Links between Geometry and

Mathematical Physics

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Workshop "Links between Geometry and Mathematical Physics"
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INTRODUCTORY LECTURE

Michael Atiyah

I. General Remarks

In the past decade there has been a remarkable interaction between Geometry and Physics. This has been due mainly to the development of Gauge Theories as a model for Elementary Particle Physics and the discovery that Topological ideas are very important in these non-linear models. The advantages of this interaction have been two-way and mathematics has greatly benefited from the new ideas, concepts and techniques coming from Physics. The most notable results in this direction have been the striking new phenomena in 4-dimensional Geometry discovered by Donaldson using instantons.

When this meeting was planned a couple of years ago there were many aspects of this interaction still in full development, but it was hard to predict two years into the future. In fact in the intervening period there have been several new developments including quite different links between Geometry and Physics so that the timing of this meeting is very topical.

2. New Developments

I will now list briefly some of the new topics which should be discussed during the meeting.

1. Elliptic Cohomology Witten has shown how the index of the Dirac-Ramond operator of the super-symmetric non-linear sigma-model gives the elliptic genus recently discovered by Landweber, Stong and others.

2. Floer Cohomology Using ideas from Yang-Mills theory Floer has introduced new cohomology groups for a 3-dimensional manifold and Donaldson has related these to his invariants in 4-dimensions.

3. Knot Polynomials Vaughan Jones has produced a new polynomial invariant for knots and links in $\mathbb{R}^3$. His method is based on ideas from Statistical Mechanics.

4. <u>Quasi-Crystals</u> The aperiodic tilings of the plane discovered by R. Penrose appear to be closely related to recently discovered "quasi-crystals" of alluminium-manganese alloys.

5. <u>String Theory</u> The theory of Riemann surfaces is intimately involved in conformal field theories which are at the basis of string theory.

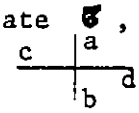
These and many other topics will be discussed during the meeting and I am pleased we have experts in all these areas.

Knots and the Yang-Baxter equation

V.F.R. Jones

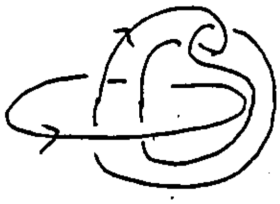
I.H.E.S.

In statistical mechanics one considers a system defined by a set of states σ , to each of which is assigned an energy $E(\sigma)$. The partition function Z is then defined by $Z = \sum_{\sigma} e^{-\beta E(\sigma)}$. It is important to find an explicit expression for Z as a function of β and any parameters involved in $E(\sigma)$. The real problem in statistical mechanics is that there are infinitely many states so Z is expressed as a limit of sums over finite regions. For what we have in mind in knot theory we will only have to consider finite systems.

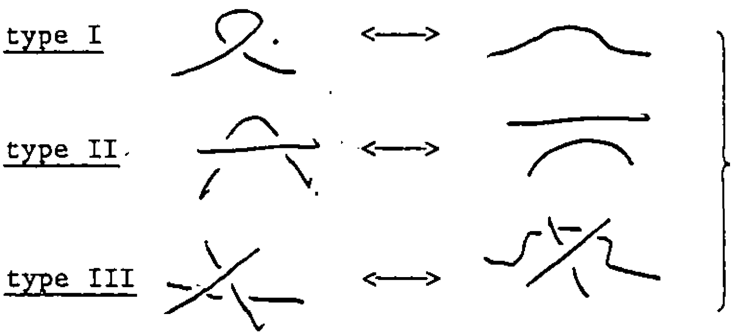
The most frequently considered examples in statistical meachnics are defined on the lattice $\mathbf{Z} \otimes \mathbf{Z} \subseteq \mathbb{R}^2$. We look at systems where the states are defined by assigning one of a finite set $\boldsymbol{\sigma}$ of real numbers ("spins") to each edge of the lattice. Thus, given a state $\boldsymbol{\sigma}$, each vertex is surrounded by four "spins" as in the figure 

. The energy of the state will then be $\sum_{\text{vertices}} W(a,b|c,d)(\Theta)$. So if $w = e^{-\beta W}$ we may write the partition function as $Z = \sum_{\text{states}} \left(\prod_{\text{vertices}} w(a,b|c,d)(\Theta) \right)$. The model may be varied by choosing different functions $W(a,b|s,d)(\Theta)$ (which are supposed to be the same at each vertex. Baxter found that he could often solve models in which the $w(a,b|c,d)(\Theta)$ satisfied a certain equation, known as the "star-triangle" or Yang-Baxter equation. It is most suggestively written as a matrix equation. One identifies $w(a,b|c,d)(\Theta)$ with an element $R(\Theta)$ of $\mathrm{End}(V \otimes V)$, V being a vector space with basis indexed by $\boldsymbol{\sigma}$. The Yang Baxter equation is then the equation in $\mathrm{End}(V \otimes V \otimes V)$: $R_{12}(\Theta)R_{13}(\gamma+\Theta)R_{23}(\gamma) = R_{23}(\gamma)R_{13}(\Theta+\gamma)R_{12}(\Theta)$ where $R_{ij}(\Theta)$ is the natural action of $R(\Theta)$ on $V \otimes V \otimes V$ on the ith and jth places.

In knot theory, one frequently pictures (oriented) knots and links in $\mathbb{R}^3$ by smooth immersed circles in $\mathbb{R}^2$ with crossing data, eg.



. The equivalence of links in $\mathbb{R}^3$ is translated into an equivalence relation on pictures using the Reidemeister moves (and 2-dimensional isotopy) of types I, II, III which are as follows

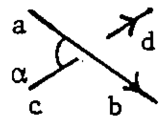


with all possible orientations and reversed crossing changes in the obvious ways.

Two pictures of the same link can be changed from one to the other by a sequence of Reidemeister moves.

One can look for topological invariants of links by devising combinatorial formulas from pictures which are invariant under Reidemeister moves. The Alexander polynomial was first defined in this way. Another possible approach is to treat the picture as a statistical mechanical system where the states are defined by assigning elements of $\mathfrak{S}$ to each edge. One allows $w(a,b|c,d)$ to be different for positive and negative crossings. The question is then: for what choices of R is the partition function invariant under Reidemeister moves? Here it seems necessary to define the partition function by

$$\sum_{\sigma \in \text{states}} \Big(\prod_{\text{crossings}} w_{\pm}(a,b|c,d)(\alpha) \Big) e^{-h \int_{\text{link}} \sigma \, d\Theta}$$

where the situation at a crossing is 

, α being the angle.

The Yang-Baxter equation turns up as the condition for invariance under type III Reidemeister moves. Further conditions are imposed by the types I and II moves.

Solutions to the Yang-Baxter equation can be found by the "Quantum group" formalism of Drinfeld, Fádeev et al. It seems that to every irreducible representation of every simple Lie algebra there is a knot-polynomial, the result having been firmly established (by myself Wenzl and Turaev) in the cases of sl_2 in all its representations and the A, B, C, D algebras in their "natural" representation. One recovers the recently discovered two-variable polynomials of Lickorish, Millett, Hoste, Ocreanu, Przyzctki, Traczyk, Freyd, Yetter, and of Kauffman. In these cases there are simple inductive formulae for calculating the invariants. In general it seems hard to calculate the polynomials.

QUASI-CRYSTALS

R. Penrose
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In 1966, Robert Berger proved a result which implies that there is no general algorithm for deciding whether or not a given finite set of polygonal shapes will tile the Euclidean plane. In the course of this work he exhibited a set of 20,426 tiles which are aperiodic, i.e., they will tile the plane but only in ways that are not periodic. This number was reduced to 6 by Raphael Robinson in 1971. Robinson's aperiodic set produces tilings which are hierarchical in nature, and in 1973, following quite different lines, I produced another set of six aperiodic tiles which also tile only according to a hierarchical scheme. Unlike Robinson's set, which were based on squares, mine were based on regular pentagons. There

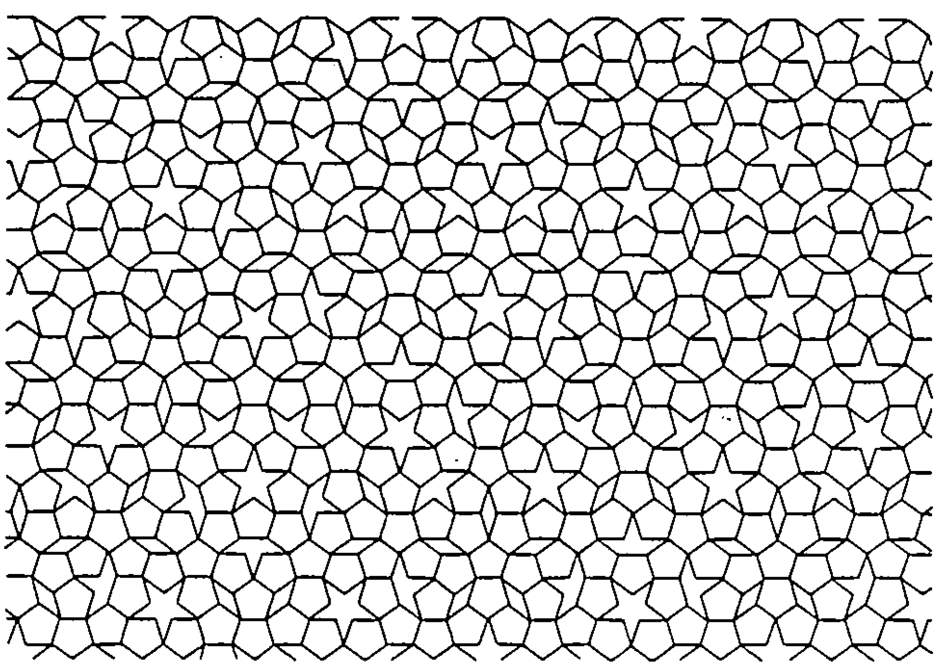


Fig. 5. Pattern with fivefold quasi-symmetry

Fig. 5,6,9 reprinted from the author's lecture in: Hermann Weyl 1885-1985, edited by K.Chandrasekharan, ETH Zürich and Springer-Verlag Berlin Heidelberg 1986

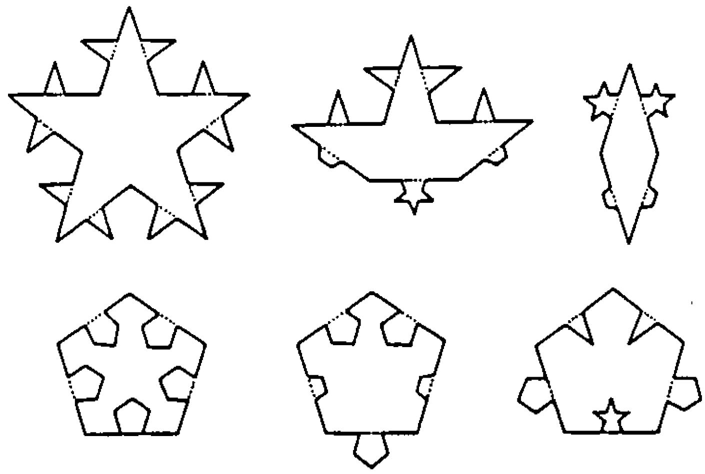


Fig. 6. Six tiles which can be assembled only according to the pattern of Fig. 5

are $2^{\aleph_0}$ distinct tilings with these shapes, and each of them exhibits a 5-fold quasi-crystallographic structure: there exist infinitely many points about which each tiling exhibits 5-fold symmetry to any preassigned degree of accuracy less than unity, and there are arbitrarily large regions with exact 5-fold symmetry; moreover every finite region of each tiling is repeated infinitely many times elsewhere in every tiling with these same shapes. In 1974 I reduced my set of 6 to a set of 2, referred to as "kites" and "darts", and also to another aperiodic pair based on

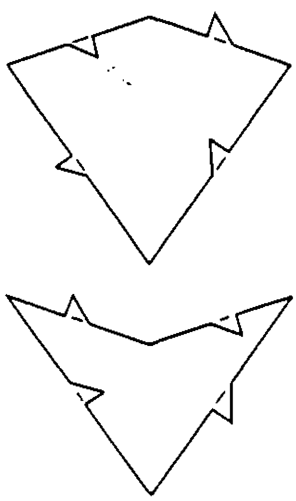


Fig. 9. Kites and darts – a non-periodic pair

rhombuses. The tilings are again all 5-fold quasi-crystollographic in the same sense, and exhibit other striking properties. For example, in the kite-dart tilings, the darts form chains which, when they close, always do so in a precisely 5-fold symmetric way. In almost all of the $2^{\aleph_0}$ tilings all chains close. At most two chains can fail to close, and in only one tiling are there two which do not close. The patterns exhibit alignments in terms of broad and narrow strips angled at 36° to one another, the width of the broad being $\tau = \frac{1}{2}(1 + \sqrt{5})$ times that of the narrow. In each direction, the pattern of broad and narrow is in accordance with a Fibonacci sequence

$$\ldots 1\tau 1\tau\tau 1\tau\tau 1\tau 1\tau\tau 1\tau \ldots$$

generated hierarchically according to the scheme

$$1 \mapsto \tau \quad , \quad \tau \mapsto 1\tau \ .$$

The different tiling patterns (and also the different Fibonacci sequences) can be labelled by infinite sequences of 0s and 1s , where no 1s appear sucessively, e.g.

$$0100101000101001\ldots \ ,$$

where two such sequences are regarded as equivalent if they differ only in a finite number of places.

