Elliptic Surfaces with Four Singular Fibres

by

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relative to this basis has the form

$$\begin{bmatrix} 0 & 0 & 0 & 0 & * \\ 0 & * & 0 & 0 & ? \\ 0 & 0 & * & 0 & ? \\ 0 & 0 & 0 & * & ? \\ * & ? & ? & ? & ? \end{bmatrix}$$

with entries equal to $2\pi\sqrt{-1}$ times sums of suitably defined residues at cusps for D of differentials such as $\partial^t\overline{w}_1/\partial c P \partial\overline{w}_1/\partial a$ and with nonzero entries at each *. See Endo [5].

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Already at the beginning of the sixties, elliptic surfaces were considered by K. Kodaira [6]; A. Kas embedded them in a projective bundle over the base curve B [5]; B. Hunt/W. Meyer introduced an estimate for the Euler number which depended on the genus of the base curve and the number of singular fibres [4]; for elliptic surfaces with three singular fibres and section over $\mathbb{P}_1\mathbb{C}$, U. Schmickler - Hirzebruch proved that there are only 36 combinations of singular fibres, subdivided in 12 cases [12].

When studying elliptic surfaces with four singular fibres, section and nonconstant $\mathcal{J}$- invariant over $\mathbb{P}_1\mathbb{C}$, as presented here, it is practical to distinguish two sets:

$T^+ = \{I_n \ (n \geq 0), II, III, IV\}$ and $T^- = \{I_n^* \ (n \geq 0), IV^*, III^*, II^*\}$,

where I_0 is a regular fibre. At least one fibre is then of type I_n ,n > 0, or I_n^* ,n > 0, see [4]. By a suitable choice of the homological invariant $\mathcal{G}$ belonging to the $\mathcal{J}$- invariant, all possible fibre combinations can be reduced such that at most one fibre is in T^-.

Theorem 6 summarises the results. Table III shows all fibre combinations and Weierstraß Models. The proof will be given by example. The notation is taken from Kodaira [6] or W. Barth/ C.Peters/ A. Van de Ven [1].

I. Naruki [10] and R. Miranda/ U. Persson [9 and 11] achieved similiar results using different methods.

For an elliptic surface $\pi : E \longrightarrow B$, where E is a two-dimensional compact complex analytic manifold, B is a compact Riemann surface of genus g and π is a proper holomorphic mapping, $E_b := \pi^{-1}(b)$ is a nonsingular curve of genus 1 for all $b \in B_0$, $B_0 := B - P$, $P := \{\rho_1, \rho_2, \ldots, \rho_n\}$, $\rho_i \in B$, $i = 1, \ldots, n$. From now on it will be assumed that E is minimal and admits a section, i.e. E has no exceptional curves of the first kind in the fibres. All singular fibres are simple, because there is a section.

The monodromy representation of $\pi : E \longrightarrow B$ is a homomorphism

$\chi : \pi_1(B_0, b) \longrightarrow SL(2, \mathbb{Z}) \quad b \in B_0$,

which is unique up to conjugation in $SL(2,\mathbb{Z})$. The image of $\pi_1(B_0,b)$ is called the monodromy group. Elements of this group are the monodromy matrices A_{β_i} corresponding to the closed paths β_i around ρ_i, $i = 1,\ldots,n$.

For each type of singular fibre F_i over ρ_i there is one $SL(2,\mathbb{Z})$ - conjugate class of monodromy matrices. In table I they are listed in normal and general form for the singular fibres.

The homological invariant $\mathcal{G}$, a sheaf over B, is equivalent to the monodromy representation. In a base point ρ with the monodromy matrix A the stalk $\mathcal{G}_\rho$ is isomorphic to $\{x \in \mathbb{Z}^2 \mid Ax = x\}$.

Each regular fibre E_ρ of an elliptic surface $\pi : E \longrightarrow B$ is isomorphic to $\mathbb{C} / \omega(\rho)\mathbb{Z} \oplus \mathbb{Z}$. $\omega : \tilde{B}_0 \longrightarrow \mathbb{H}$ with $\omega(\tilde{\beta}(\tilde{b})) = A_\beta(\omega(b))$ is a unique holomorphic function. Here A_β is the monodromy in $SL(2,\mathbb{Z})$ of the closed path β in B_0, $\sigma : \tilde{B}_0 \longrightarrow B_0$ is the universal covering of B_0, $\mathbb{H}$ the upper halfplane, $\sigma(\tilde{b}) = b$ and

$$\begin{array}{ccc} \pi_1(B_0) & \longrightarrow & \mathrm{Aut}(\tilde{B}_0) \\ \beta & \longmapsto & \tilde{\beta} \end{array}$$

is the deck transformation.

There is a mapping $\mathcal{J} : B_0 \longrightarrow SL(2,\mathbb{Z}) \backslash \mathbb{H}$, which allows the diagram to commute:

$$\begin{array}{ccccc} & & \tilde{B}_0 & \xrightarrow{\omega} & \mathbb{H} \\ & & \sigma\downarrow & & \downarrow\tilde{\mathcal{J}} \\ \dfrac{\tilde{B}_0}{\pi_1(B_0,b)} & = & B_0 & \xrightarrow{\mathcal{J}} & SL(2,\mathbb{Z}) \backslash \mathbb{H} \cong \mathbb{C}, \end{array}$$

$\tilde{\mathcal{J}}$ is the elliptic modular function.

The functional invariant of E is defined as the holomorphic continuation of $\mathcal{J}$ on B in $SL(2,\mathbb{Z}) \backslash \mathbb{H}^* \cong \mathbb{P}_1\mathbb{C}$, $\mathbb{H}^* = \mathbb{H} \cup \mathbb{P}_1\mathbb{Q}$. The values of $\mathcal{J}$ in $\rho_i \in P$, depending on the type of the singular fibre over ρ_i, are 0, 1 or ∞, except for I_0^*.

Let $P := \{\rho_i \in B \mid i = 1,\ldots,n\}$ $n \geq 2$ be the exceptional set and

$\chi : \pi_1(B_0,*) \longrightarrow \mathrm{Aut}^+(H_1(E_*,\mathbb{Z})) \cong SL(2,\mathbb{Z})$

the monodromy representation of the fundamental group, where

$$\pi_1(B_0,*) \cong \left\langle a_i,b_i,c_j \;\; \begin{smallmatrix} i = 1,\ldots,g \\ j = 1,\ldots,n \end{smallmatrix} \;\middle|\; \prod_{i=1}^{g}[a_i,b_i] \prod_{j=1}^{n} c_j \right\rangle, \text{ with}$$

$[a_i,b_i] = a_i b_i a_i^{-1} b_i^{-1}$.

$\mathcal{J}$ is the functional invariant of the elliptic surface $E \longrightarrow B$.

The extension of the homological invariant $\mathcal{G}_0$ over B_0 to $\mathcal{G}$ over B is uniquely given by the monodromy representation χ, which is determined by the $\mathcal{J}$-invariant except for its sign, i.e. there are 2^{2g+n-1} different homological invariants, depending on choice of sign for the matrices $A_i = \chi(a_i)$, $B_i = \chi(b_i)$ and $C_j = \chi(c_j)$, $i = 1,\ldots,g$; $j = 1,\ldots,n$, in the product $\prod_{i=1}^{g}[A_i,B_i] \prod_{j=1}^{n} C_j = 1$.

Definition

Two elliptic surfaces $\pi : E \longrightarrow B$ and $\pi' : E' \longrightarrow B'$ are isomorphic, if there are biholomorphic mappings f, g , so that the diagram

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ \pi \downarrow & & \downarrow \pi' \\ B & \xrightarrow{g} & B' \end{array}$$

commutes.

Let $\mathcal{F}(\mathcal{J},\mathcal{G})$ be the family of isomorphism classes of elliptic surfaces over B with only simple singular fibres with functional invariant $\mathcal{J}$ and homological invariant $\mathcal{G}$. For each such family $\mathcal{F}(\mathcal{J},\mathcal{G})$ Kodaira constructed a basic member $\mathcal{B}$, which is defined by a global holomorphic section $\sigma : B \longrightarrow E$ [6 § 8], and proved the following [6 §§ 9,10]:

Theorem 1

Let $\pi : E \longrightarrow B$ be an elliptic surface with a global section, belonging to the family $\mathcal{F}(\mathcal{J},\mathcal{G})$. Then E is isomorphic to the uniquely determined basic member $\mathcal{B}$ of the family $\mathcal{F}(\mathcal{J},\mathcal{G})$.

Kas described this using the Weierstraß Model [8].

Let $\pi : E \longrightarrow B$ be a minimal elliptic surface. K(E) and K(B) are the function fields of E and of B respectively. π induces a homomorphism π^*: K(B) $\longrightarrow$ K(E), and K(E) is a transcendental extension of K(B) of transcendence degree and genus one. The section $\sigma : B \longrightarrow E$ determines a rational point. E is birationally equivalent to the subscheme E^* in $\mathrm{Proj}(\mathcal{O} \oplus 2\mathcal{O}(L) \oplus 3\mathcal{O}(L))$, which is given by the equation

$$Y^2Z = 4X^3 - g_2XZ^2 - g_3Z^3,$$

where $\mathcal{O}$ is the structure sheaf of B, L is a line bundle and where $g_2 \in H^0(B,\mathcal{O}(4L))$ and $g_3 \in H^0(B,\mathcal{O}(6L))$ are sections with $\Delta = g_2^3 - 27\, g_3^2 \neq 0$.

Theorem 2 (Kas)

E^* is an algebraic surface with rational double points as the only singularities. E is the minimal resolution of E^*. E^* is determined uniquely by E up to a $\mathbb{C}^*$ – operation

$(g_2,g_3) \longrightarrow (\lambda^4 g_2, \lambda^6 g_3)$, $\lambda \in \mathbb{C}^* = \mathbb{C} - \{0\}$.

g_2, g_3 satisfy

(i) $\Delta = g_2^3 - 27\, g_3^2 \neq 0$

(ii) $\min(3\,\nu_p(g_2), 2\,\nu_p(g_3)) < 12$ for all $p \in B$, where $\nu_p(g_2)$, $\nu_p(g_3)$ and $\nu_p(\Delta)$ are the order of the zeroes of g_2, g_3 and Δ in p. The singular fibre in E^* over p consists of the minimal resolution of the rational double point and the rational curve, which is defined by the section σ. The type of rational double point and thereby the type of the singular fibre determines $\nu_p(g_2)$, $\nu_p(g_3)$ and $\nu_p(\Delta)$. E^* is called the Weierstraß Model of the elliptic surface.

