# ENERGY ESTIMATES AND LIOUVILLE THEOREMS

# FOR HARMONIC MAPS

by

Kensho Takegoshi

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
5300 Bonn 3
Federal Republic of Germany

Research Institute for
Mathematical Sciences
Kyoto University
Kyoto 606
Japan

MPI/88-9

# ENERGY ESTIMATES AND LIOUVILLE THEOREMS FOR HARMONIC MAPS

Kensho Takegoshi

## Introduction

This article consists of two parts. In the first section, we shall establish a method to estimate the energy of harmonic maps from a non-compact Kähler manifold into other Kähler manifolds. In spite of the importance of establishing such a method in function theory of several complex variables, up to now not much is known about the general method to estimate the energy of harmonic maps or even holomorphic maps of Kähler manifolds.

To estimate the energy of harmonic maps, our method requires that a given non-compact Kähler manifold $(M, ds_M^2)$ possesses an exhaustion function $\Phi \geq 0$ such that $\Phi$ is uniformly Lipschitz continuous and $\Phi^2$ is $C^\infty$ strongly hyper $m-1$ convex $(m = \dim_{\mathbb{C}} M)$ on $M$ relative to the Kähler metric $ds_M^2$ respectively (cf. the conditions (*) and (**) in Theorem 1) and the complex dimension $m$ of $M$ is greater than or equal to two. Fortunately there are several classes of non-compact Kähler manifolds possessing such a special exhaustion function.

From a given harmonic map $f : (M, ds_M^2) \to (N, ds_N^2)$ from a non-compact Kähler manifold $(M, ds_M^2)$ possessing the exhaustion function $\Phi$ as above into a Kähler manifold $(N, ds_N^2)$, we induce an integral inequality involving the energy $E(f,r)$ of $f$ on a sublevel set $M(r) = \{\Phi < r\}$ of $\Phi$ (cf. (1.13)), its derivative $\frac{\partial}{\partial r} E(f,r)$ and the integral $B(f,r)$ of the component of normal direction of the differential $df$ of $f$ on the boundary $\partial M(r) = \{\Phi = r\}$ (cf. (1.14)) and Lemma 1.18, (1.19)). This inequality is induced from an integral formula for vector bundle-valued differential forms on bounded domains with smooth boundary produced by Donnelly and Xavier (cf. [6] and Proposition 1.10) if $f$ is a pluriharmonic map. If $f$ is a harmonic map, then this inequality is induced by coupling the above integral formula with the semi-negativity of Riemannian curvature of the target manifold $(N, ds_N^2)$. In particular, we can obtain the integral inequality for harmonic functions on $(M, ds_M^2)$. This integral inequality plays the crucial role in this article. In fact, from this inequality, we can derive two energy estimates for the above $f$ which imply the monotone increasing property of $\frac{E(f,r)}{r^\mu}$. Here $\mu$ is the positive constant determined by the ratio of the lower bound of the strong hyper $m - 1$ convexity of $\Phi^2$ and the uniform Lipschitz constant of $\Phi$ relative to the Kähler metric $ds_M^2$.

For instance, we can obtain the following result as a corollary of our general result (cf. Theorem 1.27).

Theorem 1

Let $A \hookrightarrow \mathbb{C}^n$ be an $m \geq 2$ dimensional connected closed submanifold of $\mathbb{C}^n$ and let $\Phi$ be the restriciton of the function $\|z\| = \sqrt{\sum_{i=1}^n |z^i|^2}$, $z = (z^1,\dots,z^n) \in \mathbb{C}^n$ onto $A$ $(0 \notin A)$.

Suppose for a given Kähler metric $ds_A^2$ on $A$ the number $c_1$ defined by

$$(*) \qquad c_1 := \inf_{x \in A} \sum_{i=2}^m \epsilon_i(x)$$

is positive where $\epsilon_1 \geq \epsilon_2 \geq \dots \geq \epsilon_m$ are the eigen-values of the Levi form of $\Phi^2$ relative to $ds_A^2$ and the number $c_2$ defined by

$$(**) \qquad c_2 := \sup_{x \in A} |\partial\Phi|^2_{ds_A^2}(x)$$

is finite. (For instance, if $ds_A^2$ is the induced metric of Euclidean metric $ds_e^2$ of $\mathbb{C}^n$, then we can take $c_1 = m - 1$ and $c_2 = \frac{1}{2}$).

Then the energy $E(f,r)$ of any non-constant pluriharmonic map $f : (A, ds_A^2) \longrightarrow (N, ds_N^2)$ into a Kähler manifold $(N, ds_N^2)$ on $A(r) = \{\Phi < r\}$ possesses the following properties.

The function $H(f,r) = \frac{E(f,r)}{r^{\mu}}$ $(\mu = \frac{c_1}{c_2})$ is an increasing function of $r$ and the following estimates hold

$$H(f,r_2) - H(f,r_1) \geq \int_{r_1}^{r_2} \frac{B(f,t)}{t^{\mu}}\, dt$$

and

$$H(f,r_2) \geq H(f,r_1) \exp\left(\int_{r_1}^{r_2} \frac{B(f,t)}{E(f,t)}\, dt\right)$$

for any $r_2 > r_1 > \inf_{x \in A} \Phi(x)$.

Moreover the energy $E(f,r)$ of any non-constant harmonic map $f : (A,ds_A^2) \longrightarrow (N,ds_N^2)$ into a Kähler manifold $(N,ds_N^2)$ whose Riemannian curvature is semi-negative in the sense of Siu (cf. [22]) possesses the above properties.
In particular, the energy $E(f,r)$ of any non-constant harmonic function $f$ on $(A,ds_A^2)$ possesses the above properties.

Remark 1

In Theorem 1, if we replace the above $(A,ds_A^2)$ and $\Phi$ by an $m \geq 2$ dimensional complete Kähler manifold $(M,ds_M^2)$ with a pole $0 \in M$ whose radial curvature is non-positive and the distance function from $0 \in M$ relative to $ds_M^2$ respectively, then the same conclusion as Theorem 1 holds (cf. §1. Example 4). When $\dim_{\mathbb{C}} A = 1$ in Theorem 1, the condition (*)

is meaningless. But assuming the condition (**), we can obtain the above estimates for $\mu = 0$ and any non-constant differentiable map $f : (A, ds_A^2) \longrightarrow (N, ds_N^2)$ into any hermitian complex manifold $(N, ds_N^2)$ since $\frac{\partial}{\partial r} E(f,r) \geq B(f,r)$ for almost all $r$ (cf. (1.14)). The former estimate in Theorem 1 is called the monotonicity formula in [17].

In the second section, as an application of the result obtained in the first section, we shall show Liouville theorems on non-compact Kähler manifolds possessing the exhaustion function as above under some additional condition i.e.

$\alpha$) a non-existence theorem for non-constant bounded harmonic functions

$\beta$) a Casorati-Weierstrass theorem for holomorphic maps

$\gamma$) a non-existence theorem for bounded strictly plurisubharmonic functions.

The study of these properties is deeply related to the study of global solutions of elliptic differential equations of second order on non-compact manifolds (cf. [3], [7], [8], [9], [11], [12], [16], [31] and so on). One of the typical methods to study Liouville theorem is what we call Bochner technique which shows the vanishing of certain geometric object by coupling Weitzenböck formula with either a curvature condition or a maximum principle (cf. [29]). In particular, this method plays an important role to study Liouville theorem on non-compact manifolds with non-negative curvature (cf. [2], [4], [14], [30]). But this method is useless to non-compact manifolds with non-positive curvature. This is a motivation which an integral formula for differential forms was introduced in [6] to examine the dimension of $L^2$ harmonic forms on non-compact complete Riemannian manifolds with negative curvature (cf. also [5]).

The following theorems show that our method based on energy estimates for harmonic maps can be used to study Liouville theorem on non-compact Kähler manifolds with (asymptotically) non-positive curvature.

Theorem 2

Let $(A, ds_A^2) \overset{\iota}{\hookrightarrow} (\mathbb{C}^n, ds_e^2)$ be an $m \geq 1$ dimensional connected closed submanifold of $\mathbb{C}^n$ provided with the induced metric $ds_A^2 = \iota^* ds_e^2$ and let $\Phi$ be the restriction of $\|z\|$ onto $A$.

Suppose the function $n(A,r) = \dfrac{\mathrm{Vol}(A(r))}{r^{2m}}$ satisfies

$$\int_{\delta}^{\infty} \frac{dt}{t\,n(A,t)} = \infty \quad \text{for some } \delta > 0 \ .$$

Then $\alpha$) $(A, ds_A^2)$ admits no non-constant bounded harmonic functions.

$\beta$) Let $f : A \longrightarrow M$ be a holomorphic map into a projective algebraic variety $M$ with a very ample line bundle $L$. If the set $E_f(L) := \{\sigma \in \mathbb{P}(\Gamma(M,L)) : \mathrm{Im} f \cap \mathrm{supp}(\sigma) = \phi\}$ has positive measure, then $f$ is a constant map.

$\gamma$) Let $f : A \longrightarrow N$ be a holomorphic map into a complex manifold $N$. If $N$ admits a bounded strictly plurisubharmonic continuous function (cf. [18]), then $f$ is a constant map. In particular $A$ admits no bounded strictly plurisubharmonic continuous functions.

Theorem 3

Let $(M, ds_M^2)$ be an $m \geq 1$ dimensional complete Kähler manifold with a pole $0 \in M$ and let $\Phi$ be the distance function from $0 \in M$ relative to $ds_M^2$. Then the assertions $\alpha$), $\beta$) and $\gamma$) of Theorem 2 hold for $(M, ds_M^2)$ if the radial curvature of $ds_M^2$ satisfies one of the following conditions

(i) $$|\text{radial curvature at } x| \leq \frac{\epsilon}{(\Phi(x)+\eta)^2 \log(\Phi(x)+\eta)}$$

for a sufficiently small $\epsilon$, $0 < \epsilon = \epsilon_{m,\eta} < 1$,

$\eta > e$ and any $x \in M$.

(ii) The radial curvature of $ds_M^2$ is non-positive on $M$ and

$$0 \geq \text{radial curvature at } x \geq -\frac{\epsilon}{\Phi(x)^2 \log \Phi(x)}$$

for sufficiently small $\epsilon$, $0 < \epsilon = \epsilon_m < 1$ and any $x \in M\setminus M(r_0)$, $r_0 >> 1$.

Remark 2

In Theorem 2, it is known that $n(A,r)$ is a non-decreasing function of $r$ (cf. [20]). Moreover $n(A,r)$ is bounded if and only if $A$ is affine algebraic. This result is due to W. Stoll [25]. In this case, the assertions $\alpha$), $\beta$) and $\gamma$) are more or less known. But in the transcendental case i.e. $n(A,r)$ is unbounded, up to now there is only one result obtained by Sibony and Wong [21] in this direction. It is easy to construct examples of $A$ satisfying $\int_\delta^\infty (tn(A,t))^{-1}dt = \infty$ and being not affine algebraic (cf. [10] §1).