In December 1984, Shechtman and his associates at the National Bureau of Standards in Washington announced a quasi-crystallographic icosahedral phase of aluminimum-manganese alloy. The electron diffraction patterns closely resembled the Fourier transforms of the patterns described above, as obtained by MacKay in 1982 and Levine and Steinhardt in 1984. Electron micrograph pictures of thin portions of these materials (and also of other similar materials found subsequently) bear a significant resemblance to the above patterns. A useful test of this is to examine the pattern of broad and narrow strips that appear in the micrograpph pictures and to compare them with Fibonacci sequences, which, indeed, they closely follow. This is most easily seen using a description due to de Brujn and Pleasants involving higher-dimensional cubic (square) lattices. For the Fibonacci sequence the lattice points in 2-dimensions in a strip angled at $1 : \tau$ are taken, while for the above-mentioned (rhombus) plane tilings, they are the lattice points in a suitably angled (and positioned) slab in 5-dimensions. A suitable slab in 6-dimensions yields an icosahedral quasi-crystallographic pattern apparently resembling that of the substances studied by Shechtman.

A set of four solids which appear to be icosahedrally aperiodic were found by Robert Ammann in 1977 and could serve as a model for these substances. Ammann also found other aperiodic sets of plane tiles, one of which tiles according to an 8-fold quasi-crystallographic scheme. Nissen and colleagues in Zürich have seen an apparently 12-fold quasi-crystallographic phase of nickel-chromium alloy. Tilings with such 12-fold quasi-symmetry can be exhibited, but I do not know of an appropriate aperiodic set of tiles.

A puzzling feature of the physical existence of quasi-crystals in nature is that their assembly would appear to be necessarily non-local. To assemble such a tiling correctly it is necessary, from time to time, to examine regions of the assembled pattern which may be arbitrarily far from the assembly point. My impression is that the growth of such substances must be an essentially quantum-mechanical process.

REFERENCES

BERGER, R. (1966) The undeciability of the domino problem. Mem. Amer. Math. Soc. 66.

GARDNER, M. (1977) Extraordinary nonperiodic tiling that enriches the theory of tiles, Sci. Amer. 236, 110-121..

GRÜNBAUM, B. and SHEPHARD, G.C. (1986), Tilings and Patterns (Freeman).

LEVINE, D. and STEINHARDT, P. (1984) Quasicrystals: a new class of ordered structures, Phys. Rev. Lett. 53, 2477.

MACKAY, A.L. (1982) Crystallography and the Penrose pattern, Physica, 114A, 609.

ISHIMASA, T., NISSEN, H.-U. AND FUKANO, Y. (1985) Phys. Rev. Lett. 55 511-513.

PENROSE, R. (1974), The role of aesthetics in pure and applied mathematical research, Bull. Inst. Math. Applications 10, 266-271.

PENROSE, R. (1986), Hermann Weyl, Space-time and conformal geometry in Hermann Weyl 1885-1985. (ed. K. Chandrasekharan, ETH, Zürich & Springer-Verlag, Berlin).

ROBINSON, R.M. (1971) Undecidability and non-periodicity for tilings of the plane, Invent. Math. 12, 177-209.

SHECHTMAN, D.,BLECH, I., GRATIAS, D. and CAHN, J.W. (1984) Metallic phase with long-range orientational order and no translational symmetry, Phy. Rev. Lett. 53, 1951.

THE DEFINITIONS OF CONFORMAL FIELD THEORY

Graeme Segal

I shall propose a definition of conformal field theory in two dimensions which I believe is equivalent to that used by Friedan and others.

§ 1. The definition

The category $\mathscr{C}$ is defined as follows. There is a sequence of objects $\{C_n\}_{n\geq 0}$, where C_n is the disjoint union of a set of n parametrized circles. A morphism $C_n \to C_m$ is a Riemann surface X with boundary ∂X, together with an identification $i : C_m - C_n \to \partial X$. (We identify morphisms (X,i), (X',i') if there is an isomorphism $f : X \to X'$ such that $f \circ i = i'$.)

Example

(i) The semigroup of morphisms $C_1 \to C_1$ falls into connected components corresponding to the genus and the number of connected components of X. The identity component $\mathcal{E}$ can be identified with the space of real-analytic simple closed curves in the disc $D = \{z \in \mathbb{C} : |z| < 1\}$. $\mathcal{E}$ is a bounded domain in the complex vector space of of real-analytic maps $S^1 \to \mathbb{C}$, and the Shilov boundary of $\mathcal{E}$ is the group $\mathrm{Diff}(S^1)$ of orientation-preserving diffeomorphisms of S^1. (cf. that the semigroup $\{g \in GL_n(\mathbb{C}) : \|g\| < 1\}$ is a bounded domain in $\mathbb{C}^{n^2}$ whose Shilov boundary is U_n.) (The group $\mathrm{Diff}(S^1)$ has no complexifications: $\mathcal{E}$ is the best substitute available.)

(ii) The connected component of the space of morphisms $C_0 \to C_1$ which are represented by topological discs can be identified with $\mathrm{Diff}(S^1)$.

(iii) The semigroup of morphisms $C_0 \to C_0$ is the space of isomorphism classes of not-necessarily-connected Riemann surfaces, the operation being disjoint union.

Definition (1.1)

A conformal field theory is a representation of $\mathcal{C}$, i.e. a continuous functor T from $\mathcal{C}$ to complex Hilbert spaces such that

(i) $T(C_n) = T(C_1)^{\otimes n}$ ($= H^{\otimes n}$, say) for $n \geq 0$;

(ii) for each morphism $\xi : C_m \to C_n$ the operator $T(\xi) : H^{\otimes m} \to H^{\otimes n}$ is of trace class;

(iii) T is a *-functor in the sense that for each morphism $\xi : C_m \to C_n$ we have $T(\xi)^* = T(\bar{\xi})$, where $\bar{\xi} : C_n \to C_m$ is defined by the surface complex conjugate to ξ;

(iv) T has the collapsing property that if a morphism $\xi : C_{m+r} \to C_{n+r}$ is made into a morphism $\check{\xi} : C_m \to C_n$ by attaching r outgoing circles to r ingoing ones, then $T(\check{\xi}) = \operatorname{trace} T(\xi)$, where the trace is taken over $H^{\otimes r}$.

If $q \in \mathbb{C}$ with $0 < |q| < 1$ let ρ_q denote the morphism $C_1 \to C_1$ represented by the annulus $\{z \in \mathbb{C} : |q| \leq |z| \leq 1\}$, where the inner boundary circle is parametrized by $\theta \to qe^{i\theta}$. The ρ_q form of a sub-semigroup of $\mathcal{E}$: $\rho_{q_1}\rho_{q_2} = \rho_{q_1 q_2}$.

Definition (1.2) The partition function Z_T of the theory is defined by $Z_T(q) = \operatorname{trace} T(\rho_q)$.

Condition (iv) above implies trivially

Proposition (1.3) For any theory T the partition function Z_T is modular, i.e. if $q = e^{2\pi i\tau}$ with $\operatorname{im}(\tau) > 0$, then $Z_T(q)$ is invariant when τ is transformed by $PSL_2(\mathbb{Z})$.

§ 2 Motivation: Wick rotation

Consider a two-dimensional Minkowskian space-time X in which space is a circle. (Thus topologically $X \cong S^1 \times \mathbb{R}$.) The group of conformal diffeomorphisms of X is a $\mathbb{Z}$-fold covering group of $\mathcal{G} = \mathrm{Diff}(S^1) \times \mathrm{Diff}(S^1)$. We expect a conformal field theory to be at least a projective unitary representation H of $\mathcal{G}$. Let us restrict attention for the moment to the subgroup $G \times G$ of $\mathcal{G}$, where $G = PSL_2(\mathbb{R})$. We expect the positivity of energy to be expressed by the fact that the unitary action of $G \times G$ on H is the boundary value of a holomorphic representation by contraction operators of the semigroup $G_{\mathbb{C}}^{+} \times G_{\mathbb{C}}^{+}$ contained in the complexification of $G \times G$. Here $G_{\mathbb{C}}^{+} = \{g \in G_{\mathbb{C}} : \|g\| < 1\}$. This holomorphic contraction representation restricts to a representation T of the antidiagonal $G_{\mathbb{C}}^{+} = \{(g_1, g_2) \in G_{\mathbb{C}}^{+} \times G_{\mathbb{C}}^{+} : g_1 = \bar{g}_2\}$ with the reflection-positivity property that $T(\bar{g}) = T(g)^*$. Conversely a reflection-positive contraction representation of $G_{\mathbb{C}}^{+}$ on H "Wick-rotates" to a unitary action of $G \times G$ on H.

In our situation $G \times G$ is replaced by $\mathcal{G}$ and $G_{\mathbb{C}}^{+}$ by the semigroup $\mathcal{E}$. We have

<u>Proposition (2.1)</u> For any theory T the action of $\mathcal{E}$ on H can be continued analytically to a projective unitary representation of $\mathrm{Diff}(S^1) \times \mathrm{Diff}(S^1)$ on H.

§3 <u>Holomorphic theories: examples</u>

It would be natural to define a holomorphic field theory as a holomorphic representation of the category $\mathcal{C}$. That, however, would be too restrictive. The category $\mathcal{C}$ has a fundamental central extension $\hat{\mathcal{C}}$. The objects are the same, but a morphism $C_m \to C_n$ is $\hat{\mathcal{C}}$ is a morphism (X, i) in $\mathcal{C}$ together with a choice of a point in the determinant line bundle of the $\bar{\partial}$-operator on X with its natural ingoing and outgoing boundary conditions.

<u>Definition (3.1)</u> A holomorphic field theory is a holomorphic representation T of $\hat{\mathcal{C}}$.

Definition (3.2) The level of T is the power of $\det(\xi)$ by which $T(\xi)$ acts on $T(C_0) = \mathbb{C}$, where ξ is a morphism $C_0 \to C_0$. (Observe that a morphism $C_0 \to C_0$ is simply a compact Riemann surface.)

Remark We must allow representations with fractional level.

The partition function of a holomorphic theory of level c will be a modular form of weight c in the usual sense, for some subgroup of finite index in the modular group $PSL_2(\mathbb{Z})$.

If T is a holomorphic theory then $T \otimes \bar{T}$ is a theory in the original sense. It is conjectured that any irreducible theory is a finite sum $\bigoplus S_i \otimes \bar{T}_i$, where S_i and T_i are holomorphic theories which are permuted by the modular group.

Example

Let LG denote the loop group of a compact group G. If H_λ is the irreducible projective representation of LG with highest weight λ then there is a canonical holomorphic theory T_λ with $T_\lambda(C_1) = H_\lambda$. If H_λ is the basic representation of LG then $T_\lambda \otimes \bar{T}_\lambda$ is the non-linear sigma model (with Wess-Zumino term) based on G.

From this example we see the sense in wich an irreducible theory can give rise to some "physical space". For the action of LG on $H_\lambda \otimes \bar{H}_\lambda$ will be uniquely determined by its intertwining properties with $\mathcal{E}$, and so we automatically obtain operators parametrized by the "physical space" G.

§4 Primary fields and field operators

Definition (4.1) For any theory T a vector $\psi \in H$ is a primary field if it is an eigenvector of the semigroup $\mathcal{E}^+$ of elements $f \in \mathcal{E}$ which extend to holomorphic maps $f : \bar{D} \to \bar{D}$ such that $f(0) = 0$.

We then have $T(\rho_q) \cdot \psi = q^{-a}\bar{q}^{-b}\psi$ for some positive real numbers (a,b) such that $a - b \in \mathbb{Z}$. We call (a,b) the bidegree of ψ.

Consider a Riemann surface X whose boundary consists of two circles: $\partial X = C' - C$. Thus X is a morphism $\xi : C_1 \to C_1$. For each triple (ψ, z, dz), where $X \in X - \partial X$, dz is a holomorphic differential at z, and $\psi \in H$ is a primary field of bidegree (a,b), we can define a trace-class creation operator $\psi(z) : H \to H$ such that $\psi(z)dz^a d\bar{z}^b$ is an operator-valued differential form on X.

To construct $\psi(z)$, choose a local parameter ζ at z such that $d\zeta(z) = dz$. Let $D_\epsilon = \{x \in X : |\xi(x)| < \epsilon\}$. Then $X - D_\epsilon$ is a morphism $\eta : C_2 \to C_1$, so it induces $T(\eta) : H \otimes H \to H$. Define $\psi(z) : H \to H$ by $\psi(z) \cdot v = T(\eta)(\psi \otimes v)$. This is independent of ζ because ψ is primary. (Note that my $\psi(z)$ is usually called $T(\xi) \circ \psi(z)$.)

§5 Moduli spaces of surfaces

The space $S^1\backslash \mathrm{Diff}(S^1)$ of non-vanishing smooth probability measures on S^1 is generally believed to be a complex manifold because it can be identified with $\mathcal{E}^+\backslash\mathcal{E}$. Let us accept this.

Let $\mathfrak{M}_g$ be the moduli space of closed Riemann surfaces of genus g with a distinguished point. Let X be such a surface. Choose a holomorphic embedding $\bar{D} \to X$ of the disc, centred at the base-point, with boundary a simple closed curve C. There is a surjective holomorphic map

$$S^1\backslash \mathrm{Diff}(S^1) = \mathcal{E}^+\backslash\mathcal{E} \to \mathfrak{M}_g$$

defined by cutting X along C and inserting an element of $\mathcal{E}$.

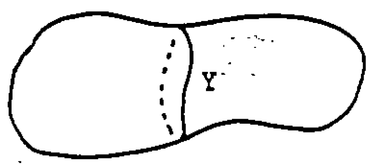
It is known that every positive energy representation H of $\mathrm{Diff}(S^1)$ extends to $\mathcal{E}$, and is the space of holomorphic sections of the bundle $(\mathcal{E} \times H_0)/\mathcal{E}^+$ on $\mathcal{E}^+\backslash\mathcal{E}$, whose fibre is the subspace H_0 of primary fields in H. This bundle comes from a holomorphic bundle on $\mathfrak{M}_g$, for the bundle $\mathcal{E} \to \mathcal{E}^+\backslash\mathcal{E}$ is pulled back from the bundle on $\mathfrak{M}_g$ whose total space is the space of pointed curves with embedded discs. Friedan has argued that holomorphic field theories correspond precisely to those holomorphic bundles on the moduli space of all disconnected surfaces with nodes which are "equivariant" under the action of removing nodes by normalization. I conjecture that such bundles are just the holomorphic representations of $\mathcal{E}$ which extend to holomorphic representations of the category $\mathcal{C}$.

