CONWAY POLYNOMIAL OF A FIBERED SOLVABLE LINK

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Abstract. We compute Kauffman's potential function (equivalent to the Conway polynomial) for any solvable fibered link. Solvable links are links that can be built from the unknot by iterated cabling and summing. Links that arise in complex algebraic geometry are generally of this type.

A solvable link is one constructed by iterated cabling and connected sum operations from the unknot. In [1] such links are encoded by their *splice diagrams* and a simple formula is given for the Alexander polynomial of the link in terms of the splice diagram. The Alexander polynomial determines the Conway polynomial up to sign (see below) and S. Yu Orevkov (private communication) conjectured the value of this sign, and hence the value of the Conway polynomial, in the case the solvable link is fibered ("generic" solvable links *are* fibered). This question arose in his study of real algebraic plane curves (Hilbert's 16-th problem, [5] [6]). In this note we prove his conjecture.

The conjecture is best expressed in terms of Kauffman's "potential function." This is an elementary modification of the Conway polynomial, see [2], Theorem 7.10 ff. It satisfies

$$\Omega(t) = \pm t^{-d}\Delta(t^2), \tag{1}$$

where $\Delta(t)$ is the Alexander polynomial and d is the degree of $\Delta(t)$. Thus the only issue is the determination of the sign of Ω.

$\Omega(t)$ is defined as

$$\Omega(t) := det(t^{-1}L - tL'), \tag{2}$$

where L is any Seifert form for the link and L' its transpose. For a fibered link, if L is the Seifert form determined by a fiber then L is an invertible matrix over $\mathbb{Z}$. Since the highest order term of the potential function is then $t^d \det(-L')$, the issue at hand reduces to "just" computing $\det(-L') = \det(-L) = \pm 1$ and verifying that it agrees with Orevkov's prediction.

An explicit computation of L for solvable knots is described in [1], and can be used to verify the prediction in this case, but a direct computation for links seems hard. Instead we use Rudolph's "enhancement to the Milnor number." For fibered links in S^3 this is an integer invariant λ which is known to satisfy

$$(-1)^\lambda = \det(-L) \tag{3}$$

(see [3] and [4]). In [4] it is computed for solvable links, which is just what we need.

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1. Main result

To state and prove the result we must assume the reader is familiar with splice diagrams ([1]). Suppose that we have been given a splice diagram Γ for our solvable link (S^3, L). Recall that Γ is a tree decorated as follows:

- some of its leaves (valency one vertices) are drawn as arrow-heads and correspond to components of L;
- each edge has an integer weight at any end where it meets a node (vertex of valency > 1);
- the edge-weights around a node are pairwise coprime.
- the arrowheads have weights $+1$ or -1 (to signify whether the corresponding component of L carries the "intrinsic" orientation determined by the splice diagram or the reverse of this orientation).

In a general splice diagram nodes also have weights ± 1, but there are permissible operations to change signs of weights in a splice diagram that allow one to arrange that all node weights are $+1$, so we shall assume this. Splice diagrams as described above encode general "graph links" (links in homology spheres with graph manifold exteriors). Solvable links are graph links for which the ambient homology sphere is S^3. The condition on a splice diagram that assures that the ambient sphere is S^3 is described in [1] but will be irrelevant to our analysis.

Associated to each non-arrowhead vertex v of Γ is a so-called "virtual component" S_v which is the additional link component that would be represented by adding a single arrow at that vertex with edge weight 1 (if v had valency 1 this is equivalent to simply changing v to an arrowhead). The linking number $l_v := \ell(S_v, L)$ is called the "multiplicity" at vertex v.

The formula of [1] for the Alexander polynomial is

$$\Delta(t) = (t-1)\prod_v (t^{|l_v|} - 1)^{\delta_v - 2}, \tag{4}$$

where the product is over all non-arrowhead vertices and δ_v is the valency of vertex v (the formula has an interpretation in [1] even if some l_v are zero). The link is fibered if all l_v are non-zero (see [1]) and then the formula we wish to prove for Kauffman's potential function is

$$\Omega(t) = (-1)^{k_-}(t - t^{-1})\prod_v (t^{l_v} - t^{-l_v})^{\delta_v - 2}, \tag{5}$$

where k_- is the number of (-1)-weighted arrowheads. As already mentioned, this is evidently correct up to sign and the only issue is to verify the sign.

We can easily compute the highest order coefficient of the above function. Let j_- be the number of non-arrowhead vertices v for which l_v is negative and δ_v is odd. Then the highest coefficient is $(-1)^{k_- + j_-}$, so by the discussion following equation (2) the result to be proved becomes:

$$\det(-L) = (-1)^{k_- + j_-}. \tag{6}$$

We shall show this is correct for fibered solvable links. It is a reasonable question also for fibered graph links in homology spheres other than S^3, but we do not answer this — see Sect. 3.

By equation (3) equation (6) can be reformulated in terms of Rudolph's λ:

$$\lambda \equiv k_- + j_- \pmod 2. \tag{7}$$

Recall that l_v is called the "multiplicity" at vertex v. The weight ± 1 associated to an arrowhead is also called "multiplicity." Thus $k_- + j_-$ counts the total number of vertices (nodes or arrowheads) of Γ with odd valency and negative multiplicity.