From Theorem 2, if $A \hookrightarrow \mathbb{C}^n$ admits a non-nonconstant bounded holomorphic function, then $\int_\delta^\infty (tn(A,t))^{-1}dt$ is finite. But we do not know whether for any given continuously increasing function $g : [0,\infty] \longrightarrow (0,\infty)$ with $\int_\delta^\infty (tg(t))^{-1}dt < +\infty$ there exists $A \hookrightarrow \mathbb{C}^n$ such that $n(A,r) = O(g(r))$ and $A$ admits a non-constant bounded holomorphic function. On the other hand for any given continuously increasing function $h : [0,\infty) \longrightarrow (0,\infty)$ we can construct $A \hookrightarrow \mathbb{C}^n$ such that $n(A,r) = O(h(r))$ and $A$ admits no non-constant bounded holomorphic functions.

Still if $\dim_{\mathbb{C}} A = 1$ and $\int_\delta^\infty (tn(A,t))^{-1}dt = \infty$, then it is known that $A$ is strongly parabolic i.e. $A$ admits no non-constant, non-negative and bounded subharmonic functions of class $C^2$. This property was proved by Karp (cf. [12] and also [3]). In account of the regularization of plurisubharmonic functions on Stein manifolds, we do not know whether $A$ admits no non-constant bounded plurisubharmonic functions under the conditions $\dim_{\mathbb{C}} A \geq 2$ and $\int_{\delta}^{\infty} (tn(A,t))^{-1}dt = \infty$ (cf. [21]).

Remark 3

In Theorem 3, if $\dim_{\mathbb{C}} M = 1$, then it is known that $(M, ds_M^2)$ satisfying the condition (i) or (ii) is conformally equivalent to the complex plane $(\mathbb{C}, dzd\bar{z})$ (cf. [9] Proposition 7.6). But in the case $\dim_{\mathbb{C}} M \geq 2$, we do not know whether $(M, ds_M^2)$ satisfying the condition (i) or (ii) for the sectional curvature of $ds_M^2$ is biholomorphic to the $m$ dimensional complex Euclidean space $(\mathbb{C}^m, ds_e^2)$ (cf. [9], [15]. [24]). In any case, by Hessian comparison theorem i.e. the estimate of solutions of Jacobi equations, we may say that Theorem 3 contains the case treated by Greene and Wu in [9] i.e. Theorem C (Quasi-isometry Theorem) (cf. [28] and Theorem 2.4).

Moreover it is not so difficult to see that $M$ admits no non-constant bounded plurisubharmonic functions in the case of Theorem 3, (ii) (cf. Remark 2.38). Recently H. Kaneko verified this property in the case of Theorem 3, (i). His method is probability theoretic.

This research was began when the author was in Research Institute for Mathematical Sciences, Kyoto University. The author thanks to Doctors A. Kasue, J. Noguchi and T. Ohsawa for their interests to this work and valuable discussions.

Especially the author expresses his hearty thanks to the hospitality and its very nice atmosphere of Max-Planck-Institut für Mathematik where the final step of this work was solved and completed.

## 1. Energy estimates for harmonic maps

Let $(M, ds_M^2)$ be an $m$ dimensional Kähler manifold with the metric tensor

$$ds_M^2 = 2\mathrm{Re}\sum_{i,j=1}^{m} g_{i\bar{j}}\,dz^i d\bar{z}^j \quad .$$

Fron now on, we always assume that $M$ is connected and non-compact.

On the space $C^{p,q}(M)$ of $C^\infty$ differential forms of $(p,q)$ type on $M$, the pointwise inner product is defined by

$$\langle u,v\rangle = 2^{p+q}\sum_{A_p, B_q} u_{A_p\bar{B}_q}\,\overline{v^{\bar{A}_p B_q}} \quad \text{for } u \text{ and } v \in C^{p,q}(M) \; .$$

The star operator $* : C^{p,q}(M) \longrightarrow C^{m-q,m-p}(M)$ relative to $ds_M^2$ is defined by

$$*u = C(m,p,q)\sum_{A_q, B_p} \mathrm{sign}\begin{bmatrix}1,\ldots,m\\ A_q A_{m-q}\end{bmatrix}\,\mathrm{sign}\begin{bmatrix}1,\ldots,m\\ B_p B_{m-p}\end{bmatrix}$$

$$\times \det(g_{i\bar{j}})\,u^{\bar{B}_p A_q} dz^{A_{m-q}} \wedge d\bar{z}^{B_{m-p}}$$

for $C(m,p,q) = (\sqrt{-1})^m (-1)^{\frac{1}{2}m(m-1)+pm}\, 2^{p+q-m}$ and $u \in C^{p,q}(M)$.

Using the star operator, the inner product on $C^{p,q}(M)$ is defined by

$$(u,v) = \int_M u \wedge * \bar{v} \quad \text{for} \quad u \quad \text{and} \quad v \in C^{p,q}(M).$$

The following relation holds

$$u \wedge * \bar{v} = \langle u,v \rangle dv_M.$$

Here $dv_M$ is the volume form of $M$ relative to $ds_M^2$ and is defined by

$$dv_M = \frac{\overset{m}{\wedge}\omega_M}{2^m m!}$$

for the Kähler form $\omega_M = \sqrt{-1} \sum_{i,j=1}^{m} g_{i\bar{j}} dz^i \wedge d\bar{z}^j$ of $ds_M^2$.

These formulae are used to determine the numerical coefficients of several integrals and operators which appear in this article.

Let $\Phi$ be a continuous function on $M$. Throughout this section, we assume the following conditions on $\Phi$

(1.1) $\Phi \geq 0$ and $\Psi := \Phi^2$ is of class $C^\infty$

(1.2) $\Phi$ is an exhaustion function of $M$ i.e. each sublevel set $M(r) := \{\Phi < r\}$ is relatively compact in $M$ for $r \geq 0$.

(1.3) $\Phi$ has only non-degenerate critical points outside a compact subset $K_*$ of $M$.

<u>Remark 1.4</u> The condition (1.3) is assumed to avoid complicated discussions and is sufficient for our purpose.

Under the condition (1.3), all critical points of $\Phi$ on $M\setminus K_*$ are isolated. Moreover if $r$ is a critical value of $\Phi$, $r > r_* := \sup_{x\in K_*} \Phi(x)$, then by (1.3), $\partial M(r) := \{\Phi = r\}$ is the union of a $2m-1$ dimensional submanifold made up of all the non-critical points in $\partial M(r)$ and a finite set of critical points. Let $x \in \partial M(r)$ be a non-critical point of $\Phi$. The volume element $dS_r$ of $\partial M(r)$ near $x$ is defined by

(1.5) $$dv_M = \frac{d\Phi}{|d\Phi|^2_{ds_M}} \wedge dS_r \quad .$$

We set

(1.6) $$\omega_r := \frac{dS_r}{|d\Phi|^2_{ds_M}} \quad .$$

For $u \in C^{s,t}(M)$, we denote by $e(u) : C^{p,q}(M) \longrightarrow C^{p+s,q+t}(M)$ the left multiplication operator by $u$ and denote by $e(u)^* : C^{p,q}(M) \longrightarrow C^{p-s,q-t}(M)$ the adjoint operator of $e(u)$ relative to the inner product $(\ ,\ )$ i.e. $e(u)^* = (-1)^{(p+q)(s+t-1)} * e(\bar{u}) *$ on $C^{p,q}(M)$.

Since $\Phi$ has only non-degenerate critical points on $M \setminus K_*$, Stokes theorem holds on $M[r] := \{\Phi \leq r\}$ for any $r > r_*$.

For a $C^\infty$ differential 1 form $\varphi$ on M, we have from (1.5) and (1.6)

$$(1.7) \qquad \int_{M(r)} d{*}\varphi = \int_{\partial M(r)} e(d\Phi)^{*}\varphi\omega_r \quad \text{for any} \quad r > r_* .$$

Here if $r$ is a critical value of $\Phi$, then the integral on the right hand-side is taken over the non-critical points of $\partial M(r)$.

For a given $C^\infty$ differential form

$$\varphi = \sum_{i=1}^{m} \varphi_i dz^i + \varphi_{\bar{i}}\, d\bar{z}^i$$

on M, we consider the tangent vector $\theta = (\theta^i, \theta^{\bar{i}})$ on M defined by $\theta^i = \sum_{j=1}^{m} g^{\bar{j}i}\varphi_{\bar{j}}$ and $\theta^{\bar{i}} = \sum_{j=1}^{m} g^{\bar{i}j}\varphi_j$ . We denote by $\nabla_i$ (resp. $\nabla_{\bar{i}}$) the i-th component of the covariant differentiation of type (1,0) (resp. (0,1)) relative to $ds_M^2$. Since $d{*}\varphi = 2(\sum_{i=1}^{m} \nabla_i\theta^i + \nabla_{\bar{i}}\theta^{\bar{i}})dv_M$, we have from (1.7)

$$(1.8) \quad 2\int_{M(r)} \Big(\sum_{i=1}^{m} \nabla_i \theta^i + \nabla_{\bar{i}} \theta^{\bar{i}}\Big) dv_M\Big) = \int_{\partial M(r)} e(d\Phi)^* \varphi \omega_r$$

for any $r > r_*$.

Let $f : (M, ds_M^2) \longrightarrow (N, ds_N^2)$ be a differentiable map into an $n$ dimensional Kähler manifolds $(N, ds_N^2)$ with the metric tensor $ds_N^2 = 2\mathrm{Re} \sum_{\alpha,\beta=1}^{n} h_{\alpha\bar{\beta}} dw^\alpha d\bar{w}^\beta$.

Let TM and TN be the complex tangent bundles of M and N respectively. Since the complexified differential df of f is regarded as an $f^*TN$-valued differential 1-form, we obtain an $f^*TN^{1,0}$-valued differential (1,0) form $\partial f$ and an $f^*TN^{1,0}$-valued differential (0,1) form $\bar{\partial} f$ by composing the mapping $\Pi^{1,0} \circ df : TM \longrightarrow TN^{1,0}$, $\Pi^{1,0} : TN \longrightarrow TN^{1,0}$ being the projection, with the inclusions $TM^{1,0}$, into TM and $TM^{0,1}$ into TM respectively (cf. [7]). Then the form $\partial f$ (resp. $\bar{\partial} f$) is represented by $(f_i^\alpha)$ (resp. $(f_{\bar{i}}^\alpha)$) locally where $f_i^\alpha = \dfrac{\partial f^\alpha}{\partial z^i}$ and so on.

