ON THE CLASSIFICATION OF MODULAR FUSION ALGEBRAS

Wolfgang Eholzer

Physikalisches Institut der
Universität Bonn
Nussallee 12

D-53115 Bonn

Germany

Max-Planck-Institut für
Mathematik
Gottfried-Claren-Straße 26

D-53225 Bonn

Germany

MPI / 94-91

HOMOTOPY TYPES

Hans-Joachim Baues

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
53225 Bonn

Germany

MPI / 94-88

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WOLFGANG EHOLZER

ABSTRACT. We introduce the notion of (nondegenerate) strong-modular fusion algebras. Here strong-modular means that the fusion algebra is induced via Verlinde's formula by a representation of the modular group $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ whose kernel contains a congruence subgroup. Furthermore, nondegenerate means that the conformal dimensions of possibly underlying rational conformal field theories do not differ by integers. Our main result is the classification of all strong-modular fusion algebras of dimension two, three and four and the classification of all nondegenerate strong-modular fusion algebras of dimension less than 24. We use the classification of the irreducible representations of the finite groups $\mathrm{SL}(2,\mathbb{Z}_{p^\lambda})$ where p is a prime and λ a positive integer. Finally, we give polynomial realizations and fusion graphs for all simple nondegenerate strong-modular fusion algebras of dimension less than 24.

CONTENTS

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1. Introduction

In the last ten years there have been several attempts for the classification of rational conformal field theories (RCFTs). However, a complete classification seems to be an impossible task since, for example, all self dual double even lattices lead to RCFTs and there is no hope to classify all such lattices of rank greater than 24. Nevertheless, it might be possible to classify all RCFTs with "small" effective central charge $\tilde{c}$. The effective central charge is given by the difference of the central charge and 24 times the smallest conformal dimension of the rational model under consideration. In particular, for $\tilde{c} \leq 1$ a classification of RCFTs can be obtained by using a theorem of Serre-Stark describing all modular forms of weight $1/2$ on congruence subgroups if one assumes that the corresponding conformal characters are modular functions on a congruence subgroup.

For $\tilde{c} > 1$ only partial results have been obtained so far. One of the possibilities is to look at RCFTs where the corresponding fusion algebra has a "small" dimension. In the special case of a trivial fusion algebra the RCFT has only one superselection sector and a classification of the corresponding modular invariant partition functions for unitary theories with $c \leq 24$ has been obtained [1]. As a next step in the classification one can try to classify the nontrivial fusion algebras of low dimension first and then investigate corresponding RCFTs. Indeed, the modular fusion algebras of dimension less than or equal to three satisfying the so-called Fuchs conditions have been classified (see e.g. [2]). In this paper we develop several tools, following the ideas of ref. [3], which enable us to classify all strong-modular fusion algebras of dimension less than or equal to four (for a definition of strong-modular fusion algebras see §2). Our approach is based on the known classification of the irreducible representations of the groups $\mathrm{SL}(2, \mathbb{Z}_{p^\lambda})$ [4].

Another possibility is to investigate theories where the corresponding fusion algebra has a certain structure but may have arbitrary or "big" dimension. Here, a classification of all selfconjugate fusion algebras which are isomorphic to a polynomial ring in one variable where the distinguished basis has a certain form and where the structure constants are less than or equal to one has been obtained (see e.g. [2][1]). Furthermore, a classification of all fusion algebras which are isomorphic to a polynomial ring in one variable and where the quantum dimension of the elementary field is smaller or equal to 2 is known (this classification contains the fusion algebras occurring in the classification of ref. [2]; for a review see e.g. [5]). With the tools developed in this paper we obtain another partial classification, namely of those strong-modular fusion algebras of dimension less than 24 where the corresponding representation ρ of the modular group is such that $\rho(T)$ has nondegenerate eigenvalues. The nondegeneracy of the eigenvalues of $\rho(T)$ means that the difference of any two conformal dimensions of a possibly underlying RCFT is not an integer. The restriction on the dimension is of purely technical nature so that it should be possible to obtain a complete classification of all nondegenerate strong-modular fusion algebras with the methods described in this paper by using systematically Galois theory.

[1]More precisely, in [2] all selfconjugate modular fusion algebras with $N_{ij}^k \leq 1$, which are isomorphic to $\mathbb{Q}[x]/ < P(x) >$ and $\Phi_0 \cong 1, \Phi_1 \cong x, \Phi_j \cong p_j(x)$ ($j = 2, \ldots, n-1$) for some polynomials P and p_j and where the degree of P is n and the degree of the p_j is j have been

This paper is organized as follows: In §2 we recall some basic properties of rational conformal field theories and give definitions of the relevant types of fusion algebras. Section 3 contains some general theorems about representations of the modular group which factor through a congruence subgroup. In the next section we give a short review about the classification of the irreducible representations of $\mathrm{SL}(2,\mathbb{Z}_{p^\lambda})$ which will be the main tool in the proof of the theorems in §5. The main results of this paper are contained in §5. Here we classify all strong-modular fusion algebras of dimension less than or equal to four and the nondegenerate strong-modular fusion algebras of dimension less than 24. Finally, we summarize our results and point out some open questions in the conclusion. Two appendices contain the explicit form of the irreducible level p^λ representations of dimension less than or equal to four as well as the fusion matrices and graphs of the simple nondegenerate strong-modular fusion algebras of dimension less than 24.

2. Rational conformal field theories and fusion algebras

2.1 Basic definitions.

Consider a chiral rational conformal field theory (or rational model) R consisting of a symmetry algebra $\mathcal{W}$ and its finitely many inequivalent irreducible modules $\mathcal{H}_i$ ($i = 0,\ldots,n-1$), i.e. R is a rational vertex operator algebra (RVOA) satisfying Zhu's finiteness condition (for RVOA see e.g. [6,7] and for the connection of RVOA to $\mathcal{W}$-algebras and rational models see [8]). Here $\mathcal{H}_0$ denotes the vacuum representation. For modules $\mathcal{H}$ of $\mathcal{W}$ there is the notion of conjugate (or adjoint or dual) modules $\mathcal{H}'$. In particular, it is conjectured that one has $(\mathcal{H}')' \cong \mathcal{H}$. Since R is rational the conjugation defines a permutation π of order two of the irreducible modules $\mathcal{H}_i' \cong \mathcal{H}_{\pi(i)}$.

The structure constants $N_{i,j}^{k}$ of the "fusion algebra" associated to R are given by the dimension of the corresponding space of intertwiners of the modules $\mathcal{H}_i \otimes \mathcal{H}_j$ and $\mathcal{H}_k$ (for a definition of intertwiners of modules of vertex operator algebras see e.g. [7]). One of the important properties of the $N_{i,j}^{k}$ which is well known in the physical literature is the fact that the numbers $N_{i,j}^{k}$ can be viewed as the structure constants of an associative commutative algebra, the fusion algebra. In the terminology of vertex operator algebras a corresponding statement is proven under certain assumptions in a recent series of papers [9]. In the abstract definition of fusion algebras the properties of all known examples associated to RCFTs are collected.

Definition. A **fusion algebra** $\mathcal{F}$ is a finite dimensional algebra over $\mathbb{Q}$ with a distinguished basis $\Phi_0 = \mathbb{I},\ldots,\Phi_{n-1}$ ($n = \dim(\mathcal{F})$) satisfying the following axioms:

(1) $\mathcal{F}$ is associative and commutative.
(2) The structure constants $N_{i,j}^{k}$ ($i,j,k = 0,\ldots,n-1$) with respect to the distinguished basis Φ_i are nonnegative integers.
(3) There exits a permutation $\pi \in S_n$ of order two such that for the structure constants in (2) one has

$$N_{i,j}^{0} = \delta_{i,\pi(j)}, \qquad N_{\pi(i),\pi(j)}^{\pi(k)} = N_{i,j}^{k}, \qquad i,j,k = 0,\ldots,n-1.$$

classified (the assumption on the degree of p_j was used implicitly in loc. cit.).

Remarks.

An isomorphism ϕ of two fusion algebras $\mathcal{F}, \mathcal{F}'$ is an isomorphism of unital algebras which maps the distinguished basis to the distinguished basis, i.e. there exists a permutation $\sigma \in S_n$ such that $\phi(\Phi_i) = \Phi'_{\sigma(i)}$ $(i = 0, \ldots, n-1)$.

The tensor product of two fusion algebras $\mathcal{F}$ and $\mathcal{F}'$ is again a fusion algebra, its distinguished basis is given by $\Phi_{i_1} \otimes \Phi'_{i_2}$ $(i_1 = 0, \ldots, \dim(F) - 1,\ i_2 = 0, \ldots, \dim(F') - 1)$.

The permutation π of order two is called charge conjugation. Fusion algebras with trivial charge conjugation are called selfconjugate.

Note that it is an open question whether two nonisomorphic fusion algebras can be isomorphic as unital algebras.

It is known in many cases that fusion algebras arising from RCFTs have additional properties. One of these additional properties is their relation to conformal characters. The conformal characters χ_i of the modules $\mathcal{H}_i$ of $\mathcal{W}$ are formal power series in q defined by

$$\chi_i(\tau) = \mathrm{tr}_{\mathcal{H}_i}(q^{L_0 - \frac{c}{24}})$$

where L_0 is the 0-th Fourier mode of the chiral energy-momentum tensor and c is the central charge or the rank of the RVOA. One can show for rational vertex operator algebras satisfying Zhu's finiteness condition [10] that the characters become holomorphic functions in the upper complex half plane by setting $q = e^{2\pi i \tau}$. Furthermore, for these RVOAs the space spanned by the finitely many conformal characters is invariant under the action of the modular group $\Gamma = \mathrm{SL}(2, \mathbb{Z})$. Indeed, it is conjectured that Zhu's finiteness condition is not necessary at all. It was conjectured in 1988 by E. Verlinde [11] that for any rational model there exists a representation $\rho : \Gamma \to \mathrm{GL}(n, \mathbb{C})$ of Γ such that

$$\begin{aligned}
\chi_i(A\tau) &= (\chi_i | A)(\tau) = \sum_{m=0}^{n-1} \rho(A)_{j,i} \chi_j(\tau) \qquad A \in \Gamma \\
N_{i,j}^0 &= \rho(S^2)_{i,j} \\
N_{i,j}^k &= \sum_{m=0}^{n-1} \frac{\rho(S)_{i,m} \rho(S)_{j,m} \rho(S^{-1})_{m,k}}{\rho(S)_{0,m}}.
\end{aligned}$$

We will refer to this formula as "Verlinde's formula" in the following. The above conjecture motivates the definition of modular fusion algebras.

Definition. A **modular fusion algebra** $(\mathcal{F}, \rho)$ is a fusion algebra $\mathcal{F}$ together with a unitary representation $\rho : \mathrm{SL}(2, \mathbb{Z}) \to \mathrm{GL}(n, \mathbb{C})$ satisfying the following additional axioms:

(1) $\rho(S)$ is a symmetric and $\rho(T)$ is a diagonal matrix.
(2) $N_{i,j}^0 = \rho(S^2)_{i,j}$,
(3) $N_{i,j}^k = \sum_{m=0}^{n-1} \frac{\rho(S)_{i,m} \rho(S)_{j,m} \rho(S^{-1})_{m,k}}{\rho(S)_{0,m}}$

where $N_{i,j}^k$ $(i, j, k = 0, \ldots, n-1)$ are the structure constants of $\mathcal{F}$ with respect to the distinguished basis.

Remarks.

Note that property (3) already implies that $\mathcal{F}$ is associative and commutative.

Two modular fusion algebras $(\mathcal{F}, \rho)$ and $(\mathcal{F}', \rho')$ are called isomorphic if: 1) $\mathcal{F}$ and $\mathcal{F}'$ are isomorphic as fusion algebras, 2) ρ and ρ' are equivalent, 3) $\rho(T)_{i,j} = \rho'(T)_{\sigma(i)\sigma(j)}$ where $\sigma \in S_n$ is the permutation defined by the isomorphism of the fusion algebras.

The tensor product of two modular fusion algebras $(\mathcal{F}, \rho), (\mathcal{F}', \rho')$ is defined by $(\mathcal{F} \otimes \mathcal{F}', \rho \otimes \rho')$ and is again a modular fusion algebra.

A (modular) fusion algebra is called composite if it is isomorphic to a tensor products of two nontrivial (modular) fusion algebras. Here a (modular) fusion algebra is called trivial if it is one dimensional. A noncomposite (modular) fusion algebra is also called simple.

Note that for a modular fusion algebra with trivial charge conjugation ($\rho(S^2) = \mathbb{I}$) the matrix $\rho(S)$ is real.

For modular fusion algebras associated to rational models the eigenvalues of $\rho(T)$ are given by the conformal dimensions h_i ($i = 0, \ldots, n-1$) of the irreducible modules $\mathcal{H}_i$ (h_i is the smallest L_0 eigenvalue in the module $\mathcal{H}_i$) and the central charge c of the theory:

$$\rho(T) = \mathrm{diag}(e^{2\pi i(h_0 - c/24)}, \ldots, e^{2\pi i(h_{n-1} - c/24)}).$$

Quite often nonisomorphic modular fusion algebras are isomorphic as fusion algebras.

In the later sections we will investigate which representations of Γ are related to modular fusion algebras.

Definition. A representation $\rho : \mathrm{SL}(2, \mathbb{Z}) \to \mathrm{GL}(n, \mathbb{C})$ of the modular group is called **conformally admissible** or simply **admissible** if there exists a fusion algebra $\mathcal{F}$ such that $(\mathcal{F}, \rho)$ is a modular fusion algebra.

It is known that modular fusion algebras associated to rational models have many additional properties. In particular, the central charge and the conformal dimensions are rational [12,13]. Furthermore, compatibility conditions between the central charge c, the conformal dimensions h_i and the fusion coefficients N_{ij}^{k} (the so-called Fuchs conditions) are satisfied (see e.g. [2][2]):

$$\frac{n(n-1)}{12} - \sum_{m=0}^{n-1} \left(h_i - \frac{c}{24}\right) \in \frac{1}{6}(\mathbb{N}\backslash\{1\}),$$

$$\sum_{m=0}^{n-1} \left((h_i + h_j + h_k + h_l) N_{i,j}^{m} N_{k,m}^{l} - h_m (N_{i,j}^{m} N_{k,m}^{l} + N_{i,k}^{m} N_{j,m}^{l} + N_{i,l}^{m} N_{k,j}^{m})\right)$$
$$- \frac{1}{2} \left(\sum_{m=0}^{n-1} N_{i,j}^{m} N_{k,m}^{l}\right) \left(1 - \sum_{m=0}^{n-1} N_{i,j}^{m} N_{k,m}^{l}\right) \in \mathbb{N}$$

[2]Note that the formula connecting the central charge with the conformal dimension in [2] contains a misprint.

However, in this paper we will not make any use of these properties.

Instead we will extensively rely on the observation that in all known examples of RCFTs the conformal characters are modular functions on some congruence subgroup of Γ. Therefore, the corresponding representation ρ factors through a representation of Γ_N. Here we have used Γ_N for the principal congruence subgroup of Γ of level N

$$\Gamma_N = \{ \ A \in \Gamma \mid A \equiv \mathbb{1} \bmod N \ \}.$$

Definition. A modular fusion algebra $(\mathcal{F}, \rho)$ is called **strong-modular** if the kernel of the representation ρ contains a congruence subgroup of Γ.

In this case ρ defines a representation of $\mathrm{SL}(2, \mathbb{Z}_N)$ and is called a level N representation of Γ (here and in the following we use $\mathbb{Z}_N$ for $\mathbb{Z}/N\mathbb{Z}$). A level N representation ρ will be called even or odd if $\rho(S^2) = \mathbb{1}$ or $\rho(S^2) = -\mathbb{1}$, respectively. Furthermore, one can show that for strong-modular fusion algebras associated to rational models the representation ρ is defined over the field K of N-th roots of unity, i.e. $\rho : \Gamma \to \mathrm{GL}(n, K)$ if the corresponding conformal characters are modular functions on some congruence subgroup [8]. Indeed, we expect that this is true for all RCFTs what motivates the following definition and conjecture.