Floer's homology groups

S.K. Donaldson

This talk described some new, unpublished, constructions of A. Floer (Courant Institute) and their application to Yang-Mills instantons over 4-manifolds.

Consider a compact manifold X devided into two pieces by a hypersurface Y :

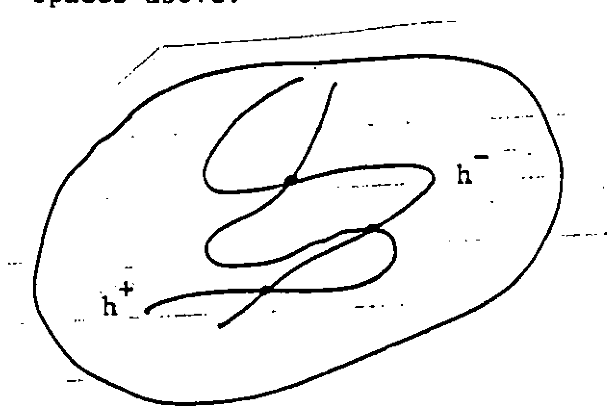


X_1 X_2

If we have a linear differential equation (1st order) on X we can define "Hardy spaces" H_+, H_- : the subspaces of the fields over Y which extend to solutions of the equation over X_1, X_2 repectively. Global solutions on X correspond to intersection points in $H_+ \cap H_-$. For example take the $\bar{\partial}$ operator over $X = S^2 = \mathbb{C} \cup \{\infty\}$. Then

$$H_+ = \{ \sum_{n \geq 0} a_n z^n \} \qquad H_+ \cap H_- = \text{constants.}$$
$$H_- = \{ \sum_{n \leq 0} a_n z^n \}$$

For a non-linear equation we can imagine a curved version of this: a space of fields over Y containing "Hardy submanifolds" h_+, h_- locally modelled on the "$\frac{1}{2}$ sized" spaces above.



We consider the case when $X = X^4$ has a Riemannian metric and the equations are the anti-self-dual equations for a Yang-Mills field over X,

$$F = - * F .$$

The gauge equivalence classes of solutions have a moduli space of dimension

$$8k - 3 - 3\,(b_2^+(X) - b_1(X))$$

(k = top. charge). Suppose, for simplicity, that this number is zero. Then under mild conditions we can define a differential topological invariant $q_X \in \mathbf{Z}$ - the number of points in the moduli space, counted with signs. This is independent of the metric on X.

Suppose $Y \subset X$ is a homology 3-sphere. Floer's theory defines groups

$$HF_{\alpha,y} \qquad \alpha = 0, 1, \ldots , 7$$

- formally the "middle dimensional" homology groups of the space $\mathcal{C}_y$ of all connections over Y. The Hardy sub-manifolds $h_+, h_- \subset \mathcal{C}_y$ carry "fundamental classes"

$$q_{X_1} = [h_+] \in HF_{\alpha,y}$$

$$q_{X_2} = [h_-] \in HF_{\beta,\bar{y}} .$$

There is a dual pairing between $HF_{\alpha,y}$, $HF_{\beta,\bar{y}}$ and the formula

$$q_X = q_{X_1} \cdot q_{X_2} \in \mathbf{Z}$$

holds. That is, the matching problem for ASD connections along Y is abstracted into the intersection pairing in Floer homology.

The groups $HF_{\alpha,y}$ are defined by generalising the Morse theory to a function:

$$\varphi : \mathcal{C}_y \longrightarrow \mathbb{R}/\mathbf{Z} .$$

$\varphi(A)$ is the Chern-Simons invariant of a connection A over Y, it's gradient vector field is $*F_A$ and the integral curves of this - the "steepest descent paths" are solutions $A(t)$ of :

$$\frac{\partial A}{\partial t} = - * F_A .$$

The key observation is that this is exactly the ASD equation for $A(t)$ regarded as a connection over $Y \times \mathbb{R}$.

The critical points of φ are the flat connections, corresponding to representations

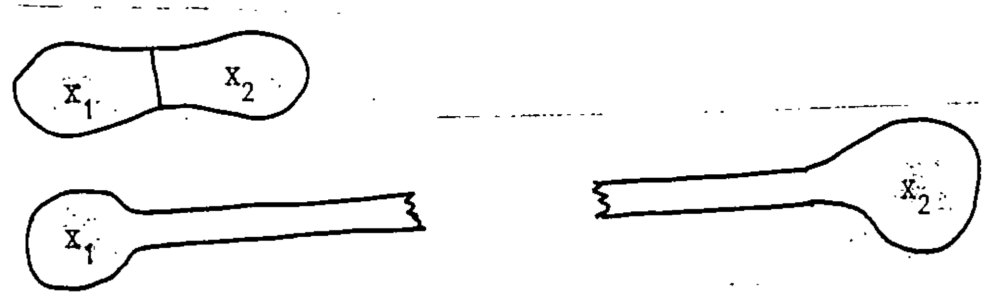
$$\rho : \pi_1(Y) \longrightarrow SU(2) .$$

We assume these form a discrete set. Floer defines a complex

$$(\oplus \mathbf{Z} < \rho >, d)$$

with generators the isomorphism classes of irreducible representations. To each pair ρ, σ we attach a <u>relative index</u> in $\mathbf{Z}/8\mathbf{Z}$. This is the Fredholm index of the linearisation of the ASD connections in the topological sector consisting of connections over $Y \times \mathbb{R}$ asymptotic to ρ at $-\infty$, to σ at ∞. If ρ, σ have relative index 1 the component of d from $< \rho >$ to $< \sigma >$ is defined to be the (signed) number of ASD solutions in this sector. Floer proves that $d^2 = 0$ and the homology groups $HF_{\alpha,y}$ of this complex are independent of the metric on Y.

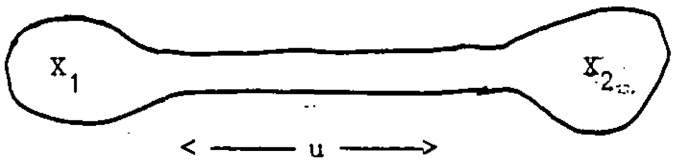
The invariants q_{X_i} are defined by choosing complete metrics on the X_i as shown:



Let n_ρ be the number of solutions to the ASD equations on X_1 asymptotic to ρ at ∞. Then

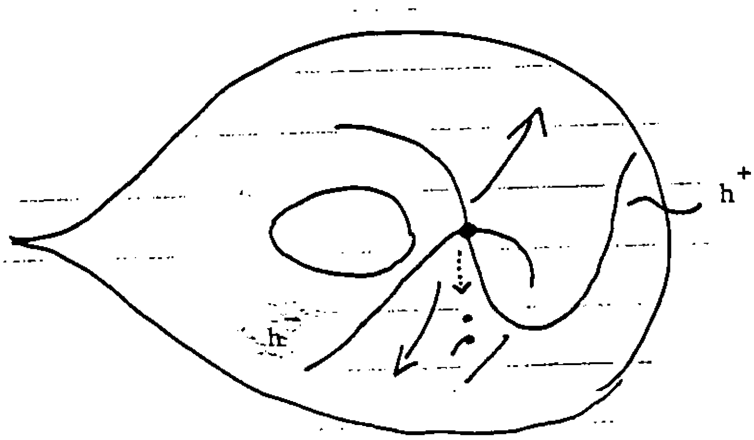
$$\sum n_\rho < \rho >$$

is a cycle in the complex whose class in $HF_{\alpha,y}$ is independent of the metric on X_1. This is defined to be q_{X_1}. Similarly for X_2. The definitions are formally related to the h^+, h^- as follows: deform the metric on X by pulling out the neck:



$u \longrightarrow \infty$

In $\mathcal{C}_y$ this corresponds to flowing h_+ down the gradient field of φ and h_- up the flow. This localises the intersection points $h_+ \cap h_-$ at the critical points ρ :



The Modular Geometry of String Theory and Conformal Field Theory

Daniel Friedan

We (S.H. Shenker and myself) have reformulated two dimensional conformal field theory, and thus string theory, as a certain kind of geometry on the universal moduli space M of stable projective curves. From the intrinsic algebraic geometry of M we construct a natural collection of noncommutative algebras A_c , without identity, indexed by one rational number c . The modular group $\pi_1(M)$ acts naturally by automorphisms of each algebra A_c . A modular geometry is defined to be a bilinear functional $H \in (\bar{A}_c \otimes A_c)^*$ ($\bar{A}_c$ being the complex conjugate algebra), which is (1) multiplicative, i.e., $H(\overline{ab}\, , cd) = H(\bar{a},c)\, H(\bar{b},c)$ (factorization); (2) invariant under $\pi_1(M)$ (modular invariance) and (3) satisfies $H(\bar{a},a) \geq 0$ whenever $a = \bar{a}$ (reflection positivity). The conformal field theories with conformal central charge c are identified with the modular geometries on A_c . The ground states of bosonic string theory are the modular geometries with $c = 26$. Fermionic string theory and superconformal field theory are formulated as perfectly analogous supermodular geometry as the supermoduli space of super Riemann surfaces.

We are motivated by two types. We want to do nonperturbative string theory by extending the modular geometries to some space of infinite genus Riemann surface. This might be accomplished by completing the algebra $(A_c, \pi_1(M))$ using the geometry H . The uniqueness of the string ground state might then be expressed as the weak equivalence of all such completions (for $c = 26$). We also hope to find an arithmetic version of modular geometry over the rational field and its localizations. The algebras A_c are constructed algebraically from the moduli space M , and M is an arithmetic variety, so the A_c are defined over the algebraic integers and an arithmetic version of modular geometry should be possible. It would then be possible to do the real (archimedean) string theory in terms of the global rational theory and its localizations at the finite primes.

Construction of the modular algebras A_c

Let $\mathfrak{M}_g$ be the moduli space of nonsingular, connected projective curves (compact, connected Riemann surfaces) of genus g. Let $\bar{\mathfrak{M}}_g$ be the moduli space of stable connected projective curves of genus g. $\bar{\mathfrak{M}}_g$ is a projective (compact) arithmetic variety. The compactification divisor $\vartheta_g = \bar{\mathfrak{M}}_g - \mathfrak{M}_g$ consists of the curves whose only singularities are nodes such that if the nodes are removed all the resulting components of genus 0 have at least three punctures. For a curve $S \in \vartheta_g$ the normalizing curve $\nu(S)$ is the nonsingular curve which is made by removing all the nodes from S and then erasing all the resulting punctures. A conformal field theory cannot distinguish S from $\nu(S)$ so we would like to identify S with $\nu(S)$. But $\nu(S)$ need not be connected, we must expand the moduli space to include the curves with more than one connected component.

The topologies of curves are described by the sequences of nonnegative integers $\vec{n} = (n_0, n_1, \ldots)$ where by n_g is the number of connected components of genus g. The sequences $\vec{n}$ must satisfy $0 < \sum_g n_g < \infty$.

Let $\mathfrak{R}(\vec{n}) = \prod_{g=0}^{\infty} \mathrm{Sym}^{n_g}(\mathfrak{M}_g)$ be the moduli space of smooth curves of topology $\vec{n}$, $\mathrm{Sym}^n(\cdot)$ being the n-fold symmetric product. Let $\bar{\mathfrak{R}}(\vec{n}) = \prod_{g=0}^{\infty} \mathrm{Sym}^{n_g}(\bar{\mathfrak{M}}_g)$ be the moduli space of stable curves of topology $\vec{n}$. $\vartheta(\vec{n}) = \bar{\mathfrak{R}}(\vec{n}) - \mathfrak{R}(\vec{n})$ is the compactification divisor. Write $\mathfrak{R} = \bigcup_{\vec{n}} \mathfrak{R}(\vec{n})$, $\bar{\mathfrak{R}} = \bigcup_{\vec{n}} \bar{\mathfrak{R}}(\vec{n})$ and $\vartheta = \bigcup_{\vec{n}} \vartheta(\vec{n})$. Normalization is then a natural map $\nu : \vartheta \longrightarrow \mathfrak{R}$. We will define the universal moduli space M to be the quotient $\bar{\mathfrak{R}} \longrightarrow M$ obtained by identifying ϑ with $\mathfrak{R}$ via ν.

Let $\vartheta(\vec{m},\vec{n}) = \vartheta(\vec{n}) \cap \nu^{-1}\mathfrak{R}(\vec{m})$, be the singular curves of topology $\vec{n}$ whose normalizing curves have topology $\vec{m}$. Write $\vec{m} \longrightarrow \vec{n}$ iff $\vartheta(\vec{m},\vec{n}) \neq \phi$. This makes the topologies into a directed system, over which we will take limits.

Write $\bar{\mathcal{R}}'(\vec{n}) = \bar{\mathcal{R}}(\vec{n}) \bigcup_{\vec{m}\to\vec{n}} \bar{\mathcal{R}}(\vec{m})$, $\mathcal{R}'(\vec{n}) = \bigcup_{\vec{m}\to\vec{n}} \mathcal{R}(\vec{m})$, $\mathcal{R}'(\vec{n}) = \mathcal{R}(\vec{n}) \bigcup_{\vec{m}\to\vec{n}} \mathcal{R}(\vec{m})$. Then $\nu(\mathcal{R}(\vec{n})) = \mathcal{R}'(\vec{n})$ so it makes sense to make the quotient $\bar{\mathcal{R}}'(\vec{n}) \longrightarrow M(\vec{n})$ by identifying $\vartheta'(\vec{n})$ with $\mathcal{R}'(\vec{n})$ via ν . $M(\vec{n})$ is compact, and there is a natural map $M(\vec{m}) \longrightarrow M(\vec{n})$ whenever $\vec{m} \longrightarrow \vec{n}$, so we can define the universal moduli space $M = \varinjlim_{\vec{n}\to\infty} M(\vec{n})$ as a direct limit of compact varieties. M is connected because each $\bar{\mathcal{R}}(\vec{n})$ is connected and for any $\vec{m}, \vec{m}'$ there exists $\vec{n}$ such that $\vec{m} \longrightarrow \vec{n}$ and $\vec{m}' \longrightarrow \vec{n}$.