The energy density $e(f)$ of $f$ is defined by

$$e(f) := e'(f) + e''(f)$$

$$e'(f) := h_{\alpha\bar{\beta}}(f)\, g^{\bar{j}i} f_i^\alpha \overline{f_j^\beta} \quad \text{and} \quad e''(f) := h_{\alpha\bar{\beta}}(f) g^{\bar{j}i} f_{\bar{j}}^\alpha \overline{f_{\bar{i}}^\beta}.$$

We denote by $\mathcal{L}(\Psi)$ the Levi form of $\Psi = \Phi^2$. We define an $f^*TN^{1,0}$-valued differential $(1,0)$ form $\mathcal{L}(\Psi)(\partial f)$ and an $f^*TN^{1,0}$-valued differential $(0,1)$ form $\mathcal{L}(\Psi)(\bar{\partial} f)$ as follows:

$$\mathcal{L}(\Psi)(\partial f) = \Big( \sum_{i,j,k=1}^{m} g^{\bar{j}k}\, \Psi_{i\bar{j}} f_k^{\alpha} dz^i \Big)_{1 \leq \alpha \leq n}$$

(1.9)

$$\mathcal{L}(\Psi)(\bar{\partial} f) = \Big( \sum_{i,j,k=1}^{m} g^{\bar{k}j}\, \Psi_{j\bar{i}}\, f_{\bar{k}}^{\alpha} d\bar{z}^i \Big)_{1 \leq \alpha \leq n}.$$

Here $\Psi_{i\bar{j}} = \dfrac{\partial^2 \Psi}{\partial z^i \partial \bar{z}^j}$

We denote by $\nabla_{1,0}$ (resp. $\nabla_{0,1}$) the covariant differentiation of type $(1,0)$ (resp. $(0,1)$) induced from the connection on $T^*M \otimes f^*TN$ relative to $ds_M^2$ and $f^*ds_N^2$. The exterior differentiation $D_{1,0}\colon C^{p,q}(M, f^*TN) \longrightarrow C^{p+1,q}(M, f^*TN)$ (resp. $D_{0,1}\colon C^{p,q}(M, f^*TN) \longrightarrow C^{p,q+1}(M, f^*TN)$) is defined by $\nabla_{1,0}$ (resp. $\nabla_{0,1}$). We denote by $D_{1,0}^*\colon C^{p,q}(M, f^*TN) \longrightarrow C^{p-1,q}(M, f^*TN)$ (resp. $D_{0,1}^*\colon C^{p,q}(M, f^*TN) \to C^{p,q-1}(M, f^*TN)$) the formal adjoint operator of $D_{1,0}$ (resp. $D_{0,1}$) (cf. [7]).

Here $C^{p,q}(M, f^*TN)$ denotes the space of $f^*TN$-valued $C^\infty$ differential forms of $(p,q)$ type.

Let $f : (M, ds_M^2) \longrightarrow (N, ds_N^2)$ be a differentiable map into a Kähler manifold $(N, ds_N^2)$. Then the following two formulae hold (cf. [6], [26]).

<u>Proposition 1.10</u>

(i) For any non-critical value $r$ of $\Phi$

(1.11)

$$\int_{M(r)} [2(\mathrm{Trace}_{ds_M^2} \mathcal{L}(\Psi) e(f) - \langle \mathcal{L}(\Psi)(\partial f), \partial f \rangle_{f^*TN} - \langle \mathcal{L}(\Psi)(\bar\partial f), \bar\partial f \rangle_{f^*TN})$$

$$+ \langle e(\partial \Psi)^* \partial f, D_{1,0}^* \partial f \rangle_{f^*TN} + \langle D_{0,1}^* \bar\partial f, e(\bar\partial \Psi)^* \bar\partial f \rangle_{f^*TN}$$

$$+ \langle e(\bar\partial \Psi)\ \partial f, D_{0,1} \partial f \rangle_{f^*TN} + \langle D_{1,0} \bar\partial f, e(\partial \Psi) \bar\partial f \rangle_{f^*TN}] dv_M$$

$$= 2r( \int_{\partial M(r)} [ |\partial \Phi|_M^2 e(f) - |e(\partial \Phi)^* \partial f|^2_{f^*TN} - |e(\bar\partial \Phi)^* \bar\partial f|^2_{f^*TN} ] \omega_r )$$

(ii)

(1.12)

$$\langle \partial f, (D_{0,1}^* D_{0,1} - D_{1,0} D_{1,0}^*)(\partial f) \rangle_{f^*TN} + \langle (D_{1,0}^* D_{1,0} - D_{0,1} D_{0,1}^*)(\bar\partial f), \bar\partial f \rangle_{f^*TN}$$

$$= - \sum R^N_{\alpha\bar\beta\gamma\bar\delta}(f) (f^{\alpha}_{\bar i} \overline{f^{\beta}_{j}} - f^{\alpha}_{\bar j} \overline{f^{\beta}_{i}}) \overline{(f^{\delta, i} \overline{f^{\gamma, \bar j}} - f^{\delta, j} \overline{f^{\gamma, \bar i}})}$$

where $\langle\ ,\ \rangle_{f^*TN}$ is the pointwise inner product on the space $C^{p,q}(M, f^*TN)$ of $f^*TN$-valued $C^\infty$ differential forms of $(p,q)$ type relative to $ds_M^2$ and $f^* ds_N^2$ and $R^N_{\alpha\bar\beta\gamma\bar\delta}$ is the Riemannian curvature tensor of $ds_N^2$.

Proof

(i) We consider the following differential 1 forms

$$\varphi_1 := e'(f)\bar{\partial}\Psi$$

$$\varphi_2 := \frac{1}{2}\sum_{\alpha,\beta,i} h_{\alpha\bar{\beta}}(f)\ (e(\partial\Psi)^*\partial f)^{\alpha}\overline{f^{\beta}_{i}}d\bar{z}^{i}$$

$$\varphi_3 := e''(f)\bar{\partial}\Psi$$

$$\varphi_4 := \frac{1}{2}\sum_{\alpha,\beta,i} h_{\alpha\bar{\beta}}(f) f^{\alpha}_{\bar{i}}\ \overline{(e(\bar{\partial}\Psi)^*\bar{\partial}f)^{\beta}}d\bar{z}^{i}.$$

Using $\varphi_k$, we define the tangent vecotors $\theta_k = (\theta^i_k, \theta^{\bar{i}}_k = 0)$ as before. We choose holomorphic normal coordinate systems $(z^i)$ around $x \in M$ and $(w^{\alpha})$ around $y = f(x) \in N$ i.e. $g_{i\bar{j}}(x) = \delta_{ij}$, $dg_{i\bar{j}}(x) = 0$ and $h_{\alpha\bar{\beta}}(y) = \delta_{\alpha\beta}$, $dh_{\alpha\bar{\beta}}(y) = 0$ respectively. Then all the Christofell symbols of $ds^2_M$ and $ds^2_N$ vanish at x and y respectively. Using these coordinate systems, the integral of the left hand-side of (1.11) can be obtained by calculating $\sum_{i=1}^{m} \nabla_i(\theta^i_1 - \theta^i_2 + \theta^i_3 - \theta^i_4)$ pointwise (cf. [26] Proposition 1.14). Substituting $\theta_1 - \theta_2 + \theta_3 - \theta_4$ and $\varphi_1 - \varphi_2 + \varphi_3 - \varphi_4$ into the left hand-side and right hand-side of (1.8) respectively, we obtain the formula (1.11).

(ii) For any point $x \in M$ and $y = f(x) \in N$, we fix the above holomorphic normal coordinate systems. Then all the Christofell symbols of $ds_M^2$ and $ds_N^2$ vanish at $x$ and $y$ respectively and it holds that $R^N_{\alpha\bar{\beta}\gamma\bar{\delta}} = \partial_\gamma \partial_{\bar{\delta}} h_{\alpha\bar{\beta}}$ at $y$. Using these properties, the formula (1.12) follows from a routine calculation.

q.e.d.

We denote $M(r_2, r_1) = \{r_1 < \Phi < r_2\}$ for $r_2 > r_1 > 0_* := \inf_{x \in M} \Phi(x)$ and $M(r, 0_*) = M(r)$ for $r > 0_*$.

For a differentiable map $f : (M, ds_M^2) \longrightarrow (N, ds_N^2)$ of Kähler manifolds, the energy $E(f, r_2, r_1)$ of $f$ on $M(r_2, r_1)$ is defined by

(1.13) $$E(f, r_2, r_1) := \int_{M(r_2, r_1)} e(f) dv_M.$$

We set $E(f, r) = E(f, r, 0_*)$ for $r > 0_*$. For some positive constant $c_0 > 0$, we set

(1.14) $$B(f, r) = c_0 \int_{\partial M(r)} [|e(\partial\Phi)^* \partial f|^2_{f^*TN} + |e(\bar{\partial}\Phi)^* \bar{\partial} f|^2_{f^*TN}] \omega_r$$

for $r > r_*$.

If $r$ is a critical value of $\Phi$, then the integral on the right hand-side of (1.14) is taken over the non-critical points of $\partial M(r)$. It is easily verified that $B(f,r)$ is finite and a continuous function of $r > r_*$ (cf. [8] p. 275).

Definition 1.15. A differentiable map $f : (M, ds_M^2) \longrightarrow (N, ds_N^2)$ of Kähler manifolds is called harmonic if $f$ satisfies the following equation

$$\mathrm{Trace}_{ds_M^2} \nabla_{1,0} \bar{\partial} f = 0$$

and $f$ is called pluriharmonic if

$$\nabla_{1,0} \bar{\partial} f = 0.$$

Clearly, any pluriharmonic map of Kähler manifolds is harmonic and any holomorphic map of Kähler manifolds is pluriharmonic.

From now on, we assume that the complex dimension $m$ of $M$ is greater than or equal to two and moreover assume the following conditons on $\Phi$.

(1.16) the constant $c_1 := \inf_{x \in M \setminus K_{**}} \sum_{i=2}^{m} \epsilon_i(x)$ is

positive, where $\epsilon_1 \geq \epsilon_2 \geq \ldots \geq \epsilon_m$ are the eigenvalues of the Levi form of $\Psi = \Phi^2$ relative to $ds_M^2$ and $K_{**}$ is a compact subset of $M$.

(1.17) the constant $c_2 := \sup_{x \in M \setminus M[0_*]} |\partial\Phi|^2_{ds^2_M}(x)$

is finite.

We show the following lemma which plays the very important role in our article.

Lemma 1.18

Let $(M, ds^2_M)$ be an $m \geq 2$ dimensional connected non-compact Kähler manifold and let $\Phi$ be a function satisfying the conditions (1.1), (1.2), (1.3), (1.16) and (1.17).

(i) For any non-constant pluriharmonic map $f : (M, ds^2_M) \longrightarrow (N, ds^2_N)$ into an n dimensional Kähler manifold $(N, ds^2_N)$ and any non-critical value $r$ of $\Phi$, $r > \max(r_0, r_*)$, the following integral inequality holds

$$r \frac{\partial}{\partial r} E(f,r,r_0) - \mu E(f,r,r_0) \geq rB(f,r) \tag{1.19}$$

for $\mu = \frac{c_1}{c_2}$ and $c_0 = \frac{1}{c_2}$ in $B(f,r)$ (cf. (1.14)), where $r_0 > r_{**} := \sup_{x \in K_{**}} \Phi(x)$ if $K_{**} \neq \phi$ or $r_0 = 0_*$ if $K_{**} = \phi$.

(ii) For any non-constant harmonic map $f : (M, ds^2_M) \longrightarrow (N, ds^2_N)$ into an $n$ dimensional Kähler manifold $(N, ds^2_N)$ whose Riemannian curvature $R^N_{\alpha\bar{\beta}\gamma\bar{\delta}}$ is semi-negative in the sence of Siu [22], i.e.

$$(1.20)\qquad R^N_{\alpha\bar{\beta}\gamma\bar{\delta}}(y)(A^\alpha\overline{B^\beta} - C^\alpha\overline{D^\beta})\overline{(A^\delta\overline{B^\gamma} - C^\delta\overline{D^\gamma})} \geq 0$$

for any $y \in N$ and complex numbers $A^\alpha$, $B^\beta$, $C^\gamma$ and $D^\delta$, the integral inequality (1.19) holds for any non-critical value $r$ of $\Phi$, $r > r_0 > \max(r_*, r_{**})$ where $r_0 = 0_*$ if $K_* = K_{**} = \phi$.