Definition. A level N representation $\rho : \mathrm{SL}(2, \mathbb{Z}) \to \mathrm{GL}(n, \mathbb{C})$ is called K-rational if it is defined over the field K of the N-th roots of unity, i.e. $\rho : \mathrm{SL}(2, \mathbb{Z}) \to \mathrm{GL}(n, K)$.

Conjecture. All modular fusion algebras associated to rational models are strong-modular fusion algebras and the corresponding representations of the modular group are K-rational.

2.2 Some simple properties of modular fusion algebras.

In this section we prove some simple lemmas about modular fusion algebras which will be needed in the proofs of the main theorems in §5.

Lemma 1. *Let $(\mathcal{F}, \rho)$ be a modular fusion algebra. Assume that $\rho(T)$ has nondegenerate eigenvalues. Then ρ is irreducible.*

Proof. Assume that ρ is reducible and $\rho(T)$ has nondegenerate eigenvalues. Then $\rho(S)$ has block diagonal form and therefore $\rho(S)_{0,m} = 0$ for some m. This is a contradiction to property (3) in the definition of modular fusion algebras.

Definition. A modular fusion algebra $(\mathcal{F}, \rho)$ is called **degenerate** or **nondegenerate** if $\rho(T)$ has degenerate or nondegenerate eigenvalues, respectively.

Lemma 2. *Let $\rho, \rho' : \Gamma \to \mathrm{GL}(n, \mathbb{C})$ be equivalent, irreducible, unitary representations of the modular group. Assume that $\rho(T) = \rho'(T)$ is a diagonal matrix with nondegenerate eigenvalues. Then there exists a unitary diagonal matrix D such that $\rho = D^{-1}\rho' D$.*

Proof. Since ρ and ρ' are equivalent there exists a matrix D' such that $\rho = {D'}^{-1}\rho' D'$. Since $\rho(T) = \rho'(T)$ is a diagonal matrix with nondegenerate eigenvalues D' is diagonal. Finally, the irreducibility of ρ implies by Schur's lemma that ${D'}^{+}D' = \alpha\mathbb{1}$ for some positive real number α so that $D = \frac{1}{\sqrt{\alpha}}D'$ satisfies the desired properties.

Lemma 3. *Let $(\mathcal{F}, \rho)$ and $(\mathcal{F}', \rho')$ be two nondegenerate modular fusion algebras. Assume that ρ is equivalent to ρ' and $\rho(T) = \rho'(T)$. Then $\mathcal{F}$ and $\mathcal{F}'$ are isomorphic as fusion algebras.*

Proof. The lemma follows directly from the definition of (modular) fusion algebras and Lemma 2.

Lemma 4. *Let $(\mathcal{F}, \rho)$ be a modular fusion algebra. Then ρ is not isomorphic to a direct sum of one dimensional representations.*

Proof. If ρ is the direct sum of one dimensional representations $\rho(S)$ is also a diagonal matrix. This implies that one cannot apply Verlinde's formula giving a contradiction since we have assumed that $(\mathcal{F}, \rho)$ is a modular fusion algebra.

Since there are exactly 12 one dimensional representations of Γ one has the following trivial lemma.

Lemma 5.

(1) *Let ρ be a one dimensional representation of Γ. Then ρ is equivalent to one of the following representations*

$$\rho(S) = e^{2\pi i \frac{3n}{4}}, \qquad \rho(T) = e^{2\pi i \frac{n}{12}}, \qquad n = 0, \ldots, 11.$$

(2) *Let $(\mathcal{F}, \rho)$ be a one dimensional modular fusion algebra. Then $(\mathcal{F}, \rho)$ is strong-modular, $\mathcal{F}$ is trivial and ρ is given by*

$$\rho(S) = (-1)^n, \qquad \rho(T) = e^{2\pi i \frac{n}{6}}, \qquad n = 0, \ldots, 5.$$

Lemma 6. *Let $(\mathcal{F}, \rho)$ be a strong-modular fusion algebra associated to a rational model. Then ρ is K-rational.*

Proof. For a rational vertex operator algebra satisfying Zhu's finiteness condition the characters are holomorphic functions on the upper complex half plane. Since we have assumed that $(\mathcal{F}, \rho)$ is strong-modular ρ is a level N representation for some N. This implies that the characters are modular functions on Γ_N. Moreover, their Fourier coefficients are positive integer so that one can apply the theorem on K-rationality of ref. [8] implying that ρ is K-rational.

Although Lemma 6 will not be used in the following it provides us with a good motivation for looking at K-rationality of level N representations.

3. Some theorems on level N representations of Γ

In this section we will consider level N representations of $\mathrm{SL}(2, \mathbb{Z})$. Firstly, we review that all irreducible representations of $\mathrm{SL}(2, \mathbb{Z}_N)$ can be obtained by those of $\mathrm{SL}(2, \mathbb{Z}_{p^\lambda})$ where p is a prime and λ is a positive integer. Secondly, we discuss the construction of level p^λ representations using Weil representations (in this part we follow ref. [4]).

Lemma 7. *Let ρ be a finite dimensional representation of* $\mathrm{SL}(2,\mathbb{Z}_N)$ *where N is a positive integer. Then the representation ρ is completely reducible. Furthermore, each irreducible component ω of ρ has a unique product decomposition*

$$\omega \cong \otimes_{j=1}^{n} \pi(p_j^{\lambda_j})$$

where $N = \prod_{j=1}^{n} p_j^{\lambda_j}$ is the prime factor decomposition of N and the $\pi(p_j^{\lambda_j})$ are irreducible representations of $\mathrm{SL}(2,\mathbb{Z}_{p_j^{\lambda_j}})$.

Proof. Since $\mathrm{SL}(2,\mathbb{Z}_N)$ is a finite group ρ is completely reducible. The second statement, namely that the irreducible representations of $\mathrm{SL}(2,\mathbb{Z}_N)$ can be written as a tensor product of irreducible representations of $\mathrm{SL}(2,\mathbb{Z}_{p_j^{\lambda_j}})$ where $N = \prod_{j=1}^{n} p_j^{\lambda_j}$ is the prime factor decomposition of N, can be seen as follows. For a proof of the second statement note that

$$\mathrm{SL}(2,\mathbb{Z}_N) = \mathrm{SL}(2,\mathbb{Z}_{p_1^{\lambda_1}}) \times \cdots \times \mathrm{SL}(2,\mathbb{Z}_{p_n^{\lambda_n}})$$

where $N = \prod_{j=1}^{n} p_j^{\lambda_j}$ (see e.g. [14]). Obviously, the tensor product of irreducible representations $\pi(p_j^{\lambda_j})$ of $\mathrm{SL}(2,\mathbb{Z}_{p_j^{\lambda_j}})$ is an irreducible representation of $\mathrm{SL}(2,\mathbb{Z}_N)$. Using now Burnside's lemma we obtain the second statement.

In order to deal with the representations of the groups $\mathrm{SL}(2,\mathbb{Z}_{p^\lambda})$ we describe their structure by the following theorem.

Theorem 1 [4, Satz 1, p. 466]. *The group $\mathrm{SL}(2,\mathbb{Z}_{p^\lambda})$ is generated by the elements*

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

and the relations

$$T^{p^\lambda} = 1\!\!1, \qquad S^2 = H(-1)$$
$$H(a)H(a') = H(aa'), \qquad H(a)T = T^{a^2}H(a), \qquad SH(a) = H(a^{-1})S$$

where $H(a) := T^{-a}ST^{-a^{-1}}S^{-1}T^{-a}S^{-1}$ and $a, a' \in \mathbb{Z}_{p^\lambda}^*$.

Remark. As elements of $\mathrm{SL}(2,\mathbb{Z}_{p^\lambda})$ the $H(a)$ ($a \in \mathbb{Z}_{p^\lambda}^*$) are given by

$$H(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}.$$

We will now describe the construction of representations of $\mathrm{SL}(2,\mathbb{Z}_{p^\lambda})$ by means of Weil representations.

Definition. Let M be a finite $\mathbb{Z}_{p^\lambda}$ module. A quadratic form Q of M is a map $Q : M \to p^{-\lambda}\mathbb{Z}/\mathbb{Z}$ such that

(1) $Q(-x) = Q(x)$ for all $x \in M$.
(2) $B(x,y) := Q(x+y) - Q(x) - Q(y)$ defines a $\mathbb{Z}_{p^\lambda}$-bilinear map from $M \times M$ to $p^{-\lambda}\mathbb{Z}/\mathbb{Z}$.

Definition. A finite $\mathbb{Z}_{p^\lambda}$ module M together with a quadratic form Q is called a quadratic module of $\mathbb{Z}_{p^\lambda}$.

Definition. Let (M, Q) be a quadratic module. Define a right action of $\mathrm{SL}(2, \mathbb{Z}_{p^\lambda})$ on the space of $\mathbb{C}$ valued functions on M by

$$
\begin{aligned}
(f|T)(x) &= e^{2\pi i Q(x)} \; f(x) \\
(f|H(a))(x) &= \alpha_Q(a)\alpha_Q(-1) \; f(x) \qquad \forall a \in \mathbb{Z}^*_{p^\lambda} \\
(f|S^{-1})(x) &= \frac{\alpha_Q(-1)}{|M|^{1/2}} \sum_{y\in M} e^{2\pi i B(x,y)} \; f(y)
\end{aligned}
$$

where $|M|$ denotes the order of M,

$$
\alpha_Q(a) = \frac{1}{|M|} \sum_{x\in M} e^{2\pi i a Q(x)}
$$

and f is any $\mathbb{C}$ valued function on M.

If this right action of $\mathrm{SL}(2, \mathbb{Z}_{p^\lambda})$ defines a representation of $\mathrm{SL}(2, \mathbb{Z}_{p^\lambda})$ it is called the Weil representation associated to the quadratic module (M, Q) and denoted by $W(M, Q)$.

Note that the above right action always defines a projective representation of Γ. A necessary and sufficient condition for it to define a proper representation is given by the following theorem.

Theorem 2 [4, Satz 2, p. 467]. *The above right action of* $\mathrm{SL}(2, \mathbb{Z}_{p^\lambda})$ *defines a representation of* $\mathrm{SL}(2, \mathbb{Z}_{p^\lambda})$ *if and only if*

$$
\alpha_Q(a)\alpha_Q(a') = \alpha_Q(1)\alpha_Q(aa') \qquad a, a' \in \mathbb{Z}^*_{p^\lambda}.
$$

4. The classification of the irreducible level p^λ representations

Although the classification of the irreducible representations of the finite groups $\mathrm{SL}(2, \mathbb{Z}_{p^\lambda})$ is contained in [4] we will give a short review here. Our main motivation for this is the fact that we will strongly rely on this classification in the proofs of the main theorems in §5. Furthermore, ref. [4] is not written in English but in German.

In the first subsection we describe how one can obtain irreducible level p^λ representations as subrepresentations of Weil representations. The second and third subsection are used to give complete lists of the corresponding representations for the cases of $p \neq 2$ and $p = 2$, respectively.

In addition to the review we investigate in some cases whether the irreducible representations are K-rational or not.

4.1 Weil representations associated to binary quadratic forms.

Most of the irreducible representations of $\mathrm{SL}(2, \mathbb{Z}_{p^\lambda})$ can be obtained as subrepresentations of Weil representations $W(M, Q)$ associated to a module M of rank one or two. The following two theorems describe the Weil representations needed in the later sections.

Theorem 3 [4, Lemma 1, Satz 3, p. 474]. *Let $p \neq 2$ be a prime. Then the following quadratic modules of $\mathbb{Z}_{p^\lambda}$ define Weil representations:*

(1)	$M = \mathbb{Z}_{p^\lambda}$,	$Q(x) = p^{-\lambda} r x^2$	$(\lambda \geq 1)$	$(R_\lambda(r))$
(2)	$M = \mathbb{Z}_{p^\lambda} \oplus \mathbb{Z}_{p^\lambda}$,	$Q(x) = p^{-\lambda} x_1 x_2$	$(\lambda \geq 1)$	(D_λ)
(3)	$M = \mathbb{Z}_{p^\lambda} \oplus \mathbb{Z}_{p^\lambda}$,	$Q(x) = p^{-\lambda}(x_1^2 - u x_2^2)$	$(\lambda \geq 1)$	(N_λ)
(4)	$M = \mathbb{Z}_{p^\lambda} \oplus \mathbb{Z}_{p^{\lambda-\sigma}}$,	$Q(x) = p^{-\lambda} r(x_1^2 - p^\sigma t x_2^2)$	$(\lambda \geq 2)$	$(R_\lambda^\sigma(r,t))$

where r, t run through $\{1, u\}$ with $\left(\frac{u}{p}\right) = -1$ ($\left(\frac{\cdot}{\cdot}\right)$ denotes the Legendre symbol), where $\sigma = 1, \ldots, \lambda - 1$ and where the last column contains the name of the corresponding Weil representation.

Theorem 4 [4, Satz 4, p. 474]. *Let $p = 2$. Then the following quadratic modules of $\mathbb{Z}_{2^\lambda}$ define Weil representations:*

(1)	$M = \mathbb{Z}_{2^\lambda} \oplus \mathbb{Z}_{2^\lambda}$,	$Q(x) = 2^{-\lambda} x_1 x_2$	$(\lambda \geq 1)$	(D_λ)
(2)	$M = \mathbb{Z}_{2^\lambda} \oplus \mathbb{Z}_{2^\lambda}$,	$Q(x) = 2^{-\lambda}(x_1^2 + x_1 x_2 + x_2^2)$	$(\lambda \geq 1)$	(N_λ)
(3)	$M = \mathbb{Z}_{2^{\lambda-1}} \oplus \mathbb{Z}_{2^{\lambda-\sigma-1}}$,	$Q(x) = 2^{-\lambda} r(x_1^2 + 2^\sigma t x_2^2)$	$(\lambda \geq 2)$	$(R_\lambda^\sigma(r,t))$

where $\sigma = 0, \ldots, \lambda - 2$, where (r, t) run through a system of representatives of the classes of pairs defined by $(r_1, t_1) \cong (r_2, t_2)$

if $t_1 \equiv t_2 \bmod \min(8, 2^{\lambda-\sigma})$ and

$$\begin{cases} r_2 \equiv r_1 \bmod 4 & \text{or} \quad r_2 \equiv r_1 t_1 \bmod 4 & \text{for} \quad \sigma = 0 \\ r_2 \equiv r_1 \bmod 8 & \text{or} \quad r_2 \equiv r_1 + 2 r_1 t_1 \bmod 8 & \text{for} \quad \sigma = 1 \\ r_2 \equiv r_1 \bmod 4 & & \text{for} \quad \sigma = 2 \\ r_2 \equiv r_1 \bmod 8 & & \text{for} \quad \sigma \geq 3 \end{cases}$$

and where the last column contains the name of the corresponding Weil representation.

All irreducible representations of $\mathrm{SL}(2, \mathbb{Z}_{p^\lambda})$ can be obtained as subrepresentations of Weil representations $W(M, Q)$. One possibility to extract subrepresentations of such representations is to use characters of the automorphism group of the quadratic form Q:

Theorem 5 (see e.g. [4]). *Let $W(M, Q)$ be a Weil representation described by Theorem 3 or 4, $\mathcal{U}$ an abelian subgroup of $\mathrm{Aut}(M, Q)$ and χ a character of $\mathcal{U}$. Then the subspace*

$$V(\chi) := \{\ f : M \to \mathbb{C} \mid f(\epsilon x) = \chi(\epsilon) f(x), \quad x \in M,\ \epsilon \in \mathcal{U}\ \}$$

of $\mathbb{C}^M$ is invariant under $\mathrm{SL}(2, \mathbb{Z}_{p^\lambda})$. The corresponding subrepresentation is denoted by $W(M, Q, \chi)$.

Remarks.

