AMPLE CARTIER DIVISORS ON NORMAL SURFACES

by

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By a *polarized normal surface* we mean a pair (Y,H) of a normal projective surface Y over $\mathbb{C}$ and an ample Cartier divisor H on it. Define a graded ring:

$$R = \bigoplus_{m\geq 0} H^0(Y, \mathcal{O}(m(K_Y+H))) .$$

Let $\kappa = \kappa(K_Y+H,Y)$, which is by definition, $\mathrm{tr.deg.}_{\mathbb{C}}R - 1$ or $-\infty$ in case $R \cong \mathbb{C}$. To state the structure theorem, we introduce an example. Let $\mathbb{F}_e = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$ and suppose $e \geq 2$. Let $\pi : \mathbb{F}_e \longrightarrow Y_e$ be the contraction of the base section b, a $(-e)$-curve, and let ℓ denote the image of a fibre f on $\mathbb{F}_e$. Then $\pi^*\ell = f + (1/e)b$ and $\pi^*K_{Y_e} = K_{\mathbb{F}_e} + (1-2/e)b$. It turns out that $H_e = e\ell$ is an ample Cartier divisor on Y_e and we have $e(K_{Y_e} + H_e) = -2H_e$.

THEOREM: *Let (Y,H) be a polarized normal surface. Then we have the following classification*:

κ	Structure of (Y,H)	Sing(Y)
$-\infty$	$(\mathbb{P}^2, \mathcal{O}(1)), (\mathbb{P}^2, \mathcal{O}(2)), (Y_e, H_e)\, e \geqq 2$	smooth or quotient
	$\mathbb{P}^1$- bundle with $Hf = 1$ for a fibre f	smooth
0	$K_Y + H \sim 0$	Gorenstein
1	Conic bundle with $\kappa = 1$	RDP of type A
2	R is finitely generated	$\mathbb{Q}$-Gorenstein
	R is not finitely generated	not $\mathbb{Q}$-Gorenstein

This type of Theorems have been obtained by Sommese [7], Lanteri-Palleschi [2] for the case in which Y is smooth and by Sommese [8] for the case in which Y is normal Gorenstein. I would like to thank A. Sommese for inspiring me by his preprint [8].

§1 PRELIMINARIES

For the basic results on normal surfaces we refer to [5] and [6]. A divisor will mean a Weil divisor. A divisor is said to be ample if its some positive multiple becomes a very ample Cartier divisor. We use the intersection theory with $\mathbb{Q}$-coefficients introduced by Mumford. We denote by $\sim$(resp. $\equiv$) the linear (resp. numerical) equivalence of divisors. A divisor D is nef if $DC \geq 0$ for all irreducible curves C and is pseudoeffective if $DP \geq 0$ for all nef divisors P . We associate to a normal surface Y a triple (X,π,Δ) where $\pi : X \longrightarrow Y$ is the minimal resolution and the Δ is an effective $\mathbb{Q}$-divisor supported on the exceptional set so that $\pi^*K_Y = K_X + \Delta$. By definition, Y is $\mathbb{Q}$-Gorenstein if some multiple of K_Y is a Cartier divisor. A rational double point will be abbreviated by RDP .

We use the following facts:

LEMMA 1. Let C be an irreducible curve on Y and let $\bar{C}$ denote its strict transform on X . Then

(i) $\overline{C}^2 \leq C^2$, <u>the equality holds if and only if</u> C <u>does not meet</u> Sing(Y) ,

(ii) $K_X\overline{C} \leq K_Y C$, <u>the equality holds if and only if</u> C <u>meets only</u> RDP's <u>in</u> Sing(Y) ,

(iii) $(K_Y + C)C \geq -2$, <u>the equality holds if and only if</u> $C \cong \mathbb{P}^1$ <u>and</u> C <u>does not meet</u> Sing(Y) .

<u>PROOF</u>: Let $A = \pi^{-1}(\mathrm{Sing}(Y))$. (i) By definition (see [5]), $\pi^*C = \overline{C} + Z$ where the Z is an effective $\mathbb{Q}$-divisor supported on A . It follows that $\overline{C}^2 = C^2 + Z^2 \leq C^2$. The equality implies that $Z = 0$ so that C does not meet Sing(Y) . For otherwise, $\overline{C}$ must meet at least one irreducible component E in A , and Z would contain E , because $(\pi^*C)E = 0$.(ii) Clearly, $K_X\overline{C} = K_Y C - \Delta\overline{C} \leq K_Y C$. The equality implies that $\Delta\overline{C} = 0$, so that C can meet only RDP's .(iii) follows from (i) and (ii).

Q.E.D.

<u>LEMMA 2</u>. <u>Let</u> (Y,H) <u>be a polarized normal surface</u>. <u>If</u> $H^0(Y, \mathcal{O}(K_Y + H)) = 0$, <u>then</u> Y <u>has only rational singularties</u>.

