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Abstract

Let G be a simple linear algebraic group over a field k and X a projective homogeneous G-variety such that G splits over $k(X)$. Such variety X is called *generically split*.

In the present note we finish the classification of generically split homogeneous varieties started in our article [PS10]. More precisely, we remove all restrictions on the characteristic of the base field k (in [PS10] we assumed that the characteristic is different from any torsion prime of the group), and complete our classification by the last missing case, namely PGO^+_{2n}. Apart from this, we give a uniform proof for all simple algebraic groups.

We encourage the reader to look at the introduction of [PS10] for history of the problem.

1 Chow rings of reductive groups

1.1. Let G_0 be a split reductive algebraic group defined over a field k. We fix a split maximal torus T in G_0 and a Borel subgroup B of G_0 containing T and defined over k. We denote by Φ the root system of G_0, by Π the set of simple roots of Φ with respect to B, and by $\widehat{T}$ the group of characters of T. Enumeration of simple roots follows Bourbaki.

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Any projective G_0-homogeneous variety X is isomorphic to G_0/P_Θ, where P_Θ stands for the (standard) parabolic subgroup corresponding to a subset $\Theta \subset \Pi$. As P_i we denote the maximal parabolic subgroup $P_{\Pi\setminus\{\alpha_i\}}$ *of type* i.

Consider the characteristic map $c\colon S(\widehat{T}) \to \mathrm{CH}^*(G_0/B)$ from the symmetric algebra of $\widehat{T}$ to the Chow ring of G_0/B given in [PS10, 2.7], and denote its image by R^*. According to [Gr58, Rem. 2°], the ring $\mathrm{CH}^*(G_0)$ can be presented as the quotient of $\mathrm{CH}^*(G_0/B)$ modulo the ideal generated by the non-constant elements of R^*.

1.2 Lemma. *The pull-back map*

$$\mathrm{CH}^*(G_0) \to \mathrm{CH}^*([G_0, G_0])$$

is an isomorphism.

Proof. Indeed, $B' = B \cap [G_0, G_0]$ is a Borel subgroup of $[G_0, G_0]$, the map

$$[G_0, G_0]/B' \to G_0/B$$

is an isomorhism, and the map $S(\widehat{T}) \to \mathrm{CH}^*(G_0/B)$ factors through the surjective map $S(\widehat{T}) \to S(\widehat{T'})$, where $T' = T \cap [G_0, G_0]$. □

Let P be a parabolic subgroup of G_0. Denote by L the Levi subgroup of P and set $H_0 = [L, L]$. We have

1.3 Lemma. *The pull-back map*

$$\mathrm{CH}^*(P) \to \mathrm{CH}^*(H_0)$$

is an isomorphism.

Proof. The quotient map $P \to L$ is Zariski locally trivial affine fibration, therefore the pull-back map $\mathrm{CH}^*(L) \to \mathrm{CH}^*(P)$ is an isomorphism. Since the composition $L \to P \to L$ is the identity map, the pull-back map $\mathrm{CH}^*(P) \to \mathrm{CH}^*(L)$ is an isomorphism as well. It remains to apply Lemma 1.2. □

1.4 Lemma. *The pull-back map*

$$\mathrm{CH}^*(G_0) \to \mathrm{CH}^*(P)$$

is surjective.

Proof. Applying [Gr58, Proposition 3] to the natural map $G_0/B \to G_0/P$ we see that the map $\mathrm{CH}^*(G_0/B) \to \mathrm{CH}^*(P/B)$ is surjective. But the map $\mathrm{CH}^*(P/B) \to \mathrm{CH}^*(P)$ is also surjective by Lemma 1.3 and fits into the commutative diagram

$$\begin{array}{ccc} \mathrm{CH}^*(G_0/B) & \twoheadrightarrow & \mathrm{CH}^*(P/B) \\ \downarrow & & \downarrow \\ \mathrm{CH}^*(G_0) & \longrightarrow & \mathrm{CH}^*(P). \end{array}$$

□

1.5 (Definition of σ)**.** Now we restrict to the situation when G_0 is simple. Let p be a prime integer. Denote $\mathrm{Ch}^*(-)$ the Chow ring with $\mathbb{F}_p$-coefficients. Explicit presentations of the Chow rings with $\mathbb{F}_p$-coefficients of split semisimple algebraic groups are given in [Kc85, Theorem 3.5].

For G_0 and H_0 they look as follows:

$$\mathrm{Ch}^*(G_0) = \mathbb{F}_p[x_1, \ldots, x_r]/(x_1^{p^{k_1}}, \ldots, x_r^{p^{k_r}}) \text{ with } \deg x_i = d_i, 1 \leq d_1 \leq \ldots \leq d_r;$$
$$\mathrm{Ch}^*(H_0) = \mathbb{F}_p[y_1, \ldots, y_s]/(y_1^{p^{l_1}}, \ldots, y_s^{p^{l_s}}) \text{ with } \deg y_m = e_m, 1 \leq e_1 \leq \ldots \leq e_s$$

for some integers k_i, l_i, d_i, and e_i depending on the Dynkin types of G_0 and H_0.