(a) The space $V(\chi)$ is spanned by $V(\chi) = < f_x(\chi) >_{x \in M}$ where

$$f_x(\chi)(y) = \sum_{\epsilon \in \mathcal{U}} \chi(\epsilon) \delta_{\epsilon x, y}, \qquad \delta_{x,y} = \begin{cases} 1 & \text{for x=y} \\ 0 & \text{otherwise.} \end{cases}$$

(b) The automorphism group of the quadratic forms in Theorem 4 contain a conjugation κ: $\kappa(x_1, x_2) = (x_2, x_1)$ in case (1) and $\kappa(x_1, x_2) = (x_1, -x_2)$ in the cases (2) and (3). In these cases the space

$$V(\chi)_\pm := \{ \ f \in V(\chi) \mid f(\kappa x) = \pm f(x), \quad x \in M \ \}$$

is invariant under $\mathrm{SL}(2, \mathbb{Z}_{2^\lambda})$. The corresponding subrepresentation is denoted by $W(M, Q, \chi)_\pm$.

From now on we will denote the trivial character $\chi \equiv 1$ by χ_1. Indeed, almost all irreducible representations of $\mathrm{SL}(2, \mathbb{Z}_{p^\lambda})$ can be obtained as subrepresentations of the Weil representations described by Theorems 3 and 4 using "primitive" characters:

Definition[3]**.** Let $W(M, Q)$ be a Weil representation described by Theorem 3 or 4 and let $\mathcal{U} = \mathrm{Aut}(M, Q)$. A character χ of $\mathcal{U}$ is called primitive iff there exists an element $\epsilon \in \mathcal{U}$ with $\chi(\epsilon) \neq 1$ such that each element of pM is a fixed point of ϵ. The set of primitive characters of $\mathcal{U}$ will be denoted by $\mathfrak{P}$.

With this definition we have:

Theorem 6 [4, Hauptsatz 1, p. 492]**.** *Let $W(M, Q)$ and $W(M', Q')$ be Weil representation described by Theorem 3 or 4 and χ, χ' primitive characters. Then one has*

(1) *$W(M, Q, \chi)$ is an irreducible level p^λ representation.*
(2) *$W(M, Q, \chi)$ and $W(M', Q', \chi')$ are isomorphic if and only if the quadratic modules (M, Q) and (M', Q') are isomorphic and $\chi = \chi'$ or $\chi = \bar{\chi}'$.*

The second main theorem of ref. [4] describes the classification of the irreducible representations of $\mathrm{SL}(2, \mathbb{Z}_{p^\lambda})$.

Theorem 7 [4, Hauptsatz 2, p. 493]**.** *The Weil representations described by the Theorems 3 and 4 contain all irreducible representations of the groups $\mathrm{SL}(2, \mathbb{Z}_{p^\lambda})$ (in general they are of the form $W(M, Q, \chi)$ for a primitive character χ) apart from 18 exceptional representations for $p = 2$. Theses exceptional representations can be obtained as tensor products of two representations contained in some $W(M, Q)$ (described by Theorem 3 or 4).*

Complete lists of irreducible representations of $\mathrm{SL}(2, \mathbb{Z}_{p^\lambda})$ will be given in §4.2 and §4.3.

[3] In the case of $M = \mathbb{Z}_{2^{\lambda-1}} \oplus \mathbb{Z}_2$ $(\lambda \geq 5)$ the definition of primitive characters is slightly different [4, p. 491]: Here $\mathcal{U} \cong < -1 > < \alpha >$ with $\alpha = \begin{cases} 1 + 4t + \sqrt{-8t} & \lambda = 5 \\ 1 - 2^{\lambda-3} + \sqrt{-2^{\lambda-2}t} & \lambda > 5 \end{cases}$ and χ is primitive if $\chi(\alpha) = -1$.

4.2 The irreducible representations of $\mathrm{SL}(2,\mathbb{Z}_{p^\lambda})$ for $p \neq 2$.

In the classification of the irreducible representations of $\mathrm{SL}(2,\mathbb{Z}_{p^\lambda})$ for $p \neq 2$ one has to distinguish the cases $\lambda = 1$ and $\lambda > 1$. Therefore, we treat them separately.

Following [4] we denote the trivial representation by C_1.

Theorem 8 [4]. *A complete set of irreducible representations of* $\mathrm{SL}(2,\mathbb{Z}_p)$ *for a prime* p *with* $p \neq 2$ *is given by the representations collected in Table 1. In Table 1 the* χ *run through the set of characters of* $\mathcal{U}$ *and* χ_{-1} *is the unique nontrivial character of* $\mathcal{U}$ *taking values in* ± 1. *Furthermore, we denote by # (here and in the following) the number of inequivalent representations.*

Table 1: Irreducible representations of $\mathrm{SL}(2,\mathbb{Z}_p)$ for $p \neq 2$

type of rep.		dimension	#
$D_1(\chi)$	$\chi \in \mathfrak{P}$	$p+1$	$\frac{1}{2}(p-3)$
$N_1(\chi)$	$\chi \in \mathfrak{P}$	$p-1$	$\frac{1}{2}(p-1)$
$R_1(r,\chi_1)$	$\left(\frac{r}{p}\right) = \pm 1$	$\frac{1}{2}(p+1)$	2
$R_1(r,\chi_{-1})$	$\left(\frac{r}{p}\right) = \pm 1$	$\frac{1}{2}(p-1)$	2
$N_1(\chi_1)$		p	1

We will denote the 3 one dimensional level 3 representations C_1, $R_1(1,\chi_{-1})$ and $R_1(2,\chi_{-1})$ by B_1, B_2 and B_3, respectively.

The explicit form of these representations is well known (see e.g. [3]) and one can address the question which of these representations are K-rational (see also [8]). Note that, in view of the results in §2.2, this question is natural in the context of admissible representations.

Lemma 8. *Let* $p \neq 2$ *be a prime.*

(1) *For* $p \equiv 1 \pmod 3$ *there is exactly one and for* $p \not\equiv 1 \pmod 3$ *there is no* K*-rational representation of type* $D_1(\chi)$.
(2) *For* $p \equiv 2 \pmod 3$ *there is exactly one and for* $p \not\equiv 2 \pmod 3$ *there is no* K*-rational representation of type* $N_1(\chi)$ $(\chi \in \mathfrak{P})$.
(3) *The representations of type* $R_1(r,\chi_{\pm 1})$ *and* $N_1(\chi_1)$ *are* K*-rational.*

Proof. Using a character table for the above representations (see e.g. [15]) one easily finds that the characters of representations of type $D_1(\chi)$ or $N_1(\chi)$ take values in the field of p-th roots of unity only if $p \equiv 1 \pmod 3$ or $p \equiv 2 \pmod 3$ and if χ is a character of order 3. Therefore, there is at most one K-rational representation of type $D_1(\chi)$ or $N_1(\chi)$ for the corresponding values of p. Using the explicit form of these representations (see e.g. [3]) one finds that these two representations are indeed K-rational. For the other two types of representations the K-rationality follows directly from the fact that $\chi_{\pm 1}$ takes values in ± 1.

Theorem 9 [4]. *A complete set of irreducible representations of* $\mathrm{SL}(2,\mathbb{Z}_{p^\lambda})$ *for* $p \neq 2$ *prime and* $\lambda > 1$ *is given by the representations in Table 2. Where* χ_{-1} *is the unique nontrivial character with values in* ± 1 *and* $R_\lambda(r,\chi_{\pm 1})_1$ *is the unique level* p^λ *subrepresentation of* $R_\lambda(r,\chi_{\pm 1})$ *which has dimension* $\frac{1}{2}(p^2-1)p^{\lambda-2}$.

Table 2: Irreducible representations of $\mathrm{SL}(2,\mathbb{Z}_{p^\lambda})$ for $p \neq 2$ and $\lambda > 1$

type of rep.		dimension	#
$D_\lambda(\chi)$	$\chi \in \mathfrak{P}$	$(p+1)p^{\lambda-1}$	$\frac{1}{2}(p-1)^2 p^{\lambda-2}$
$N_\lambda(\chi)$	$\chi \in \mathfrak{P}$	$(p-1)p^{\lambda-1}$	$\frac{1}{2}(p^2-1)p^{\lambda-2}$
$R_\lambda^\sigma(r,t,\chi)$	$(\frac{r}{p}) = \pm 1,\ (\frac{t}{p}) = \pm 1$	$\frac{1}{2}(p^2-1)p^{\lambda-2}$	$4\sum_{\sigma=1}^{\lambda-1}(p-1)p^{\lambda-\sigma-1}$
$R_\lambda(r,\chi_{\pm 1})_1$	$(\frac{r}{p}) = \pm 1$	$\frac{1}{2}(p^2-1)p^{\lambda-2}$	4

Lemma 9. *Let $p \neq 2$ be a prime and $\lambda > 1$ an integer.*

(1) *The representations of type $R_\lambda^\sigma(r,t,\chi)$ are K-rational for $p \neq 2$ and $\lambda > 1$.*
(2) *The representations of type $R_\lambda(r,\chi_{\pm 1})_1$ are K-rational for $p \neq 2$ and $\lambda > 1$. Furthermore, the image of T under these representations has nondegenerate eigenvalues only if $p = 3$ and $\lambda = 2$.*

Proof. Since the automorphism group of the quadratic form of $R_\lambda^\sigma(r,t,\chi)$ is given by [4, p. 495] $\mathcal{U} \cong \mathbb{Z}_2 \times \mathbb{Z}_{p^{\lambda-\sigma}}$ we obtain (1). In the second case one obviously has $\mathcal{U} \cong \mathbb{Z}_2$ so that the K-rationality follows directly. The statement concerning the eigenvalues of the image of T for the representations of type $R_\lambda(r,\chi_{\pm 1})_1$ is proved in Satz 4 of [4].

4.3 The irreducible representations of $\mathrm{SL}(2,\mathbb{Z}_{2^\lambda})$**.**

The classification of the irreducible representations of $\mathrm{SL}(2,\mathbb{Z}_{2^\lambda})$ is complicated since there are a lot of exceptional representations for $\lambda < 6$ [4]. Since these representations have small dimensions and we will be interested in such representations in §5 we describe them in the rest of this subsection. The Tables 3-8 list complete sets of irreducible representations of the groups $\mathrm{SL}(2,\mathbb{Z}_{2^\lambda})$ for the corresponding values of λ.

For $\lambda = 1$ there are only two irreducible representations (see Table 3). The representation C_2 is given by $C_2(S) = C_2(T) = -1$ and both level 2 representations are K-rational.

For $\lambda = 2$ there are seven irreducible representations (see Table 4). The representation C_3 is given by $C_3(S) = C_3(T) = -i$, C_4 by $C_4(S) = C_4(T) = i$ and $R_2^0(1,3)_1$ is defined by $R_2^0(1,3) \cong R_2^0(1,3)_1 \oplus C_1$. All level 4 representations are K-rational.

For $\lambda = 3$ there are 20 irreducible representations (see Table 5). Here $\hat{\chi}$ is one of the two characters of $\mathcal{U}$ of order 4 and the representation $R_3^0(1,3,\chi_1)_1$ is defined by $R_3^0(1,3,\chi_1) \cong R_3^0(1,3,\chi_1)_1 \oplus N_1(\chi_1) \oplus C_2 \oplus C_2$.

For $\lambda = 4$ there are 46 irreducible representations (see Table 6). Here the representation $R_4^2(r,3,\chi_1)_1$ is given by the equality $R_4^2(r,3,\chi_1) \cong R_4^2(r,3,\chi_1)_1 \oplus R_2^0(r,t)$.

Table 3: Irreducible representations of $\mathrm{SL}(2,\mathbb{Z}_2)$

type of rep.		dim	#
$C_2 = N_1(\chi)$	$\chi \in \mathfrak{P}$	1	1
$N_1(\chi_1)$		2	1

Table 4: Irreducible representations of $\mathrm{SL}(2,\mathbb{Z}_{2^2})$

type of rep.		dim	#
$D_2(\chi)_+$	$\chi \not\equiv 1$	3	1
$D_2(\chi)_-$	$\chi \not\equiv 1$	3	1
$R_2^0(1,3)_1$		3	1
$C_2 \otimes R_2^0(1,3)_1$		3	1
$N_2(\chi)$	$\chi \in \mathfrak{P};\ \chi \not\equiv 1$	2	1
$C_3 = R_2^0(3,1,\chi)$	$\chi \not\equiv 1$	1	1
$C_4 = R_2^0(1,1,\chi)$	$\chi \not\equiv 1$	1	1

Table 5: Irreducible representations of $\mathrm{SL}(2,\mathbb{Z}_{2^3})$

type of rep.		dim	#
$D_3(\chi)_\pm$	$\chi \in \mathfrak{P}$	6	4
$R_3^0(1,3,\chi_1)_1$		6	1
$C_3 \otimes R_3^0(1,3,\chi_1)_1$		6	1
$N_3(\chi)$	$\chi \in \mathfrak{P};\ \chi^2 \not\equiv 1$	4	2
$N_3(\chi)_\pm$	$\chi \in \mathfrak{P};\ \chi^2 \equiv 1$	2	4
$R_3^0(r,t,\hat{\chi})$	$r = 1,3;\ t = 1,5$	3	4
$R_3^0(1,t,\chi)_\pm$	$\chi \not\equiv 1;\ t = 3,7$	3	4

Table 6: Irreducible representations of $\mathrm{SL}(2,\mathbb{Z}_{2^4})$

type of rep.		dim	#
$D_4(\chi)$	$\chi \in \mathfrak{P}$	24	2
$N_4(\chi)$	$\chi \in \mathfrak{P}$	8	6
$R_4^0(r,t,\chi)$	$\chi \in \mathfrak{P};\ \chi \not\equiv 1;\ r = 1,3;\ t = 1,5$	6	4
$R_4^0(r,t,\chi)_\pm$	$\chi \in \mathfrak{P};\ \chi^2 \equiv 1;\ r = 1,3;\ t = 1,5$	3	16
$R_4^0(1,t,\chi)_\pm$	$\chi \in \mathfrak{P};\ t = 3,7$	6	8
$R_4^2(r,t,\chi)$	$\chi \not\equiv 1;\ r,t \in \{1,3\}$	6	4
$C_2 \otimes R_4^2(r,3,\chi)$	$\chi \not\equiv 1;\ r = 1,3$	6	2
$R_4^2(r,3,\chi_1)_1$	$r = 1,3$	6	2
$N_3(\chi)_+ \otimes R_4^0(1,7,\psi)_+$	$\chi \in \mathfrak{P};\ \chi^2 \equiv 1;\ \psi \not\equiv 1;$ $\psi^2 \equiv 1;\ \psi(-1) = 1$	12	2

For $\lambda = 5$ there are 92 irreducible representations (see Table 7). Here for fixed $r = 1,3$ the 2 irreducible representations of type $R_5^2(\cdot, 1, \chi)_1$ ($\chi \notin \mathfrak{P}$) are given by the 2 two dimensional irreducible level 5 subrepresentations of $R_5^2(r, 1)$.

Table 7: Irreducible representations of $\mathrm{SL}(2, \mathbb{Z}_{2^5})$

type of rep.		dim	#
$D_5(\chi)$	$\chi \in \mathfrak{P}$	48	4
$N_5(\chi)$	$\chi \in \mathfrak{P}$	16	12
$R_5^0(r,t,\chi)$	$\chi \in \mathfrak{P};\ r = 1,3;\ t = 1,5$	12	16
$R_5^0(1,t,\chi)_\pm$	$\chi \in \mathfrak{P};\ t = 3,7$	24	4
$R_5^1(r,t,\chi)_\pm$	$\chi \in \mathfrak{P};\ r,t \in \{1,5\}$ or $r = 1,3$ and $t = 3,7$	12	16
$R_5^2(r,t,\chi)_\pm$	$\chi \in \mathfrak{P};\ r = 1,3;\ t = 1,3,5,7$	6	32
$R_5^2(r,1,\chi)_1$	$\chi \notin \mathfrak{P};\ r = 1,3$	12	4
$C_3 \otimes R_5^2(r,1,\chi)_1$	$\chi \notin \mathfrak{P};\ r = 1,3$	12	4

For $\lambda > 5$ there are the following irreducible representations (see Table 8). Here χ are always primitive characters and $R_\lambda^{\lambda-3}(r,t,\chi_{\pm 1})_1$ is the unique irreducible level 2^λ subrepresentation of $R_\lambda^{\lambda-3}(r,t,\chi_{\pm 1})$ which has dimension $3 \cdot 2^{\lambda-4}$.