Proof

In the case $r_0 > 0_*$, we consider that $r_0$ is a fixed non-critical value of $\Phi$. To show the inequality (1.19), we should apply the integral formula (1.11) to the domain $M(r, r_0)$ for any non-critical value $r$ and the fixed non-critical value $r_0$ of $\Phi$, $r > r_0 > 0_*$. Since $M(r, r_0)$ has two boundaries $\partial M(r)$ and $\partial M(r_0)$, in this case two boundary integrals appear in (1.11). But the left hand-side of (1.11) is dominated by the boundary integral on $\partial M(r)$ because the boundary integral on $\partial M(r_0)$ is non-negative by Cauchy-Schwarz inequality.

Let $f : (M, ds^2_M) \longrightarrow (N, ds^2_N)$ be a non-constant pluri-harmonic map of Kähler manifolds. Then $f$ satisfies the following equations:

$$(1.21)\qquad D_{0,1}\partial f = D^*_{1,0}\partial f = D_{1,0}\bar{\partial} f = D^*_{0,1}\bar{\partial} f = 0.$$

If the compact set $K_{**}$ (cf. (1.16)) is empty, then we set $r_0 = 0_*$. Otherwise we fix a non-critical value $r_0$ of $\Phi$, $r_0 > r_{**}$.

By (1.11), (1.21) and the above consideration, we have for any non-critical value $r > \max(r_0, r_*)$

$$(1.22)\quad \int_{M(r,r_0)} (\mathrm{Trace}_{ds_M^2} \mathcal{L}(\Psi)e(f) - \langle \mathcal{L}(\Psi)(\partial f), \partial f\rangle_{f^*TN}$$

$$- \langle \mathcal{L}(\Psi)(\bar\partial f), \bar\partial f\rangle_{f^*TN}) dv_M$$

$$\leq r \int_{\partial M(r)} [|\partial\Phi|_M^2 e(f) - |e(\partial\Phi)^* \partial f|^2_{f^*TN} - |e(\bar\partial\Phi)^* \bar\partial f|^2_{f^*TN}]\omega_r.$$

For any point $x \in M \backslash K_{**}$ and $y = f(x) \in N$, we choose local coordinate systems $(z^i)$ around $x$ and $(w^\alpha)$ around $y$ so that $g_{i\bar j}(x) = \delta_{ij}$, $\Psi_{i\bar j}(x) = \epsilon_i(x)\delta_{ij}$ and $h_{\alpha\bar\beta}(y) = \delta_{\alpha\beta}$ respectively. From (1.9) and (1.16), we have at $x$

$$(1.23)\ \mathrm{Trace}_{ds_M^2} \mathcal{L}(\Psi)e(f) - \langle \mathcal{L}(\Psi)(\partial f), \partial f\rangle_{f^*TN} - \langle \mathcal{L}(\Psi)(\bar\partial f), \bar\partial f\rangle_{f^*TN}$$

$$= \sum_{\alpha=1}^{n} \sum_{i=1}^{m} \Big(\sum_{j=1}^{m} \epsilon_j(x) - \epsilon_i(x)\Big)\big(|f_i^\alpha(x)|^2 + |f_{\bar i}^\alpha(x)|^2\big)$$

$$\geq c_1 e(f)(x).$$

Then the inequality (1.19) follows from (1.14) $(c_0 = 1/c_2)$, (1.17), (1.22) and (1.23).

Next let $f : (M, ds_M^2) \longrightarrow (N, ds_N^2)$ be the non-constant harmonic map of Kähler manifolds given in (ii). Then $f$ satisfies the following equations

$$D_{1,0}^{*}\partial f = D_{0,1}^{*}\bar{\partial} f = 0. \tag{1.24}$$

If the compact sets $K_*$ and $K_{**}$ are empty, then we set $r_0 = 0_*$. Otherwise we fix a non-critical value $r_0$ of $\Phi$, $r_0 > \max(r_*, r_{**})$.

Since $D_{0,1}\partial f = D_{1,0}\bar{\partial} f$ (cf. [26] (1.8)), by (1.12), (1.24) and integration by parts, we have for any $r \geq r_0$

$$2(D_{1,0}\bar{\partial} f, D_{1,0}\bar{\partial} f)_{f^*TN, M(r)} \tag{1.25}$$

$$= - \int_{M(r)} \sum R^N_{\alpha\bar{\beta}\gamma\bar{\delta}}(f)(f^{\alpha}_{\bar{i}}\overline{f^{\beta}_{j}} - f^{\alpha}_{\bar{j}}\overline{f^{\beta}_{i}})\overline{(f^{\delta,i}\overline{f^{\gamma,\bar{j}}} - f^{\delta,j}\overline{f^{\gamma,\bar{i}}})}\, dv_M$$

$$+ \int_{\partial M(r)} [\langle e(\bar{\partial}\Phi)\partial f, D_{0,1}\partial f\rangle_{f^*TN} + \langle D_{1,0}\bar{\partial} f, e(\partial\Phi)\bar{\partial} f\rangle_{f^*TN}]\omega_r.$$

On the other hand, by (1.3) and Fubini theorem, we have

(1.26)

$$(e(\bar{\partial}\Psi)\partial f, D_{0,1}\partial f)_{f^*TN, M(r,r_0)} + (D_{1,0}\bar{\partial} f, e(\partial\Psi)\bar{\partial} f)_{f^*TN, M(r,r_0)}$$

$$= 2\int_{r_0}^{r} t\,dt \int_{\partial M(t)} [\langle e(\bar{\partial}\Phi)\partial f, D_{0,1}\partial f\rangle_{f^*TN} + \langle D_{1,0}\bar{\partial} f, e(\partial\Phi)\bar{\partial} f\rangle_{f^*TN}]\omega_t.$$

Combining (1.20) with (1.25) and (1.26), we can see that the integral (1.26) is non-negative. Hence from (1.11), (1.24) and the non-negativity of (1.26), we obtain (1.22) for the harmonic map $f$. Therefore we can obtain the inequality (1.19) similarly.

q.e.d.

From Lemma 1.18, we obtain the following energy estimates for harmonic maps.

Theorem 1.27

Let $(M, ds_M^2)$ be an $m \geq 2$ dimensional connected non-compact Kähler manifold possessing a function $\Phi$ which satisfies the conditions (1.1), (1.2), (1.3), (1.16) and (1.17)

(i) For any non-constant pluriharmonic map $f$: $(M, ds_M^2) \longrightarrow (N, ds_N^2)$ into an $n$ dimensional Kähler manifold $(N, ds_N^2)$ and any $r > \max(r_0, r_*)$, the function $H(f,r,r_0) := \dfrac{E(f,r,r_0)}{r^{\mu}}$ $(\mu = \dfrac{c_1}{c_2})$ is an increasing function of $r$ and the following estimates hold

$$(1.28)\qquad H(f,r_2,r_0) - H(f,r_1,r_0) \geq \int_{r_1}^{r_2} \frac{B(f,t)}{t^\mu}\, dt$$

$$(1.29)\qquad H(f,r_2,r_0) \geq H(f,r_1,r_0) \exp\left( \int_{r_1}^{r_2} \frac{B(f,t)}{E(f,t,r_0)}\, dt\right)$$

for any $r_2 > r_1 > \max(r_0,r_*)$ where $r_0 > r_{**}$ if $K_{**} \neq \phi$ or $r_0 = 0_*$ if $K_{**} = \phi$

(ii) For any non-constant harmonic map from $(M,ds_M^2)$ into an n dimensional Kähler manifold whose Riemannian curvature is semi-negative in the sense of Siu, the same conclusion as (i) holds for any $r > r_0$ and $r_2 > r_1 > r_0 > \max(r_*,r_{**})$, where $r_0 = 0_*$ if $K_* = K_{**} = \phi$.

In particular, the energy $E(f,r,r_0)$ of any non-constant harmonic function $f$ on $(M,ds_M^2)$ satisfies the above properties of (i).

Proof

We set ourselves in the situation of Lemma 1.18. In the case (i), we have only to show the estimates (1.28) and (1.29).

For any non-critical value $r$ of $\Phi$, $r_1 \leq r \leq r_2$, we have from (1.19)

$$(1.30)\qquad \frac{\partial}{\partial r}\left[\frac{E(f,r,r_0)}{r^\mu}\right] \geq \frac{B(f,r)}{r^\mu} \quad .$$

Since the set of critical values of $\Phi$ is discrete, integrating (1.30), we obtain (1.28).

Since $E(f,r,r_0) > 0$ for any $r > r_0$ (cf. [19] Theorem 1), we have from (1.19).

$$\frac{\mu}{r} + \frac{B(f,r)}{E(f,r,r_0)} \leq \frac{\partial}{\partial r} \log E(f,r,r_0) \quad . \tag{1.31}$$

Hence we obtain (1.29) by integrating (1.31). The case (ii) is proved quite similarly.

q.e.d.

Remark 1.32

In Theorem 1.27, when we want to estimate the energy of a given holomorphic map $f : (M_1, ds^2_{M_1}) \longrightarrow (M_2, ds^2_{M_2})$ of complex manifolds, it is sufficient to assume that the metric $ds^2_{M_i}$ is Kähler outside a compact subset of $M_i$ from the observation in the proof of Lemma 1.18. Moreover if $\mu > 1$, then it is easily verified that $E(f,r)/(r + 1)^{\mu}$ (i.e. $r_0 = 0_*$) is an increasing function of $r \geq r^*$ for some sufficiently large number $r^*$.

We call the function $\Phi$ in Theorem 1.27 a special exhaustion function of $M$ relative to $ds^2_M$. Here we point out some examples of Kähler manifold possessing such a special exhaustion function.

Example 1. An $m \geq 2$ dimensional complex Euclidean space $\mathbb{C}^m$ with Euclidean metric $ds_e^2$ has a special exhaustion function $\Phi = \|z\|$, $\|z\| = \sqrt{\sum_{i=1}^m |z^i|^2}$, $z = (z^1, \ldots, z^m) \in \mathbb{C}^m$. In this case, by $\omega_e = \sqrt{-1}\partial\bar{\partial}\Phi^2$, $c_1 = m - 1$ $(\epsilon_i \equiv 1)$ and $|\partial\Phi|^2_{ds_e^2} \equiv \frac{1}{2}$ on $\mathbb{C}^m \setminus \{0\}$ i.e. $c_2 = \frac{1}{2}$ and $K_* = K_{**} = \phi$. Hence $\mu = 2m - 2$. Moreover we can obtain (1.28) and (1.29) by equality.

Example 2. Let $(A, ds_A^2) \overset{\iota}{\hookrightarrow} (\mathbb{C}^n, ds_e^2)$ be an $m \geq 2$ dimensional connected closed submanifold of $\mathbb{C}^n$ provided with the induced metric $ds_A^2 = \iota^* ds_e^2$. If necessary, translating $(z^i) = (w^i - a^i)$, $a = (a^1, \ldots, a^n) \in \mathbb{C}^n \setminus A$, we may assume that the restriction $\Phi$ of $\|z\|$ onto $A$ has only non-degenerate critical points. $\Phi$ is a special exhaustion function relative to $ds_A^2$. In fact we have $\omega_A = \sqrt{-1}\,\partial\bar{\partial}\Phi^2$, $c_1 = m - 1$, $c_2 = \frac{1}{2}$ and $K_* = K_{**} = \phi$. Hence $\mu = 2m - 2$.