The $\mathcal{J}$ - invariant of the Model is $\mathcal{J} = \frac{g_2^3}{\Delta}$.

Meyer proved the following:
For each locally trivial fibre bundle $E \longrightarrow X$ it is possible to compute the signature of E as the signature of the E_2-term of the Leray spectral sequence of the fibration [see. 7 I.1.4 and I.2.2], i.e. for an elliptic fibration $E \longrightarrow B$:

Let $B_0 := B - \bigcup_{i=1}^{n} D_i$, with D_i being disjoint small disks around the base points ρ_i of the singular fibres. $E_0 = E_{|B_0}$ is called the "smooth" part and $E_s := E - E_0$ the "singular" part of E. The signature τ of the fibration is

$$\tau(E) = \tau(E_0) + \tau(E_s).$$

Let F_i be the singular fibre over $\rho_i \in B$, then:

$$\tau(E_s) = \sum_{i=1}^{n} \tau(F_i),$$

with $\tau(F_i) = \tau(E_{|D_i})$.

There exists a uniquely determined mapping

$$\phi : SL(2,\mathbb{Z}) \longrightarrow \frac{1}{3}\mathbb{Z},$$

so that

$$\tau(E_0) = -\sum_{i=1}^{n} \phi(\gamma_i);$$

where γ_i is the monodromy of a closed path around ρ_i (see Meyer [7]). Then:

$$\tau(E) = \sum_{i=1}^{n} (\tau(F_i) - \phi(\gamma_i)).$$

The values of $\tau(F_i)$ and $\phi(\gamma_i)$ are listed in table I:

$$\tau(F_i) + e(F_i) = \begin{cases} 1 & \text{if } F_i \text{ has type } I_n \; n > 0 \\ 2 & \text{else} \end{cases};$$

where $e(F_i)$ is the Euler number of the singular fibre F_i.

Furthermore:

Lemma 3 (Hunt)

$$|\tau(E_0)| \leq 4g - 4 + 2n;$$

where g is the genus of the base curve;

and

Theorem 4

It is known that for each minimal elliptic surface

$$\tau(E) = -\frac{2}{3} e(E).$$

Noethers formula implies that for compact complex surfaces S

$$p_a(S) = \frac{\tau(S) + e(S)}{4},$$

where $p_a(S)$ is the arithmetic genus of S and for an elliptic surface E

$$p_a(E) = \frac{1}{12} e(E).$$

Table I

Singular fibre	Euler number	Monodromy matrix normal form A	conjugate form TAT^{-1}, $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\mathbb{Z})$	Orders of zeroes $\nu_\rho(g_2)$	$\nu_\rho(g_3)$	$\nu_\rho(\Delta)$	Value of $\mathcal{J}(\rho)$	Signature of the singular fibre $\tau(F)$	$\phi(F)$
I_0	0	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	0	0	0	$\neq \infty$	0	0
I_n $n > 0$	n	$\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} =: P^n$	$\begin{bmatrix} 1-acn & a^2n \\ -c^2n & 1+acn \end{bmatrix}$ a,c relatively prime	0	0	n	Pole of order n	$1-n$	$1-\frac{n}{3}$
II	2	$\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} =: S$	$\begin{bmatrix} ad-bd-ac & a^2+b^2-ab \\ cd-c^2-d^2 & bd+ac-bc \end{bmatrix}$	≥ 1	1	2	0	0	$\frac{4}{3}$
III	3	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} =: J$	$\begin{bmatrix} -bd-ac & a^2+b^2 \\ -c^2-d^2 & ac+bd \end{bmatrix}$	1	≥ 2	3	1	-1	1
IV	4	$\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} = S^2$	$\begin{bmatrix} bc-bd-ac & a^2+b^2-ab \\ cd-c^2-d^2 & bd+ac-ad \end{bmatrix}$	≥ 2	2	4	0	-2	$\frac{2}{3}$
I_0^*	6	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$	2 ≥ 2 2	≥ 3 3 3	6 6 6	1 0 $\neq 0,1,\infty$	-4	0
I_n^* $n > 0$	$n+6$	$\begin{bmatrix} -1 & -n \\ 0 & -1 \end{bmatrix} = -P^n$	$\begin{bmatrix} -1+acn & -a^2n \\ c^2n & -1-acn \end{bmatrix}$ a,c relatively prime	2	3	$n+6$	Pole of order n	$-n-4$	$-\frac{n}{3}$
II^*	10	$\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} = S^{-1} = -S^2$	$\begin{bmatrix} ac+bd-bc & ab-a^2-b^2 \\ c^2+d^2-cd & ad-bd-ac \end{bmatrix}$	≥ 4	5	10	0	-8	$-\frac{4}{3}$
III^*	9	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = J^{-1} = -J$	$\begin{bmatrix} bd+ac & -a^2-b^2 \\ c^2+d^2 & -ac-bd \end{bmatrix}$	3	≥ 5	9	1	-7	-1
IV^*	8	$\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} = -S = S^{-2}$	$\begin{bmatrix} ac+bd-ad & ab-a^2-b^2 \\ c^2+d^2-cd & bc-bd-ac \end{bmatrix}$	≥ 3	4	8	0	-6	$-\frac{2}{3}$

Calculation of the possible fibre combinations

With the above notation, let $E \longrightarrow \mathbb{P}_1\mathbb{C}$ be a minimal elliptic surface with a section σ and nonconstant $\mathcal{J}$- invariant. The singular fibres F_i are over ρ_i, $\rho_i \neq \rho_j$ for $i \neq j$. Let
$P := \{\rho_1,\rho_2,\rho_3,\rho_4\}$ and $\chi : \pi_1(\mathbb{P}_1\mathbb{C} - P,*) \longrightarrow SL(2,\mathbb{Z})$ be the monodromy representation of the fundamental group

$$\pi_1(\mathbb{P}_1\mathbb{C} - P,*) = \langle\, a_1,a_2,a_3,a_4 \mid a_1a_2a_3a_4 = 1 \,\rangle,$$

where a_i is a closed path around ρ_i and $A_i := \chi(a_i)$, $i = 1,\dots,4$, is a monodromy matrix. The homological invariant $\mathcal{G}$ of the elliptic surface E is determined by A_i with

$$A_1A_2A_3A_4 = 1, \tag{1}$$

where A_i, $i = 1,\dots,4$, is conjugate to a matrix in $M = M^+ \cup M^-$ with

$M^+ := \{Id, P^n\ (n > 0), S, J, S^2\}$ and $M^- := \{-Id, -P^n\ (n > 0), -S, -J, -S^2\}$ (see table I).

$\mathcal{G}$ belongs to the functional invariant $\mathcal{J}$. For each functional invariant $\mathcal{J}$ and associated homological invariant $\mathcal{G}$ there is exactly one elliptic surface $\mathcal{B}$ over $\mathbb{P}_1\mathbb{C}$ with section. $\mathcal{B}$ is the basic member of $\mathcal{F}(\mathcal{J},\mathcal{G})$.
Its' Weierstraß - Model E^* will be calculated as follows:
Let $G_2 = g_2$, $g_2 \in H^0(\mathbb{P}_1\mathbb{C},\mathcal{O}(4L))$; $G_3 = 3\sqrt{3}\, g_3$, $g_3 \in H^0(\mathbb{P}_1\mathbb{C},\mathcal{O}(6L))$ where g_2,g_3 are the sections which determine the Weierstraß Model. The matrices $\tilde{A}_i = \epsilon_i A_i$, $i = 1,\dots,4$, $\epsilon_i = \pm 1$, with $\tilde{A}_1\tilde{A}_2\tilde{A}_3\tilde{A}_4 = 1$ and therefore $\epsilon_1\epsilon_2\epsilon_3\epsilon_4 = 1$, determine the homological invariant $\tilde{\mathcal{G}}$. The Model $\tilde{E}^*$ of the basic member $\tilde{\mathcal{B}} \in \mathcal{F}(\mathcal{J},\tilde{\mathcal{G}})$ can easily be calculated from E^* by " asterisking " pairs and " moving an asterisk ".

" Asterisking " a fibre over ρ_i corresponds to multiplying the monodromy matrix A_i with $-Id$. I_n, II, III and IV change to I_n^*, IV*, III* and II* respectively and vice versa (see table I).
The Euler number of the singular fibre increases or decreases by six respectively. In the Weierstraß Model the polynomials G_2, G_3 and Δ are multiplied with $(X - \rho_i Y)^{2,3 \text{ and } 6 \text{ resp.}}$, if $A_i \in M^+$ and $\epsilon_i = -1$, or divided by the same expression, if $A_i \in M^-$ and $\epsilon_i = -1$.

When " Moving the asterisk " of the singular fibre $F_i \in T^-$ over ρ_i to the singular fibre $F_j \in T^+$ over ρ_j (in short, from ρ_i to ρ_j), the monodromy matrices A_i and A_j will be multiplied with $-Id$, the polynomials G_2, G_3 and Δ with $\dfrac{(X - \rho_j Y)^{2,3 \text{ and } 6 \text{ resp.}}}{(X - \rho_i Y)^{2,3 \text{ and } 6 \text{ resp.}}}$.

So it suffices to restrict the calculation to the canonical basic member $\mathcal{B} \in \mathcal{F}(\mathcal{J},\mathcal{G})$. $\mathcal{G}$ is determined by $A_1A_2A_3A_4 = 1$, where at most one A_i is conjugate to a matrix in M^-. At least one singular fibre has to be of type I_n or I_n^* [4, page 79]. For the classification of surfaces, which have one fibre of type I_0, a regular fibre, see [12].

Lemma 5

The Euler number of an elliptic surface with four singular fibres in T^+ over $\mathbb{P}_1\mathbb{C}$ is twelve.

Proof

The monodromy satisfies

$$A_1A_2A_3A_4 = 1,$$

where all monodromy matrices are conjugate to a normal form in M^+. By " asterisking " all four singular fibres, the Euler number increases by 24. The Euler number of an elliptic surface, which depends only on the Euler number of the singular fibres is:

$$e(E) = 12\, p_a(E) = 3\,(e(E_s) + \tau(E_s) + \tau(E_0)) \leq 3\,(2n + 4g - 4 + 2n),$$

where n is the number of singular fibres and g is the genus of the base curve.
For $n = 4$, $g = 0$ this means

$$0 < e(E) \leq 36$$

and so the statement of the lemma.