<u>PROOF</u>: This has been essentially given in [8, Theorem (3.1)]. We sketch a proof. Since H is a Cartier divisor, we can write $\mathcal{O}(\pi^*H) \otimes \mathcal{O}_{[\Delta]} \cong \mathcal{O}_{[\Delta]}$ where the $[\Delta]$ is the integral part of Δ . There is an exact sequence:

$$0 \to \mathcal{O}(K_X + \pi^*H) \to \mathcal{O}(K_X + [\Delta] + \pi^*H) \to \omega_{[\Delta]} \to 0$$

and so

$$\to H^0(X,\mathcal{O}(K_X+[\Delta]+\pi^*H)) \to H^0([\Delta],\omega_{[\Delta]}) \to H^1(X,\mathcal{O}(K_X+\pi^*H)) \to$$

The hypothesis implies that $H^0(X,\mathcal{O}(K_X+[\Delta]+\pi^*H)) = 0$ (projection formula in [5]). On the other hand, $H^1(X,\mathcal{O}(K_X+\pi^*H)) = 0$. Putting these together, we get $H^0([\Delta],\omega_{[\Delta]}) = 0$, and by duality $H^1([\Delta],\mathcal{O}_{[\Delta]}) = 0$. It follows that Y has only national singularities (see [4, p. 392]).

Q.E.D.

§2 Canonical model

Let (Y,H) be a polarized normal surface. We say that (Y,H) is adjointly <u>minimal</u> (resp. adjointly <u>canonical</u>) if $(K_Y + H)C \geq 0$ (resp. $(K_Y + H)C > 0$) for all irreducible curves C with $C^2 < 0$. For brevity we omit "adjointly". In the terminology of [5] we deal with the pair $(Y, K_Y + H)$.

<u>LEMMA 3</u>. <u>(Y,H) is minimal</u>.

<u>PROOF</u>. Take an irreducible curve C with $C^2 < 0$. We have $HC \geq 1$. If $K_YC \geq 0$, of course $(K_Y + H)C \geq 1$. If $K_YC < 0$, then the strict transform $\bar{C}$ must be a (-1)-curve by Lemma 1. It follows that $K_YC \geq -1$ and hence $(K_Y + H)C \geq 0$.

Q.E.D.

An irreducible curve C with $C^2 < 0$ on Y is said to be <u>redundant</u> on (Y,H) if $(K_Y + H)C = 0$.

<u>LEMMA 4</u>. <u>Let</u> C <u>be a redundant curve on</u> (Y,H) . <u>Then</u> C <u>meets at most one singularity</u> y <u>such that</u>

(i) y <u>is an</u> RDP <u>of type</u> A_n <u>for some</u> n ,

(ii) <u>the strict transform</u> $\bar{C}$ <u>is a</u> (-1)-<u>curve meeting one of the end components of the chain of</u> (-2)-<u>curves of</u> $\pi^{-1}(y)$.

<u>In particular</u>, C <u>can be contracted to a smooth point</u>.

<u>PROOF</u>. By the proof of Lemma 3, we have $K_X\bar{C} = K_Y C = -1$, and by Lemma 1, C meets only RDP's . Note that C is an exceptional curve of the first kind on Y in the sense of [6], that is $K_Y C < 0$, $C^2 < 0$. Thus, the above description follows from [6, Example 1.2, see also 8].

Q.E.D.

Once we know that (Y,H) is minimal, we introduce the notion of a canonical model as follows. A polarized normal surface (Y_0,H_0) is a <u>canonical model</u> of (Y,H) if (i) (Y_o,H_o) is canonical, (ii) there is a birational morphism $\varphi : Y \longrightarrow Y_0$ such that $K_Y + H = \varphi^*(K_{Y_0}+H_0)$. Then it is known that $R \cong R_0$ where R_0 is the graded ring defined for (Y_0,H_0) (cf. [6]). Clearly, (Y,H) is not canonical if and only if

(Y,H) has a redundant curve. Let C be a redundant curve on (Y,H), and let $\varphi : Y \longrightarrow Y'$ be the contraction of C. Since $y' = \varphi(C)$ is a smooth point, the divisor $H' = \varphi_* H$ is again an ample Cartier divisor, and we have $K_Y + H = \varphi^*(K_{Y'} + H')$. We say that (Y',H') is obtained from (Y,H) by contracting a redundant curve C. Continuing this process, we arrive at a canonical model.

PROPOSITION 1. *Let* (Y,H) *be a polarized normal surface. Then there exists a canonical model* (Y_0,H_0) *of* (Y,H). *Furthermore,* Y_0 *is* $\mathbb{Q}$*-Gorenstein if and only if so is* Y.