The universal moduli space M is a commutative semigroup without identity, where multiplication is given by the disjoint union of curves. Clearly $M(\vec{n})\, M(\vec{n}') = M(\vec{n} + \vec{n}')$. Write $\hat{0} = (1, 0, 0, \ldots)$. The single point $M(\hat{0}) = M_0$ is a distinguished element in M , the unique curve of genus 0 , the Riemann sphere.

Next, we define the finite coverings of M . By a finite covering $M_\rho \xrightarrow{\rho} M$ we mean a directed system of finite coverings $M_\rho(\vec{n}) \longrightarrow M(\vec{n})$ with $M_\rho = \varinjlim_{\vec{n}\to\infty} M_\rho(\vec{n})$ such that M_ρ is a semigroup (<u>not</u> necessarily commutative) and ρ is of a homomorphism at semigroups. This amounts to a collection of finite coverings $\bar{\mathcal{R}}_\rho(\vec{n}) \longrightarrow \bar{\mathcal{R}}(\vec{n})$ by connected spaces $\bar{\mathcal{R}}_\rho(\vec{n})$, ramified only at $\vartheta(\vec{n})$ and at the singular locus (the curves in $\bar{\mathcal{R}}(\vec{n})$ with nontrivial automorphisms), along with lifts $\nu_\rho : \rho^{-1}(\vartheta(\vec{n})) \longrightarrow \rho^{-1}(\mathcal{R}'(\vec{n}))$ of the normalization map ν .

The universal finite covering space is the inverse limit $M_{cov} = \varprojlim M_\rho$ over the finite coverings. The inverse limit of the covering groups is the fundamental group $\pi_1(M)$, which we might call the universal modular group. M_{cov} is a connected noncommutative semigroup without identity. It is noncommutative because the covering group of $\bar{\mathcal{R}}(\vec{n})$ includes the permutation group $\prod_{g=0}^{\infty} S_{n_g}$. Going to the covering space unwinds the symmetric products $\mathrm{Sym}^{n_g}(\bar{\mathfrak{M}}_g)$.

Let λ_H be the Hodge line bundle on M, i.e. the determinant of the vector bundle of holomorphic differentials on the curves. It can be checked that λ_H is well-defined on M, because it behaves well with respect to ν. In fact, $c_1(\lambda_H)$ generates $H^2(M)$ so all line bundles on M are powers of λ_H. Let $L_c = (\lambda_H)^{c/2}$ for c a rational number. For general c, L_c is not actually a line bundle on M, but it is well-defined on suitable coverings at M, so it lifts to a line bundle L_c on M_{cov}. Since λ_H behaves well under disjoint union of curves, $L_c \longrightarrow M_{cov}$ is a homomorphism of noncommutative semigroups without identity.

Let $\Gamma(M_\rho(\vec{n}), L_c)$ be the holomorphic sections of L_c over the covering $M_\rho(\vec{n})$ of $M(\vec{n})$. This is a finite dimensional space because $M_\rho(\vec{n})$ is compact. Define $A_c(\vec{n},0) = \Gamma(M_\rho(\vec{n}), L_c)^*$ and define the modular algebra to be the double limit

$$A_c = \varprojlim_{\rho} \varinjlim_{\vec{n}} A_c(\vec{n},\rho)$$

$$= \Gamma(M_{cov}, L_c)^* .$$

A_c is a noncommutative algebra without identity because the semigroup structure of L_c makes $\Gamma(L_c, M_{cov})$ into a dual algebra. That is, the multiplication $L_c \times L_c \longrightarrow L_c$ over $M_{cov} \times M_{cov} \longrightarrow M_{cov}$ becomes a dual multiplication $\Gamma(L_c, M_{cov}) \longrightarrow \Gamma(L_c, M_{cov}) \otimes \Gamma(L_c, M_{cov})$ which becomes an (associative) multiplication law $A_c \otimes A_c \longrightarrow A_c$. A_c is bifiltered by the finite dimensional spaces $A_c(\vec{n},\rho)$ where $A_c(\vec{n},\rho)\, A_c(\vec{n},\rho) \subset A_c(\vec{n} + \vec{n}',\rho)$. The modular group $\pi_1(M)$ acts on $\Gamma(M_{cov}, L_c)$ (by permuting the sheets of the covering) so $\pi_1(M)$ acts on A_c by algebra automorphisms. There is a natural element $a_c \in A_c$ given by the restriction map $a_c : \Gamma(M_{cov}, L_c) \longrightarrow \Gamma(M(\hat{0}), L_c)$. Multiplication of sections $(L_{c_1+c_2} = L_{c_1} \otimes L_{c_2})$ provides natural homomorphisms $A_{c_1+c_2} \longrightarrow A_{c_1} \otimes A_{c_2}$.

The algebras A_c, with action of $\pi_1(M)$, provide all the data needed to describe modular geometry and thus conformal field theory. A modular geometry is a bilinear function $H \in (\bar{A}_c \otimes A_c)^*$ satisfying:

1. (Factorization) $H(\overline{ab},cd) = H(\bar{a},c)\ H(\bar{b},d)$

2. (Modular invariance) $H(\gamma\bar{a},\gamma b) = H(\bar{a},b)$ for $\gamma \in \pi_1(M)$

3. (Positivity) $H(\bar{a},a) \geq 0$ whenever $a = \bar{a}$

4. (Normalization on P') $H(\bar{a}_c,a_c) = 1$

Note that H is not assumed hermitian or positive (except on the real elements $a = \bar{a} \in A_c$). There is a natural complex conjugation $A_c \longrightarrow \bar{A}_c$ because there is a natural complex conjugation on moduli space (but over the rationals complex conjugation is only determined up to conjugation in the Galois group $Gal(\bar{Q}/Q)$. A modular geometry H can be regarded as an element in $\overline{\Gamma(M_{cov},L_c)} \otimes \Gamma(M_{cov},L_c)$ so, by multiplication of sections, determines a section $Z(H) \in \Gamma(M_{cov}, \overline{L_c} \otimes L_c)$. But invariance of H under $\pi_1(M)$ implies that $Z(H)$ really lives on M , i.e. $Z(H) \in \Gamma(M,\bar{L}_c \otimes L_c)$. $\bar{L}_c \otimes L_c$ is a well-defined real line bundle over M . $Z(H)$ is the partition function of the conformal field theory corresponding to the modular geometry H . The factorization condition on H implies the factorization of the partition function $Z_S Z_{S'} = Z_{SUS}$ for curves S, S' . The normalization condition on H is needed because the partition function in the sphere should be $Z_{P'} = 1$. The positivity condition on H is equivalent to unitarity of the corresponding conformal field theory.

It is known that conformal field theories and the modular geometries exist only for certain values of c . For example, if $c < 1$ then c must be of the form $c = 1-6/m(m+1)$ for $m \in \mathbf{Z}$, $m \geq 2$. Examples are known for all of these values of c in the discrete series, and for many values of $c \geq 1$. (All known conformal field theories have c rational.) The set of c for which modular geometries exist form an additive semigroup of the nonnegative rational numbers (including the above discrete series). The maps $A_{c_1+c_2} \longrightarrow A_{c_1} \otimes A_{c_2}$, coming from $L_{c_1+c_2} = L_{c_1} \otimes L_{c_2}$, make the set of modular geometries also a semigroup. This is just the tensor product of conformal field theories. One rather ambitious project is to find generators and relations for the semigroup of modular geometries. In addition to the discrete series, examples of modular geometry are given by the conformal field theory made from the Calabi-Yau spaces with vanishing first Chern class and those made from representations of affine algebras.

The modular geometry formulated here should be called rational modular geometry because only finite coverings of moduli space are allowed. The corresponding "rational" conformal field theories are dense in the space of all known conformal field theories, except possibly those made from smooth Calabi-Yau spaces. This motivates the following conjecture:

Conjecture: The space of rational modular geometries is an arithmetic space whose completion over the real numbers is the space of all real conformal field theories.

References

An earlier version of modular geometry was described in papers by D. Friedan and S.H. Shenker which have appeared in Physics Letters, Nuclear Physics B and Physica Scripta over the past year.

STRINGS ON ORBIFOLDS

C. Vafa

In attempts to reconcile string theory with experiments one approach is to consider a compactification scheme. Toroidal orbifolds, which are quotients of a torus by finite groups, provide examples of essentially flat compact spaces on which the string theory can be solved exactly. These provide toy models for string compactification, which could potentially describe the real world.

One can compute some interesting topological invariants for orbifolds. For example the Euler Characteristic of an orbifold is given by (computed using string theory):

$$\chi(M/G) = \frac{1}{|G|} \sum_{gh=hg} \chi(M_{g,h})$$

Where $M_{g,h}$ is the subspace of M fixed by both g and h. This agrees with the Euler characteristic of the Elliptic Cohomology computed by Hopkins and Ravenel. It is in fact the equivariant Euler characteristic of the loop space with respect to the S^1-action on the loop space which reparametrizes the loop.

In string computations on orbifolds, at the torus level, one naturally encounters interesting modular forms, such as the Klein form. These can be used to construct modular units for higher level subgroups of $SL(2)$. It is known that certain quadratic relations are needed in this construction. These relations can be understood as the condition for the vanishing of $(1/2)p_1$ of the corresponding representation of the orbifold group (in the case where the group is cyclic), by relating it to absence of global 2-d anomalies.

Probably the most interesting orbifolds are asymmetric orbifolds, for which roughly speaking, the left-movers are on one orbifold, and right-movers on a different orbifold. Strictly speaking, these correspond to string theories which cannot be thought of as propagating in a background target space. This can be shown by constructing examples in which the character valued index of Rarita-Schwinger (and similar operators) is not a constant, whereas in the target space case the character is a constant. Also, it is interesting to note that the moonshine module is the Hilbert space of an asymmetric orbifold based on the Leech lattice.

We also note that orbifolds provide examples in which one can explicitly evaluate the elliptic genus.

REFERENCES

1. L. DIXON, J. HARVEY, C. VAFA, and E. WITTEN, "Strings on Orbifolds", Nucl. Phys. B274 (1986) 285.
2. D. FREED, and C. VAFA, "Global Anomalies on Orbifolds", to appear in Comm. Math. Phys.
3. K.S. NARAIN, M.H. SARMADI, and C. VAFA, "Asymmetric Orbifolds", Harvard preprint, HUTP-86/A089.

Non commutative differential geometry

A. Connes

We first discuss the example of a non commutative topological space Y which arises as the space of labels for the Penrose tilings: Y is the quotient of $K = \{(a_\nu)_{\nu\in\mathbb{N}}, a_\nu \in \{0,1\}$, with $a_{\nu+1} = 0$ if $a_\nu = 1\}$ by the equivalence relation $a \sim b$ iff $a_\nu = b_\nu$ for all ν but finitely many. The algebra $C(Y)$ of continuous functions on Y is now replaced by a non commutative C^* algebra A, we compute its K theory as an ordered group and find $\mathbf{Z}^2$ ordered by $\{(n,m); n+m\alpha > 0\}$ where α is the golden number. This space is 0-dimensional and by the results of Bratteli, Chen, Effros, Handelman and Elliott it is classified by the above K theory group. We next discuss the non commutative torus with phase Θ whose algebra A_Θ is presented by two unitaries U and V satisfying the relation $VU = \exp 2\pi i\Theta\, UV$. The Rieffel-Powers projection is of the form $e = f_{-1}(U)V^{-1} + f_0(U) + f_1(U)V$ and the first Chern class formula $c_1(e) = \frac{1}{2i\pi}\,\tau(e(\delta_1(e)\delta_2(e) - \delta_2(e)\delta_1(e)))$ gives an integer; where δ_1 and δ_2 are the differentiations $\delta_1(U^nV^m) = 2\pi in\, U^nV^m$, $\delta_2(U^nV^m) = 2\pi im\, U^nV^m$ and where τ is the trace $\tau(U^nV^m) = 0$ if $(n,m) \neq 0$, $\tau(1) = 0$. We discuss the link, due to Bellissard, between the integrality result and the integrality of the conductivity in the Quantum Hall effect. This non commutative space has a "manifold shadow" obtained by looking at the moduli space for connexions on bundles which minimize the functional $\tau(\Theta^2)$ where Θ is the curvature. The result is:

Thm. (M. Rieffel, A.C.) 1) Let $\mathcal{E}$ be a projective module over A_θ then $\exists\, \mathcal{E}_1$ irreducible such that $\mathcal{E} = \underbrace{\mathcal{E}_1 \oplus \dots \oplus \mathcal{E}_1}_{n \text{ times}}$. 2) The moduli space for minimal connexions on $\mathcal{E}$ modulo the gauge group is the quotient of the torus $(\Pi^2)^n$ by the action of the symmetric group.

We then explain how the basic data (h, F, ε) in K homology gives rise to quantized differential forms, where ordinary differentials da are replaced by operator commutators $da = i[F,a]$. We then discuss the multiplicative analogue of K homology as a possible mathematical formulation of quantum field theory. We give evidence for that in the low dimensional case where the space is S^1 and where we construct using the V. Jones algebras associated with the numbers $4\cos^2\frac{\pi}{m}$, an analogue of the second quantization (based on Clifford algebras). The basic formulae there are the following, the Fourier components of a current are obtained by setting T_n to be the derivation of the C^* algebra generated by the e_i's , defined by: (*)

$$T_n(y) = \sum [x_k^{(n)}, y]$$

where $x_k^{(n)}$ is defined by induction, as $x_k^{(n+1)} = [x_k, x_{k+1}^{(n)}]$ and $x_k^{(1)} = \frac{1}{2} e_k(e_{k+\frac{1}{2}} - \tau)e_{k+1}$, $x_k^{(0)} = e_k$.