Since every Stein manifold $S$ can be realized as a closed submanifold of some $\mathbb{C}^n$ by a proper holomorphic map $h : S \hookrightarrow \mathbb{C}^n$, $S$ has a special exhaustion function $\Phi = h^*(\|z\|)$ relative to the Kähler metric $ds_S^2 = h^*(ds_e^2)$ and $\mu = 2m - 2$ if $\dim_{\mathbb{C}} S \geq 2$.

Example 3. Let $M$ be an $m \geq 2$ dimensional strongly pseudoconvex manifold and let $j : M \longrightarrow R$ be the Remmert reduction of $M$. Since $R$ is a normal Stein space with finitely many isolated singularities, we can embed $R$ into some $\mathbb{C}^n$ by a proper holomorphic map $h : R \hookrightarrow \mathbb{C}^n$. We set $\Phi = (h \circ j)^*(\|z\|)$. Since $j$ is biholomorphic outside a compact set of $M$, we can construct a hermitian metric $ds^2_M$ on $M$ whose fundamental form $\omega_M$ can be written $\omega_M = \sqrt{-1}\partial\bar{\partial}\Phi^2$ outside a compact subset $K_*$ $(:= K_{**})$ of $M$. Hence $\Phi$ is a special exhaustion function of $M$ relative to $ds^2_M$ and $\mu = 2m - 2$.

Example 4. Let $(M, ds^2_M)$ be an $m \geq 2$ dimensional complete Kähler manifold with a pole $0 \in M$ i.e. $\exp_0 : TM_0 \longrightarrow M$ is a diffeomorphism and let $\Phi$ be the distance function from $0 \in M$ relative to $ds^2_M$. Then $\Phi$ is an exhaustion function and satisfies $|\partial\Phi|^2_{ds^2_M} \equiv \frac{1}{2}$ on $M \setminus \{0\}$ i.e. $c_2 = \frac{1}{2}$ and $K_* = \phi$. If the radial curvature of $ds^2_M$ is non-positive, then $\Psi = \Phi^2$ is a $C^\infty$ strictly plurisubharmonic function on $M$ i.e. $K_{**} = \phi$. Moreover $c_1 = m - 1$ (cf. [9] Propositions 1.17 and 2.24) and so $\mu = 2m - 2$. Hence $\Phi$ is a special exhaustion function of $M$ relative to $ds^2_M$. Moreover in this case, it should be noted that $\Phi$ is a special exhaustion function of $M$ relative to the Kähler metric induced from $\partial\bar{\partial}\Phi^2$ and $\mu = 2m - 2$.

Though according to each example, we can restate Theorem 1.27, we omit the detail here. In the present stage, Theorem 1 stated in the introduction is clear.

Remark 1.33

Originally Donnelly and Xavier established some integral formula for differential forms with compact supports. But we applied their formula to vector bundle-valued differential forms on bounded domains with smooth boundary. By the same way, we can establish energy estimates for harmonic functions on Riemannian manifolds with certain exhaustion function. But to establish such an energy estimate for a harmonic map $f : (M, ds_M^2) \longrightarrow (N, ds_N^2)$ of Riemannian manifolds, we should assume not only the non-positivity of Riemannian curvature of $ds_N^2$ but also the non-negativity of Ricci curvature of $ds_M^2$.

Remark 1.34

From the method used to induce the integral inequality (1.19), we can also induce the following equality and inequality which are used to show the analyticity of harmonic maps respectively.

1) Let $f : (M, ds_M^2) \longrightarrow (N, ds_N^2)$ be a harmonic map of compact Kähler manifolds. Then it holds that (cf. (1.12), (1.24) and (1.25))

$$(1.35) \qquad 2(D_{1,0}\bar{\partial}f, D_{1,0}\bar{\partial}f)_{f^{*}TN,M}$$

$$= - \int_M \sum R^N_{\alpha\bar{\beta}\gamma\bar{\delta}}(f)(f^{\alpha}_{\bar{i}}\overline{f^{\beta}_{\bar{j}}} - f^{\alpha}_{\bar{j}}\overline{f^{\beta}_{\bar{i}}})\overline{(f^{\delta,i}\overline{f^{\gamma,j}} - f^{\delta,j}\overline{f^{\gamma,i}})}dv_M$$

2) Let $D \subset M$ be a bounded domain with smooth boundary $\partial D$ defined by a $C^{\infty}$ strictly hyper $m-1$ convex function $\Phi$ on a neighborhood of $\partial D$ on an $m \geq 2$ dimensional Kähler manifold $(M, ds^2_M)$. Then there exists positive constants $C$ and $\delta$ such that

$$(1.36) \qquad \|\bar{\partial}f\|^2_{D(\delta)} \leq C \int_{\partial D} [\bar{\partial}_b f]^2 \frac{dS}{|d\Phi|_M}$$

for any harmonic function $f$ on $D$ which is of $C^1$ class on $\bar{D}$, where $D(\delta) = \{-\delta < \Phi < 0\}$ and

$$[\bar{\partial}_b f]^2 := |\bar{\partial}\Phi|^2_M e''(f) - |e(\bar{\partial}\Phi)^{*}\bar{\partial}f|^2 = 2\sum_{i<j} |\Phi_{\bar{i}}f^{j} - \Phi_{\bar{j}}f^{i}|^2$$

(by Lagrange equality).

The formula (1.35) yields the alternative proof of the analyticity of harmonic maps of compact Kähler manifolds and the formula (1.36) implies that $f$ is holomorphic on $D$ if $\bar{\partial}_b f = 0$ on $\partial D$ i.e. $f$ satisfies the tangential Cauchy-Riemann equation on $\partial D$. On those topics, the reader should be refered to [1], [22], [23].

## 2. Liouville theorems for harmonic maps

In this section, first we shall show two Liouville theorems for harmonic maps. Later using these theorems, we shall give the proofs of Theorems 2 and 3 stated in the introduction.

We first state the following theorems.

Theorem 2.1

Let $(M, ds_M^2)$ be an $m \geq 1$ dimensional connected non-compact Kähler manifolds possessing a function $\Phi$ which satisfies the conditions (1.1), (1.2), (1.3), (1.16) (here we set $c_1 = 0$ if $m = 1$) and (1.17) and let $V(r)$ be the volume of $M(r)$ relative to $ds_M^2$. Suppose there exists a continuous non-decreasing function $g : [0,\infty) \longrightarrow (0,\infty)$ such that

$$(2.2) \qquad \int_\delta^\infty \frac{dt}{tg(t)} = \infty \quad \text{for some } \delta > 0$$

and

$$(2.3) \qquad \limsup_{r\to\infty} \frac{n(M,r)}{g(r)} < +\infty$$

for $n(M,r) := \dfrac{V(r)}{r^{\mu+2}}$ and $\mu = \dfrac{c_1}{c_2}$.

Then $\alpha$) $(M, ds_M^2)$ admits no non-constant bounded harmonic functions.

$\beta$) Let $f : M \longrightarrow N$ be a holomorphic map into a projective algebraic variety $N$ with a very ample line bundle $L$. If the set $E_f(L) := \{\sigma \in \mathbb{P}(\Gamma(N,L)) : \mathrm{Im} f \cap \mathrm{supp}(\sigma) = \phi\}$ ($(\sigma)$ is the divisor defined by $\sigma$) has positive measure, then $f$ is a constant map.

$\gamma$) Let $f : M \longrightarrow N$ be a holomorphic map into a complex manifold $N$. If $N$ admits a bounded strictly plurisubharmonic continuous function in the sense of Richberg [18], then $f$ is a constant map. In particular, $M$ admits no bounded strictly plurisubharmonic continuous functions.

We introduce the following functions $g_n$ $(n \geq 0)$ defined by $g_n(r) := \prod_{i=0}^{n} L_i(r)$, $L_0(r) \equiv 1$, $L_1(r) = \log r$ and $L_{i+1}(r) = L_i(\log r)$ for $i \geq 1$. We should note that $\int_{1_n}^{\infty} \frac{dt}{t g_n(t)} = \infty$ for any $n \geq 0$ and some $1_n >> 0$.

Theorem 2.4

Let $(M, ds_M^2)$ be an $m \geq 1$ dimensional complete Kähler manifold with a pole $0 \in M$ and let $\Phi$ be the distance function from $0$ relative to $ds_M^2$. Suppose there exists a continuously increasing function $h : [r_*, \infty) \longrightarrow (1, \infty)$ such that

$$(2.5) \quad \max_{1 \leq i \leq m} |\epsilon_i(x) - 1| \leq \frac{1}{h(\Phi(x))} \quad \text{for any } x \in M \setminus M(r_*)$$

where $\epsilon_i$ are the eigenvalues of the Levi form of $\Psi = \Phi^2$ relative to $ds_M^2$ and

$$(2.6) \qquad \limsup_{r\to\infty} \frac{\exp((4m-2)\int_{r_*}^{r} \frac{dt}{th(t)})}{g_n(r)} < +\infty$$

for some $n \geq 0$.

Then the assertions $\alpha$), $\beta$) and $\gamma$) of Theorem 2.1 hold for the above $(M, ds_M^2)$.

Remark 2.7. In Theorem 2.4 from the condition (2.6), $h(r)$ is unbounded. When $\int_{r_*}^{\infty} \frac{dt}{th(t)} < +\infty$ i.e. $n = 0$, the assertion $\alpha$) has been verified in some cases (cf. [13], [28]).

Proof of Theorem 2.1

$\alpha$) Let $f : (M, ds_M^2) \longrightarrow (\mathbb{C}, dzd\bar{z})$ be a non-constant bounded harmonic function i.e. $\Delta_M f \equiv 0$ and $0 \leq |f| \leq C < +\infty$ for some $C > 0$.

We set ourselves in the situation of Lemma 1.18, (ii). First we obtain the following inequality.

$$(2.8) \qquad E(f,r,r_0)^2 \leq c \frac{\partial}{\partial r} V(r)\ B(f,r)$$

for any non-critical value $r$ of $\Phi$, $r > r_0$ and $c > 0$. By the harmonicity of $f$ and Stokes theorem, we have

$$2E(f,r,r_0) \leq (df,df)_{M(r)} = \int_{\partial M(r)} \langle f, e(d\Phi)^* df \rangle \omega_r$$

by the boundedness of $|f|$ $\leq c_3 \int_{\partial M(r)} |e(d\Phi)^* df| \omega_r$

by Chauchy-Schwarz inequality $\leq c_4 \left(\frac{\partial}{\partial r} V(r) B(f,r)\right)^{\frac{1}{2}}$.

Hence we have (2.8).

By (2.3), we have

(2.9) $\quad n(M,r) \leq c_5 g(r)$ for any $r > r_1^* > r_0$ .

We set $H(r) := \dfrac{E(f,r,r_0)}{r^\mu}$ for $r > r_0$. From (1.30) and (2.8), we have

$$H(r)^2 \leq c \frac{\frac{\partial}{\partial r} V(r) \frac{\partial}{\partial r} H(r)}{r^\mu} .$$

Hence we have

(2.10) $$\int_{r_1}^{r_2} \frac{t^\mu}{\frac{\partial}{\partial t} V(t)} dt \leq c_6 \left(\frac{1}{H(r_1)} - \frac{1}{H(r_2)}\right)$$

for any $r_2 > r_1 > r_0$.