Table 8: Irreducible representations of $\mathrm{SL}(2, \mathbb{Z}_{2^\lambda})$ for $\lambda > 5$

type of rep.[4]		dim	#
$D_\lambda(\chi)$		$3 \cdot 2^{\lambda-1}$	$2^{\lambda-3}$
$N_\lambda(\chi)$		$2^{\lambda-1}$	$3 \cdot 2^{\lambda-3}$
$R_\lambda^0(1,7,\chi)$	$t = 3,7$	$3 \cdot 2^{\lambda-2}$	$2^{\lambda-3}$
$R_\lambda^\sigma(r,t,\chi)$	$\begin{cases} r = 1,3;\ t = 1,5 & \text{for } \sigma = 0 \\ r,t \in \{1,5\} \text{ or} & \\ r = 1,3 \text{ and } t = 3,7 & \text{for } \sigma = 1 \\ r = 1,3;\ t = 1,3,5,7 & \text{for } \sigma = 2 \end{cases}$	$3 \cdot 2^{\lambda-3}$	$5 \cdot 2^{\lambda-2}$
$R_\lambda^\sigma(r,t,\chi)$	$\sigma = 3,\dots,\lambda-3;\ r,t \in \{1,3,5,7\}$	$3 \cdot 2^{\lambda-4}$	$4 \cdot \sum_{\sigma=3}^{\lambda-3} 2^{\lambda-\sigma}$
$R_\lambda^{\lambda-2}(r,t,\chi)$	$r = 1,3,5,7;\ t = 1,3$	$3 \cdot 2^{\lambda-4}$	16
$R_\lambda^{\lambda-3}(r,t,\chi_{\pm 1})_1$	$r = 1,3,5,7;\ t = 1,3$	$3 \cdot 2^{\lambda-4}$	16

[4] For $\lambda = 6$ one has to use representation of type $R_6^4(r,t,\chi_1)_1$ and $C_2 \otimes R_6^4(r,t,\chi_1)_1$ ($r = 1,3$) instead of those of type $R_\lambda^{\lambda-3}(r,t,\chi_{\pm 1})_1$. The representations $R_6^4(r,t,\chi_1)_1$ are the unique level 6 subrepresentations of $R_6^4(r,t,\chi_1)$ with dimension 12.

5. Results on the classification of strong-modular fusion algebras

5.1 Classification of the strong-modular fusion algebras of dimension less then or equal to four.

In this section we consider all two, three and four dimensional level N representations of Γ and investigate whether they are admissible.

Main theorem 1. *Let $(\mathcal{F}, \rho)$ be a two dimensional strong-modular fusion algebra. Then $(\mathcal{F}, \rho)$ is isomorphic to the tensor product of a one dimensional modular fusion algebra with one of the modular fusion algebras in Table 9.*

Table 9: Two dimensional strong-modular fusion algebras

$\mathcal{F}$	$\rho(S)$	$\frac{1}{2\pi i}\log(\rho(T)) \bmod \mathbb{Z}$
$\Phi_1 \cdot \Phi_1 = \Phi_0$ ($\mathbb{Z}_2$)	$\frac{1}{\sqrt{2}}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$	$\begin{cases} \mathrm{diag}(\frac{1}{8}, \frac{3}{8}) \\ \mathrm{diag}(\frac{7}{8}, \frac{5}{8}) \end{cases}$
$\Phi_1 \cdot \Phi_1 = \Phi_0 + \Phi_1$ ("(2,5)")	$\frac{2}{\sqrt{5}}\begin{pmatrix} -\sin(\frac{\pi}{5}) & -\sin(\frac{2\pi}{5}) \\ -\sin(\frac{2\pi}{5}) & \sin(\frac{\pi}{5}) \end{pmatrix}$	$\begin{cases} \mathrm{diag}(\frac{19}{20}, \frac{11}{20}) \\ \mathrm{diag}(\frac{1}{20}, \frac{9}{20}) \end{cases}$
	$\frac{2}{\sqrt{5}}\begin{pmatrix} -\sin(\frac{2\pi}{5}) & \sin(\frac{\pi}{5}) \\ \sin(\frac{\pi}{5}) & \sin(\frac{2\pi}{5}) \end{pmatrix}$	$\begin{cases} \mathrm{diag}(\frac{3}{20}, \frac{7}{20}) \\ \mathrm{diag}(\frac{17}{20}, \frac{13}{20}) \end{cases}$

Proof. Let $(\mathcal{F}, \rho)$ be a two dimensional strong-modular fusion algebra. Lemma 4 implies that ρ is irreducible. Therefore, we have to consider all irreducible two dimensional representations of Γ which factor through a congruence subgroup. By Lemma 7 we know that these representations can be obtained by taking the tensor products of all irreducible two dimensional level p^λ representations with all one dimensional representations of Γ.

There are exactly 11 inequivalent irreducible two dimensional level p^λ representations. Their explicit form is given in Appendix A. We are interested in the classification of the two dimensional strong-modular fusion algebras up to tensor products with one dimensional fusion algebras. Therefore, we can restrict our investigation to one of the two dimensional representations of level 2, 2^3, 3 and the two representations of level 5 (see Appendix A). For the remaining 5 two dimensional representations the eigenvalues of the image of T are nondegenerate. Hence, Lemma 2 implies that the corresponding matrix representations are unique up to conjugation with unitary diagonal matrices and permutation of the basis elements. One can easily apply Verlinde's formula and check whether the resulting coefficients $N_{i,j}^k$ have integer absolute values for the two possible choices of the basis element Φ_0 corresponding to the vacuum (conjugation with a unitary diagonal matrix does not change the absolute value of $N_{i,j}^k$). In particular for the level 2 representation $N_1(\chi_1)$ and the level 3 representation $N_1(\chi)$ we obtain for both possible choices of the distinguished basis elements Φ_0 and Φ_1

$$|N_{1,1}^1| = \begin{cases} \frac{2}{\sqrt{3}}, & \text{for } N_1(\chi_1),\ p = 2 \\ \frac{1}{\sqrt{2}}, & \text{for } N_1(\chi),\ p = 3. \end{cases}$$

Since $|N_{1,1}^1|$ is not an integer we can exclude these two representations. For the level 2^3 and 5 representations one obtains integer values for the $N_{i,j}^k$. Moreover, in all three cases both possible choices of the distinguished basis elements Φ_0 and Φ_1 lead to isomorphic fusion algebras. We conclude that the representation of the modular group given by a two dimensional strong-modular fusion algebra is isomorphic to the tensor product of a one dimensional representation and $N_3(\chi)_+$ ($p^\lambda = 2^3$) or $R_1(r, \chi_{-1})$ ($r = 1, 2; p^\lambda = 5$). Using that $\rho(S^2)$ should be a matrix consisting of nonnegative integers one can determine the one dimensional representation of Γ up to an even one dimensional representation. Therefore, $(\mathcal{F}, \rho)$ is determined up to tensor products with one dimensional modular fusion algebras. The resulting representations and fusion algebras are collected in Table 9. □

Remark. The two fusion algebras in Table 9 are called $\mathbb{Z}_2$ and "(2,5)" fusion algebras, respectively. The first name is evident since this fusion algebra is isomorphic to the group algebra of $\mathbb{Z}_2$ with the distinguished basis given by the group elements. We will call the fusion algebra given by the group algebra of $\mathbb{Z}_N$ in the following $\mathbb{Z}_N$ fusion algebra. The second name results from the fact that the Virasoro vertex operator algebra is rational for $c = c(p,q) = 1 - 6\frac{(p-q)^2}{pq}$ ($p, q > 1$, $(p,q) = 1$) [16,17] (these models are called Virasoro minimal models) and the corresponding fusion algebra are denoted by "(p,q)" fusion algebra. In particular the "(2,5)" fusion algebra is isomorphic to the fusion algebra in the second row of Table 9.

Main theorem 2. *Let $(\mathcal{F}, \rho)$ be a three dimensional strong-modular fusion algebra. Then $(\mathcal{F}, \rho)$ is isomorphic to the tensor product of a one dimensional modular fusion algebra with one of the modular fusion algebras in Table 10.*

Proof. Let $(\mathcal{F}, \rho)$ be a three dimensional strong-modular fusion algebra. By Lemma 7, ρ is either irreducible or isomorphic to a sum of a two dimensional and a one dimensional irreducible representation. We will now consider these two cases separately.

Firstly, assume that ρ is irreducible. By Lemma 7, ρ is isomorphic to the tensor product of a one dimensional representation and one of the three dimensional irreducible level p^λ representations. There are exactly 33 inequivalent irreducible 3 dimensional level p^λ representations. Their explicit form is given in Appendix A. We are interested in the classification up to tensor products with one dimensional modular fusion algebras. Therefore, we can restrict our investigation to a set of irreducible representations which are not related via tensor products with one dimensional representations. This means that we have to consider one representation of level 3 and 2^2, two representations of level 5 and 7 and, finally, four representations of level 2^4 (see Appendix A).

For these representations the eigenvalues of the image of T are nondegenerate so that we can proceed now as in the proof of the Main theorem 1.

Using Verlinde's formula for the representation $N_1(1, \chi_1)$ ($p = 3$) we obtain $|N_{1,1}^1| = \frac{1}{2}$ for all possible choices of the distinguished basis. In the same way one finds for $R_1(r,\chi_1)$ $(r=1,2; p=5)$ that

$$\begin{cases} |N_{1,1}^2| = \frac{1}{\sqrt{2}} & \text{for } \rho(T) = \mathrm{diag}(1, e^{2\pi i \frac{r}{5}}, e^{2\pi i \frac{4r}{5}}) \\ & \text{or } \rho(T) = \mathrm{diag}(1, e^{2\pi i \frac{4r}{5}}, e^{2\pi i \frac{r}{5}}) \\ |N_{1,1}^1| = \frac{1}{\sqrt{2}} & \text{for } \rho(T) = \mathrm{diag}(e^{2\pi i \frac{r}{5}}, 1, e^{2\pi i \frac{4r}{5}}) \\ |N_{1,1}^1| = \frac{1}{\sqrt{2}} & \text{for } \rho(T) = \mathrm{diag}(e^{2\pi i \frac{4r}{5}}, 1, e^{2\pi i \frac{r}{5}}). \end{cases}$$

Here the different cases correspond to the different possible choices of the distinguished basis. We conclude that ρ cannot be isomorphic to a tensor product of a one dimensional representation and $N_1(1,\chi_1)$ $(p=3)$ or $R_1(r,\chi_1)$ $(r=1,2;p=5)$.

Table 10: Three dimensional strong-modular fusion algebras

$\mathcal{F}$	$\rho(S)$	$\frac{1}{2\pi i}\log(\rho(T)) \bmod \mathbb{Z}$
$\Phi_1\cdot\Phi_1=\Phi_2$ $\Phi_1\cdot\Phi_2=\Phi_0$ $\Phi_2\cdot\Phi_2=\Phi_1$ ($\mathbb{Z}_3$)	$\frac{1}{\sqrt{3}}\begin{pmatrix}1 & 1 & 1\\ 1 & e^{2\pi i\frac{1}{3}} & e^{2\pi i\frac{2}{3}}\\ 1 & e^{2\pi i\frac{2}{3}} & e^{2\pi i\frac{1}{3}}\end{pmatrix}$	$\mathrm{diag}(\frac{1}{4},\frac{7}{12},\frac{7}{12})$
$\Phi_1\cdot\Phi_1=\Phi_0+\Phi_2$	$\frac{2}{\sqrt{7}}\begin{pmatrix}-s_2 & -s_1 & s_3\\ -s_1 & -s_3 & -s_2\\ s_3 & -s_2 & s_1\end{pmatrix}$	$\begin{cases}\mathrm{diag}(\frac{4}{7},\frac{1}{7},\frac{2}{7})\\ \mathrm{diag}(\frac{3}{7},\frac{6}{7},\frac{5}{7})\end{cases}$
$\Phi_1\cdot\Phi_2=\Phi_1+\Phi_2$	$\frac{2}{\sqrt{7}}\begin{pmatrix}-s_3 & -s_1 & s_2\\ -s_1 & -s_2 & -s_3\\ s_2 & -s_3 & s_1\end{pmatrix}$	$\begin{cases}\mathrm{diag}(\frac{1}{7},\frac{4}{7},\frac{2}{7})\\ \mathrm{diag}(\frac{6}{7},\frac{3}{7},\frac{5}{7})\end{cases}$
$\Phi_2\cdot\Phi_2=\Phi_0+\Phi_1+\Phi_2$ ("(2, 7)")	$\frac{2}{\sqrt{7}}\begin{pmatrix}s_1 & s_2 & s_3\\ s_2 & -s_3 & s_1\\ s_3 & s_1 & -s_2\end{pmatrix}$ $s_j=\sin(\frac{j\pi}{7})$	$\begin{cases}\mathrm{diag}(\frac{2}{7},\frac{1}{7},\frac{4}{7})\\ \mathrm{diag}(\frac{5}{7},\frac{6}{7},\frac{3}{7})\end{cases}$
$\Phi_1\cdot\Phi_1=\Phi_0$ $\Phi_1\cdot\Phi_2=\Phi_2$ $\Phi_2\cdot\Phi_2=\Phi_0+\Phi_1$ ("(3, 4)")	$\frac{1}{2}\begin{pmatrix}1 & 1 & \sqrt{2}\\ 1 & 1 & -\sqrt{2}\\ \sqrt{2} & -\sqrt{2} & 0\end{pmatrix}$	$\begin{cases}\mathrm{diag}(\frac{8-n}{16},\frac{16-n}{16},\frac{n}{8})\\ \mathrm{diag}(\frac{16-n}{16},\frac{8-n}{16},\frac{n}{8})\\ n=0,\ldots,7\end{cases}$

An analogous calculation shows that for the representations of type $R_1(r,\chi_{-1})$ one has $|N_{i,j}^k|\in\mathbb{N}$ for all 3 possible choices of the distinguished basis. For the remaining representations one also has $|N_{i,j}^k|\in\mathbb{N}$ for the two possible choices of the distinguished basis (here the matrix $\rho(S)$ contains a zero so that there are only two possible choices of the distinguished basis).

Hence, ρ is isomorphic to a tensor product of a one dimensional representation with one of these 7 representations. Using that for a modular fusion algebra $\rho(S^2)_{i,j}$ equals $N_{i,j}^0$ one can determine the possible one dimensional representations. The corresponding strong-modular fusion algebras are contained in Table 10 in the second and third row.

Secondly, assume that ρ decomposes into a direct sum of two irreducible representations $\rho \cong \rho_1 \oplus \rho_2$ with $\dim(\rho_j) = j$. Then ρ_2 is isomorphic to the tensor product of a one dimensional representation with one of the two dimensional irreducible level p^λ representations contained in Table A1.

Using Lemma 1 we conclude that $\rho(T)$ has degenerate eigenvalues so that $\rho_2(T)$ must have an eigenvalue of the form $e^{2\pi i \frac{n}{12}}$. Hence, ρ_2 cannot be isomorphic to the tensor product of a one dimensional representation and one of the two dimensional irreducible level 5 and 2^3 representations in Table A1. Using once more that $\rho(T)$ has degenerate eigenvalues we obtain that ρ is isomorphic to the tensor product of a one dimensional representation with either $N_1(\chi_1) \oplus C_j$ $(j = 1, 2; p = 2)$ or $N_1(\chi) \oplus B_j$ $(j = 2, 3; p = 3)$. In order find out whether these four representations are admissible we have to look for distinguished bases.