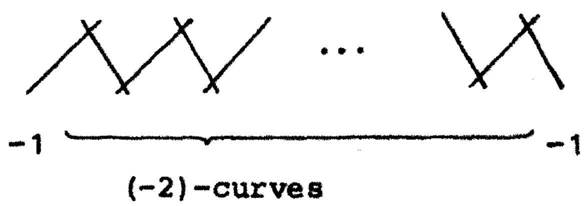
PROOF. The latter part is clear from the construction.

A morphism of Y onto a smooth curve is called a *ruled fibration* if the general fibre is isomorphic to $\mathbb{P}^1$ (see [6]). We say that (Y,H) is a *conic bundle* if Y has a ruled fibration such that $Hf = 2$ for a fibre f.

PROPOSITION 2. *Let* (Y,H) *be a conic bundle. Then*

(i) $(K_Y + H)^2 = 0$,

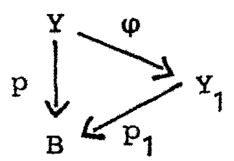
(ii) *each singular fibre consists of two irreducible components and obtained by contracting all* (-2)*-curves in the following chain of* $\mathbb{P}^1$*'s on* X :



In particular, Y _has only_ RDP _of type_ A .

PROOF. Let $p : Y \longrightarrow B$ be the conic fibration of (Y,H) . We examine the singular fibre. If f is an irreducible fibre of p ,then $f^2 = 0$ and $K_Y f = -Hf = -2$, and so by Lemma 1, we have $f \cong \mathbb{P}^1$ and f does not meet $\mathrm{Sing}(Y)$. A reducible fibre consists of two irreducible components, because $Hf = 2$ for a fibre f . Take a reducible fibre $f = f' + f''$. We must have $f'^2 < 0$ and $f''^2 < 0$. Hence, by the minimality of (Y,H) , we get $(K_Y + H)f' = (K_Y + H)f'' = 0$. Thus, both f' and f'' are redundant curves on (Y,H) .

We now contract one component of each reducible fibre. Then we obtain a conic bundle (Y_1,H_1) with a commutative diagram:



Since every fibre of p_1 is irreducible, the above argument shows that Y_1 is smooth. Hence Y_1 is a $\mathbb{P}^1$ - bundle over B . To see (i), from $K_Y + H = \varphi^*(K_{Y_1} + H_1)$, it suffices to check it for (Y_1,H_1) , which is immediate (cf. [2,p.20]). We infer from the construction $(Y,H) \longrightarrow (Y_1,H_1)$ that each reducible fibre has the form (ii).

Q.E.D.

COROLLARY. For a conic bundle (Y,H), we have $\kappa = 1$ except in the following cases: $\kappa = -\infty$ in case $(Y_1,H_1) = (\mathbb{P}^1 \times \mathbb{P}^1, p_1^* \mathcal{O}_{\mathbb{P}^1}(2) \otimes p_2^* \mathcal{O}_{\mathbb{P}^1}(1))$, $\kappa = 0$ in case $(Y_1,H_1) =$ either $(\mathbb{P}^1 \times \mathbb{P}^1, p_1^* \mathcal{O}_{\mathbb{P}^1}(2) \otimes p_2^* \mathcal{O}_{\mathbb{P}^2}(2))$ or $(\mathbb{F}_1, -K_{\mathbb{F}_1})$.

REMARK. (Y_1,H_1) is a canonical model of (Y,H) unless $(Y_1,H_1) = (\mathbb{F}_1, -K_{\mathbb{F}_1})$.

§3 PROOF OF THE THEOREM

Applying Theorem (7.4) in [5] to $K_Y + H$, we see that $K_Y + H$ is nef if and only if $K_Y + H$ is pseudoeffective. There are four distinguished numerical types (cf.[3]) : (a) $K_Y + H$ is not nef, (b) $K_Y + H \equiv 0$, (c) $(K_Y + H)^2 = 0$, $K_Y + H \not\equiv 0$, (d) $(K_Y + H)^2 > 0$.

PROPOSITION 3. The invariant κ is determined by the numerical type of $K_Y + H$ as:

Numerical Type	a	b	c	d
κ	$-\infty$	0	1	2

PROOF. Put $P = \pi^*(K_Y + H)$. From Table II in [3], it is sufficient to consider types (b) and (c). If P is nef and $P^2 = 0$, then $PK_X = -P(\pi^*H) \leq 0$. But, in case $P(\pi^*H) = 0$, we would have $P \equiv 0$. Therefore, if $K_Y + H$ is of type (c), then we must have $PK_X < 0$. It follows from a result in [3] that $\kappa = \kappa(P,X) = 1$. It now remains to show that if $K_Y + H$ is of type (b), then $\kappa = 0$. This is asserted by the following:

PROPOSITION 4. $K_Y + H \equiv 0 \Leftrightarrow K_Y + H \sim 0$.