Theorem 6

Let $E \longrightarrow \mathbb{P}_1\mathbb{C}$ be a minimal elliptic surface with section, nonconstant $\mathcal{J}$-invariant and four singular fibres, of which at most one is in T^-. Up to permutation and " Moving the asterisk " there are only those combinations of singular fibres which are listed in table III.

(i) If one singular fibre is in T^-, three in T^+, the Weierstraß Model depends on a parameter. Given four different base points, in the case $I_1^*\ I_1\ I_1$ III there exist four, in the cases $I_1^*\ I_1\ I_2$ II and $I_1\ I_1$ II IV^* there exist two elliptic surfaces, depending on the $\mathcal{J}$-invariant, and for all other fibre combinations there exists precisely one elliptic surface.

ii) If all four singular fibres are in T^+, then the Weierstraß Models are determined uniquely up to isomorphism, except for one combination. In the case $I_1\ I_6$ II III there are two nonisomorphic models.

In table III the Weierstraß Models including the $\mathcal{J}$- invariant and cross ratio of the base points for $\Delta = G_2^3 - 27\ G_3^2$ are listed.

Corollary 7

All elliptic surfaces with four singular fibres can be deduced from table III by the following methods:

(i) " Asterisking " the singular fibres in pairs

(ii) " Moving the asterisk " of singular fibres.

Proof

1. If one singular fibre is of type I_0^*, a surface with three singular fibres is obtained by " moving the asterisk ". So these elliptic surfaces are easily calculated [12, pages 120 ff., cases 6 – 12].

In the following it is assumed that $n > 0$ for all fibres of type I_n or I_n^*.

2. Determination of all possible fibre combinations

Because of (1), it follows for the monodromy matrices $A_i \in SL(2,\mathbb{Z})$, $i = 1,...,4$ that:

$$\text{trace}\ (A_1A_2) = \text{trace}\ ((A_3A_4)^{-1}) = \text{trace}\ (A_3A_4). \qquad (2)$$

The trace is preserved under conjugation. So let A_2 and A_4 be in normal form, A_1 and A_3 be conjugate to $\pm P^n, S, J, S^2$ (see table I). Table II lists the trace (A_iA_{i+1}) for different fibre combinations.

In the following the calculation will be separate according to the occurence of a fibre $F_1 \in T^-$ and the number of fibres of type I_n.

2.1. One singular fibre in T^-

Assume that this fibre F_1 is of type I_n^*.

2.1.1. F_3 of type I_n, $n > 0$; $F_2, F_4 \in T^+ - \{I_n\}$

See table II. There is trace $(A_1A_2) \geq 0$ and trace $(A_3A_4) \leq 0$ with " = " exactly for $F_2 = F_4 = II$. It follows that

$$-1 + n_1\ (a_1^2 + a_1c_1 + c_1^2) = 1 - n_3\ (a_3^2 + a_3c_3 + c_3^2).$$

Because of $a_i^2 + a_ic_i + c_i^2 > 0$, we have $n_1 = n_3 = 1$ and the combination is $I_1^*\ I_1\ II\ II$.

Table II

Singular fibre	trace (A_iA_{i+1})	
$I_{n_i}I_{n_{i+1}}$	$2 - c_i^2 n_i n_{i+1} \leq 2$	
I_{n_i} II	$1 - n_i\,(a_i^2 + a_ic_i + c_i^2) \leq 0$	a_i, c_i relatively prime
I_{n_i} III	$- n_i\,(a_i^2 + c_i^2) \leq -1$	a_i, c_i relatively prime
I_{n_i} IV	$-[1 + n_i\,(a_i^2 + a_ic_i + c_i^2)] \leq -2$	a_i, c_i relatively prime
II II	$-[(b_i - \frac{1}{2}a_i + \frac{1}{2}d_i)^2 + (c_i + \frac{1}{2}a_i - \frac{1}{2}d_i)^2 + \frac{1}{2}(a_i^2 + d_i^2)] \leq -1$	
II III	$-(a_i^2 - a_ib_i + b_i^2 + c_i^2 - c_id_i + d_i^2) \leq -2$	
II IV	$-[(a_i - \frac{1}{2}b_i + \frac{1}{2}c_i)^2 + (d_i + \frac{1}{2}b_i - \frac{1}{2}c_i)^2 + \frac{1}{2}(b_i^2 + c_i^2)] \leq -2$	
III III	$-(a_i^2 + b_i^2 + c_i^2 + d_i^2) \leq -2$	
III IV	$-(a_i^2 + a_ic_i + c_i^2 + b_i^2 + b_id_i + d_i^2) \leq -2$	
IV IV	$-[(b_i - \frac{1}{2}a_i + \frac{1}{2}d_i)^2 + (c_i + \frac{1}{2}a_i - \frac{1}{2}d_i)^2 + \frac{1}{2}(a_i^2 + d_i^2)] \leq -1$	
$I^*_{n_i}I_{n_{i+1}}$	$-2 + c_i^2 n_i n_{i+1} \geq -2$	
$I^*_{n_i}$ II	$-1 + n_i\,(a_i^2 + a_ic_i + c_i^2) \geq 0$	a_i, c_i relatively prime
$I^*_{n_i}$ III	$n_i\,(a_i^2 + c_i^2) \geq 1$	a_i, c_i relatively prime
$I^*_{n_i}$ IV	$1 + n_i\,(a_i^2 + a_ic_i + c_i^2) \geq 2$	a_i, c_i relatively prime

2.1.2. $F_2, F_3, F_4 \in T^* - \{I_n\}$

Table II shows that the equation (2) cannot be satisfied.

2.2. Four singular fibres in T^*

2.2.1. F_1, F_2, F_3, F_4 of type I_n, $n > 0$

Equation (2) is equivalent to $c_1^2 n_1 n_2 = c_3^2 n_3 n_4$ see table II. If $c_1 = c_3 = 0$, one easily deduces a contradiction $A_1A_2A_3A_4 \neq 1$ to equation (1). So equation (2) is now equivalent to

$$n_1 n_2 n_3 n_4 = \frac{c_3^2}{c_1^2}\; n_3^2 n_4^2.$$

So $\prod_{i=1}^{4} n_i$ is a square and $\sum_{i=1}^{4} n_i = 12$. Only the fibre combinations, which are listed in table III, exist up to permutation.

2.2.2. F_1, F_3 of type I_n, $n > 0$; $F_2, F_4 \in T^* - \{I_n\}$

Lemma 5 shows

$$n_1 + n_3 = 12 - e(F_2) - e(F_4).$$

I_5 II I_3 II and I_3 II I_3 IV are excluded, because of

$$0 \equiv 5\,(a_1^2 + a_1 c_1 + c_1^2) \not\equiv 3\,(a_3^2 + a_3 c_3 + c_3^2) \text{ modulo } 5$$

and

$$0 \equiv 3\,(a_1^2 + a_1 c_1 + c_1^2) \not\equiv 3\,(a_3^2 + a_3 c_3 + c_3^2) + 2 \text{ modulo } 3,$$

see table II and (2). The remaining fibre combinations, up to permutation of the fibres, are those which are listed in table III, and the combination I_3 I_1 IV IV. Explicit calculation of the Weierstraß Model shows, that the last combination is impossible.

2.2.3. F_1 of type I_n, $n > 0$; $F_2, F_3, F_4 \in T^* - \{I_n\}$

Lemma 5 shows that the Euler number is twelve. Only the combinations listed in table III and I_1 III IV IV, I_2 II IV IV can be possible up to permutation. In the last two cases G_2 and G_3 must have the degree ≥ 5 and ≥ 6 or ≥ 5 and 5 respectively (see table I). This however is impossible.

2.2.4. If there are three fibres of type I_n, one gets all combinations of table III and I_4 I_3 I_1 IV as in 2.2.2.. This fibre combination can be excluded by explicit calculation.

3. Calculation of the polynomials G_2, G_3 and Δ in homogeneous coordinates (X,Y) of $\mathbf{P}_1\mathbf{C}$:

Equation $\Delta = G_2^3 - G_3^2$ gives a nonlinear system of equations for the coefficients of G_2, G_3. Common factors of $G_2^3 - G_3^2$ and Δ will be cancelled.

Note

Let $\bar{\Delta} = \dfrac{G_2^3 - G_3^2}{\gcd(G_2^3, G_3^2)} = \dfrac{\Delta}{\gcd(G_2^3, G_3^2)}$ (see table I) and let C_i be the coefficient of

$X^{k-i}Y^i$ in $\bar{\Delta}$, where k is the sum of the n_j over the numbers of the fibres of types I_{n_j} and $I^*_{n_j}$ of the surface with $0 \leq i \leq k$. The base points are written as quadruple $(\rho_1,\rho_2,\rho_3,\rho_4)$.

3.1. One singular fibre in T⁻

$I_1^* I_1$ II II

It may be assumed that the singular fibres are over $(0,\infty,1,\rho_4)$. The orders of zeroes at the base points have to be:

ρ	$\nu_\rho(G_2)$	$\nu_\rho(G_3)$	$\nu_\rho(\Delta)$
0	2	3	7
∞	0	0	1
1	≥ 1	1	2
ρ_4	≥ 1	1	2
sum	≥ 4	5	12

The equation $\Delta = G_2^3 - G_3^2$ with

$$G_2(X,Y) = \mu X^2 (X - Y)(X - \rho_4 Y)$$

$$G_3(X,Y) = \nu X^3 (X - Y)(X - \rho_4 Y)(X + BY)$$

$$\Delta (X,Y) = \sigma \mu^3 X^7 Y(X - Y)^2(X - \rho_4 Y)^2 ,$$

where $\mu,\nu,\sigma \in \mathbb{C}^*$, produces the following system of equations with $\mu^3 - \nu^2 = 0$:

$$C_1 = -(\rho_4 + 1 + 2B) = \sigma$$

$$C_2 = \rho_4 - B^2 = 0.$$

It follows that $\rho_4 = B^2 \neq 0$ and $\sigma = -(B + 1)^2$. Consequently one gets

$$G_2(X,Y) = \mu X^2 (X - Y)(X - B^2 Y)$$

$$G_3(X,Y) = \nu X^3 (X - Y)(X - B^2 Y)(X + BY)$$

$$\Delta (X,Y) = -(B + 1)^2 \mu^3 X^7 Y(X - Y)^2(X - B^2 Y)^2$$

where $B \neq -1,1$.