Let us first consider the case $\rho \cong C \otimes (N_1(\chi) \oplus B_j)$ $(j = 2, 3; p = 3)$ where C is a one dimensional representation. Here $\rho(S^2)$ has two different eigenvalues since $N_1(\chi)$ is odd and the representations B_j are even. Since the vacuum is selfconjugate, i.e. $\rho(S^2)_{00} = 1$ the representation C has to be odd. Without loss of generality we choose $C = C_4$ for $j = 2$ and $C = C_3$ for $j = 3$. Furthermore, the fact that $\rho(S^2)$ has two different eigenvalues implies that we must have

$$\rho(S^2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Using these two conditions it follows that in a basis in which $\rho(S^2)$ has this form and $\rho(T)$ is diagonal we must have

$$\rho(S) = \frac{1}{\sqrt{3}} \begin{pmatrix} \epsilon & \epsilon & \epsilon \\ \epsilon & e^{2\pi i \frac{1}{3}} & e^{2\pi i \frac{2}{3}} \\ \epsilon & e^{2\pi i \frac{2}{3}} & e^{2\pi i \frac{1}{3}} \end{pmatrix}, \qquad \epsilon^2 = 1$$

and

$$\rho(T) = \begin{cases} \operatorname{diag}(e^{2\pi i \frac{5}{12}}, e^{2\pi i \frac{1}{12}}, e^{2\pi i \frac{1}{12}}) & \text{or} \\ \operatorname{diag}(e^{2\pi i \frac{7}{12}}, e^{2\pi i \frac{11}{12}}, e^{2\pi i \frac{11}{12}}) \end{cases}$$

up to conjugation with a unitary diagonal matrix (the two possibilities for $\rho(T)$ correspond to the two possible choices of the distinguished basis).

Applying now Verlinde's formula leads to a modular fusion algebra iff $\epsilon = 1$ for both choices of the distinguished basis. The corresponding fusion algebra, $\rho(S)$ and $\rho(T)$ are listed in the first row of Table 10.

Finally, consider the case $\rho \cong C \otimes (N_1(\chi_1) \oplus C_j)$ $(j = 1,2)$. Since $N_1(\chi_1)$ $(p = 2)$ and C_j $(j = 1,2)$ are even ρ has to be even, too. Therefore, C is even and w.l.o.g. we choose $C = C_1$ for $j = 1$ and $C = C_2$ for $j = 2$. Since ρ is even one must have $\rho(S^2) = \mathbb{I}$ and, therefore, $\rho(S)$ is real (c.f. the second remark in §2). Plugging this in we find (up to permutation of the basis elements) that

$$\rho(S) = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3}a & \sqrt{3}b \\ -\sqrt{3}a & 2-3a^2 & 3ab \\ \sqrt{3}b & 3ab & 3a^2-1 \end{pmatrix}, \qquad \rho(T) = (-1)^j \operatorname{diag}(1,-1,-1)$$

where $a, b \in \mathbb{R}$ and $a^2 + b^2 = 1$. Using Verlinde's formula we obtain as conditions for ρ to be admissible

$$\begin{cases} \frac{(1-3a^2)(3a^2-2)}{\sqrt{3}a} \in \mathbb{N} & \text{for } \rho(T) = (-1)^j \operatorname{diag}(1,-1,-1) \\ \frac{1}{\sqrt{3}a(3a^2-2)}, \frac{3a^2-1}{\sqrt{3}a(3a^2-2)} \in \mathbb{N} & \text{for } \rho(T) = (-1)^j \operatorname{diag}(-1,-1,1). \end{cases}$$

The first case implies that $a^2 = \frac{1}{3}$ or $a^2 = \frac{2}{3}$ and the second one $a^2 = \frac{1}{3}$, respectively. Inserting these values of a in the explicit form of $\rho(S)$ above we indeed obtain modular fusion algebras if we choose the signs of a and b correctly. The resulting modular fusion algebras are contained in the third row of Table 10. As fusion algebras they are of type "(3,4)", also called Ising fusion algebra.

This completes the proof of the Main theorem 2. □

Main theorem 3. *Let $(\mathcal{F}, \rho)$ be a four dimensional strong-modular fusion algebra. Then $(\mathcal{F}, \rho)$ is either isomorphic to the tensor product of 2 two dimensional strong-modular fusion algebras or isomorphic to the tensor product of a one dimensional modular fusion algebra with one of the modular fusion algebras in Table 11.*

Proof. Let $(\mathcal{F}, \rho)$ be a strong-modular fusion algebra. Then, by Lemma 4, we have the following possibilities for ρ:

(1) ρ is irreducible,
(2) $\rho \cong \rho_1 \oplus \rho_2$ with $\dim(\rho_1) = 3$, $\dim(\rho_2) = 1$,
(3) $\rho \cong \rho_1 \oplus \rho_2$ with $\dim(\rho_1) = \dim(\rho_2) = 2$,
(4) $\rho \cong \rho_1 \oplus \rho_2 \oplus \rho_3$ with $\dim(\rho_1) = 2$, $\dim(\rho_2) = \dim(\rho_3) = 1$

where ρ_i $(i = 1,2,3)$ are irreducible representations.

(1) ρ is irreducible

Assume that ρ is irreducible. Then ρ is either isomorphic to the tensor product of 2 two dimensional representations of coprime levels or it is isomorphic to the tensor product of a one dimensional representation with a four dimensional irreducible level p^λ representation. In the first case we obviously have that ρ is only admissible iff both two dimensional representations are admissible (look at Table A1). In this case the corresponding modular fusion algebra is a tensor product of two fusion algebras contained in Table 9. Let us now consider the other case, namely that $\rho \cong C \otimes \rho_1$ where C is a one dimensional representation and ρ_1 is a four dimensional irreducible level p^λ representation. In this case ρ_1 is given by one of the 9 representations in Table A3. Note that for all of these representations the eigenvalues of the image of

T are nondegenerate so that we can use the argumentation used in the proof of the Main theorem 1.

Table 11: Four dimensional simple strong-modular fusion algebras

$\mathcal{F}$	$\rho(S)$	$\frac{1}{2\pi i}\log(\rho(T)) \bmod \mathbb{Z}$
$\Phi_1^2 = \Phi_2,\quad \Phi_1 \cdot \Phi_2 = \Phi_3,$ $\Phi_2^2 = \Phi_0,\quad \Phi_1 \cdot \Phi_3 = \Phi_0,$	$\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & -1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$	$\begin{cases} \mathrm{diag}(\frac{7}{8},\frac{1}{4},\frac{3}{8},\frac{1}{4}) \\ \mathrm{diag}(\frac{3}{8},\frac{1}{4},\frac{7}{8},\frac{1}{4}) \end{cases}$
$\Phi_3^2 = \Phi_2,\quad \Phi_2 \cdot \Phi_3 = \Phi_1,$ $(\ \mathbb{Z}_4\ )$	$\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & -1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}$	$\begin{cases} \mathrm{diag}(\frac{5}{8},\frac{3}{4},\frac{1}{8},\frac{3}{4}) \\ \mathrm{diag}(\frac{1}{8},\frac{3}{4},\frac{5}{8},\frac{3}{4}) \end{cases}$
$\Phi_1^2 = \Phi_0,\quad \Phi_1 \cdot \Phi_2 = \Phi_3,$ $\Phi_2^2 = \Phi_0,\quad \Phi_1 \cdot \Phi_3 = \Phi_2,$ $\Phi_3^2 = \Phi_0,\quad \Phi_2 \cdot \Phi_3 = \Phi_1$ $(\ \mathbb{Z}_2 \otimes \mathbb{Z}_2\ )$	$\frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}$	$\begin{cases} \mathrm{diag}(0,0,0,\frac{1}{2}) \\ \mathrm{diag}(\frac{1}{2},0,0,0) \end{cases}$
$\Phi_1^2 = \Phi_0 + \Phi_3$ $\Phi_1 \cdot \Phi_2 = \Phi_1 + \Phi_3$	$\frac{2}{3}\begin{pmatrix} -s_4 & s_1 & s_3 & -s_2 \\ s_1 & s_2 & s_3 & s_4 \\ s_3 & s_3 & 0 & -s_3 \\ -s_2 & s_4 & -s_3 & s_1 \end{pmatrix}$	$\begin{cases} \mathrm{diag}(\frac{7}{36},\frac{19}{36},\frac{1}{12},\frac{31}{36}) \\ \mathrm{diag}(\frac{29}{36},\frac{17}{36},\frac{11}{12},\frac{5}{36}) \end{cases}$
$\Phi_1 \cdot \Phi_3 = \Phi_2 + \Phi_3$ $\Phi_2^2 = \Phi_0 + \Phi_2 + \Phi_3$	$\frac{2}{3}\begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ s_2 & -s_4 & s_3 & -s1 \\ s_3 & s_3 & 0 & -s_3 \\ s_4 & -s_1 & -s_3 & s_2 \end{pmatrix}$	$\begin{cases} \mathrm{diag}(\frac{31}{36},\frac{7}{36},\frac{1}{12},\frac{19}{36}) \\ \mathrm{diag}(\frac{5}{36},\frac{29}{36},\frac{11}{12},\frac{17}{36}) \end{cases}$
$\Phi_2 \cdot \Phi_3 = \Phi_1 + \Phi_2 + \Phi_3$ $\Phi_3^2 = \Phi_0 + \Phi_1 + \Phi_2 + \Phi_3$ $(\ "(2,9)"\ )$	$\frac{2}{3}\begin{pmatrix} s_2 & -s_4 & s_3 & -s_1 \\ -s_4 & s_1 & s_3 & -s_2 \\ s_3 & s_3 & 0 & -s_3 \\ -s_1 & -s_2 & -s_3 & -s4 \end{pmatrix}$ $s_j = \sin(\frac{j\pi}{9})$	$\begin{cases} \mathrm{diag}(\frac{19}{36},\frac{31}{36},\frac{1}{12},\frac{7}{36}) \\ \mathrm{diag}(\frac{17}{36},\frac{5}{36},\frac{11}{12},\frac{29}{12}) \end{cases}$

For the representation $N_1(\chi)$ ($\chi^3 \not\equiv 1; p = 5$) we find by Verlinde's formula

$$|N_{1,1}^1| = \sqrt{3}, \qquad \text{for } \rho(T) = \mathrm{diag}(e^{2\pi i \frac{n}{5}}, e^{2\pi i \frac{3n}{5}}, e^{2\pi i \frac{2n}{5}}, e^{2\pi i \frac{4n}{5}}) \qquad (n = 1, \ldots, 4)$$

where again the different possibilities for $\rho(T)$ correspond to the different possible distinguished basis. This shows that ρ_1 cannot be isomorphic to this representation.

Since the representation $N_1(\chi)$ ($\chi^3 \equiv 1; p = 5$) is isomorphic to the tensor product of the two different level 5 representations in Table A1 it is clear that this representation is admissible. Since the image of T under this representation has nondegenerate eigenvalues the corresponding modular fusion algebras are isomorphic to the tensor product of 2 two dimensional modular fusion algebras (as fusion algebras they are of type "(2,5)").

Consider now the representations $R_1(r,\chi_1)$ ($r = 1,2; p = 7$). Here Verlinde's formula implies that

$$|N_{1,1}^1| = \frac{1}{\sqrt{2}} \quad \text{for} \quad \rho(T) = \mathrm{diag}(e^{2\pi i \frac{n}{7}}, 1, \cdot, \cdot) \quad (n = 1, \ldots 6)$$

and

$$|N_{1,1}^2| = \frac{1}{\sqrt{2}} \quad \text{for} \quad \rho(T) = \begin{cases} \mathrm{diag}(1, e^{2\pi i \frac{2}{7}}, e^{2\pi i \frac{4}{7}}, e^{2\pi i \frac{1}{7}}) & \text{or} \\ \mathrm{diag}(1, e^{2\pi i \frac{5}{7}}, e^{2\pi i \frac{3}{7}}, e^{2\pi i \frac{6}{7}}). \end{cases}$$

As above this removes these representations from the list of candidates leading to modular fusion algebras.

For the representation $N_3(\chi)$ ($\chi^3 \not\equiv 1; p = 2^3$) one has

$$|N_{1,1}^1| = \sqrt{\frac{4}{3}} \quad \text{for} \quad \rho(T) = \mathrm{diag}(e^{2\pi i \frac{2n+1}{8}}, e^{2\pi i \frac{2n+5}{8}}, \cdot, \cdot) \quad (n = 1, \ldots, 4)$$

so that this representation is also excluded.

Consider now the representations $R_2^1(r,1,\chi)$ ($r = 1,2; \chi^3 \not\equiv 1; p = 3^2$). Here one has

$$|N_{1,1}^1| = \frac{1}{\sqrt{3}} \quad \text{for} \quad \rho(T) = \mathrm{diag}(e^{2\pi i \frac{rn^2}{9}}, e^{2\pi i \frac{r}{3}}, \cdot, \cdot) \quad (n = 1,2,3).$$

The basis element in the representation space corresponding to the $\rho(T)$ eigenvalue of order three cannot correspond to Φ_0 since in the corresponding row of $\rho(S)$ contains a zero.

Finally, the only remaining four dimensional irreducible level p^λ representations that might lead to modular fusion algebras are those of type $R_2^1(r,1,\chi)$ ($r = 1,2; \chi^3 \equiv 1; p^\lambda = 3^2$). Indeed, these representations lead to modular fusion algebras. To be more precise one has to consider the tensor product of an odd one dimensional representation with them because the $R_2^1(r,1,\chi)$ ($\chi^3 \equiv 1$) are odd themselves. The corresponding fusion algebras are of type "$(2,9)$" and the explicit form is given in Table 11. The different modular fusion algebras result from the two different representations and the fact that the distinguished basis can be chosen in different ways.

$\rho \cong \rho_1 \oplus \rho_2$ with $\dim(\rho_1) = 3$, $\dim(\rho_2) = 1$

Assume that ρ is isomorphic to the direct sum of a one dimensional and an irreducible three dimensional representation. Then one has $\rho \cong C \otimes (\rho_1 \oplus D)$ where C and D are one dimensional representations and ρ_1 is one of the three dimensional irreducible level p^λ representations in Table A2. By Lemma 1 we know that $\rho(T)$ has degenerate eigenvalues. Therefore, ρ_1 is of type $N_1(\chi_1)$ ($p = 3$), $R_1(r,\chi_1)$ ($r = 1,2; p = 5$), $D_2(\chi)_+$ ($p^\lambda = 2^2$) or $R_3^0(1,3)_\pm$ ($p^\lambda = 2^3$).

Consider first the representation $N_1(\chi_1)$ ($p = 3$). In this case we can have $D = B_j$ ($j = 1,2,3$). Since B_j and $N_1(\chi_1)$ are even we can choose without loss of generality $C = C_1$. Using Verlinde's formula we find that

$$|N_{1,1}^1| = \frac{1}{2} \quad \text{for} \quad \rho(T) = \mathrm{diag}(e^{2\pi i \frac{i+1}{3}}, e^{2\pi i \frac{i+2}{3}}, e^{2\pi i \frac{i}{3}}, e^{2\pi i \frac{i}{3}})$$

giving a contradiction for these choices of the distinguished basis. For $\rho(T) = \mathrm{diag}(e^{2\pi i\frac{i}{3}}, e^{2\pi i\frac{i}{3}}, e^{2\pi i\frac{i+1}{3}}, e^{2\pi i\frac{i+2}{3}})$ the line of reasoning is a little bit more involved. Here $N^0_{i,j} = \rho(S^2)_{i,j} = \delta_{i,j}$ implies that $\rho(S)$ is given by

$$\rho(S) = \frac{1}{3}\begin{pmatrix} 4b^2-1 & 4ab & 2a & 2a \\ 4ab & 3-4b^2 & -2b & -2b \\ 2a & -2b & -1 & 2 \\ 2a & -2b & 2 & -1 \end{pmatrix}$$

up to conjugation with an orthogonal diagonal matrix, with $a, b \in \mathbb{R}$ and $a^2+b^2 = 1$. With the explicit form of $\rho(S)$ we find as conditions for ρ to be admissible

$$N^1_{1,1} = \frac{1}{2a(3-4a^2)} \in \mathbb{Z}, \qquad N^2_{1,1} = \frac{2a^2-1}{2a(3-4a^2)} \in \mathbb{Z}.$$

However, the only solutions that satisfy these two conditions are those a which equal $\frac{1}{2m}$ for an integer m and satisfy $m^3 \equiv 0 \bmod 3m^2-1$. It follows that $m \equiv 0 \bmod 3m^2-1$ which gives a contradiction . Therefore, the representations $N_1(\chi_1) \oplus B_j$ $(p=3)$ do not lead to modular fusion algebras.

