij}) \leq 1/2(\delta_{ij})$

and the norm of a function $u \in C^{k,\alpha}_\delta$ is given by the supremum of the $C^{k,\alpha}$-norms of $(r^2+1)^{\delta/2}u$ with respect to the coordinates x. Then we can apply the interior Schauder estimates as in [BK]. Note that on a cone-like manifold the Sobolev inequality holds.

Definition. For the Hermitian holomorphic vector bundle (E,h) on (X,ω) with fast decreasing curvature $|F| = O(r^{-2-\epsilon})$, $\epsilon > 0$, we can define a holonomy at infinity as follows. Take a complex disk D in $\bar{X}$ which transversally intersects D at a point o and the circle S_r of radius r centered at o. Then the equivalence class of the holonomy of S_r converges as r tends to zero. It is easy to see that the equivalence class is independent of the choice of the disk, and we call it the holonomy at infinity.

Theorem 4. *Let E a holomorphic vector bundle on $\bar{X}$. If its restriction $E\big|_D$ to D is poly-stable and degree zero with respect to $[D]$, then the restriction $E\big|_X$ admits an Einstein-Hermitian metric with respect to the metric ω which satisfies the curvature decay condition*

$$|F| = O(r^{-2}).$$

Moreover, if $E\big|_D$ is flat, then the Einstein-Hermitian metric satisfies

$$|F| = O(r^{-2-\epsilon}),$$

and has trivial holonomy at infinity.

Proof. By assumption, $E\big|_D$ admits an Einstein-Hermitian metric h_0 with respect to the Kähler metric $\theta\big|_D$ (cf. [N-S], [D1–3], [U-Y1–2], [Sim], [Siu], [K]). We smoothly extend it over $\bar{X}$ and get a Hermitian metric, we still call it h_0, on E. Then it is easy to see that with respect to the metric ω, its curvature F_0 satisfies with some $0 < \epsilon < 1$

$$\begin{aligned} &|F_0| = O(r^{-2}), \qquad (|F_0| = O(r^{-2-\epsilon}), \quad \text{if } E\big|_D \text{ is flat}), \\ &|\Lambda F_0| = O(r^{-2-\epsilon}), \end{aligned}$$

together with the corresponding estimates for their covariant derivatives. Now we solve the following heat equation on Hermitian metric h.

$$\begin{aligned} \frac{dh}{dt}h^{-1} &= -\sqrt{-1}\Lambda F, \\ h\big|_{t=0} &= h_0. \end{aligned}$$

As shown by Simpson [Sim], for any compact smooth subdomain D in X, we have a unique solution until infinite time with the boundary condition

$$h\big|_{\partial D} = h_0\big|_{\partial D}.$$

Then it satisfies

$$\begin{aligned}
\frac{d\Lambda F}{dt} &= \Box \Lambda F, \\
\frac{d|\Lambda F|}{dt} &\leq \Box |\Lambda F|, \\
|\Lambda F|(t,x) &\leq \int_D H_D(t,x,y)|\Lambda F_0|(y).
\end{aligned}$$

Here $\Box$ and $H_D(t,x,y)$ are the complex (crude) Laplacian with respect to ω and its heat kernel on D with the Dirichlet boundary condition. Since the metric ω is cone-like, X admits the heat kernel $H(t,x,y)$ and the Green function $G(x,y) = \int_0^\infty H(t,x,y)dt$. Then

$$\begin{aligned}
\int_0^\infty |\frac{dh}{dt}h^{-1}|(x)dt &\leq \int_0^\infty dt \int_D H_D(t,x,y)|\Lambda F_0|(y) \\
&\leq \int_0^\infty dt \int_X H(t,x,y)|\Lambda F_0|(y) \\
&= \int_X G(x,y)|\Lambda F_0|(y).
\end{aligned}$$

Applying the argument of [B-K] to the function $u(x) = \int_X G(x,y)|\Lambda F_0|(y)$ which satisfies $\Box u = -|\Lambda F_0|(y) = O(r^{-2-\epsilon})$, we can show the estimate $u = O(r^{-\epsilon})$. Thus taking the limit of $t \longrightarrow \infty$ and $D \longrightarrow X$, the solution metrics conveges to an Einstein-Hermitian metric. Now we call it h and it holds that $|h - h_0| = O(r^{-\epsilon})$. Then the argument of [B-K] and the proof of Proposition 1 in [B-S] gives the higher order estimates of $h - h_0$ and hence the desired curvature estimates. The triviality of the holonomy of h at infinity follows from that for h_0 and the estimate of $h - h_0$.

If $n = 2$ and $E\big|_D$ is flat, then it holds

$$2c_2(E) - c_1(E)^2 = (8\pi^2)^{-1} \int_X (|F|^2 - |\Lambda F|^2).$$

Since $\Lambda F = 0$, we get

$$2c_2(E) - c_1(E)^2 = (8\pi^2)^{-1} \int_X |F|^2 \geq 0.$$

This shows the first part of Corollary 3, and that the equality implies the flatness of E on X. To show the flatness on $\bar{X}$ we need results in the next section.