(*) here the e_i's are the V. Jones projections, $i \in \frac{1}{2}\mathbf{Z}$, with $e_i e_{i\pm\frac{1}{2}} e_i = \tau e_i$, $e_i e_j = e_j e_i$ if $|i-j| \geq 1$.

"On the local euclidean path connectivity of configuration spaces for quantum gravity"

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Abstract:

It is argued that local euclidean path connectivity for the configuration space of a quantum theory, namely the property that two sufficiently close points can be connected by a solution to the euclidean field equations having them as boundary values, is important for a path integral formulation of the theory and thus should be checked for any candidate configuration space.

It is shown the existence of at least one candidate configuration space for quantum gravity with the above property. This is not only done to indicate the type of analysis that should be carried out for each candidate, but also to study some of the properties we expect to be generic for configuration spaces with the above property.

I. Introduction:

Inspite of intensive work for several decades we still lack a quantum theory of gravity. I personally blame for this failure the lack of any experimental evidence to guide us, but realizing that this is a very unproductive stand I shall advocate here that perhaps we should blame the lack of some deeper understanding of space-time and that perhaps by trying different ways of describing it we could find one where quantization is possible. Using our present understanding of the quantization process I shall advocate then that it is a question of choosing the right configuration space. In this spirit it is then important to find suitable properties to impose on the possible configuration spaces and thus to reduce the member of bailable candidates.

In the path integral formulation of quantum field theories the probability of a system to evolve from a point A in configuration space to a point B is given by an "integral" of the exponential of the action functional over all path connecting A with B. In general such an integral is not well defined and so a related one, where the usual action is replaced by the euclidean one, is employed. Calculations are then usually performed using a saddle point approximation which essentially consists of an expansion around a classical euclidean path connecting A with B, (that is a solution to the classical euclidean field equations with points A and B as boundary conditions), provided such a path exists.

The existence of such a path is not only important for computational purposes but also for the formal aspects of the theory since in general we know how to make sense of the euclidean path integral only in the sense of the above approximation. Thus we would like to impose the existence of such paths as a primary property to be satisfied for the candidate configuration spaces. As it stands this property is probably too strong (For non-linear theories we expect, for large enough field configurations, the appearance of singularities.) and so we shall impose a weaker one, namely the local euclidean path connectivity, which only demands the existence of such paths for close enough points in configuration space.

In what follows we explore this property for a particular configuration space for quantum gravity. The aim is not so much to advocate for this particular configuration space as the appropriate one for quantization but rather to indicate the type of analysis that should be carried out for each candidate. In section II we define a set of configuration variables and use them to reduce the field equations to an elliptic system. We then, in section III, apply the theory of elliptic equations to show existence of solutions with these variables as boundary data. We only consider boundary data sufficiently close to the trivial one, (that is we are only considering the above property for a small enough neighborhood of the flat configuration), other cases require a more elaborated treatment but the same fundamental analysis.

II. Elliptic Reduction for the Euclidean Einstein Equations

Following a similar procedure to the usual one for the Lorentzian case we fix a flat positive definite four-metric e_{ab} in M, a slab region (w.r.t. e_{ab}) in $^4\mathbb{R}$, and define for any other metric g_{ab} in M: $\phi^{ab} := e^{ab} - \sqrt{g}\, g^{ab}$ and $\psi^a := \nabla_b\phi^{ab}$, where $\sqrt{g}$ is defined by $\epsilon_{abcd}(g) = \sqrt{g}\, \epsilon_{abcd}(e)$, and ∇_c is the covariant derivative associated with e_{ab}, $\nabla_c e_{ab} = 0$.

In terms of ϕ^{ab} and ψ^b the Einstein tensor becomes,

(1) $\sqrt{g}\, G^{ab}(g) := E^{ab}(\phi) + (\text{terms in } \nabla_c\psi^e, \psi^e)$.

The advantage of this decomposition is that E^{ab} is an elliptic map, and as we shall see by imposing certain boundary conditions on ϕ^{ab} we can make ψ^b to vanish on M which implies that any solution to $E^{ab}(\phi) = 0$ satisfying certain boundary conditions is also a solution to the Euclidean Einstein equations.

Reduction Theorem:
Let (M, e_{ab}) be an Euclidean slab region of $^4\mathbb{R}$. Then there exists a neighborhood, v, of zero in $H_{3,-1}(M)$, such that for any $\phi^{ab} \in V$ satisfying, in a weak sense, $E^{ab}(\phi) = 0$, $\tau(n^a n_b \nabla_a \psi^b) = 0$, and $\tau(h^a{}_b\psi^b) = 0$ we also have $G^{ab}(g(\phi)) = 0$.

Here τ denotes restriction of fields in M to fields in ∂M. The vector n^a is the translation w.r.t e_{ab} normal to ∂M, and $h_{ab} := e_{ab} - n_a n_b$. $H_{3,-1}(M)$ denotes a weighted Sobolev space [Y. Choquet-Bruhat & D. Christodoulou].

Proof:
We must show that the terms in ψ, and $\nabla\psi$ in (1) vanish. This is done by showing that ψ itself vanishes in M. Using Bianchi's identity and the vanishing of $E^{ab}(\phi)$ we obtain a linear, elliptic equation for ψ^b, $L^b{}_{(\phi)}(\psi) = 0$. Since $(\phi^{ab}, \psi^c) \to L^b{}_{(\phi)}(\psi)$ is a continuous map from $H_{3,-1}(M) \times H_{2,0}(M)$ to $H_{0,2}(M)$, $L^b{}_{(\phi=0)}(\psi) = \Delta_e\psi^b$ is an infective map for the boundary conditions $\tau(n_b n^a \nabla_a \psi^b) = \tau(h^a{}_b\psi^b) = 0$, and the space of injective operators is open, we conclude there exist a neighborhood of zero, $V \subset H_{3,-1}(M)$ such that if $\phi \in V$, then L_ϕ is also injective and therefore $\psi^b = 0$ on M.

We could have choosen for the above theorem any other boundary condition ensuring the injectivity of the flat Laplacian. The relevance of this choice will be appreciated in the next section.

III. Existence of Solutions

Here we use the implicit function theorem and the theory of linear elliptic equations to show existence of solutions to the reduced system for small enough boundary data.

Existence Theorem:

Let (M, e_{ab}) be an Euclidean slab region of $\mathbb{R}^4$. There exists a neighborhood of zero, $V \in H_{5/2,-1}(\partial M)$ such that for any $\hat{\phi}^{ab} \in V$, a symmetric 2-tensor in ∂M, with $D_a \hat{\phi}^{ab} = 0$, there exists a unique $\phi^{ab} \in H_{3,-1}(M)$ satisfying,

(2)
$$\text{a)} \quad E^{ab}(\phi) = 0$$
$$\text{b)} \quad \tau(n_b n^a \nabla_a \psi^b) = 0$$
$$\text{c)} \quad \tau(h^a{}_b \psi^b) = 0$$
$$\text{d)} \quad \tau(h^a{}_c h^b{}_d \phi^{cd}) - \hat{\phi}^{ab} = 0.$$

Here D_a is the invariant derivative in ∂M associated with the flat induced metric h_{ab}. In equation 2.b) all second order normal derivatives have been substituted using 2.a). Condition $D_a \hat{\phi}^{ab} = 0$ is no loss of generality.

The set of all symmetric two-tensors $\hat{\phi}^{ab}$ in $H_{5/2,-1}(S)$, where S is one of the sides of the slab, such that $D_a \hat{\phi}^{ab} = 0$ is the particular configuration space we mentioned in the introduction.

Here we merely sketch the proof of the above theorem. A detailed proof can be found in [O. Reula]. Consider the system of equations (2) as a function of ϕ^{ab} and $\hat{\phi}^{ab}$, $F(\phi, \hat{\phi})$ into a certain product of Hilbert spaces Y. We have $F(0,0) = 0$ and it is not difficult to show, using certain properties of weighted Sobolev spaces that F is a c^1 function of both arguments. It can also be shown, using the theory of linear elliptic equations that the differential of F with respect to ϕ at $\phi = \hat{\phi} = 0$, $DF_\phi(0,0)$, is an isomorphism between $H_{3,-1}(M)$ and Y. But then the implicit function theorem accerts the existence of a neighborhood of $H_{5/2,-1}(\partial M)$, V, such that there exist a C^1 function $\hat{\phi} \rightarrow \phi(\hat{\phi})$ from V to $H_{3,-1}(M)$ satisfying $F(\phi(\hat{\phi}), \hat{\phi}) = 0$, and the theorem is proven.

IV. Conclusion

We have presented a configuration space that at least in a neighborhood of the trivial configuration satisfy local euclidean path connectivity. Can this result be generalized to arbitrary points in configuration space? This configuration space is in some sense the tangent space to the one of interest and it is for this second one that the question should be formulated. For this one we believe that the answer is afirmative. Is this configuration space big enough, in the sense that any sufficiently small solution to the euclidean field equations can be obtained as a path connecting two points on it? If we restrict the class of solutions to be asymptotically Schwarzschild up to order $1/r$ then that is the case. On the other hand, if we assume that our solutions admit a "powers of r" asymptotic expansion then one can show they have this asymptotic Schwarzschild behaviour. This fact also suggest that we cannot connect to three-metrics with different masses, in the ADM sense, and so for a configuration space to be a bailable candidate it must be that its elements do not have a definite mass.

This configuration variables, namely the set of all divergence free, symmetric 2-tensors in $H_{5/2,-1}(S)$, have been stripped of almost all its gauge freedom, and so it is a gauge dependent space. The remaining gauge freedom can be fixed by requiring for example that the tensors be trace free. This fixing requires gauge transformations (diffeomorphisms) which also move the boundaries and so they are related to the time slicing or parametrization of the theory. It does not seem to be possible to obtain, from these variables, gauge independent, i.e. fully geometrical quantities. They can be obtained once a whole classical solution is computed, but for that we need two points in configuration space. It is clear then that in the quantum domain we could obtain them only in a semiclassical approximation. It would be nice to obtain a configuration space consisting of geometrical objects and with similar properties to the one above.

In the existence theorems above we have ignored the constraints, what happened with them? The equations that we call constraints in the hyperbolic case still are there in the elliptic case, but they are not constraints anymore. They can constrain initial data, for they have up to first order normal derivatives, but not boundary values. As in the hyperbolic case they can be obtained in the reduced picture as normal derivatives of ψ^b. Thus, $\tau(n_a n^c \nabla_c \psi^a)$ is the scalar constraint and $\tau(n^c h^b_d \nabla_c \psi^d)$ the vectorial one.

The scalar one is part of the reduced system, equation 2.b), while the satisfied! Note that the reason one can solve at all for the scalar constraint is that it contains only quadratic terms in the normal derivates (no linear ones), and since we take the differential at zero normal derivative they do not appear. This point is important, for if we would try to show existence for other regions where we could not choose zero normal derivatives (as is the case for a ball), we would have to choose different boundary variables.

Acknowledgement

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References

Y. Choquet-Bruhat and D. Christodoulou, Acta Mathematica, Vol. 146, 1981

O. Reula, "A Configuration Space for Quantum Gravity and Solutions to the Euclidean Einstein Equations in a Slab Region", to appear in Classical and Quantum Gravity.

Remarks on the Landweber-Stong elliptic genus

Don Zagier

In algebraic topology one studies genera, which are ring homomorphisms from the oriented bordism ring Ω_*^{SO} to a $\mathbb{Q}$-algebra R (commutative, with 1). To a genus $\varphi\colon \Omega_*^{SO} \to R$ one associates the following three power series with coefficients in R:

(i) $g(x)$, an odd power series with leading term x, the logarithm of the formal group law of φ (this means that the formal group, whose definition we do not repeat here, equals $g^{-1}(g(x)+g(y))$). It is given explicitly by $g(x) = \sum_{n=0}^{\infty} \varphi(\mathbb{CP}^{2n}) \frac{x^{2n+1}}{2n+1}$ and hence, since the classes of the $\mathbb{CP}^{2n}$ generate $\Omega_*^{SO} \otimes \mathbb{Q}$, determines φ completely.

(ii) $P(u)$, an even power series with leading term 1, the Hirzebruch characteristic power series of φ. This means that if $\mathcal{P}$ denotes the stable $H^*(\cdot;R)$-valued exponential characteristic class on oriented bundles characterized by $\mathcal{P}(\xi) = P(c_1(\xi))$ if ξ is a complex line bundle (regarded as a real 2-plane bundle), then the genus of an arbitrary oriented manifold M is obtained by evaluating $\mathcal{P}(TM)$ on the homology fundamental class of M.

(iii) $F(y)$, a power series with leading term 1, the KO-theory characteristic power series of φ. This means that if $\mathcal{F}$ denotes the stable $KO(\cdot)\otimes R$-valued exponential characteristic class on oriented bundles characterized by $\mathcal{F}(\xi) = F(\xi-[2])$ for as above (this makes sense because $\xi-[2]$ is nilpotent in $KO(B_\xi)\otimes R$, as one sees by applying the complexified Chern character), then the genus of an arbitrary Spin manifold is obtained by evaluating $\mathcal{F}(TM)$ on the KO-homology fundamental class of M.

These three power series determine one another by the formulas

$$(1) \qquad \frac{u}{g^{-1}(u)} = P(u) = \frac{u/2}{\sinh u/2}\, F(e^u + e^{-u} - 2) \,,$$

where g^{-1} denotes the inverse power series of g.

Recently, a particular class of genera has come into prominence through the work of Landweber, Stong, Ochanine, Witten and others. These genera are characterized topologically by the property that $\varphi(M)$ vanishes if M is the total space of the complex projective bundle associated to an even-dimensional complex vector bundle over a closed oriented manifold, and numerically by the property that the power series $g'(x)^{-2}$ is a polynomial of degree ≤ 4, i.e., that

$$(2) \qquad g(x) = \int_0^x \frac{dt}{\sqrt{1-2\delta t^2+\varepsilon t^4}} \qquad \text{for some } \delta,\varepsilon \in R .$$

(The equivalence of these two definitions is due to Ochanine [3].) Since this is an elliptic integral, such φ are called elliptic genera. Landweber and Stong [2] discovered that there is a particular elliptic genus with values in the power series ring $R = \mathbb{Q}[[q]]$ satisfying:

(a) For $r \geq 1$ the coefficient of y^r in $F(y)$ belongs to $q^{2r-1}R$.