Next we consider the representations $R_1(r, \chi_1)$ $(r = 1, 2; p = 5)$. In this case the one dimensional representation D has to be the trivial one. Since these two representations are even we can choose without loss of generality $C = C_1$, too. Using that $N^0_{i,j} = \delta_{i,j}$ we find that the matrix which describes the basis in the two dimensional eigenspace corresponding to the eigenvalue 1 of $\rho(T)$ is orthogonal. Furthermore, by looking at suitable $N^k_{i,j}$ we find that there are only two possibilities for this matrix. In the corresponding basis we indeed find modular fusion algebra given by the tensor product of two modular fusion algebras of type "$(2,5)$". That ρ is admissible can also be interfered from the equality $R_1(r,\chi_1) \oplus C \cong R_1(r, \chi_{-1}) \otimes R_1(r, \chi_{-1})$ $(r = 1, 2; p = 5)$.

Finally, we have to consider $D_2(\chi)_+$ $(p^\lambda = 2^2)$ and $R^0_3(1,3,\chi)_\pm$ $(p^\lambda = 2^3)$. The corresponding possibilities for ρ are $C_3 \otimes D_2(\chi)_+ \oplus C_j$ $(j = 1,3,4)$, $C_4 \otimes R^0_3(1,3,\chi)_+ \oplus C_3$ or $C_3 \otimes R^0_3(1,3,\chi)_- \oplus C_4$. For the case $\rho \cong C_3 \otimes D_2(\chi)_+ \oplus C_1$ we obtain a modular fusion algebra given by the tensor product of two $\mathbb{Z}_2$ fusion algebras. This can also be seen by looking at the identity

$$C_3 \otimes D_2(\chi)_+ \oplus C_1 \cong D_2(\chi)_+ \otimes D_2(\chi)_+.$$

For $C_4 \otimes R^0_3(1,3,\chi)_+ \oplus C_3$ or $C_3 \otimes R^0_3(1,3,\chi)_- \oplus C_4$ we obtain $\mathbb{Z}_4$ type fusion algebras (see Table 11). The other two representations ($C_3 \otimes D_2(\chi)_+ \oplus C_j$ $(j = 3, 4)$) are not admissible as one can easily check by applying Verlinde's formula.

$\rho \cong \rho_1 \oplus \rho_2$ with $\dim(\rho_1) = \dim(\rho_2) = 2$

Assume that ρ decomposes into a direct sum of 2 two dimensional irreducible representations. In this case we have $\rho = C \otimes (\rho_1 \oplus D \otimes \rho_2)$ where C and D are one dimensional representations and ρ_1, ρ_2 are some level p^λ representations contained in Table A1. Since ρ is reducible we know that $\rho(T)$ has degenerate eigenvalues. This together with the parity of the representations in Table A1 implies that ρ

equals (up to a tensor product with an even one dimensional representation) one of the following representations:

$$\begin{aligned}
&N_1(\chi_1) \oplus N_1(\chi_1)\\
&C_3 \otimes (N_1(\chi) \oplus B_i \otimes N_1(\chi)) \quad (i = 1,2)\\
&C_4 \otimes (R_1(r,\chi_{-1}) \oplus R_1(r,\chi_{-1})) \quad (r = 1,2)\\
&C_4 \otimes (N_3(\chi)_+ \oplus N_3(\chi)_+).
\end{aligned}$$

In all cases we have that $\rho(S)$ is conjugate to a matrix of block diagonal form. More precisely, this matrix consists of two identical two by two matrices. A simple calculation shows now that conjugation of $\rho(S)$ with a matrix which leaves $\rho(T)$ diagonal leads to a matrix which has at least one zero element in every row. This is a contradiction since we have assumed that ρ is admissible and one can apply Verlinde's formula.

$\rho \cong \rho_1 \oplus \rho_2 \oplus \rho_3$ with $\dim(\rho_1) = 2$, $\dim(\rho_2) = \dim(\rho_3) = 1$

Assume that ρ decomposes into a direct sum of an irreducible two dimensional and 2 one dimensional representations. Then, again by Lemma 1, $\rho(T)$ has degenerate eigenvalues and a simple parity argument shows that the only possibilities for ρ are (up to a tensor product with an even one dimensional representation):

$$N_1(\chi_1) \oplus C_1 \oplus C_1 \qquad \text{or} \qquad N_1(\chi_1) \oplus C_1 \oplus C_2$$

where $N_1(\chi_1)$ is the level 2 representation in Table A1. We have to consider these two cases separately.

Firstly, let ρ be conjugate to $N_1(\chi_1) \oplus C_1 \oplus C_1$. Then the requirements that $\rho(S)$ has to be symmetric and real and that $\rho(T)$ has to be diagonal imply that (up to permutation of the basis elements and conjugation with an orthogonal diagonal matrix):

$$\rho(S) = -\frac{1}{2}\begin{pmatrix} -1 & \sqrt{3}a & \sqrt{3}b & \sqrt{3}c \\ \sqrt{3}a & 3a^2-2 & 3ab & 3ac \\ \sqrt{3}b & 3ab & 3b^2-2 & 3bc \\ \sqrt{3}c & 3ac & 3bc & 3c^2-2 \end{pmatrix}$$

where $a, b, c \in \mathbb{R}$ with $a^2 + b^2 + c^2 = 1$ and $\rho(T) = \mathrm{diag}(-1, 1, 1, 1)$.

Fixing the distinguished basis such that Φ_0 corresponds to the eigenvector of $\rho(T)$ with eigenvalue -1 we obtain

$$\begin{aligned}
&N_{11}^1 = \frac{(2-3a^2)(1-3a^2)}{\sqrt{3}a}, \quad N_{22}^2 = \frac{(2-3b^2)(1-3b^2)}{\sqrt{3}b}, \quad N_{33}^3 = \frac{(2-3c^2)(1-3c^2)}{\sqrt{3}c}\\
&N_{11}^2 = \sqrt{3}(3a^2-1)b, \qquad N_{11}^3 = \sqrt{3}(3a^2-1)c\\
&N_{22}^1 = \sqrt{3}(3b^2-1)a, \qquad N_{22}^3 = \sqrt{3}(3b^2-1)c.
\end{aligned}$$

This implies that $a^2 = b^2 = c^2 = \frac{1}{3}$. The resulting structure constants indeed define a fusion algebra, namely the tensor product of two fusion algebras of type $\mathbb{Z}_2$. As a modular fusion algebra this fusion algebra is **simple**, i.e. it is not a tensor product of two nontrivial modular fusion algebras. The resulting modular fusion algebra is contained in Table 11.

For the other choice of the distinguished basis where Φ_0 corresponds to an eigenvector $\rho(T)$ with eigenvalue 1 we find

$$N_{33}^1 = \frac{(3a^2-1)b}{a(3a^2-2)}, \qquad N_{33}^2 = \frac{(3a^2-1)c}{a(3a^2-2)},$$
$$N_{33}^3 = \frac{3a^2-1}{\sqrt{3}a(3a^2-2)}, \qquad N_{22}^3 = \frac{1-3b^2}{\sqrt{3}a(3a^2-2)}$$

where the basis was chosen such that $\rho(T) = \mathrm{diag}(1,1,1,-1)$. Let now $n := (N_{33}^1)^2 + (N_{33}^2)^2$ and $m := (N_{33}^3)^2$. It is now easy to verify that n and m satisfy the equation

$$m^3 + (1-5n)m^2 + (4n^2+7n)m + 4n^2 - 3n^3 = 0.$$

By Lemma 10 in §5.2 below the only nonnegative integer solution of this equation is given by $n = m = 0$. Therefore, we find as the only possible solution $a^2 = b^2 = c^2 = \frac{1}{3}$. The resulting structure constants define a fusion algebra isomorphic to the tensor product of two $\mathbb{Z}_2$ fusion algebras. However, analogous to the case of the other distinguished basis discussed above this modular fusion algebra is **simple** and contained in Table 11.

Secondly, assume that ρ is conjugate to $N_1(\chi_1) \oplus C_1 \oplus C_2$. Requiring that $\rho(S)$ is a symmetric real matrix and that $\rho(T)$ is diagonal implies (up to a permutation of the basis elements and conjugation with an orthogonal diagonal matrix)

$$\rho(S) = \frac{1}{2}\begin{pmatrix} 3b^2-1 & -3ab & -\sqrt{3}ac & \sqrt{3}ad \\ -3ab & 3a^2-1 & -\sqrt{3}bc & \sqrt{3}bd \\ -\sqrt{3}ac & -\sqrt{3}bc & 3c^2-2 & -3cd \\ \sqrt{3}ad & \sqrt{3}bd & -3cd & 3d^2-2 \end{pmatrix}$$

where $a,b,c,d \in \mathbb{R}$ and $a^2+b^2=1, c^2+d^2=1$ and $\rho(T) = \mathrm{diag}(1,1,-1,-1)$. Using Verlinde's formula we obtain for the choice of the distinguished basis in which Φ_0 corresponds to the eigenvector of $\rho(T)$ with eigenvalue 1

$$(N_{11}^1)^2 = \frac{(3a-1)^2(6a-5)^2}{9a^2(1-a^2)(3a-2)^2}, \quad (N_{11}^2)^2 = \frac{c^2}{3a^2(3a^2-2)}, \quad (N_{11}^3)^2 = \frac{d^2}{3a^2(3a^2-2)}.$$

For the other choice of the distinguished basis (Φ_0 corresponding to eigenvalue -1) one finds the same expressions with a and c exchanged.

Let $n := (N_{11}^2)^2 + (N_{11}^3)^2$ and let $m := (N_{11}^1)^2$. It is easy to verify that the following equation for n and m holds true

$$\begin{aligned}(1-3n)m^3 &+ (12-37n+31n^2)m^2 + (48-152n+155n^2-53n^3)m \\ &+ 64 - 208n + 249n^2 - 130n^3 + 25n^4 = 0.\end{aligned}$$

By Lemma 10 in §5.2 below the only nonnegative integer solution of this equation is given by $m = 0, n = 1$. This is a contradiction to the explicit form of n and m in terms of a above. Hence the representation $N_1(\chi_1) \oplus C_1 \oplus C_2$ is not admissible.

This proves the Main theorem 3. □

5.2 Proof of a Lemma on diophantic equations.

Lemma 10[5]**.** *Let n be a nonnegative integer, m a square of an integer and n, m solutions of*

(1) $m^3 + (1-5n)m^2 + (4n+7n^2)m + 4n^2 - 3n^3 = 0$ *or*
(2) $(1-3n)m^3 + (12-37n+31n^2)m^2 + (48-152n+155n^2-53n^3)m + 64 - 208n + 249n^2 - 130n^3 + 25n^4 = 0$

Then either $n = m = 0$ for (1) or $m = 0, n = 1$ for (2).

Proof. Firstly, consider the equation (1). It can be written in the form

$$(3n-m)(m-n)^2 = (m+2n)^2.$$

If $n = m$ then $m = n = 0$. Otherwise, set $t = \frac{m+2n}{m-n}$ implying

$$m = \frac{(t+2)t^2}{2t-5}, \qquad n = \frac{(t-1)t^2}{2t-5}.$$

If m and n are integral then also t has to be integral (any prime factor of the denominator of t would divide the denominator of m and n). Then $N = 2t-5$ divides $(t-1)t^2 = \frac{1}{8}(N+5)^2(N+3)$ so that N divides $3 \cdot 5^2$. None of the resulting 12 possibilities leads to a nonnegative integer solution of n, m where $m \neq n$ and m is a square.

Secondly, consider the equation (2). Set $k = m - n + 4$, then (2) is equivalent to

$$k^3 + 2k^2n - 3k^3n + 125n^2 - 92kn^2 + 22k^2n^2 - 11n^3 = 0.$$

If $k = 0$ then $n = 0$ and $m = -4$ is not a square. Otherwise, (2) is equivalent to

$$(-3t+22t^2)k^2 + (1+2t-92t^2-11t^3)k + 125t^2 = 0, \quad k \neq 0$$

where $t = \frac{n}{k}$. This equation has discriminant $(1+18t+t^2)(1-7t+11t^2)^2$ and this must be a square. Setting $\frac{p}{q} := (1-t-(1+18t+t^2)^{1/2})/(10t) \in \mathbb{Q}$ (with coprime p, q and $q > 0$) we get

$$t = \frac{q(p+q)}{p(5p+q)}.$$

Hence, using the quadratic equation in k we finally have

$$m = \frac{(2p+q)^2(p-q)^2}{p^2(2q-p)^2(p+q)}, \qquad n = \frac{q^3}{p^2(2q-p)}.$$

The parametrization of n implies that $p = \pm 1$ and, furthermore, that $q^3 \equiv 0 \bmod (2q-p)$. Therefore, we have $p^3 \equiv 0 \bmod (2q-p)$ so that $2q - p = \pm 1$. From the resulting four possiblities only $p = q = 1$ satisfies the desired properties and leads to $m = 0, n = 1$. □

Remark. Note that the proof of Lemma 10 relies essentially on the fact that the curves defined by the two above equations are rational.

[5] I would like to thank D. Zagier for discussion on this lemma [18]

5.3 Classification of the nondegenerate strong-modular fusion algebras with dimension less than 24.

In this section we classify all strong modular fusion algebras $(\mathcal{F}, \rho)$ of dimension less than 24 for which $\rho(T)$ has nondegenerate eigenvalues. The main tool used in the proof is the classification of the irreducible representations of the groups $\mathrm{SL}(2, \mathbb{Z}_{p^\lambda})$ described in section 4.

Main theorem 4. *Let $(\mathcal{F}, \rho)$ be a simple nondegenerate strong-modular fusion algebra. Furthermore, assume that the dimension of $\mathcal{F}$ is less than 24. Then ρ is isomorphic to the tensor product of an even one dimensional representation of Γ with one of the representations in Table 12. Moreover, $\mathcal{F}$ is isomorphic to $\mathbb{Q}[x]/ < P(x) >$ with distinguished basis $p_j(x)$ $(j = 0, \ldots, n-1)$. Here P and p_j are the unique polynomials satisfying*

$$P(x) = \det(\mathcal{N}_1 - x)$$

$$p_0(x) = 1, \qquad p_1(x) = x, \qquad p_j(x) = \sum_{k=0}^{n-1} (\mathcal{N}_1)_{j,k} \; p_k(x).$$

where the $(\mathcal{N}_1)_{j,k} := N_{1,j}^{k}$ are the fusion matrices given in Appendix B.