By the previous lemmas the pull-back $\varphi\colon \mathrm{Ch}^*(G_0) \to \mathrm{Ch}^*(H_0)$ is surjective. For a graded ring S^* denote by S^+ the ideal generated by the non-constant elements of S^*. The induced map

$$\mathrm{Ch}^+(G_0)/\,\mathrm{Ch}^+(G_0)^2 \to \mathrm{Ch}^+(H_0)/\,\mathrm{Ch}^+(H_0)^2$$

is also surjective. Moreover, for any m with $e_m > 1$ there exists a unique i such that $d_i = e_m$. We denote $i =: \sigma(m)$. The surjectivity implies that

$$\varphi(x_{\sigma(m)}) = cy_m + \text{lower terms}, \quad c \in \mathbb{F}_p^\times.$$

2 Generically split varieties

For a semisimple group G and a prime number p denote by

$$J_p(G) = (j_1(G), \ldots, j_r(G))$$

its *J-invariant* defined in [PSZ08].

2.1 Theorem. *Let G_0 be a split simple algebraic group over k, $G = {}_{\gamma}G_0$ be the twisted form of G_0 given by a 1-cocycle $\gamma \in H^1(k, G_0)$, $X = {}_{\gamma}(G_0/P)$ be the twisted form of G_0/P, and $Y = {}_{\gamma}(G_0/B)$ be the twisted form of G_0/B. The following conditions are equivalent:*

1. *X is generically split;*
2. *The composition map*
$$\overline{\mathrm{CH}}^*(Y) \to \mathrm{CH}^*(G_0) \to \mathrm{CH}^*(P)$$
is surjective;
3. *For every prime p the composition map*
$$\overline{\mathrm{Ch}}^1(Y) \to \mathrm{Ch}^1(G_0) \to \mathrm{Ch}^1(P)$$
is surjective, and
$$j_{\sigma(m)}(G) = 0 \text{ for all } m \text{ with } d_m > 1.$$

Proof. 1⇒2. The same argument as in the proof of Lemma 1.4 (with Y instead of G_0/B and X instead of G_0/P).

2⇒3. Clearly, the composition

$$\overline{\mathrm{Ch}}^*(Y) \to \mathrm{Ch}^*(G_0) \to \mathrm{Ch}^*(P)$$

is surjective for every p. In particular, when $d_m > 1$ $\overline{\mathrm{Ch}}^{d_m}(Y)$ contains an element of the form $x_{\sigma(m)} + a$, where a is decomposable, hence $j_{\sigma(m)}(G) = 0$.

3⇒1. $G_{k(X)}$ has a parabolic subgroup of type P; denote the derived group of its Levi subgroup by H. We want to prove that H is split. By [PS10, Proposition 3.9(3)] it suffices to show that $J_p(H)$ is trivial for every p.

Denote the variety of complete flags of H by Z. It follows from the commutative diagram

$$\begin{array}{ccc} \mathrm{Ch}^*(Y_{k(X)}) & \longrightarrow & \mathrm{Ch}^*(Z) \\ \downarrow & & \downarrow \\ \mathrm{Ch}^*(\overline{G}) & \longrightarrow & \mathrm{Ch}^*(\overline{H}) \end{array}$$

that $j_m(H) \le j_{\sigma(m)}(G)$ if $d_m > 1$. Therefore

$$j_m(H) \le j_{\sigma(m)}(G_{k(X)}) \le j_{\sigma(m)}(G) = 0$$

when $d_m > 1$. It remains to show that $\mathrm{Ch}^1(\overline{Z})$ is rational. But this follows from the commutative diagram

$$\begin{array}{ccccccc}
\mathrm{Ch}^1(Y) & \longrightarrow & \mathrm{Ch}^1(Y_{k(X)}) & \longrightarrow & \mathrm{Ch}^1(Z) & & \\
 & & \downarrow & & \downarrow & & \\
 & & \mathrm{Ch}^1(\overline{G}) & \longrightarrow & \mathrm{Ch}^1(\overline{H}) & = & \mathrm{Ch}^1(P).
\end{array}$$

□

2.2 Remark.

- If all $e_m > 1$, then the condition on $\overline{\mathrm{Ch}}^1(Y)$ is void.
- If G_0 is different from PGO^+_{2n} and $e_1 = 1$ (resp. $G_0 = \mathrm{PGO}^+_{2n}$ and $e_1 = e_2 = 1$), then in view of [PS10, Proposition 4.2] it is equivalent to the fact that all Tits algebras of G are split. The latter is also equivalent to the fact that $j_1(G) = 0$ (resp. $j_1(G) = j_2(G) = 0$).
- If $G_0 = \mathrm{PGO}^+_{2n}$ and there is exactly one m with $e_m = 1$, then there are exactly two fundamental weights among $\bar\omega_1, \bar\omega_{n-1}, \bar\omega_n$ whose image with respect to the composition $\mathrm{Ch}^1(\overline{Y}) \to \mathrm{Ch}^1(\overline{G}) \to \mathrm{Ch}^1(\overline{H})$ equals y_1. Then the condition on $\overline{\mathrm{Ch}}^1(Y)$ is equivalent to the fact that at least one of the Tits algebras corresponding to these fundamental weights in the preimage of y_1 is split.