The cross ratio of the base points $CR(I_1^* \, I_1 \mid II \; II)$ is $\frac{1}{B^2}$. If the base points are given, there exist two different Weierstraß Models, depending on the choice of the $\mathcal{J}$- invariant. Let $\tilde{\Delta} = 27\,\Delta$ and $\tilde{G}_2 = 3\,G_2$. Table III lists the surface for $\mu = 1$, $\nu = 1$, $\tilde{\Delta}$ as Δ and $\tilde{G}_2$ as G_2 in abuse of the notation. G_2 and G_3 are uniquely determined up to a transformation $(G_2,G_3) \longrightarrow (h^4G_2,h^6G_3)$, $h \in \mathbb{C}^*$. Consequently in this calculation, as in the following ones, there are values given for μ and ν, so that one arrives at the polynomials G_2,G_3 and Δ as above which are listed in table III.

3.2. Four singular fibres in T^+

3.2.1. Calculation by using the common divisor of G_2,G_3 and Δ

$I_3 \, I_3 \, I_3 \, I_3$

The orders of zeroes have to be:

ρ	$\nu_\rho(G_2)$	$\nu_\rho(G_3)$	$\nu_\rho(\Delta)$
ρ_1	0	0	3
ρ_2	0	0	3
ρ_3	0	0	3
ρ_4	0	0	3
sum	0	0	12

(3)

Therefore

$$F^3 = \Delta = G_2^3 - G_3^2 \qquad (4)$$

with $F, G_2 \in H^0(\mathbb{P}_1\mathbb{C}, \mathcal{O}(4L))$, $G_3 \in H^0(\mathbb{P}_1\mathbb{C}, \mathcal{O}(6L))$.

It follows from (3) that G_2, G_3 and F are relatively prime. (4) is equivalent to

$$G_3^2 = G_2^3 - F^3 = (G_2 - F)(\eta^2 G_2 - \eta F)(\eta G_2 - \eta^2 F) \qquad \eta = e^{\frac{2\pi i}{3}}.$$

As mentioned above, the single factors in this decomposition are relatively prime in pairs (3). They are squares. Let

$$H_1^2 := G_2 - F$$
$$H_2^2 := \eta^2 G_2 - \eta F$$
$$H_3^2 := \eta G_2 - \eta^2 F.$$

If H_i has the appropriate sign it follows that

$$G_3 = H_1 \cdot H_2 \cdot H_3$$

where $H_i \in H^0(\mathbb{P}_1\mathbb{C}, \mathcal{O}(2L)) \quad i = 1,2,3$

$$0 = H_1^2 + H_2^2 + H_3^2 \quad \Leftrightarrow \quad -H_1^2 = H_2^2 + H_3^2 = (H_2 + iH_3)(H_2 - i H_3).$$

Both factors are relatively prime and squares. Let

$$J_1^2 := H_2 + i H_3$$
$$J_2^2 := H_2 - i H_3$$

where $J_1, J_2 \in H^0(\mathbb{P}_1\mathbb{C}, \mathcal{O}(L))$. It can then be assumed that

$$i H_1 = J_1 \cdot J_2.$$

J_1, J_2 are relatively prime. By suitable choice of the coordinates on $\mathbb{P}_1\mathbb{C}$, it is possible to choose $J_1 = X$ and $J_2 = Y$. Therefore

$$H_1 = -i XY$$
$$H_2 = \frac{1}{2}(X^2 + Y^2)$$
$$H_3 = \frac{1}{2i}(X^2 - Y^2)$$

with $\omega = \sqrt{3}$

$$G_2(X,Y) = \frac{1}{\eta^2 - \eta}(-\eta H_1^2 + H_2^2) = \frac{i}{4\omega}(X^4 + 2 i\omega X^2Y^2 + Y^4)$$

$$G_3(X,Y) = H_1 \cdot H_2 \cdot H_3 = -\frac{1}{4} X \cdot Y (X^2 + Y^2)(X^2 - Y^2)$$

$$F(X,Y) = G_2 - H_1^2 = \frac{i}{4\omega}(X^4 - 2 i\omega X^2Y^2 + Y^4)$$

$$\Delta(X,Y) = -\frac{i}{192\omega}(X^4 - 2 i\omega X^2Y^2 + Y^4)^3.$$

The cross ratio is $CR(I_3 I_3 \mid I_3 I_3) = -\eta$. Table III gives the surface after the transformation $(X,Y) \longrightarrow (\varphi_1(X + \psi_1 Y), \varphi_2(X + \psi_2 Y))$ with $\varphi_{1,2} = \sqrt{-2\,(\zeta^3 \mp \zeta^{10})}$, $\psi_{1,2} = \frac{1}{2}\frac{1 \mp \zeta^3}{1 \mp \zeta^7}$ and $\zeta = e^{\frac{2\pi i}{12}}$.

Using this method, it is also possible to calculate the Weierstraß Models of $I_4 I_4 I_2 I_2$, $I_4 I_4$ II II , $I_3 I_3$ III III [2].

3.2.2. All other fibre combinations are calculated using the same method as in 3.1. [2].

$I_7 I_1 I_1$ III , $I_6 I_2 I_1$ III and I_6 II I_1 III

An $Aut(\mathbb{P}_1\mathbb{C})$ - operation transforms the singular fibres over the base points $(\infty, \rho_2, \rho_3, 0)$. The fibres over ∞, ρ_2 are either of type $I_7 I_1$ and $I_6 I_2$ or of type $I_6 I_2$ and I_6 II respectively, therefore the three calculations differ by a common factor of $G_2^3 - G_3^2$ and Δ only.

$$\text{Let}\quad i := \begin{cases} 0 & \text{at } I_7\, I_1\, I_1\, \text{III} \\ 1 & \text{at } I_6\, I_2\, I_1\, \text{III and } I_6\, \text{II}\, I_1\, \text{III}. \end{cases}$$

The orders of zeroes have to be:

ρ	$\nu_\rho(G_2)$	$\nu_\rho(G_3)$	$i = 0$ $\nu_\rho(\Delta)$	$i = 1$ $\nu_\rho(\Delta)$
∞	0	0	7	6
ρ_2	$0\,(\geq 1)$	$0\,(1)$	1	2
ρ_3	0	0	1	1
0	1	≥ 2	3	3
sum	$1\,(\geq 2)$	$\geq 2\,(3)$	12	12

The orders of zero for the fibre over ρ_2 in $T^{+}-\{I_n\}$ are in brackets.

The equation $\Delta = G_2^3 - G_3^2$ with

$$G_2(X,Y) = \mu X(X^3 + A_1X^2Y + A_2XY^2 + A_3Y^3)$$

$$G_3(X,Y) = \nu X^2(X^4 + B_1X^3Y + B_2X^2Y^2 + B_3XY^3 + B_4Y^4)$$

$$\Delta(X,Y) = \sigma \mu^3 X^3Y^{7-i}(X-\rho_2Y)^{1+i}(X-\rho_3Y)$$

where $\mu,\nu,\sigma \in \mathbb{C}^*$; $i = 0,1$ produces the following system of equations with $\mu^3 - \nu^2 = 0$:

$$C_1 = 3\,A_1 - 2\,B_1 = 0$$

$$C_2 = 3\,A_1^2 + 3\,A_2 - B_1^2 - 2\,B_2 = 0$$

$$C_3 = A_1^3 + 3\,A_3 + 6\,A_1A_2 - 2\,B_3 - 2\,B_1B_2 = 0$$

$$C_4 = 3\,A_2^2 + 3\,A_1^2A_2 + 6\,A_1A_3 - B_2^2 - 2\,B_4 - 2\,B_1B_3 = 0$$

$$C_5 = 3\,A_1A_2^2 + 3\,A_1^2A_3 + 6\,A_2A_3 - 2\,B_1B_4 - 2\,B_2B_3 = 0$$

$$C_6 = A_2^3 + 3\,A_3^2 + 6\,A_1A_2A_3 - B_3^2 - 2\,B_2B_4 = \begin{cases} 0 & i=0 \\ \sigma & i=1 \end{cases}$$

$$C_7 = 3\,A_1A_3^2 + 3\,A_2^2A_3 - 2\,B_3B_4 = \begin{cases} \sigma & i=0 \\ -(2\rho_2+\rho_3)\sigma & i=1 \end{cases}$$

$$C_8 = 3\,A_2A_3^2 - B_4^2 = \begin{cases} -(\rho_2+\rho_3)\sigma & i=0 \\ (2\,\rho_2\rho_3 + \rho_2^2)\sigma & i=1 \end{cases}$$

$$C_9 = A_3^3 = \begin{cases} \rho_2\rho_3\sigma & i=0 \\ -\rho_2^2\rho_3\sigma & i=1\,. \end{cases}$$

To fulfil $C_1 = C_2 = C_3 = 0$, let

$$A_1 = 2\alpha \qquad B_1 = 3\alpha$$

$$A_2 = 2\beta - \alpha^2 \qquad B_2 = 3\beta$$

$$A_3 = 2\gamma \qquad B_3 = 3\gamma - 2\alpha^3 + 3\alpha\beta$$

where $\alpha, \beta, \gamma \in \mathbb{C}$.

From $C_4 = 0$ it follows that:

$$B_4 = \frac{3}{2}[(\beta - \alpha^2)^2 + 2\alpha\gamma]$$

and from $C_5 = 0$:

$$3(\beta - \alpha^2)[2\gamma - \alpha(\beta - \alpha^2)] = 0.$$

1) $\beta - \alpha^2 = 0$

The result is

$$C_6 = 3\gamma^2 = \begin{cases} 0 & i = 0 \\ \sigma & i = 1 \end{cases}$$

$$C_7 = 6\alpha\gamma^2 = \begin{cases} \sigma & i = 0 \\ -(2\rho_2 + \rho_3)\sigma & i = 1 \end{cases}$$

$$C_8 = 3\alpha^2\gamma^2 = \begin{cases} -(\rho_2 + \rho_3)\sigma & i = 0 \\ (2\rho_2\rho_3 + \rho_2^2)\sigma & i = 1 \end{cases}$$

$$C_9 = 8\gamma^3 = \begin{cases} \rho_2\rho_3\sigma & i = 0 \\ -\rho_2^2\rho_3\sigma & i = 1. \end{cases}$$

γ may not equal zero, so $i = 1$. Consequently the only solution is

$$\gamma = \frac{1}{18}\alpha^3$$

$$\rho_2 = -\frac{1}{3}\alpha$$

$$\rho_3 = -\frac{4}{3}\alpha$$

$$\sigma = \frac{1}{108}\alpha^3.$$

If $\alpha = -3$:

$G_2(X,Y) = \mu X(X^3 - 6X^2Y + 9XY^2 - 3Y^3)$

$G_3(X,Y) = \frac{1}{2}\nu X^2(2X^4 - 18X^3Y + 54X^2Y^2 - 63XY^3 + 27Y^4)$

$\Delta(X,Y) = \frac{27}{4}\mu^3 X^8Y^6(X-Y)^2(X-4Y)$

with $\mu^3 - \nu^2 = 0 \quad \mu,\nu \in \mathbb{C}^*$.