Table 12: Simple nondegenerate strong-modular fusion of dimension less than 24 (q is a prime satisfying $q < 47$)

fusion	dim	ρ
$\mathbb{Z}_2$	2	$C_4 \otimes N_3(\chi)_\pm$, $(p^\lambda = 2^3)$
"$c(3,4)$" Ising	3	$C_4 \otimes D_2(\chi)_+$, $(p^\lambda = 2^2)$ $C_4 \otimes R_3^0(1,3,\chi)_\pm$, $(p^\lambda = 2^3)$ $C_4 \otimes R_4^0(r,3,\chi)_\pm$, $(r = 1,2; p^\lambda = 2^4)$
"$(2,q)$"	$\frac{1}{2}(q-1)$	$C_4^{\frac{q+1}{2}} \otimes R_1(r, \chi_{-1})$, $((\frac{r}{p}) = \pm 1; p^\lambda = q)$
"$(2,9)$"	4	$C_4 \otimes R_2^1(r,1,\chi)$, $(r = 1,2; \chi^3 \equiv 1; p^\lambda = 3^2)$
B_9	6	$N_2(\chi)$, $(\chi^3 \equiv 1; p^\lambda = 3^2)$
B_{11}	10	$N_1(\chi)$, $(\chi^3 \equiv 1; p^\lambda = 11)$
G_{17}	16	$N_1(\chi)$, $(\chi^3 \equiv 1; p^\lambda = 17)$
E_{23}	22	$N_1(\chi)$, $(\chi^3 \equiv 1; p^\lambda = 23)$

Remark. For all fusion algebras in Table 12 apart from B_9 there indeed exist RCFTs where the associated fusion algebras are isomorphic to the ones in Table 12: The fusion algebra in the first row occurs in the so-called $\mathbb{Z}_2$ model, the ones in row 2, 3 and 4 in the corresponding Virasoro minimal models (see also the remark at the end of the Main Theorem 1) and, finally, the ones in row 6, 7 and 8 occur as fusion algebras of certain rational models, so-called minimal models of Casimir $\mathcal{W}$-algebras, namely for $\mathcal{W}B_2$ and $c = -\frac{444}{11}$, $\mathcal{W}G_2$ and $c = -\frac{1420}{17}$ and $\mathcal{W}E_7$ and $c = -\frac{3164}{23}$ [3]. The fusion algebras of type B_9 seems to be related to $\mathcal{W}B_2$ and $c = -24$. However, in this case the model is not rational.

Proof. Let $(\mathcal{F}, \rho)$ be a simple nondegenerate strong-modular fusion algebra of dimension less than 24. Lemma 1 implies that ρ is irreducible. Furthermore, since $(\mathcal{F}, \rho)$ is strong-modular we have to consider all irreducible representations of $\mathrm{SL}(2, \mathbb{Z}_N)$ of dimension less than 24. Since $(\mathcal{F}, \rho)$ is simple and nondegenerate simple Lemma 7 shows that we can restrict our investigation to irreducible representations of $\mathrm{SL}(2, \mathbb{Z}_{p^\lambda})$. Once again, since $(\mathcal{F}, \rho)$ is nondegenerate we can follow the line of reasoning in the proof of the Main theorem 1.

Therefore, we can directly apply Verlinde's formula to any such matrix representation $\hat{\rho}$ and look whether the resulting coefficients $N_{i,j}^k$ have integer absolute values for the different choices of the basis element corresponding to Φ_0. If the resulting numbers $N_{i,j}^k$ do not have integer absolute values we can conclude that there exists no nondegenerate strong-modular fusion algebra $(\mathcal{F}, \rho)$ where ρ is conjugate to the tensor product of a one dimensional representation of Γ and $\hat{\rho}$. We have investigated this for all irreducible representations of $\mathrm{SL}(2, \mathbb{Z}_{p^\lambda})$ of dimension less than 24 by constructing them explicitly[6].

The proof of the theorem will consist of three separate cases: We consider representations of $\mathrm{SL}(2, \mathbb{Z}_p)$ and $\mathrm{SL}(2, \mathbb{Z}_{p^\lambda})$ and $\mathrm{SL}(2, \mathbb{Z}_{2^\lambda})$ separately.

Firstly, let ρ be isomorphic to a tensor product of a one dimensional representation and an irreducible representation $\hat{\rho}$ of $\mathrm{SL}(2, \mathbb{Z}_p)$ $(p \neq 2)$. Note that this case was already discussed in [3].

For the representations of type $D_1(\chi)$ the matrix $\rho(T)$ has degenerate eigenvalues so that we can leave out this type of representation.

For the representations of type $N_1(\chi)$ we find modular fusion algebras only for $p = 5, 11, 17$ and 23 and $\chi^3 \equiv 1$. For $p = 5$ the modular fusion algebra is not simple but equals the tensor product of two modular fusion algebras where the corresponding fusion algebras are of type "$(2,5)$" (cf. also the proof of the Main theorem 3). The modular fusion algebras corresponding to $p = 11, 17, 23$ are contained in the last three rows of Table 12. As was already mentioned in [3] these four representations are probably the only admissible ones of type $N_1(\chi)$. However, we do not have a proof of this statement but numerical checks show that there is no other admissible representation of this type for $p < 167$ [3].

The representations of type $R_1(r, \chi_1)$ and $N_1(\chi_1)$ do not lead to modular fusion algebras [3].

For all $\hat{\rho}$ of type $R_1(r, \chi_{-1})$ we obtain modular fusion algebras. Here $\rho \cong (C_4)^{\frac{p+1}{2}} \otimes R_1(r, \chi_{-1})$ is admissible for all odd primes p. The corresponding modular fusion algebras are of type "$(2,p)$". They are contained in the third row of Table 12.

Secondly, let ρ be isomorphic to a tensor product of a one dimensional representation and a irreducible representation $\hat{\rho}$ of $\mathrm{SL}(2, \mathbb{Z}_{p^\lambda})$ $(p \neq 2,\ \lambda > 1)$.

For the representations of type $D_\lambda(\chi)$ the matrix $\rho(T)$ has degenerate eigenvalues excluding these representations from our investigation.

The only representations of type $N_\lambda(\chi)$ which have dimension less than 24 are those corresponding to $(p = 3; \lambda = 2, 3)$ and $(p = 5; \lambda = 2)$. A calculation shows that exactly one of these representations leads to a modular fusion algebra. This

[6] Here we have used the computer algebra system PARI-GP [20].

is the representation with $(p = 3; \lambda = 2)$ and $\chi^3 \equiv 1$. The corresponding strong-modular fusion algebra is contained in Table 12.

Only those representations of type $R^{\sigma}_{\lambda}(r, t, \chi)$ and $R_{\lambda}(r, \chi_{\pm 1})_1$ with $(p = 3; \lambda = 2, 3)$ or $(p = 5; \lambda = 2)$ have dimension less than 24. The representations $R^1_2(r, 1, \chi)$ $(p^{\lambda} = 3^2; \chi^3 \equiv 1)$ lead to nondegenerate modular fusion algebras (cf. the proof of the Main theorem 3). All other representations of the above two types do not lead to nondegenerate modular fusion algebras.

Thirdly, consider the irreducible representations of $\mathrm{SL}(2, \mathbb{Z}_{2^{\lambda}})$. All irreducible representations of dimension less than or equal to 4 have been considered in the Main theorems 1 to 3. The corresponding admissible representations with nondegenerate eigenvalues of $\rho(T)$ are contained in Table 12.

For $\lambda = 1, 2$ all irreducible representations have dimension less then or equal to 3.

For $\lambda = 3$ we have to consider the representations of type $R^0_3(1, 3, \chi_1)_1$ and $D_3(\chi)_{\pm}$. The former representation does not lead to a modular fusion algebra but the representations $D_3(\chi)_{\pm}$ lead to modular fusion algebras of type $\mathbb{Z}_2 \otimes$ "(3, 4)". The corresponding modular fusion algebras are composite and therefore not contained in Table 12.

For $\lambda = 4$ only the irreducible representations of type $R^0_4(r, t, \chi)_{\pm}$, $R^2_4(r, 3, \chi_1)_1$ and $R^2_4(r, t, \chi)$ lead to modular fusion algebras. The first one leads to a fusion algebra of type "(3, 4)" (see Main theorem 2). The other two representations lead to composite modular fusion algebras. These fusion algebras are of type $\mathbb{Z}_2 \otimes$"(3, 4)" and are not contained in Table 12.

For $\lambda = 5, 6$ there are no irreducible representation of dimension less than 24 leading to modular fusion algebras (some of them correspond to "fermionic fusion algebras" of $N = 1$-Super-Virasoro minimal models which we do not discuss here). □

6. Conclusions

In this paper we have classified all strong-modular fusion algebras of dimension less than or equal to four and all nondegenerate strong-modular fusion algebras of dimension less than 24. In order to obtain our results we have used the classification of the irreducible representations of the groups $\mathrm{SL}(2, \mathbb{Z}_{p^{\lambda}})$. Not all modular fusion algebras in our classification show up in known RCFTs. However, all corresponding fusion algebras are realized in known RCFTs apart from the fusion algebra of type B_9. This fusion algebra can formally be related to the Casimir $\mathcal{W}$-algebra $\mathcal{W}B_2$ at $c = -24$ and seems to be an analogue of the fusion algebra formally associated to the Virasoro algebra with central charge $c = c(3, 9)$.

The fact that we do not know examples of RCFTs for all of the modular fusion algebras in our classification can be understood as follows. The classification of the strong-modular fusion algebras implies restrictions on the central charge and the conformal dimensions of possibly underlying RCFTs. In Table 13 we have collected the possible values of c and the h_i for the simple strong-modular fusion algebras of dimension less than or equal to four. Note, however, that these restrictions are not as strong as the ones in [19] for the two dimensional case or in [2] for the two and three dimensional case. A natural way to obtain stronger restrictions than the ones presented in Table 13 is to look whether there exist vector valued modular functions transforming under the corresponding representation of the modular group which have the correct pole order at $i\infty$. This can be done using the methods developed in [8] and indeed leads to much stronger restrictions on c and the h_i as we will discuss elsewhere. Of course, we expect that for any RCFT the corresponding characters are modular functions so that these stronger restrictions have to be valid explaining that our classification contains modular fusion algebras for which we do not know of any realization in RCFTs.

Table 13: Central charges and conformal dimensions of simple strong-modular fusion algebras

$\mathcal{F}$	c (mod4)	h_i (mod$\mathbb{Z}$)
$\mathbb{Z}_2$	1	$0, \frac{1}{4}$
	3	$0, \frac{3}{4}$
$\mathbb{Z}_3$	2	$0, \frac{1}{3}, \frac{1}{3}$ or $0, \frac{2}{3}, \frac{2}{3}$
$\mathbb{Z}_4$	1	$0, \frac{1}{8}, \frac{1}{2}, \frac{1}{8}$ or $0, \frac{5}{8}, \frac{1}{2}, \frac{5}{8}$
	3	$0, \frac{3}{8}, \frac{1}{2}, \frac{3}{8}$ or $0, \frac{7}{8}, \frac{1}{2}, \frac{7}{8}$
$\mathbb{Z}_2 \otimes \mathbb{Z}_2$	0	$0, 0, 0, \frac{1}{2}$ or $0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$
"(2,5)"	$\frac{6}{5}$	$0, \frac{3}{5}$
	$\frac{14}{5}$	$0, \frac{2}{5}$
	$\frac{2}{5}$	$0, \frac{1}{5}$
	$\frac{18}{5}$	$0, \frac{4}{5}$
"(2,7)"	$\frac{16}{7}$	$0, \frac{4}{7}, \frac{5}{7}$
	$\frac{12}{7}$	$0, \frac{3}{7}, \frac{2}{7}$
	$\frac{4}{7}$	$0, \frac{3}{7}, \frac{1}{7}$
	$\frac{24}{7}$	$0, \frac{4}{7}, \frac{6}{7}$
	$\frac{8}{7}$	$0, \frac{6}{7}, \frac{2}{7}$
	$\frac{20}{7}$	$0, \frac{1}{7}, \frac{5}{7}$
"(2,9)"	$\frac{10}{3}$	$0, \frac{1}{3}, \frac{2}{3}, \frac{2n}{9}$
	$\frac{2}{3}$	$0, \frac{1}{3}, \frac{2}{3}, \frac{n}{9}$
		$n = 1, 4, 7$
"(3,4)"	$\frac{3n}{2}$	$0, \frac{1}{2}, \frac{n}{16}$
	$n = 0, \ldots, 15$	

From our considerations it is clear that a complete classification of all simple non-degenerate strong-modular fusion algebras is a purely number theoretical problem which can probably be solved. However, we do not expect any new "exceptional" fusion algebras (of course there exist those of type "$(2,q)$" with prime q greater than 47).

Unfortunately, the methods used in this paper seem to be not sufficient for obtaining a complete classification of strong-modular fusion algebras. For those strong-modular fusion algebras which are not nondegenerate the corresponding representations of the modular group are in general reducible and therefore there is a lot freedom for possible choices of the distinguished basis in the representation space. In the main theorems we have shown how one can deal with this freedom in the case of two, three and four dimensional fusion algebras. However, we do not know a general method to overcome this problem for arbitrary dimensions so that new methods have to be developed.

Finally, we would like stress that the main assumption for obtaining our classifications, namely that fusion algebras are induced by representations of $\mathrm{SL}(2, \mathbb{Z}_N)$, is valid for all known examples of rational conformal field theories. Nevertheless, the question whether all fusion algebras associated to RCFTs are strong-modular is not yet answered.

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We have used for many calculations the computer algebras system PARI-GP [20].

7. Appendix A: The irreducible level p^λ representations of dimension less than or equal to four

Using the results in §4 one obtains as a complete list of two dimensional irreducible level p^λ representations

$$\begin{array}{ll} p^\lambda = 2^1, & N_1(\chi_1) \\ p^\lambda = 3^1, & N_1(\chi_1) \otimes B_i \\ p^\lambda = 5^1, & R_1(1,\chi_{-1}),\ R_1(2,\chi_{-1}) \\ p^\lambda = 2^2, & N_1(\chi_1) \otimes C_3 \\ p^\lambda = 2^3, & N_3(\chi)_+ \otimes C_j \\ \text{where } i = 1,2,3;\ j = 1,\dots,4. & \end{array}$$

The explicit form of the representations which are not related by tensor products with B_i or C_j is given in Table A1.

Table A1: Two dimensional irreducible level p^λ representations

level	type of rep.	$\rho(S)$	$\frac{1}{2\pi i}\log(\rho(T))$
2	$N_1(\chi_1)$	$\frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$	$\mathrm{diag}(0,\frac{1}{2})$
3	$N_1(\chi)$	$-\frac{i}{\sqrt{3}}\begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}$	$\mathrm{diag}(\frac{1}{3},\frac{2}{3})$
5	$R_1(1,\chi_{-1})$	$\frac{2i}{\sqrt{5}}\begin{pmatrix} -\sin(\frac{\pi}{5}) & \sin(\frac{2\pi}{5}) \\ \sin(\frac{2\pi}{5}) & \sin(\frac{\pi}{5}) \end{pmatrix}$	$\mathrm{diag}(\frac{1}{5},\frac{4}{5})$
	$R_1(2,\chi_{-1})$	$\frac{2i}{\sqrt{5}}\begin{pmatrix} -\sin(\frac{2\pi}{5}) & -\sin(\frac{\pi}{5}) \\ -\sin(\frac{\pi}{5}) & \sin(\frac{2\pi}{5}) \end{pmatrix}$	$\mathrm{diag}(\frac{2}{5},\frac{3}{5})$
2^3	$N_3(\chi)_+$	$\frac{i}{\sqrt{2}}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$	$\mathrm{diag}(\frac{3}{8},\frac{5}{8})$

Similarly, one obtains as a complete list of three dimensional irreducible level p^λ representations

$$\begin{array}{ll} p^\lambda = 3^1, & N_1(\chi_1) \\ p^\lambda = 5^1, & R_1(1,\chi_1),\ R_1(2,\chi_1) \\ p^\lambda = 7^1, & R_1(1,\chi_{-1}),\ R_1(2,\chi_{-1}) \\ p^\lambda = 2^2, & D_2(\chi)_+ \otimes C_j \\ p^\lambda = 2^3, & R_3^0(1,3,\chi)_+ \otimes C_j,\ R_3^0(1,3,\chi)_- \otimes C_j \\ p^\lambda = 2^4, & R_4^0(1,1,\chi)_+ \otimes C_j,\ R_4^0(1,1,\chi)_- \otimes C_j, \\ & R_4^0(3,1,\chi)_+ \otimes C_j,\ R_4^0(3,1,\chi)_- \otimes C_j \\ \text{where } j = 1,\dots,4. & \end{array}$$

The explicit form of the representations which are not related by tensor products with C_j is given in Table A2.