(This, or rather the weaker statement that the coefficient of y^r is divisible by q^{r+1} for $r \geq 2$, arises from a certain natural property of the above-mentioned KO-characteristic class $\mathcal{F}$ which we do not formulate here.) Based on numerical computations, they conjectured that condition (a) characterizes the genus in question up to a reparametrization (i.e., up to replacing q by $aq+bq^2+\dots$ with $a \neq 0$) and that with a suitable choice of parameter one has

(b) $F(y)$ has coefficients in $\mathbb{Z}[[q]]$.

By what was said in (iii), this means that the genus takes on values in $\mathbb{Z}[[q]]$ for all Spin manifolds. These facts were proved by D. and G. Chudnovsky [1], whose formulas show that with a suitable normalization one also has

(c) The leading term of the coefficient of y^r in $F(y)$ for $r \geq 1$ is $-q^{2r-1}$, and

(d) The genus takes values in the subring $M_*^{\mathbb{Q}}(\Gamma_0(2)) \subset \mathbb{Q}[[q]]$ of modular forms on $\Gamma_0(2)$ with rational Fourier coefficients.

(We recall basic definitions about modular forms below.) In particular, the δ and ε of equation (2) are certain modular forms (of weights 2 and 4); since $M_*^{\mathbb{Q}}(\Gamma_0(2))$ is known to be the free polynomial algebra on δ and ε, it follows that the Landweber-Stong genus is universal for all elliptic genera. This universal, modular form-valued elliptic genus has been the object of considerable interest; it gives rise to new cohomology theories (the "elliptic cohomology" of Landweber, Stong, and Ravenel) and to connections with index theory, string theory, etc. The purpose of the lecture was to describe a variety of formulas for the power series g, P, and F associated to the Landweber-Stong genus (and in particular, easy proofs of the properties (a)-(d)). The proofs use ideas from the theory of elliptic functions and modular forms.

THEOREM. Let $R=\mathbb{Q}[[q]]$. Then the following five formulas define the same power series $P(u) \in R[[u]]$:

$$(3) \qquad P(u) = 1 - \sum_{\substack{k>0 \\ 2|k}} \frac{G_k^*}{2^{k-2}(k-1)!} u^k ,$$

$$(4) \qquad P(u) = \exp\Bigl(\sum_{\substack{k>0 \\ 2|k}} \frac{2\widetilde{G}_k}{k!} u^k\Bigr) ,$$

(5) $$P(u) = \frac{u}{g^{-1}(u)}$$ with g given by (2) ,

(6) $$P(u) = \frac{u/2}{\sinh u/2} \prod_{n=1}^{\infty} \left[\frac{(1-q^n)^2}{(1-q^n e^u)(1-q^n e^{-u})}\right]^{(-1)^n} ,$$

(7) $$P(u) = \frac{u/2}{\sinh u/2} \cdot \left[1 - \sum_{r=1}^{\infty} a_r \, (e^u + e^{-u} - 2)^r \right] ,$$

where G_k^*, $\widetilde{G}_k$, δ, ε and $a_r \in R$ are defined by

$$G_k^* = G_k^*(q) = \frac{2^{k-1}-1}{2k} B_k + \sum_{n\geq 1} \Big(\sum_{\substack{d|n \\ 2\nmid d}} d^{k-1} \Big) q^n ,$$

$$\widetilde{G}_k = \widetilde{G}_k(q) = -\frac{1}{2k} B_k + \sum_{n\geq 1} \Big(\sum_{d|n} (-1)^{n/d} d^{k-1} \Big) q^n ,$$

$$\delta = -3\,G_2^* = 3\,\widetilde{G}_2 = -\frac{1}{8} - 3 \sum_{n\geq 1} \Big(\sum_{\substack{d|n \\ 2\nmid d}} d \Big) q^n ,$$

$$\varepsilon = -\frac{1}{6}(G_4^* + 7\,\widetilde{G}_4) = \sum_{n\geq 1} \Big(\sum_{\substack{d|n \\ 2\nmid n/d}} d^3 \Big) q^n ,$$

(8) $$a_r = \sum_{m\geq 1} \frac{q^{m(2r-1)}(1+q^{2m})}{(1-q^{2m})^{2r}} = \sum_{n\geq 1} \sum_{\substack{d|n \\ 2\nmid d}} \left[\binom{\frac{1}{2}(d-1)+r}{2r-1} + \binom{\frac{1}{2}(d-3)+r}{2r-1} \right] q^n$$

(here $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, ... are Bernoulli numbers and $\sum_{d|n}$ denotes a sum over positive divisors of n). The genus with characteristic power series $P(u)$ satisfies properties (a)-(d).

Each of the five formulas in the theorem describes some aspect of the genus with characteristic power series $P(u)$: (3) and (4) describe the genus in cohomology and make the modularity property (d) clear (since G_k^* and $\widetilde{G}_k$ are the Fourier expansions of well-known Eisenstein series, as recalled below), (5) shows that the genus is elliptic, and (6) and (7) describe the genus in K-theory and (both) make the properties (a)-(c) evident. (To deduce (a) and (c) from (6) one has to split off the terms $n=1$ and $n=2$ from the infinite product.) Formula (6) was given by the Chudnovsky's, but with a different proof. It has been generalized by Witten [4] to get other genera whose coefficients are modular forms, and in this form interpreted by him, using ideas from quantum field theory, as the equivariant index formula (Atiyah-Bott-Singer fixed point theorem) for a Dirac operator on the free loop space of a manifold.

References

[1] D.V. Chudnovsky, G.V. Chudnovsky:
Elliptic modular functions and elliptic genera
to appear in Topology

[2] P.S. Landweber, R.E. Stong:
Circle actions on Spin manifolds and characteristic numbers
To appear in Topology

[3] Serge Ochanine:
Sur les genres multiplicatifs définis par des intégrales elliptiques
Topology 26 (1987), 143-151

[4] E. Witten:
The index of the Dirac operator in loop space
To appear in Lecture Notes in Mathematics

Some remarks on elliptic genera of almost parallelizable manifolds

F. Hirzebruch

Let M^{4k} $(k \geq 2)$ be a compact combinatorial almost-smooth and almost-parallelizable manifold, i.e. M^{4k} is smooth and parallelizable outside a point. Then the Pontrjagin numbers are well-defined as rational numbers, they vanish except possibly $p_k[M^{4k}]$. There exists an M^{4k} with signature $\pm 8a$ (using E_8-plumbing) or one can use

$$(*) \qquad z_1^2 + \ldots + z_{2k-1}^2 + z_{2k}^3 + z_{2k+1}^{6a-1} = \varepsilon \;, \quad \varepsilon \neq 0 \text{ and small}$$

$$\sum z_j \bar{z}_j \leq 1$$

whose boundary is a topological sphere which can be collapsed to a point (see E. Brieskorn, Beispiele zur Differentialtopologie von Singularitäten, Invent. math. 2 (1966), 1-14, compare F. Hirzebruch und K.H. Mayer, O(n)-Mannigfaltigkeiten, exotische Sphären und Singularitäten, Lecture Notes in Math. 57, Springer-Verlag 1968). We have

$$\hat{A}(M^{4k}) = 2^{-4k} A(M^{4k}) =$$

$$= -\operatorname{sign}(M^{4k})/2^{2k+1}(2^{2k-1}-1)$$

(see F. Hirzebruch, Neue topologische Methoden ... 1956, § 1).

The elliptic genus (Landweber-Stong, Ochanine, E. Witten) of M^{4k} can be calculated. We take the one of E. Witten which relates to modular forms with respect to the full modular group (cf. preceding lecture of D. Zagier) and obtain (for the non-stable genus)

$$q^{-k/6} \cdot \hat{A}(M^{4k}, \prod_{n=1}^{\infty} \sum_{r=0}^{\infty} q^{nr} S^r T) = \hat{A}(M^{4k}) \cdot \frac{E_{2k}}{\Delta^{k/6}}$$

where T is the complex extension of the tangent bundle of M^{4k}, and $\Delta = q \prod_{n=1}^{\infty} (1-q^n)^{24}$, $E_{2k} = 1 - \frac{4k}{B_{2k}} \sum_n (\sum_{d|n} d^{2k-1})q^n$. If

sign $M^{4k} = -2^{2k+1}(2^{2k-1}-1)$ and hence $\hat{A}(M^{4k}) = 1$, we get

$$q^{-k/6} \cdot \hat{A}(M^{4k}, \prod_{n=1}^{\infty} \sum_{r=0}^{\infty} q^{nr} S^r T) = \frac{E_{2k}}{\Delta^{k/6}} .$$

The smallest signature $8a$ (in absolute value) for which the elliptic genus or equivalently $\hat{A}(M^{4k}) \cdot E_{2k}$ has integral coefficients is

$$2^{2k+1}(2^{2k-1}-1) \quad \text{numerator} \quad (\frac{B_{2k}}{4k}) .$$

In fact the $\hat{A}$-genus is integral and divisible by a_k ($=2$ if k is odd, $=1$ if k is even) if M^{4k} is smooth. The boundary sphere is standard if and only if

$$a \equiv 0 \mod a_k 2^{2k-2}(2^{2k-1}-1) \quad \text{numerator} \quad (\frac{B_{2k}}{4k}) .$$

The number on the right side of the congruence equals the order of the Kervaire-Milnor group bP^{4k} of those exotic spheres which bound a parallelizable manifold. If we take $a = |bP^{4k}|$, then we obtain the Kervaire-Milnor manifold M_0^{4k} (smooth, almost parallelizable with smallest $|\text{signature}| \neq 0$) . We have

$$\hat{A}(M_0^{4k}) = \pm a_k \quad \text{numerator} \quad (\frac{B_{2k}}{4k})$$

(first occurrence of M_0^{4k} in John W. Milnor and Michel A. Kervaire, Bernoulli numbers, homotopy groups, and a theorem of Rohlin, Proc. Intern. Congress Math. 1958).

For M_0^8 with $\hat{A}(M_0^8) = 1$ the above elliptic genus equals

$$\sqrt[3]{j} = q^{-1/3}(1 + 248q + 4124q^2 + \dots)$$

$$\hat{A}(M_0^8) = 1 \ , \ \hat{A}(M_0^8,T) = 248 \ ,$$

$$\hat{A}(M_0^8,S^2T) = 3876 = 1 + 3875 \ .$$

The elliptic genus for M^{4k} has integral coefficients if and only if $\hat{A}(M^{4k})$ and $\hat{A}(M^{4k},T)$ are integral. This is the case if and only if $\hat{A}(M^{4k})$ and $ch(T)[M^{4k}]$ are integral where ch is the Chern character. For smooth almost-parallelizable manifolds, the numbers $\hat{A}(M^{4k})$ and $ch(T)[M^{4k}]$ are integral (for k odd, both numbers are even). For $ch(T)[M^{4k}]$ this follows from Bott periodicity. Kervaire's and Milnor's result on the order of bP_{4k} implies, conversely, that M^{4k} is smoothable if and only if the two numbers are integral (and even, if k is odd). Thus M^{4k} has an elliptic genus with integral coefficients (even coefficients, if k is odd) if and only if M^{4k} is smoothable.

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Virasoro Representations and Modular Invariance

P. Goddard

The critical behavior of two-dimensional statistical systems is described (in suitable cases) by a two-dimensional Euclidean conformal quantum field theory [BPZ]. Acting in the space of states of such a theory we have two commuting copies $\{L_n\}$, $\{\bar{L}_n\}$ of the Virasoro algebra $\hat{V}$ possessing the same value of the central charge c

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m,-n} ,$$

$(m,n \in \mathbf{Z})$ with $L_n^+ = L_{-n}$ and the spectrum of L_o bounded below. The value of c is characteristic of the theory. The representation of $\hat{V}$ is a direct sum of highest weight representations built up from states $|\hbar>$ satisfying

$$L_o|\hbar> = h|\hbar> ; \; L_n|\hbar> = 0 \; , \; n > 0 \; .$$

Such representations are labelled by (c,h). The analysis of for which (c,h) unitary representations exist started with the Kac determinant formula. From this it was shown [FQS] that for a unitary representation (in a positive inner product space) it is necessary that <u>either</u> $c \geq 1$, $h \geq 0$ <u>or</u>

$$c = 1 - \frac{6}{(m+2)(m+3)} \; , \; m = 0,1,2,\ldots \; ; \; h = h_{p,q} = \frac{[(m+3)p-(m+2)q]^2-1}{4(m+2)(m+3)}$$

$p = 1,\ldots,m+1$; $q = 1,\ldots,p$. The former (continuous) series are easily seen to exist and the construction of the latter (discrete) series is sketched below.