The cross ratio is $CR(I_6\, I_2 \mid I_1\, III) = -\frac{1}{3}$. Table III shows the surface for $\mu = 4$, $\nu = 8$.

2) $\beta - \alpha^2 \neq 0$

$C_5 = 0$ leads to

$$\gamma = \frac{1}{2}\alpha(\beta - \alpha^2).$$

After the substitution of $\delta = \beta - \alpha^2$, it follows for C_6, C_7, C_8 and C_9 that:

	i = 0	i = 1	
$C_6 = -\delta^2(\delta - \frac{3}{4}\alpha^2) =$	0	σ	
$C_7 = -\frac{3}{2}\alpha\delta^2(\delta - \alpha^2) =$	σ	$-(2\rho_2 + \rho_3)\sigma$	(5)
$C_8 = -\frac{3}{4}\delta^2(3\delta + \alpha^2)(\delta - \alpha^2) =$	$-(\rho_2 + \rho_3)\sigma$	$(2\rho_2\rho_3 + \rho_2^2)\sigma$	
$C_9 = \alpha^3\delta^3 =$	$\rho_2\rho_3\sigma$	$-\rho_2^2\rho_3\sigma$.	

i) $i = 0 \quad I_7\, I_1\, I_1\, III$

Because $\sigma \neq 0$, this gives:

$$\delta = \frac{3}{4}\alpha^2,$$

$$C_7 = \frac{3^3}{2^7}\alpha^7 = \sigma$$

$$C_8 = \frac{3^3 \cdot 13}{2^{10}}\alpha^8 = -(\rho_2 + \rho_3)\sigma$$

$$C_9 = \frac{3^3}{2^6}\alpha^9 = \rho_2\rho_3\sigma\ ,$$

$$\rho_2 + \rho_3 = -\frac{13}{2^3}\alpha$$

$$\rho_2 \cdot \rho_3 = 2\alpha^2$$

and

$$\rho_2 = \frac{1}{8}\alpha\left[\frac{1 \pm i\sqrt{7}}{2}\right]^7$$

$$\rho_3 = \frac{1}{8}\alpha\left[\frac{1 \mp i\sqrt{7}}{2}\right]^7.$$

If $\alpha = 2$, we get:

$$G_2(X,Y) = \mu X(X^3 + 4X^2Y + 10XY^2 + 6Y^3)$$

$$G_3(X,Y) = \frac{1}{2}\nu X^2(2X^4 + 12X^3Y + 42X^2Y^2 + 70XY^3 + 63Y^4)$$

$$\Delta(X,Y) = \frac{27}{4}\mu^3 X^3Y^7(4X^2 + 13XY + 32Y^2)$$

with $\mu^3 - \nu^2 = 0$ $\mu,\nu \in \mathbb{C}^*$.

The cross ratio is $CR(I_7\, I_1 \mid I_1\, III) = \dfrac{(1 - i\sqrt{7})^7}{(1 + i\sqrt{7})^7 - (1 - i\sqrt{7})^7}$. Table III shows the surface for $\mu = 4$, $\nu = 8$.

ii) $i = 1$

There is a double zero of Δ at ρ_2. The discriminant of

$$-\frac{1}{4}\delta^2[(4\delta - 3\alpha^2)X^3 + 6\alpha(\delta - \alpha^2)X^2Y + 3(3\delta^2 - 2\alpha^2\delta - \alpha^4)XY^2 - 4\alpha^3\delta Y^3]$$

vanishes (see (5)), i.e.

$$-\frac{27}{16}\delta(\delta^2 - \alpha^2\delta + \frac{1}{3}\alpha^4)^3 = 0.$$

Because $\delta = \frac{1}{2}\alpha^2(1 \pm \frac{1}{3}\omega)$ with $\omega = i\sqrt{3}$, it follows from (5) that:

$$C_6 = \frac{1}{72}\alpha^6(1 \pm \omega)(3 \mp 2\omega) = \sigma$$

$$C_7 = \frac{1}{24}\alpha^7(1 \pm \omega)(3 \mp \omega) = -(2\rho_2 + \rho_3)\sigma$$

$$C_8 = \frac{1}{48}\alpha^8(1 \pm \omega)(9 \mp \omega) = (2\rho_2\rho_3 + \rho_2^2)\sigma$$

$$C_9 = \frac{1}{36}\alpha^9(1 \pm \omega)(3 \pm \omega) = -\rho_2^2\rho_3\sigma.$$

The result of the system of equations is:

$$\rho_2 = \frac{\alpha(9 + \omega)}{2(2\omega - 3)}$$

$$\rho_3 = -\frac{4\alpha\omega}{2\omega - 3}\ .$$

Let $\alpha = -1 + \omega$. After the transformation $(X,Y) \longrightarrow (X, \frac{1}{2}Y)$:

$$G_2(X,Y) = \frac{1}{6}\mu X(X - Y)[6X^2 + 6\omega XY - (3 + \omega)Y^2]$$

$$G_3(X,Y) = \frac{1}{4}\nu X^2(X - Y)[4X^3 - 2(1 - 3\omega)X^2Y - 4(2 + \omega)XY^2 + (5 - \omega)Y^3]$$

$$\Delta(X,Y) = -\frac{1}{2^3 \cdot 3^2}\mu^3 X^3Y^6(X - Y)^2[(9 + \omega)X + 8\omega Y]$$

with $\mu^3 - \nu^2 = 0$.

The cross ratio is $CR(I_6\ II \mid I_1\ III) = \frac{3 - 2\omega}{9}$. After the transformation $(G_2,G_3) \longrightarrow (\frac{1}{9}G_2, \frac{1}{27}G_3)$ table III shows the surface for $\mu = 36$, $\nu = 216$.

Notes to table III

Table III lists the Weierstraß Models of the fibre combinations with the base points $(\rho_1,\rho_2,\rho_3,\rho_4)$ of the fibres. G_2 and G_3 appear as follows: The discriminant is $\Delta = G_2^3 - 27G_3^2$. All polynomials can be chosen to have integer coefficients except for the combination $I_6\ I_1\ II\ III$. (G_2,G_3) are determined up to $(\lambda^4 G_2, \lambda^6 G_3)$ $\lambda \in \mathbb{C}^*$ only.

If there is a singular fibre in T^-, then table III lists in addition those values of the the cross ratio $CR(\rho_1\ \rho_2 \mid \rho_3\ \rho_4) = \frac{\rho_1 - \rho_3}{\rho_2 - \rho_3} : \frac{\rho_1 - \rho_4}{\rho_2 - \rho_4}$, which are excluded.

All surfaces with four singular fibres, section and nonconstant $\mathcal{J}$- invariant $\mathcal{J} = \frac{G_2^3}{\Delta}$ can easily be calculated from the models by " moving the asterisk " and " asterisking " the fibres (see page 7). They are uniquely determined up the operation of $Aut(\mathbb{P}_1\mathbb{C})$.

Table III

Fibre combination	Weierstraß Model	j – invariant	Cross ratio of the base points
$I_4\ I_1\ I_1\ I_0^*$ $(1,\infty,0,\rho_4)$	$G_2 = 3\,(X-\rho_4 Y)^2(X^2+14\,XY+Y^2)$ $G_3 = (X-\rho_4 Y)^3(X^3-33\,X^2Y-33\,XY^2+Y^3)$ $\Delta = 2^2\cdot 3^6\,XY(X-\rho_4 Y)^6(X-Y)^4$	$\frac{1}{108}\,\frac{(X^2+14\,XY+Y^2)^3}{XY(X-Y)^4}$	$\frac{1}{1-\rho_4} \neq 0,1,\infty$
$I_2\ I_2\ I_2\ I_0^*$ $(1,\infty,0,\rho_4)$	$G_2 = 12\,(X-\rho_4 Y)^2(X^2-XY+Y^2)$ $G_3 = 4\,(X-\rho_4 Y)^3(2\,X^3-3\,X^2Y-3\,XY^2+2\,Y^3)$ $\Delta = 2^4\cdot 3^6\,X^2Y^2(X-\rho_4 Y)^6(X-Y)^2$	$\frac{4}{27}\,\frac{(X^2-XY+Y^2)^3}{X^2Y^2(X-Y)^2}$	$\frac{1}{1-\rho_4} \neq 0,1,\infty$
$I_1\ I_1\ I_1\ I_3^*$ $(u_1,u_2,\infty,0)$	$G_2 = 12\,X^2(X^2+2\,aXY+Y^2)$ $G_3 = 4\,X^3(2\,X^3+3\,(a^2+1)X^2Y+6\,aXY^2+2\,Y^3)$ $\Delta = -2^4\cdot 3^3\,(a-1)^2X^9Y[12\,X^2+3\,(3\,a^2+6\,a-1)XY+4\,(a+2)Y^2]$ $a \neq -2,\ -\frac{5}{3},\ 1 \qquad u_{1,2} = -\frac{1}{16}\,[3\,a^2+6\,a-1 \pm \sqrt{\frac{1}{3}\,(a-1)(3\,a+5)^3}]$	$-\frac{4}{(a-1)^2}\,\frac{(X^2+2\,aXY+Y^2)^3}{X^3Y[12\,X^2+3\,(3\,a^2+6\,a-1)XY+4\,(a+2)Y^2]}$	$\frac{u_2}{u_1} \neq 0,1,\infty$
$I_1\ I_1\ I_2\ I_2^*$ $(u_1,u_2,\infty,0)$	$G_2 = 12\,X^2(X^2+aXY+Y^2)$ $G_3 = 4\,X^3(2\,X^3+3\,aX^2Y+3\,aXY^2+2\,Y^3)$ $\Delta = 2^4\cdot 3^3\,(2-a)^2X^8Y^2[3\,X^2+2\,(2\,a-1)XY+3\,Y^2]$ $a \neq -1,\ 2 \qquad u_{1,2} = -\frac{1}{3}\,(2\,a-1 \pm 2\,\sqrt{a^2-a-2})$	$\frac{4}{(2-a)^2}\,\frac{(X^2+aXY+Y^2)^3}{X^2Y^2[3\,X^2+2\,(2\,a-1)XY+3\,Y^2]}$	$\frac{u_2}{u_1} \neq 0,1,\infty$
$I_2\ I_1\ II\ I_0^*$ $(0,\infty,1,\rho_4)$	$G_2 = 3\,(X-\rho_4 Y)^2(X-Y)(X-9\,Y)$ $G_3 = (X-\rho_4 Y)^3(X-Y)(X^2+18\,XY-27\,Y^2)$ $\Delta = -2^6\cdot 3^3\,X^3Y(X-Y)^2(X-\rho_4 Y)^6$	$-\frac{1}{64}\,\frac{(X-Y)(X-9\,Y)^3}{X^3Y}$	$\frac{1}{\rho_4} \neq 0,1,\infty$
$I_2\ I_1\ III\ I_0^*$ $(0,\infty,1,\rho_4)$	$G_2 = 3\,(X-\rho_4 Y)^2(X-Y)(X-4\,Y)$ $G_3 = (X-\rho_4 Y)^3(X-Y)^2(X+8\,Y)$ $\Delta = -3^6\,X^2Y(X-Y)^3(X-\rho_4 Y)^6$	$-\frac{1}{27}\,\frac{(X-4\,Y)^3}{X^2Y}$	$\frac{1}{\rho_4} \neq 0,1,\infty$
$I_1\ I_1\ IV\ I_0^*$ $(0,\infty,1,\rho_4)$	$G_2 = 3\,(X-\rho_4 Y)^2(X-Y)^2$ $G_3 = (X-\rho_4 Y)^3(X-Y)^2(X+Y)$ $\Delta = -108\,XY(X-Y)^4(X-\rho_4 Y)^6$	$-\frac{1}{4}\,\frac{(X-Y)^2}{XY}$	$\frac{1}{\rho_4} \neq 0,1,\infty$