We first show equation (7) is invariant under the allowed changes of sign in a splice diagram. To see this, we recall from [1] what changes of sign are allowed, and also how linking numbers are computed via a splice diagram. First the latter:

Computing Linking Numbers. Given two components (virtual or genuine) for the link, corresponding to vertices v and w of the diagram, their linking number is the product of all splice diagram weights directly adjacent to but not on the shortest path in Γ connecting these vertices. The weights in question are the edge weights adjacent to nodes that the path passes through and arrowhead weights ± 1 for whichever of v and w are arrow-heads.

Allowable changes of signs of weights in a splice diagram are as follows: one may change signs of an even number of edge weights adjacent to each vertex so long as one also adjusts the arrowhead weights so that mutual linking numbers between the components represented by arrowheads remain unchanged. Such changes can be generated by "elementary changes" of the following type: change the edge weights on two edges adjacent to a node v and simultaneously change the weights at all arrowheads of the following subgraph Γ_0 of Γ: the full subgraph on vertices of Γ whose shortest path to v passes through one of the changed weights. Using the above description of linking numbers, we see that the effect of this on the vertex multiplicity weights of Γ is to change the signs of all of them in Γ_0 except at v itself and leave them unchanged off Γ_0. Since the total number of vertices of odd valency in Γ_0 is even (this is a general property of graphs), the parity of the number of vertices of odd valency with negative multiplicity is unchanged by this.

Thus, if equation (7) holds after an allowable change of signs of weights of Γ then it holds for Γ.

Any solvable link can be iteratively constructed by cabling and summing operations for which only (p,q)-cabling operations have been allowed with $p > 0$, and then adjusting orientations of components of the link at the end. For the splice diagram Γ corresponding to such a construction, Rudolph's enhancement λ is computed in [4] Theorem 6.1 as the linking number $\lambda = \ell(P \cup K_+, N \cup K_-)$ where:

- K_+ and K_- are the unions of link components corresponding to arrowheads with multiplicities $+1$ and -1 respectively;
- P is the union of copies of virtual link components S_v having positive l_v, each such S_v being taken a total of $2 - \delta_v$ times (if $2 - \delta_v$ is negative this means: take it $\delta_v - 2$ times but with reversed orientation). Similarly for N, but using virtual link components with negative l_v.

Modulo 2 this linking number is unchanged if we replace P by the union J_+ of one virtual link component for each node v with odd valency and with $l_v > 0$ and replace N by the union J_- of virtual link components for nodes v with odd valency and with $l_v < 0$. Thus we need to show

$$\ell(J_+ \cup K_+, J_- \cup K_-) = \#(J_- \cup K_-). \tag{8}$$

We shall see that this equation depends only on the fact that the vertices corresponding to $J_+ \cup K_+$ and $J_- \cup K_-$ partition the vertices of odd valency in Γ.

Let Γ' be the result of changing the arrowheads of Γ corresponding to K_+ to nodes and adding arrowheads at nodes to represent the components in J_-. Thus

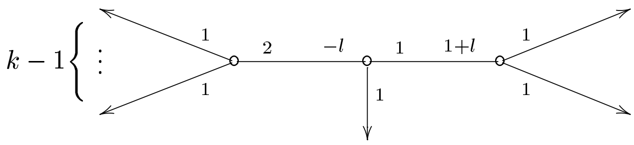
Γ' is a splice diagram for the link $J_- \cup K_-$. By [1] the euler characteristic of a minimal Seifert surface for this link is given by

$$\chi = \sum_v (2 - \delta'_v)|\ell(S_v, J_- \cup K_-)| \tag{9}$$

where δ'_v is valency in Γ' and the sum is over all nodes v of Γ'. Since $2 - \delta'_v$ is odd precisely for v corresponding to a component in $J_+ \cup K_+$, this formula reduces modulo 2 to the left side of equation (8). Thus we just need to show that χ equals the number of components of $J_- \cup K_-$ modulo 2. But this is true, since modulo 2 the euler characteristic of a compact oriented surface equals the number of its boundary components. This completes the proof.

2. Example

The example that Orevkov uses in [6] has the following splice diagram.



By the algorithm for computing linking numbers above, the three nodes have multiplicities $2k+1$, $k-3l-1$, and $k+l+1$ respectively, so we see that the potential function is

$$(t - t^{-1})(t^{2k+1} - t^{-2k-1})^{k-2}(t^{k-3l-1} - t^{-k-3l+1})(t^{k+l+1} - t^{-k-l-1}) \tag{10}$$

If k and l are positive then the computation of λ described earlier gives $\lambda = 0$ for $3l \leq k - 1$ and $\lambda = 2l - 1$ for $3l > k - 1$.

3. Comments

We do not know if formula (5) (or the equivalent formula (6)) holds for general fibered graph links and/or for non-fibered links. For general fibered links in homology spheres Rudolph's invariant λ depends on a choice of stable almost complex structure on the ambient homology sphere, and, as discussed in [4], there is a natural such choice for graph links that makes the analogous computation of λ to that described above remain valid. Thus (5) would again follow from (3). But both of the known proofs of (3) depend heavily on the ambient sphere being S^3. It would be interesting if (3) holds for general graph links, and more generally, if it holds for some preferred coset of stable almost complex structures on any homology 3-sphere.

References

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