PROOF. Suppose that $K_Y + H \equiv 0$. It suffices to show that $H^0(Y,\mathcal{O}(K_Y+H)) \neq 0$. Assume $H^0(Y,\mathcal{O}(K_Y+H)) = 0$. Viewing $-K_Y \equiv H$, by the vanishing theorem on Y , we get $H^1(Y,\mathcal{O}_Y) = H^2(Y,\mathcal{O}_Y) = 0$ (see [5]). As we have seen in Lemma 2, Y has only rational singularities. Hence $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_Y)$ and so $\chi(\mathcal{O}_X) = 1$. By the Riemann-Roch theorem and vanishing results on X , we get

$$\begin{aligned} \dim H^0(X,\mathcal{O}(K_X+\pi^*H)) &= \chi(\mathcal{O}(K_X+\pi^*H)) \\ &= \frac{1}{2}(K_X+\pi^*H)(\pi^*H)+\chi(\mathcal{O}_X) \\ &= 1 , \end{aligned}$$

because $K_X+\pi^*H \equiv -\Delta$ and hence $(K_X+\pi^*H)(\pi^*H) = 0$. This contradicts the fact: $\dim H^0(Y,\mathcal{O}(K_Y+H)) \geq \dim H^0(X,\mathcal{O}(K_X+\pi^*H))$.

Q.E.D.

PROOF OF THE THEOREM, continued.

Type (a): The argument in [2] combined with the Mori-theory for normal surfaces ([5]) proves the existence of an extremal rational curve ℓ such that $(K_Y+H)\ell < 0$. By the minimality of (Y,H) , we get $\ell^2 \geq 0$. As in [5] we have two subcases:

(i) $\rho(Y) = 1$ and $-(K_Y+H)$ is numerically ample,

(ii) $\rho(Y) = 2$ and Y has a $\mathbb{P}^1$-fibration, i.e., a ruled fibration of which every fibre is irreducible, and ℓ is a fibre.

First, we examine the case (i). Since Y has only rational singularities (Lemma 2), $-(K_Y+H)$ is ample. From a result of [4], we infer that X is rational. We show that X contains no (-1)-curves. Indeed, if not, take a (-1)-curve $\overline{C}$ on $\overline{X}$, and let C denote the image of $\overline{C}$ by π . Since $\rho(Y) = 1$, C can also be a generator of the divisor group of Y with $\mathbb{Q}$-coefficients. So $(K_Y+H)C < 0$. It follows that $K_YC < -1$ and hence $K_X\overline{C} < -1$, a contradiction. Consequently, X is isomorphic to one of $\mathbb{P}^2$ and $\mathbb{F}_e$ $(e \neq -1)$, and so Y is among $\mathbb{P}^2$ and Y_e . For $\mathbb{P}^2$, H could be either $\mathcal{O}_{\mathbb{P}^2}(1)$ or $\mathcal{O}_{\mathbb{P}^2}(2)$. For Y_e it is easy to verify that $H = H_e$. We now consider the case (ii). For any fibre f of the $\mathbb{P}^1$-fibration, we have $K_Yf < -Hf \leq -1$. Using Lemma 1, we conclude that $f \cong \mathbb{P}^1$ and f does not meet $\mathrm{Sing}(Y)$. Thus, Y is a $\mathbb{P}^1$-bundle.

Type (b): The assertion follows from Proposition 4.

Type (c): As in the proof of Proposition 3, the complete linear system $|mR|$ for a suitable positive integer m such that mP is integral defines a ruled fibration ([3]). Take a fibre f. Then $(K_Y+H)f = 0$ and so $Hf = 2$. Thus, (Y,H) is a conic bundle. The structure of Sing(Y) has been given in Proposition 2.

Type (d): Let (Y_0,H_0) be a canonical model of (Y,H). By definition, $K_{Y_0}+H_0$ is numerically ample. It is ample if and only if Y_0 is $\mathbb{Q}$-Gorenstein. On the other hand, as is remarked for a general setting in [5], we know that R is finitely generated if and only if $K_{Y_0}+H_0$ is ample. Therefore, by Proposition 1, we conclude that R is finitely generated if and only if Y is $\mathbb{Q}$-Gorenstein.

Q.E.D.

CONCLUDING REMARK. Given a polarized normal surface (Y,H), we define the genus: $g(H) = \frac{1}{2}(K_Y+H)H+1$. It is easy to see that $g(H) = 0 \Leftrightarrow \kappa = -\infty$ and X is rational, $g(H) = 1 \Leftrightarrow$ either $\kappa = 0$ or $\kappa = -\infty$ and X is a $\mathbb{P}^1$-bundle of genus 1. Our theorem together with the classification of normal Gorenstein surfaces with ample anticanonical divisors describes the cases $g(H) = 0$ and 1. These cases have been discussed by Bădescu [1].

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