$I_1\ I_1\ I_1\ III^*$ $(\nu_1,\nu_2,\infty,0)$	$G_2 = 3X^3(X+aY)$ $G_3 = X^5(X+Y)$ $\Delta = 27X^9Y[(3a-2)X^2+(3a^2-1)XY+a^3Y^2]$ $a \neq -\frac{1}{3}, 0, \frac{2}{3}, 1 \qquad \nu_{1,2} = -\frac{1}{6a-4}\left[3a^2-1 \pm \sqrt{(3a+1)(1-a)^3}\right]$	$\dfrac{(X+aY)^3}{Y[(3a-2)X^2-(3a^2-1)XY+a^3Y]}$	$\dfrac{\nu_2}{\nu_1} \neq 0,1,\infty$
$I_1\ I_1\ I_2\ IV^*$ $(\nu_1,\nu_2,\infty,0)$	$G_2 = 3X^3(X+2aY)$ $G_3 = X^4(X^2+3aXY+Y^2)$ $\Delta = 27X^8Y^2[(3a^2-2)X^2+2a(4a^2-3)XY-Y^2]$ $a \neq 0, \pm\sqrt{\frac{1}{2}}, \pm\sqrt{\frac{2}{3}} \qquad \nu_{1,2} = -\frac{1}{3a^2-2}\left[a(4a^2-3) \pm \sqrt{2(2a^2-1)^3}\right]$	$\dfrac{X(X+2aY)^3}{Y^2[(3a^2-2)X^2+2a(4a^2-3)XY-Y^2]}$	$\dfrac{\nu_2}{\nu_1} \neq -1,0,\frac{1}{2},1,2,\infty$
$I_1\ II\ III\ I_0^*$ $(\infty,0,1,\rho_4)$	$G_2 = 3X(X-\rho_4Y)^2(X-Y)$ $G_3 = X(X-\rho_4Y)^3(X-Y)^2$ $\Delta = 27X^2Y(X-Y)^3(X-\rho_4Y)^6$	$\dfrac{X}{Y}$	$\rho_4 \neq 0,1,\infty$
$I_2\ II\ II\ I_0^*$ $(\infty,0,1,\rho_4)$	$G_2 = 12X(X-Y)(X-\rho_4Y)^2$ $G_3 = 4X(X-\rho_4Y)^3(X-Y)(2X-Y)$ $\Delta = -2^4\cdot 3^3X^2Y^2(X-Y)^2(X-\rho_4Y)^6$	$-4\dfrac{X(X-Y)}{Y^2}$	$\rho_4 \neq 0,1,\infty$
$I_1\ I_1\ II\ IV^*$ $(0,\infty,1,a^2)$	$G_2 = 3(X-Y)(X-a^2Y)^3$ $G_3 = (X-Y)(X-a^2Y)^4(X+aY)$ $\Delta = -27(a+1)^2XY(X-Y)^2(X-a^2Y)^8$ $a \neq -1,0,1,\infty$	$-\dfrac{1}{(a+1)^2}\dfrac{(X-Y)(X-a^2Y)}{XY}$	$\dfrac{1}{a^2} \neq 0,1,\infty$
$I_1\ I_1\ I_1\ I_9$ $(1,\epsilon,\epsilon^2,\infty)$	$G_2 = 3X(9X^3-8Y^3)$ $G_3 = 27X^6-36X^3Y^3+8Y^6$ $\Delta = 2^6\cdot 3^3Y^9(X^3-Y^3)$ $\epsilon = e^{\frac{2\pi i}{3}}$	$\dfrac{1}{64}\dfrac{X^3(9X^3-8Y^3)^3}{Y^9(X^3-Y^3)}$	$-\epsilon$
$I_1\ I_1\ I_2\ I_8$ $(-1,1,0,\infty)$	$G_2 = 3(16X^4-16X^2Y^2+Y^4)$ $G_3 = 64X^6-96X^4Y^2+30X^2Y^4+Y^6$ $\Delta = 2^2\cdot 3^6X^2Y^8(X+Y)(X-Y)$	$\dfrac{1}{108}\dfrac{(16X^4-16X^2Y^2+Y^4)^3}{X^2Y^8(X+Y)(X-Y)}$	-1

$I_1\ I_2\ I_3\ I_6$ $(4, -\frac{1}{2}, 0, \infty)$	$G_2 = 12\ (X^4 - 4\ X^3Y + 2\ XY^3 + Y^4)$ $G_3 = 4\ (2\ X^6 - 12\ X^5Y + 12\ X^4Y^2 + 14\ X^3Y^3 + 3\ X^2Y^4 + 6\ XY^5 + 2\ Y^6)$ $\Delta = 2^4 \cdot 3^6\ X^3Y^6(2\ X + Y)^2(X - 4\ Y)$	$\frac{4}{27}\ \frac{(X^4 - 4\ X^3Y + 2\ XY^3 + Y^4)^3}{X^3Y^6(2\ X + Y)^2(X - 4\ Y)}$	-8
$I_1\ I_1\ I_5\ I_5$ $(\omega_1, \omega_2, 0, \infty)$	$G_2 = 3\ (X^4 - 12\ X^3Y + 14\ X^2Y^2 + 12\ XY^3 + Y^4)$ $G_3 = X^6 - 18\ X^5Y + 75\ X^4Y^2 + 75\ X^2Y^4 + 18\ XY^5 + Y^6$ $\Delta = 2^6 \cdot 3^6\ X^5Y^5(X^2 - 11\ XY - Y^2)$ $\omega_{1,2} = \left[\frac{1 \pm \sqrt{5}}{2}\right]^5$	$\frac{1}{2^4 \cdot 3^3}\ \frac{(X^4 - 12\ X^3Y + 14\ X^2Y^2 + 12\ XY^3 + Y^4)^3}{X^5Y^5(X^2 - 11\ XY - Y^2)}$	$\left[\frac{1 + \sqrt{5}}{1 - \sqrt{5}}\right]^5$
$I_2\ I_2\ I_4\ I_4$ $(-1, 1, 0, \infty)$	$G_2 = 12\ (X^4 - X^2Y^2 + Y^4)$ $G_3 = 4\ (2\ X^6 - 3\ X^4Y^2 - 3\ X^2Y^4 + 2\ Y^6)$ $\Delta = 2^4 \cdot 3^6\ X^4Y^4(X + Y)^2(X - Y)^2$	$\frac{4}{27}\ \frac{(X^4 - X^2Y^2 + Y^4)^3}{X^4Y^4(X + Y)^2(X - Y)^2}$	-1
$I_3\ I_3\ I_3\ I_3$ $(1, \rho, \rho^2, \infty)$	$G_2 = 3\ Y(8\ X^3 + Y^3)$ $G_3 = 8\ X^6 + 20\ X^3Y^3 - Y^6$ $\Delta = -\ 2^6 \cdot 3^3\ X^3(X^3 - Y^3)^3$ $\rho = e^{\frac{2\pi i}{3}}$	$-\frac{1}{64}\ \frac{Y^3(8\ X^3 + Y^3)^3}{X^3(X^3 - Y^3)^3}$	$-\rho$
$I_1\ I_1\ I_8\ II$ $(\omega_1, \omega_2, \infty, 0)$	$G_2 = 12\ X(X^3 - 6\ X^2Y + 15\ XY^2 - 12\ Y^3)$ $G_3 = 4\ X(2\ X^5 - 18\ X^4Y + 72\ X^3Y^2 - 144\ X^2Y^3 + 135\ XY^4 - 27\ Y^5)$ $\Delta = -\ 2^4 \cdot 3^6\ X^2Y^8(3\ X^2 - 14\ XY + 27\ Y^2)$ $\omega_{1,2} = -\frac{1}{3}\ (1 \pm i\ \sqrt{2})^4$	$-\frac{4}{27}\ \frac{X(X^3 - 6\ X^2Y + 15\ XY^2 - 12\ Y^3)^3}{Y^8(3\ X^2 - 14\ XY + 27\ Y^2)}$	$\left[\frac{1 - i\ \sqrt{2}}{1 + i\ \sqrt{2}}\right]^4$
$I_1\ I_2\ I_7\ II$ $(-\frac{9}{4}, \frac{8}{9}, \infty, 0)$	$G_2 = 12\ X(9\ X^3 + 36\ X^2Y + 42\ XY^2 + 14\ Y^3)$ $G_3 = 12\ X(18\ X^5 + 108\ X^4Y + 234\ X^3Y^2 + 222\ X^2Y^3 + 87\ XY^4 + 8\ Y^5)$ $\Delta = -\ 2^4 \cdot 3^8\ X^2Y^7(9\ X + 8\ Y)^2(4\ X + 9\ Y)$	$-4\ \frac{X(9\ X^3 + 36\ X^2Y + 42\ X^2Y + 14\ Y^3)^3}{Y^7(9\ X + 8\ Y)^2(4\ X + 9\ Y)}$	$-\frac{32}{81}$
$I_1\ I_4\ I_5\ II$ $(-10, 0, \infty, \frac{1}{8})$	$G_2 = 3\ (8\ X - Y)(8\ X^3 + 87\ X^2Y + 96\ XY^2 - 64\ Y^3)$ $G_3 = (8\ X - Y)(64\ X^5 + 2^4 \cdot 5 \cdot 13\ X^4Y + 5^2 \cdot 157\ X^3Y^2 + 100\ X^2Y^3 + 2^7 \cdot 5^2\ XY^4 - 2^9\ Y^5)$ $\Delta = -\ 2^3 \cdot 3^{15}\ X^4Y^5(8\ X - Y)^2(X + 10\ Y)$	$-\frac{1}{2^3 \cdot 3^{12}}\ \frac{(8\ X - Y)(8\ X^3 + 87\ X^2Y + 96\ XY^2 - 64\ Y^3)^3}{X^4Y^5(X + 10\ Y)}$	$\frac{1}{81}$
$I_2\ I_3\ I_5\ II$ $(-\frac{5}{9}, 0, \infty, 3)$	$G_2 = 3\ (X - 3\ Y)(81\ X^3 - 9\ X^2Y - 53\ XY^2 - 27\ Y^3)$ $G_3 = (X - 3\ Y)(3^6\ X^5 - 3^5 \cdot 5\ X^4Y - 2 \cdot 3^3 \cdot 5^2\ X^3Y^2 - 350\ X^2Y^3 - 3^3 \cdot 5^2\ XY^4 - 243\ Y^5)$ $\Delta = -\ 2^{14} \cdot 3^4\ X^3Y^5(X - 3\ Y)^2(9\ X + 5\ Y)^2$	$-\frac{1}{2^{14} \cdot 3}\ \frac{(X - 3\ Y)(81\ X^3 - 9\ X^2Y - 53\ XY^2 - 27\ Y^3)^3}{X^3Y^5(9\ X + 5\ Y)^2}$	$\frac{27}{32}$