By Chauchy-Schwarz inequality, we have

$$(2.11) \qquad (r_2 - r_1)^2 \leq \int_{r_1}^{r_2} \frac{\frac{\partial}{\partial t} V(t)}{t^\mu} dt \int_{r_1}^{r_2} \frac{t^\mu}{\frac{\partial}{\partial t} V(t)} dt$$

By (2.9), we have

$$(2.12) \qquad \int_{r_1}^{r_2} \frac{\frac{\partial}{\partial t} V(t)}{t^\mu} dt \leq c_7 r_2^2 g(r_2)$$

for any $r_2 > r_1 > r_1^*$.

From (2.10), (2.11) and (2.12), we have

$$(2.13) \qquad \frac{c_8}{g(r_2)} \left(1 - \frac{r_1}{r_2}\right)^2 \leq \frac{1}{H(r_1)} - \frac{1}{H(r_2)}$$

for any $r_2 > r_1 > r_1^*$. We consider a sequence $\{r_n\}_{n\geq 1}$ so that $r_{n+1} = 2r_n$ and $r_1 = 2r_1^*$. Substituting $r_1 = r_n$ and $r_2 = r_{n+1}$ into (2.13), we have

$$(2.14) \qquad \frac{c_9}{g(r_{n+1})} \leq \frac{1}{H(r_n)} - \frac{1}{H(r_{n+1})}$$

for any $n \geq 1$.

Hence we have from (2.14)

$$\int_{r_2}^{\infty} \frac{dt}{tg(t)} \leq \frac{c_{10}}{H(r_1)} < +\infty .$$

This contradicts to (2.2).

$\beta$) Let $N$ be a projective algebraic variety with a very ample line bundle $L$. We assume that $N$ is reduced and irreducible. The space $\Gamma(N,L)$ of global sections of $L$ is a finite dimensional vector space. We set $V := \Gamma(N,L)$ and $\dim_{\mathbb{C}} V = n + 1$.

Let $h : M \longrightarrow N$ be a holomorphic map into $N$ so that $E_h(L) := \{\sigma \in \mathbb{P}_n(V) : \mathrm{Im}\, h \cap \mathrm{supp}(\sigma) = \phi\}$ ($(\sigma)$ is the divisor defined by the section $\sigma$) has positive measure in $\mathbb{P}_n(V)$. We shall induce a contradiction by assuming that $h$ is non-constant.

Since $L$ is very ample, we have an embedding $j : N \hookrightarrow \mathbb{P}_n(V^*)$ into the $n$ dimensional projective space $\mathbb{P}_n(V^*)$ ($V^*$ is the dual space of $V$). We consider the holomorphic map $f := j \circ h : M \longrightarrow \mathbb{P}_n(V^*)$. Since $L = j^*H$ ($H$ is the hyperplane bundle over $\mathbb{P}_n(V^*)$) and $\mathbb{P}_n(V) = \mathbb{P}_n(V^*)^*$ (the dual projective space of $\mathbb{P}_n(V^*)$), setting $\mathbb{P}_n = \mathbb{P}_n(V^*)$, we may assume that $f : M \longrightarrow \mathbb{P}_n$ is non-constant and $E_f = \{\xi \in \mathbb{P}_n^* : \mathrm{Im} f \cap \mathrm{supp}(\xi) = \phi\}$ has positive measure in $\mathbb{P}_n^*$. Under this assumption, we have only to show the estimate (2.8) in account of the proof of $\alpha$).

Let $\sigma = (\sigma_0 : \sigma_1 : \ldots : \sigma_n) \in \mathbb{P}_n$ (resp. $\xi = (\xi^0 : \xi^1 : \ldots : \xi^n) \in \mathbb{P}_n^*$) be the homogeneous coordinates of $\mathbb{P}_n$ (resp. $\mathbb{P}_n^*$). We denote by $\omega$ (resp. $\omega^*$) the Kähler form of the Fubini study metric of $\mathbb{P}_n$ (resp. $\mathbb{P}_n^*$). We denote $<\sigma,\xi> = \sum_{i=0}^{n} \sigma_i \xi^i$ and $\|\sigma\|^2 = \sum_{i=0}^{n} |\sigma_i|^2$ for $\sigma \in \mathbb{P}_n$ and $\xi \in \mathbb{P}_n^*$. We define a positive function $\Lambda$ on $\mathbb{P}_n \times \mathbb{P}_n^*$ by

$$\Lambda(\sigma,\xi) := \frac{\|\sigma\|\|\xi\|}{|<\sigma,\xi>|} \quad \text{for } \sigma \in \mathbb{P}_n \text{ and } \xi \in \mathbb{P}_n^*.$$

It is easily verified that the function $\Lambda$ satisfies the following properties:

($\beta$.1) For any $\sigma \in \mathbb{P}_n$, the functions $\log \Lambda(\sigma, \ )$ and $\Lambda(\sigma, \ ) \geq 1$ are integrable on $\mathbb{P}_n^*$ and

$$A_1 := \int_{\xi \in \mathbb{P}_n^*} \log \Lambda(\sigma,\xi)\omega^{*n}$$

$$A_2 := \int_{\xi \in \mathbb{P}_n^*} \Lambda(\sigma,\xi)\omega^{*n}$$

are positive constants not depending on $\sigma \in \mathbb{P}_n$ ($\omega^{*k} = \overset{k}{\Lambda}\, \omega^*$ and so on).

($\beta$.2) For any subset $E \subset \mathbb{P}_n^*$ with $\int_E \omega^{*n} > 0$ if $f(\partial M(r)) \cap \mathrm{supp}(\xi) = \phi$ for any $\xi \in E$, then the functions

$\log f^{*}\Lambda$ and $f^{*}\Lambda$ are integrable on $\partial M(r) \times E$ (Here $(\xi)$ is the hyperplane defined by $\langle\sigma,\xi\rangle$).

($\beta$.3) There exists a positive constant $C_{*}$ not depending on $(\sigma,\xi) \in \mathbb{P}_n \times \mathbb{P}_n^{*}$ such that

$$|\partial_{\sigma} \log \Lambda(\sigma,\xi)|_{\omega} \leq C_{*}\Lambda(\sigma,\xi)$$

for any $\xi \in \mathbb{P}_n^{*}$ and any $\sigma \in \mathbb{P}_n \setminus \mathrm{supp}(\xi)$.

Using these properties of $\Lambda$, we show the estimate (2.8) for $f : M \longrightarrow \mathbb{P}_n$.

We set $\eta := \int_{E_f} \omega^{*n}$. By assumption, we have $\eta > 0$. For any $\xi \in \mathbb{P}_n^{*}$, it holds that

$$\text{(2.15)} \qquad \omega = 2dd_c \log \Lambda(\sigma,\xi) \quad \text{on} \quad \mathbb{P}_n \setminus \mathrm{supp}(\xi).$$

Here $d_c = i(\bar{\partial} - \partial)/2$. We set ourselves in the situation of Lemma 1.18, (i). Since $f$ is holomorphic, for any $\xi \in E_f$ and any non-critical value $r$ of $\Phi$, $r > r_0$,

$$E(f,r,r_0) = c \int_{M(r,r_0)} f^{*}\omega \wedge \omega_M^{m-1} \quad (c = c_m > 0)$$

by (2.15) $\qquad \leq 2c \int_{M(r)} dd_c \log f^{*}\Lambda(\sigma,\xi) \wedge \omega_M^{m-1}$

by Stokes theorem $\qquad = 2c \int_{\partial M(r)} d_c \log f^{*}\Lambda(\sigma,\xi) \wedge \omega_M^{m-1}$

by Chauchy-Schwarz inequality

$$\leq c_3 \int_{\partial M(r)} |\partial_\sigma \log \Lambda(f(z),\xi)|_\omega |e(\partial\Phi)^* \partial f|_{f^* T\mathbb{P}_n} \omega_r$$

by ($\beta$.3)

$$\leq c_4 \int_{\partial M(r)} \Lambda(f(z),\xi) |e(\partial\Phi)^* \partial f|_{f^* T\mathbb{P}_n} \omega_r$$

Hence by ($\beta$.2) and Fubini theorem we have

$$\eta\ E(f,r,r_0) \leq \int_{\partial M(r)} \left( \int_{\xi\in E_f} \Lambda(f(z),\xi)\omega^{*n}\right) |e(\partial\Phi)^* \partial f|_{f^* T\mathbb{P}_n} \omega_r$$

by ($\beta$.1)

$$\leq c_5 \int_{\partial M(r)} |e(\partial\Phi)^* \partial f|_{f^* T\mathbb{P}_n} \omega_r .$$

Therefore applying Chauchy-Schwarz inequality to the right hand-side, we have (2.8).

$\gamma$) Let $N$ be a complex manifold possessing a bounded strictly plurisubharmonic continuous function $\psi$ in the sense of Richberg [18]. By the approximation theorem [18], Satz 4.2, we may assume that $\psi$ is of class $C^\infty$ and bounded on $N$. Then $N$ admits a Kähler metric $ds_N^2$ whose Kähler form $\omega_N$ can be written as follows:

($\gamma$.1) There exists a $C^\infty$ function $\chi$ on $N$ such that

(i) $$\omega_N = \sqrt{-1}\ \partial\bar{\partial}\chi$$

(ii) $$0 \leq \chi \leq \log 2 \quad \text{and} \quad |\partial\chi|^2_{ds^2_N} \leq 2.$$

In fact, we may assume $\sup_{x \in N} \psi(x) = 0$. We set $\lambda := 1 - \frac{e^{\psi}}{2}$. Then $\frac{1}{2} \leq \lambda \leq 1$ and $-\lambda$ is strictly plurisubharmonic on $N$. Hence $\chi := -\log \lambda$ is strictly plurisubharmonic and $\partial\bar{\partial}\chi \geq \partial\chi \wedge \bar{\partial}\chi$ on $N$. Hence the assertion ($\gamma$.1) has been verified.

Let $f : (M, ds^2_M) \longrightarrow (N, ds^2_N)$ be a non-constant holomorphic map into $(N, ds^2_N)$. Then using ($\gamma$,1), we can obtain the estimate (2.8) for $f$ similarly to the case $\beta$).

This completes the proof of Theorem 2.1.

Proof of Theorem 2.4

To show this theorem in the case $m \geq 2$, we need to modify the way of energy estimates for harmonic maps in the first section.

We take a value $r_0$ of $\Phi$ with $r_0 > r_*$ so that $h(r_0) \geq 2$. First we show the following energy estimates for harmonic maps which are trivial in the one dimensional case (cf. Remark 1 in the introduction).

For any non-constant pluriharmonic map $f : (M, ds^2_M) \longrightarrow (N, ds^2_N)$ into a Kähler manifold $(N, ds^2_N)$ and any $r > r_1 > r_0$, it holds that

(2.16) $r^{2m-2}\exp(\chi_f(r) - (2m-2)\int_{r_1}^{r}\frac{dt}{th(t)} \leq cE(f,r,r_0)$

for $\chi_f(r) := \int_{r_1}^{r}\frac{B(f,t)}{E(f,t,r_0)}\,dt$ ($c_0 = \frac{1}{c_2}$ if $m = 1$ or $c_0 = 1$ if $m \geq 2$).

Moreover the same estimate as (2.16) holds for any non-constant harmonic function $f$ on $(M, ds_M^2)$.