Remark. By the similar argument we can show the following existence theorem for harmonic mappings. Let M be a not necessarily complete Riemannian manifold with the Green function $G(x, y) \geq 0$ and N a complete Riemannian manifold with non-positive sectional curvature. We denote the distance function on N by d. For a mapping $f : M \longrightarrow N$, we define

$$u_f(x) = \int_M G(x, y)|\Delta f|(y).$$

Theorem. *If the integral u_f converges and defines a continuous function on M, then f can be deformed by the heat equation to a harmonic mapping h which satisfies*

$$d(h(x), f(x)) \leq u_f(x).$$

2. Einstein-Hermitian bundles with fast curvature decay

By Theorem 4, we get the correspondence stated in Theorem 1 in one direction. Here we work in the converse direction. Let (E, h) be an Einstein-Hermitian holomorphic vector bundle on (X, ω) whose curvature F decreases rapidly such that with $0 < \epsilon < 1$

$$|F| = O(r^{-2-\epsilon}).$$

F satisfies the following equation.

$$(*) \qquad \Box F = F * R + F * F,$$

where R is the curvature tensor of the metric ω and $*$'s stand for some bilinear pairings.

The following Lemma 5 is standerd. For instance, refer to [B-K-N].

Lemma 5. *Let u, f and g be non-negative functions and τ a constant such that*

$$\Box u \geq -fu - g, \qquad f = O(r^{-2}), \quad g = O(r^{-2-\tau}),$$
$$\frac{1}{r^{2n}} \int_{B(x,\delta r)} u^2 = O(r^{-\tau}),$$

where $B(x, \delta r)$ is the ball of radius δr centered at x with some fixed number $0 < \delta < 1$. Then u satisfies

$$u = O(r^{-\tau}).$$

Lemma 6. *For any non-negative integar k, we have*

$$|\nabla^k F| = O(r^{-2-k-\epsilon}).$$

Proof. We only show the case $k = 1$. The general case is done by induction. The equation $(*)$ implies

$$(**) \qquad \Box |F|^2 \geq |\nabla F|^2 - C(|R||F|^2 + |F|^3).$$

Here and hereafter C stands for a general constant which may change in different appearance. Fix a small $0 < \delta < 1$ and take a cut-off function $\phi \geq 0$ such that $\phi = 1$ on $B(x, \delta r)$, $d(\text{supp}\phi, o) \geq \delta r$ and $\Box \phi \leq Cr^{-2}$.

Multiply the inequality $(**)$ by ϕ and integrate the result by parts, then we get

$$\int_{B(x,\delta r)} |\nabla F|^2 \leq \int \Box\phi |F|^2 + Cr^{-2}\int \phi|F|^2 \leq Cr^{2n-6-2\epsilon}.$$

Differentiate the equation $(*)$ and get

$$\Box\nabla F = R * \nabla F + F * \nabla F + \nabla R * F$$
$$\Box|\nabla F| \geq -C(|R| + |F|)|\nabla F| - C|\nabla R||F|.$$

Then we apply Lemma 5 and conclude that

$$|\nabla F| = O(r^{-3-\epsilon}).$$

From now on we work locally. We take a local coordinates system $(z', z) = (z^1, z^2, \ldots, z^{n-1}, z^n)$ at an arbitrary fixed point $p \in D$ such that $D = \{z^n = 0\}$. By calculation one can show the following Lemmas.

Lemma 7. *With respect to the flat metric $|dz'|^2 + |z^n|^{-2}|dz^n|^2$, the curvature F admits the following estimates. For any non-zero integer k*

$$|\nabla^k F| = O(|z|^{a\epsilon}).$$

We take an m-covering $\phi_m : (w', w^n) \longrightarrow (z', z^n)$ such that $z' = w'$ and $z^n = (w^n)^m$ with large positve integer m.

Lemma 8. *We pull back the bundle (E, h) to w-space by ϕ_m, then with respect to the flat metric $|dw|^2$*

$$|\nabla^k F| = O(|w|^{a\epsilon m-k-2}).$$

Now we put the assumption of trivial holonomy at infinity. On the w-space we have C^l-bound on the curvature tensor for any fixed l taking m large, the connection extends over the set $D_m = \{w^n = 0\}$ smoothly up to C^l-order (cf. [BKN]). Since outside D_m, the Hermitian connection satisfies the integrability condition, it remains so over D_m and defines a Hermitian holomorphic vector bundle E_m on the w-space. The deck transformation group $G_m = \{\tilde{\rho} : (w', w^n) \longrightarrow (w', \rho w^n) \mid \rho^m = 1\}$ lifts to a group of holomorphic bundle maps of E_m. We recover the original bundle E as the invariant subspace $E = E_m^{G_m}\big|_{\{w^n \neq 0\}}$. Since by assumption the isotropy group of G_m at D_m is trivial, the natural extension $\hat{E} = E_m^{G_m}$ of E over D is again a Hermitian holomorphic vector bundle. Note that $\hat{E}\big|_D$ and $E_m\big|_{D_m}$ is isomorphic and the later has vanishing curvature. Hence $\hat{E}\big|_D$ is a flat bundle. This completes the proof of the converse direction of Theorem 1. The proof also shows the last part of Corollary 3.

Corollary 2 follows from the results in [B-K-N, §4].

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