For a simple group G we denote by A_l its Tits algebra corresponding to $\bar\omega_l$ (see [Ti71]).

2.3 Theorem. *Let G be a group given by a 1-cocycle from $H^1(k, G_0)$, where G_0 stands for the split adjoint group of the same type as G, and let X be the variety of the parabolic subgroups of G of type i.*

The variety X is generically split if and only if

G_0	i	*conditions on G*

PGL_n	*any* i	$\gcd(\exp A_1, i) = 1$
PGSp_{2n}	*any* i	i *is odd or* G *is split*
O^+_{2n+1}	*any* i	$j_m(G) = 0$ *for all* $1 \leq m \leq \left[\frac{n+1-i}{2}\right]$
PGO^+_{2n}	i *is odd,* $i < n-1$	$[A_{n-1}] = 0$ *or* $[A_n] = 0$, *and* $j_m(G) = 0$ *for all* $2 \leq m \leq \left[\frac{n+2-i}{2}\right]$
PGO^+_{2n}	i *is even,* $i < n-1$	$j_m(G) = 0$ *for all* $1 \leq m \leq \left[\frac{n+2-i}{2}\right]$
PGO^+_{2n}	$i = n-1$ *or* $i = n$, n *is odd*	*none*
PGO^+_{2n}	$i = n-1$, n *is even*	$[A_1] = 0$ *or* $[A_n] = 0$
PGO^+_{2n}	$i = n$, n *is even*	$[A_1] = 0$ *or* $[A_{n-1}] = 0$
E_6	$i = 3, 5$	*none*
E_6	$i = 2, 4$	$J_3(G) = (0, *)$
E_6	$i = 1, 6$	$J_2(G) = (0)$
E_7	$i = 2, 5$	*none*
E_7	$i = 3, 4$	$J_2(G) = (0, *, *, *)$
E_7	$i = 6$	$J_2(G) = (0, 0, *, *)$ ($J_2(G) = (0, 0, 0, 0)$ *if* $\mathrm{char}\, k \neq 2$)
E_7	$i = 1$	$J_2(G) = (0, 0, 0, *)$ ($J_2(G) = (0, 0, 0, 0)$ *if* $\mathrm{char}\, k \neq 2$)
E_7	$i = 7$	$J_3(G) = (0)$ *and* $J_2(G) = (*, 0, *, *)$ ($J_2(G) = (*, 0, 0, 0)$ *if* $\mathrm{char}\, k \neq 2$)
E_8	$i = 2, 3, 4, 5$	*none*
E_8	$i = 6$	$J_2(G) = (0, *, *, *)$ ($J_2(G) = (0, 0, 0, *)$ *if* $\mathrm{char}\, k \neq 2$)
E_8	$i = 1$	$J_2(G) = (0, 0, *, *)$ ($J_2(G) = (0, 0, 0, *)$ *if* $\mathrm{char}\, k \neq 2$)
E_8	$i = 7$	$J_3(G) = (0, *)$ *and* $J_2(G) = (0, *, *, *)$ ($J_3(G) = (0, 0)$ *if* $\mathrm{char}\, k \neq 3$, $J_2(G) = (0, 0, 0, *)$ *if* $\mathrm{char}\, k \neq 2$)
E_8	$i = 8$	$J_3(G) = (0, *)$ *and* $J_2(G) = (0, 0, 0, *)$ ($J_3(G) = (0, 0)$ *if* $\mathrm{char}\, k \neq 3$)
F_4	$i = 1, 2, 3$	*none*
F_4	$i = 4$	$J_2(G) = (0)$
G_2	*any* i	*none*

("$*$" means "any value").

Proof. Follows immediately from Theorem 2.1 and [PSZ08, Table 4.13]. □

This theorem allows to give a shortened proof of the main result of [Ch10]:

2.4 Corollary. Let G be a group of type E_8 over a field k with $\operatorname{char} k \neq 3$. If the 3-component of the Rost invariant of G is zero, then G splits over a field extension of degree coprime to 3.

Proof. Let K/k be a field extension of degree coprime to 3 such that the 2-component of the Rost invariant of G_K is zero.

Consider the variety X of parabolic subgroups of G_K of type 7. The Rost invariant of the semisimple anisotropic kernel of $G_{K(X)}$ is zero. Therefore $G_{K(X)}$ splits, and, thus, X is generically split.

By Theorem 2.3 $J_3(G_K) = (0, 0)$, hence by [PS10, Proposition 3.9(3)] G_K splits over a field extension of degree coprime to 3. This implies the corollary. □

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