We have only to show the case $m \geq 2$. Let $f : (M, ds_M^2) \longrightarrow (N, ds_N^2)$ be the non-constant pluriharmonic map as above. We set

$$E_*(f,r,r_0) := \int_{M(r,r_0)}(1 - \frac{1}{h(\Phi)})e(f)dv_M \quad \text{for any} \quad r > r_0.$$

Since $|\partial\Phi|_M^2 \equiv \frac{1}{2}$ on $M\setminus\{0\}$, by (1.22), (1.23) and (2.5), we have for any $r > r_0$

(2.17) $(2m-2)E_*(f,r,r_0) \leq \frac{rh(r)}{h(r)-1}\frac{\partial}{\partial r}E_*(f,r,r_0) - 2rB(f,r)$.

Since $h(r_0) \geq 2$, we have from (2.17)

(2.18) $(2m-2)(\frac{1}{r} - \frac{1}{rh(r)}) + \frac{B(f,r)}{E_*(f,r,r_0)} \leq \frac{\partial}{\partial r}\log E_*(f,r,r_0)$

for any $r > r_0$.

Since $E_*(f,r,r_0) \leq E(f,r,r_0)$, from (2.18), we obtain (2.16) for any $r > r_1 > r_0$. The proof of (2.16) for harmonic functions is now clear in view of the proof of Lemma 1.18, (ii).

Next we need the following estimates.

$$\text{(2.19)} \qquad \frac{\partial}{\partial r}V(r) \leq c_3\, r^{2m-1} \exp(2m \int_{r_1}^{r} \frac{dt}{th(t)})$$

$$\text{(2.20)} \qquad V(r) \leq c_4 r \frac{\partial}{\partial r} V(r)$$

for any $r > r_1 > r_0$.

By a standard calculation (cf. [8] p. 273-274), we have

$$\text{(2.21)} \qquad \frac{\partial}{\partial r} \int_{\partial M(r)} |d\Phi|_M^2 \omega_r = \int_{\partial M(r)} - \Delta_M \Phi \omega_r$$

for $\Delta_M = -4 \sum_{i,j=1}^{m} g^{\bar{j}i} \partial_i \partial_{\bar{j}}$ by the Kählerity of $ds_M^2$. Since $|d\Phi|_M \equiv 1$ on $M\setminus(0)$, by the assumption (2.5) and (2.21), we have for any $r > r_0$

$$\frac{\partial^2}{\partial r^2} V(r) \leq \frac{1}{r} (2m - 1 + \frac{2m}{h(r)}) \frac{\partial}{\partial r} V(r).$$

Hence we have (2.19).

Applying $\varphi = d\Phi^2$ to (1.7), we have

(2.22) $$\int_{M(r)} -\Delta_M \Phi^2 dv_M = 2r \int_{\partial M(r)} |d\Phi|_M^2 \omega_r .$$

By the assumption (2.5), $-\Delta_M \Phi^2$ is bounded from below some positive constant. Since $|d\Phi|_M \equiv 1$ on $M \setminus \{0\}$, we have (2.20).

At the present stage, we can begin the proofs of $\alpha$), $\beta$) and $\gamma$).

$\alpha$) Let $f$ be a non-constant bounded harmonic function on $(M, ds_M^2)$. Then we can obtain the following two inequalities.

(2.23) $$E(f,r,r_0)^2 \leq c_3 \frac{\partial}{\partial r} V(r)\ B(f,r)$$

(2.24) $$\frac{E(f,r,r_0)}{r^{2m-2}} \leq c_4 n(M,2r)$$

for any $r > 2r_0$.

Here $n(M,r) := \frac{V(r)}{r^{2m}}$. (2.23) is nothing but (2.8). Hence we have only to show (2.24). Since $\Phi$ is a uniformly Lipschitz continuous exhaustion function on $M$, by Stampaccia's inequality (cf. [27] Theorem 1.2), we have

(2.25) $$(df,df)_{M(r)} \leq \frac{c_5}{r^2} \int_{M(2r)} |f|^2 dv_M \quad \text{for any } r > 0.$$

Since $|f|$ is bounded, we have (2.24) from (2.25).

From (2.6), (2.16), (2.19) and (2.23), we have

$$\frac{c_6 e^{\chi_f(r)}}{r g_n(r)} \leq \frac{B(f,r)}{E(f,r,r_0)} \quad \text{for any} \quad r \geq r_1.$$

Hence we have

$$(2.26) \qquad c_6 \int_{r_1}^{r} \frac{e^{\chi_f(t)}}{t g_n(t)}\, dt \leq \chi_f(r) \quad \text{for any} \quad r > r_1.$$

From (2.26), we can obtain the following assertion inductively.

There exists positive constants $\{C_{(k)}\}_{0 \leq k \leq n}$ and a sequence of real numbers $\{r_{(k)}\}_{0 \leq k \leq n}$, $r_{(k)} < r_{(k+1)}$ and $r_{(0)} = r_1$ such that

$$C_{(k)} \int_{r_{(k)}}^{r} \frac{dt}{t g_{n-k}(t)} \leq \chi_f(r)$$

for any $r > r_{(k)}$ and $0 \leq k \leq n$.
Finally we obtain

$$(2.27) \qquad C_{(n)} \log r \leq \chi_f(r) + O(1) \quad \text{for any} \quad r > r_{(n)}.$$

On the other hand, from (2.6), (2.16), (2.19), (2.20) and (2.24), we have

(2.28) $\chi_f(r) \leq c_7 \log g_n(r) + 0(1)$ for any $r > r_1$.

From (2.27) and (2.28), we obtain a contradiction.

$\beta$) We set ourselves in the situation of the proof of Theorem 2.1, $\beta$). In account of the proofs of Theorem 2.1,$\beta$) and Theorem 2.4,$\alpha$), we have only to show the estimate (2.24) for the holomorphic map $f : M \longrightarrow \mathbb{P}_n$ in the proof of Theorem 2.1,$\beta$).

By $h(r_0) \geq 2$ and (2.5), $\Phi$ is subharmonic on $M \backslash M(r_0)$. For any $\xi \in E_f$ and any $r > r_0$,

$$\int_{r_0}^{r} E(f,t,r_0)dt = c \int_{r_0}^{r} dt \int_{M(t,r_0)} dd_c \log f^* \Lambda(\sigma,\xi) \wedge \omega_M^{m-1}$$

$$\leq c \int_{M(r,r_0)} d\Phi \wedge d_c \log f^* \Lambda(\sigma,\xi) \wedge \omega_M^{m-1}$$

$$= c \int_{M(r,r_0)} d \log f^* \Lambda(\sigma,\xi) \wedge d_c \Phi \wedge \omega_M^{m-1}$$

$$\leq c_3 \int_{\partial M(r)} \log f^* \Lambda(\sigma,\xi) \, |\partial\Phi|_M^2 \omega_r \quad .$$

The last step is done by Stokes theorem and the subharmonicity of $\Phi$ on $M(r,r_0)$. Using ($\beta$.1) and ($\beta$.2), we have

$$(2.29)\qquad \int_{r_0}^{r} E(f,t,r_0)dt \leq c_4 \int_{\partial M(r)} |\partial\Phi|_M^2 \omega_r$$

for any $r > r_0$.

Since $-\Delta_M \Phi^2$ is bounded from above by (2.5), from (2.22) and (2.29), we can obtain (2.24) for $f : M \longrightarrow \mathbb{P}_n$.

This completes the proof of $\beta$).

$\gamma$) We set ourselves in the situation of the proof of Theorem 2.1,$\gamma$). We have only to show the estimate (2.24) for the holomorphic map $f : (M, ds_M^2) \longrightarrow (N, ds_N^2)$ in the proof of Theorem 2.1,$\gamma$). But this is done by the same procedure as the case $\beta$) in account of ($\gamma$,1).

This completes the proof of Theorem 2.4.

Proof of Theorem 2

Since $n(A,r) = V(A(r))/r^{2m}$ ($\mu = 2m - 2$) is a continuously non-decreasing function, Theorem 2 follows from Theorem 2.1 immediately.

q.e.d.

Proof of Theorem 3

To prove this theorem, we should estimate the eigenvalues of the Levi form of $\Psi = \Phi^2$ relative to $ds_M^2$ by using Hessian comparison theorem.

(i) We put $\eta = e^4$ and fix a positive number $\epsilon_*$ with $0 < 8\epsilon_* < \frac{1}{(4m-2)(\eta+1)}$. We set $\epsilon = 8\epsilon_1$ for some constant $\epsilon_1$

with $0 < \epsilon_1 \leq \epsilon_*$. We consider a $C^\infty$ function $k_1 : [0,\infty) \longrightarrow (0,\infty)$ defined by

$$k_1(r) = \frac{\epsilon}{8(r+\eta)^2 \log(r+\eta)}$$

We assume

(2.30) $\quad |\text{radial curvature at } x \in M, \ \Phi(x) = r| \leq k_1(r)$

for any $r \geq 0$.

Next we consider a $C^\infty$ function $k_2 : [0,\infty) \longrightarrow (0,\infty)$ defined by

$$k_2(r) = \frac{\epsilon}{2(r+\eta)^2 \log(r+\eta)}\left(1 - \frac{1}{\log(r+\eta)}\right)$$

We consider the solutions $f_1$ and $f_2$ of the following Jacobi equations:

$$f_1''(r) = -k_1(r)f_1(r), \quad f_1(0) = 0 \quad \text{and} \quad f_1'(0) = 1$$
$$f_2''(r) = k_2(r)f_2(r), \quad f_2(0) = 0 \quad \text{and} \quad f_2'(0) = 1.$$

Then the solutions $f_1$ and $f_2$ satisfy the following property respectively

(2.31) $\quad f_1(r) > 0$ and $f_1'(r) > 0$ for $r > 0$

(2.32) $\quad f_2(r) > 0$ and $f_2'(r) > 0$ for $r > 0$

(2.32) follows from [9], Proposition 4.2. We show (2.31). We consider a $C^\infty$ funciton $f_3 : [0,\infty) \longrightarrow [0,\infty)$ defined by

$$f_3(r) = r(\log(r+\eta))^{-\epsilon} \quad \text{for} \quad r \geq 0.$$

Then it holds that $f_1(0) = f_3(0) = 0$, $f_3'(0) < f_1'(0)$ and $f_3''(r)/f_3(r) < f_1''(r)/f_1(r)$ for $r > 0$. Hence we have $f_1(r) > f_3(r) > 0$ for $r > 0$ and moreover

(2.33) $$0 < f_3'(r) < f_1'(r) \quad \text{for} \quad r > 0.$$

Hence we have (2.31).

Let $(M_i, ds^2_{M_i})$ be a 2m dimensional model whose radial curvature function is $k_i$ (cf. [9] Proposition 4.2) and let $r_i$ be the distance function of $M_i$ from some fixed point in $M_i$ $(i = 1.2)$. By (2.30) and $-k_2 \leq -k_1$, we obtain the following assertion from Hessian compariosn theorem concerning $r_i$ and $\Psi$ (cf. [9] Theorem A, Lemma 1.13, Proposition 2.20 and [28]).

(2.34) $$\frac{rf_1'(r)}{f_1(r)} \leq \mathcal{L}(\Psi)(v,\bar{v}) \leq \frac{rf_2'(r)}{f_2(r)}$$

for any $v \in TM_x^{1,0}$, $\Phi(x) = r > 0$ and $|v|_M = 1$.