In a conformal field theory we have states $|A>$ which are highest weight states for both $\{L_n\}$, $\{\bar{L}_n\}$, with $\hbar = \hbar_A$, $\bar{\hbar}_A$, resp., from which all states are generated by $\{L_n\}$, $\{\bar{L}_n\}$. Corresponding to each such state is a primary field $\varphi_A|z>$ satisfying

$$[L_n,\varphi_A(z)] = z^{n+1}\frac{d}{dz}\varphi_A + (n+1)h_A z^n\varphi_A\ ,\ \lim_{z\to 0}\varphi_A(z)|0> = |A> .$$

(The vacuum $|0>$ has $\bar{L}_n|0> = L_n|0> = 0,\ n \geq -1$, and $\varphi_0(z) \leq 1$)
$h_A + \bar{h}_A$ = scaling dimension of φ_A and $h_A - \bar{h}_A$ = spin of φ_A . Experimentally recovered (or theoretical calculated) critical exponents are simple linear combinations of these scaling dimensions. In a given theory (with given c) a certain set of $(h_A,\bar{h}_A)$ will occur, with $h_A,\bar{h}_A$ chosen from the permitted list if $c < 1$, possibly with nonzero multiplicity. Which sets are possible? E.g. the Ising model has $c = \frac{1}{2}$ and the 3 primary fields correspond to $(0,0), (\frac{1}{16},\frac{1}{16}),\ (\frac{1}{2},\frac{1}{2})$. The criterion for determining possible sets is modular invariance [C]. By considering the theory on a torus, it is argued that

$\mathcal{Z}(\tau) = \mathrm{Tr}(q^{L_0-c/24}(q^*)^{\bar{L}_0-c/24}),\ q = e^{2\pi i\tau}$ should be invariant under the modular transformations $\tau \to \frac{a\tau+b}{c\tau+d}$, $\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL(2,\mathbb{Z})$. There is evidence for the conjecture that, for each c in the discrete series, there exist only 1,2 or 3 possible independent sets of $(h,\bar{h})$ such that the sum $\mathcal{Z}(\tau) = \sum \chi_{ch}(\tau)\chi_{c\bar{h}}(\tau)^*$, with $\chi_{ch}(\tau) = q^{-c/24}\ \mathrm{Tr}(q^{L_0})$, possibly with nonzero multiplicities for the $(h,\bar{h})$, is modular invariant [CTZ]. There is always the trivial invariant $\sum_h \chi_{ch}(\tau)\chi_{c\bar{h}}(\tau)^*$ summed over all possible h .

To construct the discrete series (and hence show the FQS conditions are too sufficient) [GKO] start with a four-dim. Lie algebra g (assumed simple for ease) $[t^a,t^b] = if^{abc}t^c$ and consider the affine algebra $\hat{g}$, $[T^a_m, T^b_n] = if^{abc}T^c_{m+n} + km\,\delta^{ab}\delta_{m,-n}$. A unitary highest weight irreducible rep. of this has $x = 2k/\psi^2 \in \mathbb{Z}, \geq 0$ where ψ = a long root of g . Then the Segal/Sugawara operators $\mathcal{L}^g_n = \frac{1}{2k+Q_g}\sum_m {}^x_x T^a_m T^a_{n-m} {}^x_x$ [where Q_g = quadratic Casimir of g in adjoint representation] satisfy Virasoro alg. with

$$c = c^g = \frac{2k\dim g}{2h + Q_g} = \frac{x\dim g}{x + h_g}\ ,\ h_g = \text{dual coxeter number} .$$

Then $\dim g \geq c^g \geq \mathrm{rank}\, g \geq 1$. To get $c^g < 1$ consider subalgebra $h \subset g$ and so $\hat{h} \subset \hat{g}$. Then as $[\mathcal{L}^g_m, T^a_n] = [\mathcal{L}^h_m, T^a_n] = -n\,T^a_{m+n}$ if $t^a \in \hat{h}$ we have $[K_m, \hat{h}] = 0$ if $K_m = \mathcal{L}^g_m - \mathcal{L}^h_n$ and K_m satisfies Virasoro algebras with $c = c_K = c^g - c^h$. This extends easily to the cases with g semisimple, h compact. We can get all the discrete series from considering $g = su(2)\times su(2)$,

$h = su(2)$ (diagonal). (All cases with $c_K < 1$ are listed [BG]). To check we get all values of h from this construction we compare characters, writing the characters of a unitary highest weight repr. of $\hat{g}$ as a sum of characters of representations of $\hat{h} \times \hat{V}$. (This decomposition is finite iff $c_K < 1$ and $\hat{g}$ is finitely reducible with respect to $\hat{h}$ iff $c_K = 0$.) Using $g = su(2) \times su(2)$, $h = su(2)$, we relate the transformation of $\hat{V}$ characters to that of $\widehat{su}(2)$ characters, reproducing a result first obtained by inspection [G]. It also follows that we get a $\hat{V}$ invariant for $c = 1 - \frac{b}{(m+2)(m+3)}$ from level $m, m+1$ $\widehat{su}(2)$ invariants. [CIZ] conjecture that all $\hat{V}$ invariants come this way.

If $c_k = 0$ we can relate the transformation of $\hat{h}$ characters to that of $\hat{g}$ characters. A "trivial invariant" for $\hat{g}$ characters (i.e. the sum of the squares of the moduli) at a given level may produce non-trivial $\hat{h}$ invariants. It has been shown [N] that all of a list of $\widehat{su}(2)$ invariants given by [CIZ], and conjectured to be complete, can be obtained this way.

[BG] P. Bowcock and P. Goddard, ITP preprint Santa Barbara Dec. 1986.

[BPZ] A.A. Belavin, A.M. Polyakov, A.B. Zamolodchikov, Nucl. Phys. B241 (1984) 333.

[C] J.L. Cardy, Nucl. Phys. B270 (1986), 186.

[CIZ] A. Cappelli, C. Itzykson and J-B. Zuber, Nucl. Phys. to appear.

[FQS] D. Friedan, Z. Qiu and S. Shenker, Phys. Rev. Lett. 52 (1984) 1575.

[G] D. Grepner, Princeton preprint June 1986, Nucl. Phys. to appear.

[GKO] P. Goddard, A. Kent, D.Olive, Phys. Lett. 152B (1985) 88, Commun. Math. Phys. 103 (1986) 105.

[N] W. Nahm, SUNY preprint ITP-SB-87-7 (1987).

Review Articles:

[GO] P. Goddard and D. Olive, Int. J. Mod. Phys. A1 (1986) 303.

[C'] J. Cardy "Conformal Invariance" to appear in Phase Transitions and Critical Phenomena (ed. C. Domb, J.L. Lebowitz, Academic Press) Vol. 11.

Einstein metrics

S.T. Yau

I will report on some recent progress on constructing Einstein metrics on Kähler manifolds M. An obvious necessary condition for the existence of such metric on M is either $c_1(M) < 0$, $= 0$ or > 0. It turns out that the first two conditions are sufficient. The last condition is not sufficient . There are two known obstructions.

(1) The automorphism group of M has to be reductive. Fuhaka also defined a character on $\mathrm{Aut}(M)$ which has to be trivial.

(2) The tangent bundle of M is stable.

It is not clear that these are the only obstructions. The equation for the existence of Kähler Einstein metric is

$$\det(g_{ij} + \frac{\partial^2\varphi}{\partial z_i \partial \bar{z}_j}) = e^{-\varphi+F} \det(g_{ij}) .$$

In general, the solution is not unique, it was proved by Futaki and Berger that they are unique up to an automorphism.

Recently Tian made a breakthrough on this equation. He defined an invariant in the following way. Let $P(g_{ij}) = \{\varphi | g_{ij} + \varphi_{ij} > 0 , \sup_M \varphi = 0\}$.

Let $\alpha > 0$ be defined such that $\int_M e^{-\alpha\varphi} < \infty$ for all $\varphi \in P(g_{ij})$. Let $\alpha(M,\omega)$ be $\sup \alpha$. It depends only on the Kähler class of M . Tian proved that if $\omega = c_1(M)$ and if $\alpha > \frac{n}{n+1}$ where $n = \dim M$, then M admits a Kähler Einstein metric. In general one can prove that a positive lower estimate of α can give rise to a lower estimate of the Ricci curvature of M . (In particular an estimate of c_1^n).

Relative Index Theorems

and

Supersymmetric Scattering Theory

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This talk reports on results obtained in joint collaboration with N.V. Borisov and W. Müller [1].

A general formulation of a supersymmetric scattering theory is given which allows to derive relative index theorems on open manifolds. Consider a Hilbert space $\mathcal{H}$ with a unitary involution τ and a selfadjoint operator Q, called a supercharge and which anticommutes with τ. The $\pm$ eigenspaces $\mathcal{H}^{\pm}$ of τ are called the bosonic and fermionic sector respectively. $H = Q^2$ is called a supersymmetric Hamiltonian [2]. If $\exp - tH$ is of trace class, then the supertrace $\operatorname{Trace} \tau \exp - tH$ is t-independent $(t > 0)$ and an integer, the so called Witten index. Standard index problems on compact spaces may be obtained in this way: Given an operator $L : \mathcal{H}^{+} \longrightarrow \mathcal{H}^{-}$ mapping square integrable sections of one bundle into another such space, simply set $Q = \begin{pmatrix} 0 & L^* \\ L & 0 \end{pmatrix}$ on $\mathcal{H} = \mathcal{H}^{+} \oplus \mathcal{H}^{-}$.

On open spaces one considers pairs Q, Q_0 of supercharges such that $(\exp - tQ^2)-(\exp - tQ_0^2)$ is of trace class. The question arises for what situations $\operatorname{Trace} \tau ((\exp - tQ^1)-(\exp - tQ_0^2))$ is independent of t and in fact an integer. In that case this quantity may be interpreted as a relative index in the sense of Gromov and Lawson [3]. Supersymmetric scattering theory, which gives an answer to this question, is defined by the following condition: The Møller operators $\Omega^{\pm}(H, H_0) = s - \lim\limits_{t \to \pm\infty} e^{itH} e^{-itH_0}$ exist on the absolute continuous subspace of $H_0 = Q_0^2$, are unitary there and are intertwining operators for Q and Q_0. A sufficient condition for this to be satisfied is that $(Q \exp - tH)-(Q_0 \exp - tH_0)$ is of trace class. On the energy shell E the S-matrix $S = (\Omega^{+})^{*} \Omega^{-}$ has a decomposition $S^{\pm}(E)$ into bosonic and fermionic part and by the above intertwining relation the total phase shifts $\delta^{\pm}(E)$, defined by $\det S^{\pm}(E) = \exp 2i\, \delta^{\pm}(E)$, satisfy

$\pi^{-1}(\delta^+(E) - \delta^-(E)) = u(E) \in \mathbf{Z}$. If the left hand side is continuous in E , this integer is E-independent on each connected component of the absolute continuous spectrum of H_0 . Using Krein's spectral shift function [4], which is essentially equal to the phase shift, this may be exploited to discuss Trace $\tau((\exp - tH) - (\exp - tH_0))$. As an example we discuss the de Rham complex. For the case of an obstacle in $\mathbb{R}^n$, it is shown that the absolute or relative Euler characteristic can be obtained form the scattering data if the Hamiltonian is chosen to be the Laplace operator with relative or absolute boundary conditions respectively. Also our method may be applied to manifolds which are flat at infinity, leading to a sharpened version of the Chern-Gauss-Bonnet theorem due to Stern for the L^2-Euler characteristic of spaces which are asymtotically flat [5].

References:

[1] N.V. Borisov, W. Müller, R. Schrader, submitted to Commun. Math. Phys.

[2] E. Witten, Supersymmetry and Morse theory, J. Diff. Geom. 17, 661-692 (1982),

[3] N. Gromov, H.B. Lawson, Positive scalar curvature and the Dirac operator on complete Riemannian manifolds, Publ. Math. IHES 58, 295-408 (1983).

[4] H. Baumgärtel, M. Wollenberger, Mathematical scattering theory, Birkhäuser, Basel-Boston-Stuttgart 1983.

[5] M.A. Stern, L^2-index theorems on warped products, Dissertation Princeton Univ., 1984.

Quantum Hall Effect

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In 1980 v. Klitzing measured the Hall conductivity σ in a two-dimensional interface at low temperature and high magnetic field B. He discovered that $2\pi\sigma$ is an integer and decreases as a function of B monotonically (Figure 1). The high accuracy came as surprise $(1:10^8)$. The plateaus seemed to get wider with increasing density of impurities. In clean material however the fractional Quantum Hall Effect was discovered; for certain values of B conductivity is rational, $2\pi\sigma = p/q$, p and q small integers, q odd.

The theory which I want to present here [2] is formulated in the framework of non relativistic quantum mechanics. The main ingredient is of topological nature: Configuration space Λ of the particles has two holes (Figure 2a). This is an abstraction of the experimental situation including the measuring devices (Figure 2b). The battery is replaced by a time dependent magnetic flux $\phi_1 = -Vt$ (Faraday's law). The Ampère meter is substituted by a second flux through loop 2. It stands for the magnetic flux induced by the current around the second loop. The theory does not depend on any particular form of the interaction between the particles or the particles and the impurities. The external magnetic field acts as a U(1)-gauge field.

The main result is the following: If the groundstate of the Schrödinger operator H which describes the dynamics of the system is q-fold degenerate then the flux averaged conductivity $<\sigma>$ is $\frac{1}{2\pi q}$ times an integer. (Averaging over ϕ has a good physical interpretation).

Generically H has a non-degenerate groundstate due to a theorem of Wigner and v. Neumann. If however configuration space is a two torus and the interaction translationally invariant (no impurities) H is non generic.

In fact if the filling factor ν is rational, $\nu = p/_q$, the groundstate is q-fold degenerate [3]. (The filling factor is defined by $\nu := 2\pi \cdot$ Number of Particles/magnetic flux through the torus).

The main line of proof is as follows: The Schrödinger operator H acts on sections of the trivial Hilbert bundle $\mathbb{R}^2 \times H$, where H denotes the space of square integrable functions of the particle coordinates $(x_1, \ldots, x_N) \in \Lambda^N$. $H(\phi)$ - the operator which acts on the fiber over ϕ - turns out to be periodic in both variables with respect to the lattice $\Gamma \subset \mathbb{R}^2$. If $P(\phi)$ denotes the spectral projector of the q-fold degenerate discrete groundstate, then $\frac{1}{2\pi} \int_T \text{Trace } P\, dP\, dP$ is an integer. (PdPdPP is the curvature of the canonical connection PdP). T denotes the two torus $\mathbb{R}^2/_\Gamma$. It turns out that this number is q times $< 2\pi\sigma >$. This is Kubo's formula which can be established rigorously in this context using the adiabatic theorem [4]. This is natural because conductivity is defined as the derivative of current with respect to voltage V for $V = 0$. Hence the limit of large time scale $1/_V$ is relevant.

If this theory is specialized to a one particle theory with a torus as configuration space one recovers the results of Thouless, Kohomoto, Nightingale and de Nijs [5]. The fractional Hall effect cannot be understood in a one particle theory. Correlation between particles are necessary. Laughlin made a clever guess of multi-particle wave functions which suggests a q-fold degeneracy of the groundstate [6].