$I_1\ I_1\ I_7\ III$ $(a_1, a_2, \infty, 0)$	$G_2 = 12\ X(X^3 + 4\ X^2Y + 10\ XY^2 + 6\ Y^3)$ $G_3 = 4\ X^2(2\ X^4 + 12\ X^3Y + 42\ X^2Y^2 + 70\ XY^3 + 63\ Y^4)$ $\Delta = 2^4 \cdot 3^6\ X^3Y^7(4\ X^2 + 13\ XY + 32\ Y^2)$ $a_{1,2} = \frac{1}{4}\left[\frac{1 \pm i\sqrt{7}}{2}\right]^7$	$\frac{4}{27}\ \frac{(X^3 + 4\ X^2Y + 10\ XY^2 + 6\ Y^3)^3}{Y^7(4\ X^2 + 13\ XY + 32\ Y^2)}$	$\left[\frac{1 - i\sqrt{7}}{1 + i\sqrt{7}}\right]^7$
$I_1\ I_2\ I_6\ III$ $(4, 1, \infty, 0)$	$G_2 = 12\ X(X^3 - 6\ X^2Y + 9\ XY^2 - 3\ Y^3)$ $G_3 = 4\ X^2(2\ X^4 - 18\ X^3Y + 54\ X^2Y^2 - 63\ XY^3 + 27\ Y^4)$ $\Delta = 2^4 \cdot 3^6\ X^3Y^6(X - Y)^2(X - 4\ Y)$	$\frac{4}{27}\ \frac{(X^3 - 6\ X^2Y + 9\ XY^2 - 3\ Y^3)^3}{Y^6(X - Y)^2(X - 4\ Y)}$	$\frac{1}{4}$
$I_1\ I_3\ I_5\ III$ $(-\frac{25}{3}, 0, \infty, \frac{1}{5})$	$G_2 = 75\ (5\ X - Y)(5\ X^3 + 45\ X^2Y + 39\ XY^2 - 25\ Y^3)$ $G_3 = 25\ (5\ X - Y)^2(25\ X^4 + 340\ X^3Y + 2 \cdot 3 \cdot 181\ X^2Y^2 + 100\ X^3Y + 5^4\ Y^4)$ $\Delta = -\ 2^{14} \cdot 3^6 \cdot 5^4\ X^3Y^5(5\ X - Y)^3(3\ X + 25\ Y)$	$-\frac{25}{2^{14} \cdot 3^3}\ \frac{(5\ X^3 + 45\ X^2Y + 39\ XY^2 - 25\ Y^3)^3}{X^3Y^5(3\ X + 25\ Y)}$	$\frac{3}{128}$
$I_2\ I_3\ I_4\ III$ $(-\frac{1}{3}, 0, \infty, 1)$	$G_2 = 3\ (X - Y)(16\ X^3 - 3\ XY^2 - Y^3)$ $G_3 = (X - Y)^2(64\ X^4 + 32\ X^3Y + 6\ X^2Y^2 + 5\ XY^3 + Y^4)$ $\Delta = 2^2 \cdot 3^6\ X^3Y^4(X - Y)^3(3\ X + Y)^2$	$\frac{1}{108}\ \frac{(16\ X^3 - 3\ XY^2 - Y^3)^3}{X^3Y^4(3\ X + Y)^2}$	$\frac{3}{4}$
$I_1\ I_1\ I_6\ IV$ $(1, -1, \infty, 0)$	$G_2 = 3\ X^2(9\ X^2 - 8\ Y^2)$ $G_3 = X^2(27\ X^4 - 36\ X^2Y^2 + 8\ Y^4)$ $\Delta = 2^6 \cdot 3^3\ X^4Y^6(X - Y)(X + Y)$	$\frac{1}{64}\ \frac{X^2(9\ X^2 - 8\ Y^2)^3}{Y^6(X - Y)(X + Y)}$	-1
$I_1\ I_2\ I_5\ IV$ $(-\frac{27}{4}, -\frac{1}{2}, \infty, 0)$	$G_2 = 12\ X^2(X^2 + 8\ XY + 10\ Y^2)$ $G_3 = 4\ X^2(2\ X^4 + 24\ X^3Y + 78\ X^2Y^2 + 56\ XY^3 + 27\ Y^4)$ $\Delta = -\ 2^4 \cdot 3^6\ X^4Y^5(2\ X + Y)^2(4\ X + 27\ Y)$	$-\frac{4}{27}\ \frac{X^2(X^2 + 8\ XY + 10\ Y^2)^3}{Y^5(2\ X + Y)^2(4\ X + 27\ Y)}$	$\frac{2}{27}$
$I_2\ I_3\ I_2\ IV$ $(\infty, 0, -1, 1)$	$G_2 = 3\ (X - Y)^2(9\ X^2 + 14\ XY + 9\ Y^2)$ $G_3 = (X - Y)^3(27\ X^4 + 36\ X^3Y + 2\ X^2Y^2 + 36\ XY^3 + 27\ Y^4)$ $\Delta = -\ 2^{12} \cdot 3^3\ X^3Y^3(X - Y)^4(X + Y)^2$	$-\frac{1}{2^{12}}\ \frac{(X - Y)^2(9\ X^2 + 14\ XY + 9\ Y^2)^3}{X^3Y^3(X + Y)^2}$	-1