Using (2.34), we shall show the following assertion

(2.35) $$-\frac{\epsilon}{\log(r+\eta)} < \mathcal{L}(\Psi)(v,\bar{v}) - 1 < \frac{\epsilon}{\log(r+\eta)}$$

for any $v \in TM_x^{1,0}$, $\Phi(x) = r > 0$ and $|v|_M = 1$.

If (2.35) was proved, then setting $h(r) = \log(r+\eta)/\epsilon$, the conditions (2.5) and (2.6) of Theorem 2.4 are verified.

Since $f_1(r) \leq r$, setting $\phi_1(r) = f_1(r)/f_1'(r)$, we have from (2.33)

$$(2.36) \qquad \phi_1(r) < 2\,(\log(r+\eta))^{\epsilon} r \quad \text{for} \quad r > 0.$$

By (2.36) and $\phi_1'(r) = 1 + k_1(r)\phi_1(r)^2$, we have

$$r \leq \phi_1(r) \leq r + \frac{\epsilon}{2} I_1(r) \quad \text{for} \quad r > 0.$$

Here $I_1(r) = \int_0^r (\log(t+\eta))^{2\epsilon-1} dt$. Since $I_1(r) \leq \frac{4}{3} r$ for $r \geq 0$, we have

$$(2.37) \qquad r \leq \phi_1(r) \leq c_* r \quad \text{for} \quad r > 0.$$

Here $c_* = 1 + \frac{2}{3}\epsilon$. Again by $\phi_1'(r) = 1 + k_1(r)\phi_1(r)^2$ and (2.37), we have

$$\phi_1(r) \leq r + \frac{\epsilon c_*^2}{8} I_2(r) \quad \text{for} \quad r \geq 0.$$

Here $I_2(r) = \int_0^r (\log(t+\eta))^{-1} dt$. Since $I_2(r) \leq 4r/3 \log(r+\eta)$, we have

$$\phi_1(r) \leq r + \frac{\epsilon c_*^2 r}{6\log(r+\eta)} .$$

Hence we have

$$r \leq \phi_1(r) \leq r + \frac{\epsilon r}{\log(r+\eta)} \quad \text{for any} \quad r \geq 0.$$

Finally we have

$$1 - \frac{rf_1'(r)}{f_1(r)} < \frac{\epsilon}{\log(r+\eta)}.$$

By (2.34), this means the left hand-side of (2.35).

Next we show the right hand-side of (2.35). Setting $\phi_2(r) = f_2(r)/f_2'(r)$, we have $\phi_2'(r) = 1 - k_2(r)\phi_2(r)^2$. So we have

$$r \geq \phi_2(r) \geq r - \frac{\epsilon r}{2\log(r+\eta)} \quad \text{for} \quad r \geq 0.$$

Hence we have

$$\frac{rf_2'(r)}{f_2(r)} - 1 < \frac{\epsilon}{\log(r+\eta)} \quad \text{for} \quad r > 0.$$

Therefore the proof of (2.35) completes.

(ii) We fix a number $r_1 > \max(r_0, e^2)$. For some fixed positive constant $\epsilon$, $0 < \epsilon < \frac{1}{12(2m-1)}$, we consider the following $C^\infty$ function $k_2 : [0,\infty) \longrightarrow [0,\infty)$ defined by

$$k_2(r) = \frac{2\epsilon}{r^2 \log r}\left(1 - \frac{1}{\log r}\right) \quad \text{for} \quad r > r_1$$

and assume

$$0 \geq \text{radial curvature on } \partial M(r) \geq -k_2(r)$$
$$\text{for any } r \geq 0.$$

We consider the solutions $f_1$ and $f_2$ of the following Jacobi equations.

$$f_1''(r) \equiv 0, \qquad f_1(0) = 0 \quad \text{and} \quad f_1'(0) = 1$$
$$f_2''(r) = k_2(r) f_2(r), \qquad f_2(0) = 0 \quad \text{and} \quad f_2'(0) = 1.$$

Here we consider $k_1(r) \equiv 0$ because the radial curvature of $ds_M^2$ is non-positive on M. Clearly $f_1(r) \equiv r$, $f_2(r) > 0$ and $f_2'(r) > 0$, $r > 0$. Since $-k_2(r) \leq -\epsilon/r^2 \log r$ for $r > r_1$, by the same procedure as (i), we have only to estimate $\frac{r f_2'(r)}{f_2(r)} - 1$.

Setting $\phi_2(r) = f_2(r)/f_2'(r)$, $r \geq 0$, we have

$$r \geq \phi_2(r) \geq r - \frac{2\epsilon r}{\log r} - c_{**} \quad \text{for} \quad r > r_1.$$

Here $c_{**} = r_1 - \phi_2(r_1) \geq 0$. We take a number $r_* > r_1$ so that $c_{**}\frac{\log r}{r} < \epsilon < \frac{\log r}{6}$ for $r > r_*$. Hence we have for $r > r_*$

$$\frac{rf_2'(r)}{f_2(r)} - 1 < \frac{6\epsilon}{\log r}.$$

Hence setting $h(r) = \log r/6\epsilon$, the conditions (2.5) and (2.6) of Theorem 2.4 are verified.

q.e.d.

Remark 2.38

In the case of Theorem 3, (ii), $M$ admits no non-constant bounded plurisubharmonic functions. If $M$ admits it, say $\psi$, then we may assume that $\psi$ is a bounded $C^\infty$ plurisubharmonic function on $M$ by the usual regularization method (since $M$ is realized as a closed submanifold of $\mathbb{C}^{2m+1}$) . Setting $\Psi = e^\psi$, $\Psi$ satisfies the same properties as $\psi$. Since $\Phi^2$ is strictly plurisubharmonic and $\log \Phi^2$ is plurisubharmonic on $M$, the function $F(r) = \int_{M(r)} dd_c\Psi \wedge (dd_c\Phi^2)^{m-1}/r^{2m-2}$ is a non-decreasing funciton of $r$ i.e. $F(r) \geq c_3 > 0$ for any $r \geq 1$. By Stokes theorem, we have

$$(***) \qquad \int_1^r \frac{F(t)}{t}\,dt \leq c_4 \sup_{z\in M} \Psi(z) \cdot n(M,r)$$

for $n(M,r) := V(r)/r^{2m}$ and $r > 1$. Since $n(M,r) \sim (\log r)^\delta$, $0 < \delta < 1$ (cf. (2.19), (2.20), $h(r) = \log r/6\epsilon$ and $0 < 6\epsilon < 1/4m-2$), we have from (***)

$$F(r) \leq c_5(\log r)^{\delta-1}.$$

This means that $\Psi$ is constant.

## REFERENCES

[1] BEDFORD, E., TAYLOR, B.A., Two applications of a non-linear integral formula to analytic functions, Indiana Univ. Math. J. 29, 463-465 (1980).

[2] CHENG, S.Y., Liouville theorem for harmonic maps, Geometry of the Laplace operator, Proc. Sym. Pure Math., 36, 147-151(1980).

[3] CHENG, S.Y., YAU, S.T., Differential equations on Riemannian manifolds and their geometric applications, Comm. Pure Appl. Math. 28, 333-354 (1975).

[4] DONNELLY, H., Bounded harmonic functions and positive Ricci curvature, Math. Z., 191, 559-565 (1986).

[5] DONNELLY, H., FEFFERMAN, C., $L^2$ cohomology and index theorem for the Bergman metric, Ann. of Math., 118, 593-618 (1983).

[6] DONNELLY, H., XAVIER, F., On the differential form spectrum of negatively curved Riemannian manifolds, Amer. J. Math. 106, 169-185 (1984).

[7] EELLS, J., LEMAIRE, L., A report on harmonic maps, Bull. London Math. Soc. 10, 1-68 (1978).

[8] GREENE, R.E., WU. H., Integral of subharmonic functions on manifolds of non-negative curvature, Invent. Math. 27. 265-298 (1974).

[9] GREENE, R.E., WU, H., Funciton theory on manifolds which possesses a pole, Lecture Notes in Mathematics, 699, Springer Verlag 1979.

[10] HAYMAN, W.K., Meromorphic functions, Clarendon Press, Oxford, 1964.

[11] HILDEBRANDT, S., Liouville theorems for harmonic mappings and an application to Bernstein theorems, Seminar on differential geometry, Ann. of Math. Studies 102, 107-131 (1982).

[12] KARP, L., Subharmonic functions on real and complex manifolds, Math. Z., 179, 534-554 (1982).

[13] KASUE, A., On Riemannian manifolds admitting certain strictly convex function, Osaka J. Math. 18, 577-582. (1981).

[14] LI, P. TAM, L.F., Positive harmonic functions on complete manifolds with non-negative curvature outside a compact set, Ann. of Math. 125, 171-207 (1987)

[15] MOK, N., SIU, Y.T., YAU, S.T., The Poincare-Lelong equation on complete Kähler manifolds, Compositio Math. 44, 183-218 (1981).

[16] MOSER, J., On Harnack's theorem for elliptic differential equations, Comm. Pure Appl. Math. 14, 577-591 (1961).

[17] PRICE, P., A monotonicity formula for Yang-Mills fields, Manuscripta Math. 43, 131-166 (1983).

[18] RICHBERG, R., Stetige streng pseudokonvexe Funktionen, Math. Ann. 175, 257-286 (1968).

[19] SAMPSON, J.H., Some properties and applications of harmonic mappings, Ann. Ecolenorm, Sup. 11, 211-228 (1978).

[20] SCHABAT, B.V., Distribution of values of holomorphic mappings, Translation of Mathematical Monographs, Amer. Math. Soc. 61, 1985.

[21] SIBONY, N., WONG, P.M., Some remarks on the Casorati-Weierstrass theorem, Ann. Polonici Math. 39, 165-174 (1981).

[22] SIU, Y.T., The complex analyticity of harmonic maps and the strong rigidity of compact Kähler manifolds, Ann. of Math. 112, 73-111 (1980).

[23] SIU, Y.T., Complex analyticity of harmonic maps, vanishing and Lefschetz theorems, J. of Diff. Geometry, 17, 55-138 (1982).

[24] SIU, Y.T., YAU, S.T., Complete Kähler manifolds with non-positive curvature of faster than quadratic decay, Ann. of Math. 105, 225-264 (1977).

[25] STOLL, W., The growth of area of a transcendental analytic set I, II, Math. Ann. 156, 47-78, 144-170 (1964).

[26] TAKEGOSHI, K., A non-existence theorem for pluriharmonic maps of finite energy, Math. Z. 192, 21-27 (1986).

[27] VESENTINI, E., Lectures on Levi convexity of complex manifolds and cohomology vanishing theorems, Tata Institue of Fundamental Research, Bombay, 1967.

[28] WU, H., On a problem concerning the intrinsic characterization of $\mathbb{C}^n$, Math. Ann. 246, 15-22 (1979).

[29] WU, H., The Bochner technique in differential geometry, Mathematical Report Series, Harwood Academic Press, 1987.

[30] YAU, S.T., Harmonic functions on complete Riemannian manifolds, Comm. Pure Appl. Math. 28, 201-228 (1975).

[31] YAU, S.T., A general Schwarz lemma for Kähler manifolds, Amer. J. Math. 100, 197-204 (1978).