Table A2: Three dimensional irreducible level p^λ representations

level	type of rep.	$\rho(S)$	$\frac{1}{2\pi i}\log(\rho(T))$
3	$N_1(1,\chi_1)$	$\frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$	$\mathrm{diag}(\frac{1}{3},\frac{2}{3},0)$
5	$R_1(1,\chi_1)$	$\frac{2}{\sqrt{5}}\begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -s_1 & s_2 \\ \frac{1}{\sqrt{2}} & s_2 & -s_1 \end{pmatrix}$	$\mathrm{diag}(0,\frac{1}{5},\frac{4}{5})$
	$R_1(2,\chi_1)$	$\frac{2}{\sqrt{5}}\begin{pmatrix} -\frac{1}{2} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -s_2 & s_1 \\ -\frac{1}{\sqrt{2}} & s_1 & -s_2 \end{pmatrix}$ $s_j = \cos(\frac{j\pi}{5})$	$\mathrm{diag}(0,\frac{2}{5},\frac{3}{5})$
7	$R_1(1,\chi_{-1})$	$\frac{2}{\sqrt{7}}\begin{pmatrix} s_1 & s_2 & s_3 \\ s_2 & -s_3 & s_1 \\ s_3 & s_1 & -s_2 \end{pmatrix}$	$\mathrm{diag}(\frac{2}{7},\frac{1}{7},\frac{4}{7})$
	$R_1(2,\chi_{-1})$	$- -$ " $- -$ $s_j = \sin(\frac{j\pi}{7})$	$\mathrm{diag}(\frac{5}{7},\frac{6}{7},\frac{3}{7})$
2^2	$D_2(\chi)_+$	$\frac{i}{2}\begin{pmatrix} 0 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & -1 & 1 \\ \sqrt{2} & 1 & -1 \end{pmatrix}$	$\mathrm{diag}(\frac{1}{4},\frac{1}{2},0)$
2^3	$R_3^0(1,3,\chi)_+$	$\frac{i}{2}\begin{pmatrix} 0 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & 1 & -1 \\ \sqrt{2} & -1 & 1 \end{pmatrix}$	$\mathrm{diag}(\frac{1}{2},\frac{5}{8},\frac{1}{8})$
	$R_3^0(1,3,\chi)_-$	$(-1)\cdot(- -$ " $- -)$	$\mathrm{diag}(\frac{1}{2},\frac{7}{8},\frac{3}{8})$
2^4	$R_4^0(1,1,\chi)_+$	$\frac{i}{2}\begin{pmatrix} 0 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & 1 & -1 \\ \sqrt{2} & -1 & 1 \end{pmatrix}$	$\mathrm{diag}(\frac{5}{8},\frac{1}{16},\frac{9}{16})$
	$R_4^0(1,1,\chi)_-$	$- -$ " $- -$	$\mathrm{diag}(\frac{1}{8},\frac{5}{16},\frac{13}{8})$
	$R_4^0(3,1,\chi)_+$	$\frac{i}{2}\begin{pmatrix} 0 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & -1 & 1 \\ \sqrt{2} & 1 & -1 \end{pmatrix}$	$\mathrm{diag}(\frac{7}{8},\frac{3}{16},\frac{11}{16})$
	$R_4^0(3,1,\chi)_-$	$- -$ " $- -$	$\mathrm{diag}(\frac{3}{8},\frac{15}{16},\frac{7}{16})$

Table A3: Four dimensional irreducible level p^λ representations

level	type of rep.	$\rho(S)$	$\frac{1}{2\pi i}\log(\rho(T))$
5	$N_1(\chi),\ \chi^3 \not\equiv 1$	$\frac{2i}{5}\begin{pmatrix} \eta_- & \sqrt{3}s_2 & \eta_+ & \sqrt{3}s_4 \\ \sqrt{3}s_2 & -\eta_+ & \sqrt{3}s_4 & \eta_- \\ \eta_+ & \sqrt{3}s_4 & -\eta_- & -\sqrt{3}s_2 \\ \sqrt{3}s_4 & \eta_- & -\sqrt{3}s_2 & \eta_+ \end{pmatrix}$ $s_j = \sin(\frac{j\pi}{5}),\ \eta_\pm = s_2 \pm s_4$	$\mathrm{diag}(\frac{3}{5}, \frac{4}{5}, \frac{2}{5}, \frac{1}{5})$
	$N_1(\chi),\ \chi^3 \equiv 1$	$-\frac{2}{5}\begin{pmatrix} \xi_1 & -\xi_2 & \xi_1 & -\xi_3 \\ -\xi_2 & -\xi_1 & \xi_3 & \xi_1 \\ \xi_1 & \xi_3 & \xi_1 & \xi_2 \\ -\xi_3 & \xi_1 & \xi_2 & -\xi_1 \end{pmatrix}$ $r_j = \cos(\frac{j\pi}{5}),\ \xi_1 = r_1 - r_4 - \frac{1}{2},$ $\xi_2 = 3r_2 + 2r_4,\ \xi_3 = 2r_2 + 3r_4$	$\mathrm{diag}(\frac{3}{5}, \frac{4}{5}, \frac{2}{5}, \frac{1}{5})$
7	$R_1(1,\chi_1)$	$\sqrt{\frac{2}{7}i}\begin{pmatrix} -\frac{1}{\sqrt{2}} & -1 & -1 & -1 \\ -1 & \xi_1 & \xi_2 & \xi_3 \\ -1 & \xi_2 & \xi_3 & \xi_1 \\ -1 & \xi_3 & \xi_1 & \xi_2 \end{pmatrix}$	$\mathrm{diag}(0, \frac{1}{7}, \frac{4}{7}, \frac{2}{7})$
	$R_1(2,\chi_1)$	$(-1)\cdot \quad (--\text{''}--)$ $s_j = \sqrt{\frac{2}{7}}\sin(\frac{j\pi}{7}),$ $\xi_1 = 2s_2 - s_4,\ \xi_2 = 2s_4 + s_6$ $\xi_2 = 2s_4 + s_6,\ \xi_3 = -2s_6 - s_2$	$\mathrm{diag}(0, \frac{6}{7}, \frac{3}{7}, \frac{5}{7})$
2^3	$N_3(\chi),\ \chi^3 \not\equiv 1$	$\frac{i}{\sqrt{8}}\begin{pmatrix} 1 & 1 & \sqrt{3}i & -s_1\sqrt{3}i \\ 1 & -1 & -\sqrt{3}i & -s_1\sqrt{3}i \\ -\sqrt{3}i & \sqrt{3}i & 1 & s_1 \\ s_2\sqrt{3}i & s_2\sqrt{3}i & s_2 & -1 \end{pmatrix}$ $s_j = e^{2\pi i \frac{j}{3}}$	$\mathrm{diag}(\frac{3}{8}, \frac{5}{8}, \frac{1}{8}, \frac{7}{8})$
3^2	$R_2^1(1,1,\chi),\ \chi^3 \equiv 1$	$\frac{2i}{3}\begin{pmatrix} -s_8 & -s_4 & -s_2 & -s_6 \\ -s_4 & s_2 & -s_8 & s_6 \\ -s_2 & -s_8 & s_4 & s_6 \\ -s_6 & s_6 & s_6 & 0 \end{pmatrix}$	$\mathrm{diag}(\frac{4}{9}, \frac{1}{9}, \frac{7}{9}, \frac{1}{3})$
	$R_2^1(2,1,\chi),\ \chi^3 \equiv 1$	$(-1)\ \cdot\ (--\text{''}--)$	$\mathrm{diag}(\frac{2}{9}, \frac{5}{9}, \frac{8}{9}, \frac{2}{3})$
	$R_2^1(1,1,\chi),\ \chi^3 \not\equiv 1$	$\frac{2}{3}\begin{pmatrix} s_1 & s_5 & s_7 & s_6 \\ s_5 & -s_7 & -s_1 & s_6 \\ s_7 & -s_1 & s_5 & -s_6 \\ s_6 & s_6 & -s_6 & 0 \end{pmatrix}$	$\mathrm{diag}(\frac{4}{9}, \frac{1}{9}, \frac{7}{9}, \frac{1}{3})$
	$R_2^1(2,1,\chi),\ \chi^3 \not\equiv 1$	$--\text{''}--$ $s_j = \sin(\frac{\pi j}{18})$	$\mathrm{diag}(\frac{5}{9}, \frac{8}{9}, \frac{2}{9}, \frac{2}{3})$

Similarly, one obtains as a complete list of four dimensional irreducible level p^λ representations

$$\begin{array}{ll}
p^\lambda = 5^1, & N_1(\chi)\ (\chi^3 \not\equiv 1), N_1(\chi)\ (\chi^3 \equiv 1), \\
p^\lambda = 7^1, & R_1(1,\chi_1),\ R_1(2,\chi_1) \\
p^\lambda = 2^3, & N_3(\chi),\ C_4 \otimes N_3(\chi) \\
p^\lambda = 3^2, & B_i \otimes R_2^1(1,1,\chi), B_i \otimes R_2^1(2,1,\chi)
\end{array}$$

where $i = 1,2,3$ and for $p^\lambda = 3^2$ the character χ is a primitive character of order 3 or 6 (so there are 12 four dimensional irreducible level 3^2 representations).

The explicit form of the representations which are not related by tensor products with C_j or B_i is given in Table A3.

8. Appendix B: Fusion matrices and graphs of the nondegenerate strong-modular fusion algebras of dimension less than 24

The fusion matrices $\mathcal{N}_1$ which define the distinguished basis of the simple non-degenerate strong-modular fusion algebras of dimension less than 24 are given by:

$$\mathbb{Z}_2: \qquad \mathcal{N}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$"(3,4)": \qquad \mathcal{N}_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$"(2,q)": \qquad \mathcal{N}_1 = \left.\begin{pmatrix} 0 & 1 & & & \\ 1 & \ddots & \ddots & & \\ & \ddots & & 0 & 1 \\ & & & 1 & 1 \end{pmatrix}\right\}\tfrac{q-1}{2}$$

$$\mathrm{B}_9: \qquad \mathcal{N}_1 = \begin{pmatrix} 0 & 1 & 0 & & & \\ 1 & 0 & 1 & 1 & & \\ 0 & 1 & 0 & 0 & 1 & \\ & 1 & 0 & 1 & 1 & 0 \\ & & 1 & 1 & 1 & 1 \\ & & & 0 & 1 & 1 \end{pmatrix}$$

$$\mathrm{B}_{11}: \qquad \mathcal{N}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & & & & & & \\ 1 & 0 & 1 & 0 & 0 & & & & & \\ 0 & 1 & 0 & 0 & 1 & 0 & & & & \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & & & \\ & 0 & 1 & 1 & 0 & 1 & 0 & 1 & & \\ & & 0 & 0 & 1 & 0 & 0 & 0 & 1 & \\ & & & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ & & & & 1 & 0 & 1 & 1 & 1 & 0 \\ & & & & & 1 & 0 & 1 & 1 & 1 \\ & & & & & & 0 & 0 & 1 & 1 \end{pmatrix}$$

and for G_{17} we have

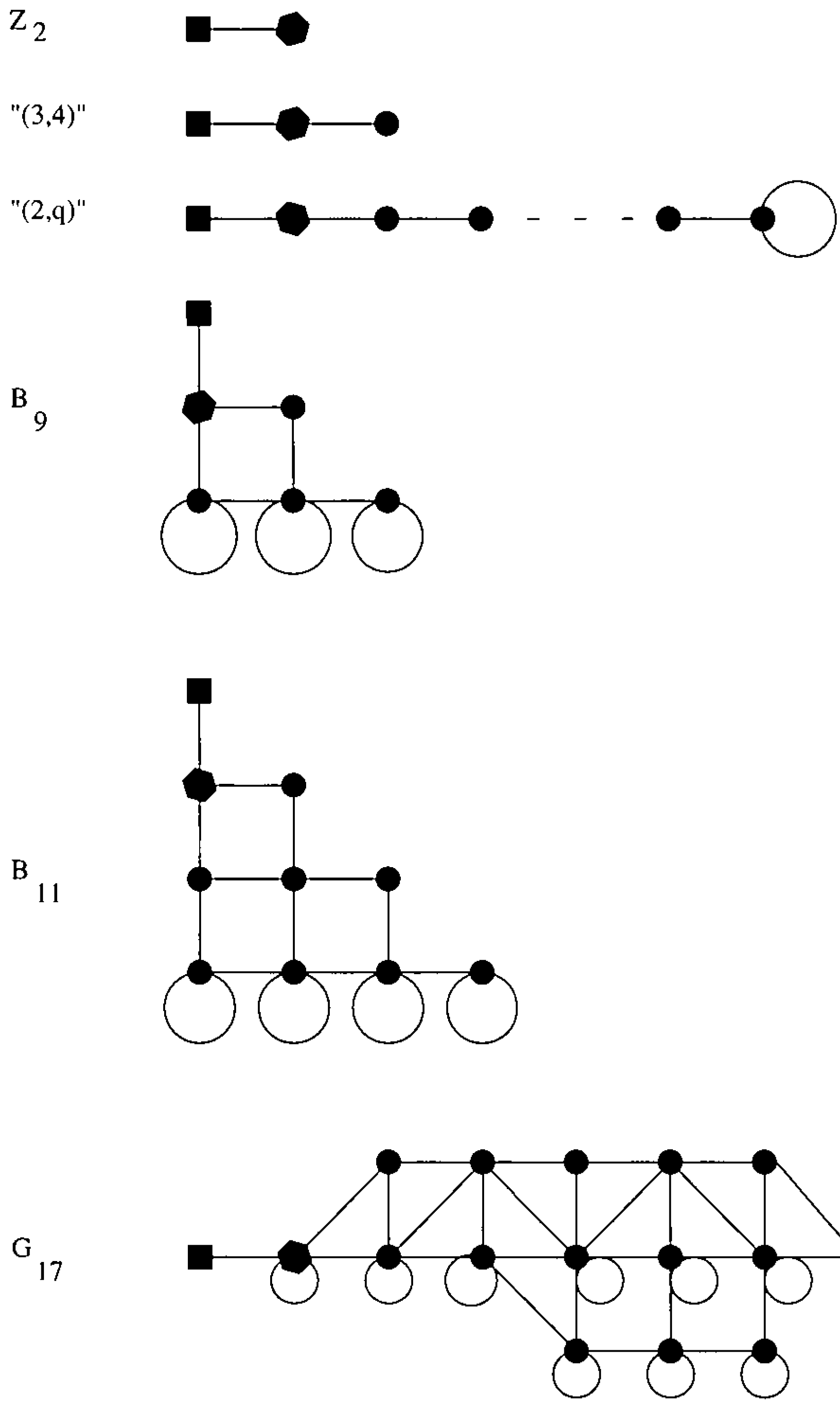
$$\mathcal{N}_1 = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & & \\
1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & & & & & & \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & & & & & \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & & & & \\
0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & & & \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
 & & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
 & & & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\
 & & & & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
 & & & & & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
 & & & & & & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
 & & & & & & & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}$$

and finally for E_{23} the matrix $\mathcal{N}_1$ is given by

$$\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & & \\
1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & & & & \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & & \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\
 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
 & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
 & & & & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
 & & & & & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\
 & & & & & & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
 & & & & & & & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}.$$

The fusion graphs corresponding to the fusion matrices $\mathcal{N}_1$ can be found on the next page.

= vacuum ("0"), = first field ("1"), = all other fields ("j", j=2,...,n-1)



We have omitted the fusion graph of the fusion algebra of type E_{23} since it is not possible to draw it without intersections in a plane.

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MAX-PLANCK-INSTITUT FÜR MATHEMATIK BONN, GOTTFRIED-CLAREN-STRASSE 26, 53225 BONN, GERMANY
E-mail address: eholzer@mpim-bonn.mpg.de

PHYSIKALISCHES INSTITUT DER UNIVERSITÄT BONN, NUSSALLEE 12, 53115 BONN, GERMANY
E-mail address: eholzer@avzw01.physik.uni-bonn.de