$I_1\ I_7\ II\ II$ $(0,\infty,\nu_1,\nu_2)$	$G_2 = 3\ (X^2 - 13\ XY + 49\ Y^2)(X^2 - 5\ XY + Y^2)$ $G_3 = (X^2 - 13\ XY + 49\ Y^2)(X^4 - 14\ X^3Y + 63\ X^2Y^2 - 70\ XY^3 - 7\ Y^4)$ $\Delta = -\ 2^6\cdot 3^3\ XY^7(X^2 - 13\ XY + 49\ Y^2)^3$ $\nu_{1,2} = -\frac{1}{4}\ (-1 \pm 3\ i\ \sqrt{3})^2$	$-\frac{1}{2^6\cdot 3^3}\ \frac{(X^2 - 13\ XY + 49\ Y^2)(X^2 - 5\ XY + Y^2)^3}{XY^7}$	$\left[\frac{-1 + 3\ i\ \sqrt{3}}{-1 - 3\ i\ \sqrt{3}}\right]^2$
$I_2\ I_6\ II\ II$ $(0,\infty,1,-1)$	$G_2 = 3\ (X - Y)(X + Y)(9\ X^2 - Y^2)$ $G_3 = (X - Y)(X + Y)(27\ X^4 - 18\ X^2Y^2 - Y^4)$ $\Delta = -\ 2^6\cdot 3^3\ X^2Y^6(X - Y)^2(X + Y)^2$	$-\frac{1}{64}\ \frac{(X - Y)(X + Y)(9\ X^2 - Y^2)^3}{X^2Y^6}$	-1
$I_4\ I_4\ II\ II$ $(\nu_1,\nu_2,\frac{1}{2},-4)$	$G_2 = 12\ XY(2\ X - Y)(X + 4\ Y)$ $G_3 = 2\ (2\ X - Y)(X + 4\ Y)(X^4 + 4\ X^3Y + 8\ XY^3 - 4\ Y^4)$ $\Delta = -\ 108\ (2\ X - Y)^2(X + 4\ Y)^2(X^2 + 2\ XY - 2\ Y^2)^4$ $\nu_{1,2} = -1 \pm \sqrt{3}$	$-16\ \frac{X^3Y^3(2\ X - Y)(X + 4\ Y)}{(X^2 + 2\ XY - 2\ Y^2)^4}$	$(-2 + \sqrt{3})^3$
$I_1\ I_6\ II\ III$ $(\nu,\infty,1,0)$	$G_2 = 2\ X(X - Y)[6\ X^2 + 6\ \zeta\ XY - (3 + \zeta)Y^2]$ $G_3 = 2\ X^2(X - Y)[4\ X^3 - 2\ (1 - 3\ \zeta)X^2Y - 4\ (2 + \zeta)XY^2 + (5 - \zeta)Y^3]$ $\Delta = 24\ X^5Y^6(X - Y)^3[(9 + \zeta)X + 8\ \zeta\ Y]$ $\nu = -\frac{2}{7}\ (3\ \zeta + 1) \qquad \zeta = \pm\ i\ \sqrt{3}$	$\frac{1}{3}\ \frac{(X - Y)[6\ X^2 + 6\ \zeta\ XY - (3 + \zeta)Y^2]^3}{Y^6[(9 + \zeta)X + 8\ \zeta\ Y]}$	$\frac{3}{8}(3 - \zeta)$
$I_2\ I_5\ II\ III$ $(\frac{125}{14},\infty,0,\frac{27}{2})$	$G_2 = 3\ X(2\ X - 27\ Y)(2\ X^2 - 35\ XY + 140\ Y^2)$ $G_3 = X(2\ X - 27\ Y)^2(2\ X^3 - 39\ X^2Y + 222\ XY^2 - 250\ Y^3)$ $\Delta = 2^2\cdot 3^4\ X^2Y^5(2\ X - 27\ Y)^3(14\ X - 125\ Y)^2$	$\frac{1}{108}\ \frac{X(2\ X^2 - 35\ XY + 140\ Y^2)^3}{Y^5(14\ X - 125\ Y)^2}$	$-\frac{125}{64}$
$I_3\ I_4\ II\ III$ $(0,\infty,-27,1)$	$G_2 = 3\ (X - Y)(X + 27\ Y)(16\ X^2 + 80\ XY - 243\ Y^2)$ $G_3 = (X - Y)^2(X + 27\ Y)(64\ X^3 + 2^5\cdot 43\ X^2Y + 2\cdot 3^5\ XY^2 + 3^8\ Y^3)$ $\Delta = -\ 2^2\cdot 3^6\cdot 7^7\ X^3Y^4(X - Y)^3(X + 27\ Y)^2$	$-\frac{1}{2^2\cdot 3^3\cdot 7^7}\ \frac{(X + 27\ Y)(16\ X^2 + 80\ XY - 243\ Y^2)^3}{X^3Y^4}$	-27
$I_1\ I_8\ II\ IV$ $(-\frac{16}{3},\infty,3,0)$	$G_2 = 3\ X^2(X - 3\ Y)(X + 5\ Y)$ $G_3 = X^3(X - 3\ Y)(X^3 + 6\ X^2Y - 3\ XY^2 - 32\ Y^3)$ $\Delta = -\ 2^6\cdot 3^3\ X^4Y^8(X - 3\ Y)^2(3\ X + 16\ Y)$	$-\frac{1}{64}\ \frac{X^2(X - 3\ Y)(X + 5\ Y)^3}{Y^8(3\ X + 16\ Y)}$	$\frac{25}{16}$
$I_2\ I_4\ II\ IV$ $(\frac{1}{9},\infty,1,0)$	$G_2 = 36\ X^2(X - Y)(3\ X - Y)$ $G_3 = 4\ X^2(X - Y)(54\ X^3 - 54\ X^2Y + 9\ XY^2 - Y^3)$ $\Delta = -\ 2^4\cdot 3^4\ X^4Y^4(X - Y)^2(9\ X - Y)^2$	$-108\ \frac{X^2(X - Y)(3\ X - Y)^3}{Y^4(9\ X - Y)^2}$	-8

$I_1\ I_5$ III III $(-\frac{11}{2},\infty,i,-i)$	$G_2 = 3\ (X^2 + Y^2)(X^2 + 6\ XY + 4\ Y^2)$ $G_3 = (X^2 + Y^2)^2(X^2 + 9\ XY + 19\ Y^2)$ $\Delta = -\ 3^6\ Y^6(X^2 + Y^2)^3(2\ X + 11\ Y)$	$-\frac{1}{27}\ \frac{(X^2 + 6\ XY + 4\ Y^2)^3}{Y^3(2\ X + 11\ Y)}$	$\left[\frac{1 + 2\ i}{1 - 2\ i}\right]^3$
$I_2\ I_4$ III III $(0,\infty,1,-1)$	$G_2 = 3\ (X - Y)(X + Y)(4\ X^2 - Y^2)$ $G_3 = (X - Y)^2(X + Y)^2(8\ X^2 + Y^2)$ $\Delta = 3^6\ X^2Y^4(X - Y)^3(X + Y)^3$	$\frac{1}{27}\ \frac{(4\ X^2 - Y^2)^3}{X^2Y^4}$	-1
$I_3\ I_3$ III III $(\omega_1,\omega_2,0,\infty)$	$G_2 = 3\ XY(X^2 + 6\ XY - 3\ Y^2)$ $G_3 = 6\ X^2Y^2(X^2 + 3\ Y^2)$ $\Delta = 27\ X^3Y^3(X^2 - 6\ XY - 3\ Y^2)^3$ $\omega_{1,2} = 3 \pm 2\sqrt{3}$	$\frac{(X^2 + 6\ XY - 3\ Y^2)^3}{(X^2 - 6\ XY - 3\ Y^2)^3}$	$-(2 + \sqrt{3})^2$
$I_1\ I_4$ III IV $(-\frac{27}{5},\infty,1,0)$	$G_2 = 12\ X^2(X - Y)(X + 5\ Y)$ $G_3 = 4\ X^2(X - Y)^2(2\ X^2 + 16\ XY + 27\ Y^2)$ $\Delta = 2^4 \cdot 3^6\ X^4Y^4(5\ X + 27\ Y)(X - Y)^3$	$\frac{4}{27}\ \frac{X^2(X + 5\ Y)^3}{Y^4(5\ X + 27\ Y)}$	$\frac{32}{27}$
$I_2\ I_3$ III IV $(\frac{1}{5},\infty,1,0)$	$G_2 = 3\ X^2(X - Y)(9\ X - 5\ Y)$ $G_3 = X^2(X - Y)^2(27\ X^2 - 9\ XY + 2\ Y^2)$ $\Delta = 108\ X^4Y^3(5\ X - Y)^2(X - Y)^3$	$\frac{1}{4}\ \frac{X^2(9\ X - 5\ Y)^3}{Y^3(5\ X - Y)^2}$	-4
$I_2\ I_2$ IV IV $(0,\infty,1,-1)$	$G_2 = 3\ (X - Y)^2(X + Y)^2$ $G_3 = (X - Y)^2(X + Y)^2(X^2 + Y^2)$ $\Delta = -\ 108\ X^2Y^2(X - Y)^4(X + Y)^4$	$-\frac{1}{4}\ \frac{(X - Y)^2(X + Y)^2}{X^2Y^2}$	-1

I_2 IV III III $(\infty, 0, 1, -1)$	$G_2 = 3\,X^2(X - Y)(X + Y)$ $G_3 = X^2(X - Y)^2(X + Y)^2$ $\Delta = 27\,X^4Y^2(X - Y)^3(X + Y)^3$	$\frac{X^2}{Y^2}$	-1
I_3 III III III $(\infty, \rho, \rho^2, 1)$	$G_2 = 3\,X\,(X^3 - Y^3)$ $G_3 = (X^3 - Y^3)^2$ $\Delta = 27\,Y^3(X^3 - Y^3)^3$ $\rho = e^{\frac{2\pi i}{3}}$	$\frac{X^3}{Y^3}$	$-\rho^2$
I_3 II III IV $(\infty, -3, 1, 0)$	$G_2 = 3\,X^2(X - Y)(X + 3\,Y)$ $G_3 = X^2(X - Y)^2(X + 3\,Y)(X + 2\,Y)$ $\Delta = 108\,X^4Y^3(X + 3\,Y)^2(X - Y)^3$	$\frac{1}{4}\,\frac{X^2(X + 3\,Y)}{Y^3}$	$\frac{3}{4}$
I_4 IV II II $(\infty, 0, 1, -1)$	$G_2 = 12\,X^2(X - Y)(X + Y)$ $G_3 = 4\,X^2(X - Y)(X + Y)(2\,X^2 - Y^2)$ $\Delta = -\,2^4\cdot 3^3\,X^4Y^4(X - Y)^2(X + Y)^2$	$-4\,\frac{X^2(X - Y)(X + Y)}{Y^4}$	-1
I_4 II III III $(\infty, -5, \omega_1, \omega_2)$	$G_2 = 3\,(X^2 + 2\,Y^2)(X + 5\,Y)(X + Y)$ $G_3 = (X^2 + 2\,Y^2)^2(X + 5\,Y)(X + 4\,Y)$ $\Delta = -\,3^6\,Y^4(X + 5\,Y)^2(X^2 + 2\,Y^2)^3$ $\omega_{1,2} = \pm\, i\,\sqrt{2}$	$-\frac{1}{27}\,\frac{(X + 5\,Y)(X + Y)^3}{Y^4}$	$\left[\frac{1 + i\,\sqrt{2}}{1 - i\,\sqrt{2}}\right]^2$
I_5 III II II $(\infty, 0, \omega_1, \omega_2)$	$G_2 = 3\,X(X^2 + 11\,XY + 64\,Y^2)(X + 3\,Y)$ $G_3 = X^2(X^2 + 11\,XY + 64\,Y^2)(X^2 + 10\,XY + 45\,Y^2)$ $\Delta = 2^6\cdot 3^6\,X^3Y^5(X^2 + 11\,XY + 64\,Y^2)^2$ $\omega_{1,2} = \frac{1}{8}\,(1 \pm i\,\sqrt{15})^3$	$\frac{1}{2^6\cdot 3^3}\,\frac{(X^2 + 11\,XY + 64\,Y^2)(X + 3\,Y)^3}{Y^5}$	$\left[\frac{1 - i\,\sqrt{15}}{1 + i\,\sqrt{15}}\right]^3$
I_6 II II II $(\infty, \rho, \rho^2, 1)$	$G_2 = 12\,X(X^3 - Y^3)$ $G_3 = 4\,(X^3 - Y^3)(2\,X^3 - Y^3)$ $\Delta = -\,2^4\cdot 3^3\,Y^6(X^3 - Y^3)^2$ $\rho = e^{\frac{2\pi i}{3}}$	$-4\,\frac{X^3(X^3 - Y^3)}{Y^6}$	$-\rho^2$

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