Figures

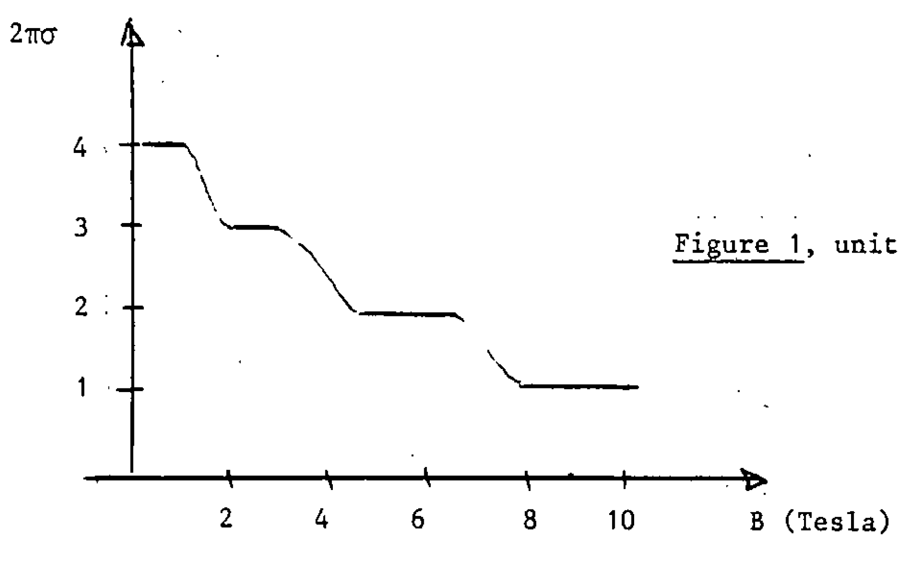

Figure 1, units are $e = \hbar = m_e = 1$.

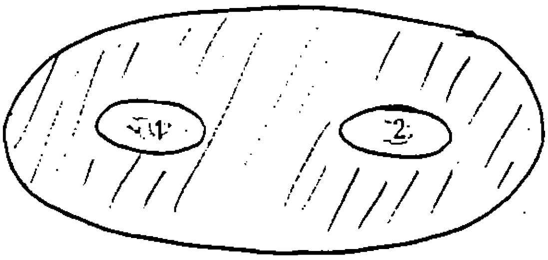

Figure 2a

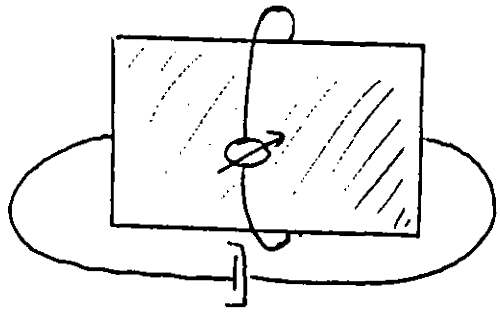

Figure 2b

References

[1] v. Klitzing, Dorda and Pepper (1980); Phys. Rev. Lett. 45, 494.

[2] Avron and Seiler (1985), Phys. Rev. Lett. 54, 259. Avron, Seiler and Shapiro (1986); Nucl. Phys. B265.

[3] Knabe, private communication.

[4] Avron, Seiler and Yaffee (1987), Commun.Math.Phys. 110, 33.

[5] Thouless, Kohomoto, Nightingale and de Nijs (1982); Phys. Rev. Lett. 49.

[6] Laughlin (1983), Phys. Rev. Lett. 50, 1395; Knabe (1986), Technische Universität Berlin, Preprint.

Bosonization on Compact Riemann Surfaces

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Bosonization is a property of two-dimensional conformal quantum field theory. Roughly it is the assertion that all the correlation functions of a fermionic field theory with fields of any spin can be exactly reproduced by those of a suitable bosonic theory. We formulate the appropriate bosonic theory on a compact Riemann surface X and express the mathematical content of bosonization in a set of identities involving various special functions associated to X.

Let ξ be a holomorphic bundle of degree $2\lambda(g-1)$ on a Riemann surface X. Consider the fermionic field theory with fields b, c taking values in $\xi, K \otimes \xi^{-1}$ respectively. K is the canonical bundle. We also introduce fields $\bar{b}, \bar{c}$ in the complex-conjugate bundles, and an action

$$S_f = \int_X (b\bar{\partial}c + \bar{b}\partial\bar{c}) .$$

The fermionic moments of e^{S_f} are called "correlation functions". Computing them by manipulations of fermionic Gaussian integrals we have

$$(*) \quad \langle b(P_1)\bar{b}(P_1)\ldots b(P_p)\bar{b}(P_p)C(Q_1)\bar{C}(Q_1)\ldots C(Q_q)\bar{C}(Q_q) \rangle_f$$

$$= \frac{\det{}'\bar{\partial}_\xi^{+}\bar{\partial}_\xi}{\det(u_i,u_j)} \left| \det \begin{bmatrix} u_1(P_1) & \ldots & u_k(P_1) & G(P_1,Q_1) & \ldots & G(P_1,Q_q) \\ \vdots & & & & & \\ u_1(P_p) & \ldots & \ldots & \ldots & \ldots & G(P_p,Q_q) \end{bmatrix} \right|^2 .$$

Here we assume that the Cauchy-Riemann operator $\bar{\partial}_\xi$ coupled to ξ has k zero modes: $u_1,\ldots u_k$ span $H^0(X;\xi)$, while $\bar{\partial}_\xi^{+}$ has no zero modes: $H^1(X;\xi) = 0$. The integers p, q satisfy $p-q = k = (2\lambda-1)(g-1)$ where g is the genus of X, $\det'$ denotes the zeta-regulated determinant. (u_i,u_j) is the matrix of L_2 inner products of u_i. $G(P,Q)$ is a Green function for $\bar\partial_\xi$. These three constructions require that we place a metric on X . We can also use this metric to convert both sides of the expression to real functions of X, ξ , the metric, and $P_1,\ldots,Q_q$.

Bosonization is the assertion that (*) equals a "bosonic correlation function". The appropriate theory has a real field φ defined modulo $\frac{1}{2}\mathbb{Z}$ and action

$$S_b = 4\pi i \int_X \partial_\varphi \bar\partial_\varphi + 2 \int_X R_{\mathcal{L}}\varphi + 4\pi i \sum_{i=1}^{g} [\oint_{b_i} d\varphi H(a_i;\mathcal{L}) - \oint_{a_i} d\varphi H(b_i;\mathcal{L})]$$

$$+ i\pi\sigma\,(L_0 \otimes F(d\varphi)) \ .$$

Here L_0 is any even spin bundle on X . $F(d\varphi)$ is the flat bundle corresponding to the class of $d\varphi$ in $H^1(X;\mathbb{R})$. $\sigma(L)$ is the parity of a spin bundle L . $\mathcal{L}$ is the bundle

$$\mathcal{L} = \xi^{-1} \otimes L_0 \ .$$

$H(\gamma;\mathcal{L})$ is the holonomy of $\mathcal{L}$ in its hermitian connection about the curve γ . $\{a_1,\ldots,a_g,\ldots,b_g\}$ is a set of curves representing a canonical homology basis and all intersecting at one point. These curves dissect X into a polygon, on which φ is continuous. $R_{\mathcal{L}}$ is the hermitian curvature of $\mathcal{L}$ inherited from the given metric on X . Taking moments as before, a standard Gaussian integral gives

$$< e^{4\pi i\varphi(P_1)} \ldots e^{4\pi i\varphi(P_p)} e^{-4\pi i\varphi(Q_1)} \ldots e^{-4\pi i\varphi(Q_q)} >$$

(**)

$$= \left(\frac{\det'\Delta}{(\det \operatorname{Im}\Omega)\cdot A}\right)^{-1/2} N(z) \frac{\prod_{i,j} G(P_i,P_j) \prod_{i,j} G(Q_i,Q_j)}{\prod_{i,j} G(P_i,Q_j)^2} \ .$$

Here Ω is the period matrix of X in the given marking and A is the area in the given metric. z is the image by the Jacobi map given by the marking of the degree-zero bundle

$$\xi \otimes L_0^{-1} \otimes \mathcal{O}\left(\sum_1^p P_i - \sum_1^q Q_i\right) .$$

L_0 is now the Riemann class in Pic_{g-1} associated to the marking. $N(z) = \exp[-2\pi(\mathrm{Im}\, z)(\mathrm{Im}\,\Omega)^{-1}(\mathrm{Im}\, z)]\cdot|\vartheta(z)\Omega)|^2$, so that in fact (**) is independent of the choice of marking. G is a Green function satisfying

$$(+) \qquad \partial_P\bar{\partial}_P \log G(P,Q) = i\pi[\mu(P)-\delta_Q(P)] ,$$

where $\mu = [4\pi i(1-g)]^{-1}R_K$ and δ_Q is the delta-function, both (1,1) forms. We also require $\int_X \mu(P) \cdot \log G(P,Q) = 0$.

(**) is zero as it stands due to factors of $G(P_i,P_i)$. We must replace these coincident factors by "regulated" ones. If we make a convenient choice for the metric on X , however, the coincident factors will be equal to one and can be dropped. This choice is the Arakelov metric characterized by the following property: the canonical isomorphism

$$[K \otimes \mathcal{O}(P)]\big|_P \cong \mathbb{C}$$

is an isometry when K has the Arakelov metric and $\mathcal{O}(P)$ has the metric

$$(\ddagger) \qquad \| 1_{\mathcal{O}(P)} \| (Q) = G(P,Q) ,$$

where G is defined by (+) with the Arakelov curvature.

Bosonization states that (*) = (**), up to an overall constant dependant only on the integers g, $\deg \xi$, and p . For example when $\deg \xi = g-1$ then generically $\bar{\partial}_\xi$ and $\bar{\partial}_\xi^+$ each have no zero modes and one has the "spin $\lambda = 1/2$ bosonization formula"

$$(++) \qquad \det \bar{\partial}_\xi^+ \bar{\partial}_\xi = C(g)\left(\frac{\det'\Delta}{(\det \mathrm{Im}\Omega)A}\right)^{-1/2} \cdot N(\xi \otimes L_0^{-1}) .$$

This formula is well-known in genus $g = 1$.

The bosonization formulae (*) = (**) can be proved (up to an overall multiplicative constant) using the notion of Quillen metric (cf. the talk of Bismut). Indeed, they can be reformulated as asserting that some "natural" isomorphisms of determinant line bundles are isometries, when these line bundles are equipped with the right Quillen metrics. The two basic facts used to prove these isometries are:

(1) the spin-1/2 bosonization formula (++), which asserts that the isomorphism between $\mathrm{DET}\ \bar\partial_{\theta}$ and $(\mathrm{DET}\ \bar\partial_{\mathcal{L}})^{\otimes(-2)}$ given by the multiplication by the theta function $\vartheta(z,\Omega)^2$ is an isometry when these spaces are equipped with Quillen's metrics.

(2) the insertion theorem, which relates, for any line bundle ξ on X and any point P in X, the Quillen metrics on $\mathrm{DET}\ \bar\partial_{\xi}$ and $\mathrm{DET}\ \bar\partial_{\xi \otimes \mathcal{O}(-P)}$.

These facts are direct consequences of the expression, â la Riemann-Roch-Grothendieck, of the curvature of determinant line bundles equipped with Quillen metrics (at least when $g > 2$).

The properties (1) and (2) are closely related to the work of Faltings on arithmetic surfaces. The property (2) asserts for instance that the metric $\|\cdot\|_F$ on $\det \bar\partial_{\xi}$ defined by Faltings is related to the Quillen metric $\|\cdot\|_Q$ by the formula:

$$\|\cdot\|_F = C(g,d)\left(\frac{\det \Delta}{A}\right)^{1/2} \|\cdot\|_Q$$

(g = genus of X; d = degree of ξ).

The "spin-1/2 bosonization formula"(++) allows then to prove that the invariant $\delta(X)$ introduced by Faltings is related to the analytic torsion of X equipped with the Arakelov metric by:

$$\delta(X) = C(g) - 6 \log \frac{\det' \Delta}{A}.$$

The bosonization formulae are also related to the classical work of Klein on prime forms and to the theory of abelian functions on Riemann surfaces (Fay's trisecant identities, etc...).

References:

- J.-B. BOST and P. NELSON,
  "Spin-1/2 bosonization on compact surfaces"
  Phys. Rev. Lett. 57 (1986) .

- L. ALVAREZ-GAUMÉ, J.-B. BOST, G. MOORE, P. NELSON and C. VAFA,
  "Bosonization in arbitrary genus"
  Phys. Lett. B 178 (1986) 41 .

- L. ALVAREZ-GAUMÉ, J.-B. BOST, G. MOORE, P. NELSON and C. VAFA,
  "Bosonization on higher genus Riemann surfaces"
  CERN-HUTP Preprint .

ANALYTIC TORSION AND HOLOMORPHIC DETERMINANTS

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This is an account of a joint work with C. Soulé (IHES) and H. Gillet (Chicago) [BGS].

Let $M \underset{\pi}{\to} B$ a fibration of complex manifolds with holomorphic map, with smooth compact connected fiber Z.

Let ξ be a holomorphic vector bundle on M.

Let λ^{KM} be the determinant bundle on B obtained by the direct image construction. By a construction of Grothendieck, Knudsen and Mumford, λ^{KM} is canonically defined as a holomorphic line bundle on B. The fiber λ_y^{KM} is canonically isomorphic to

$$(\det H_y^0)^{-1} \otimes \det H_y^1 \otimes \ldots$$

On the other hand, when the fibers Z and the bundle ξ are endowed with metrics, the fibers λ_y^{KM} can be endowed with the Quillen metric using the Ray-Singer analytic torsion.

In this situation, it is a priori not clear that λ^{KM} can be endowed with the Quillen metric as a C^∞ metric.

From now on, I assume the fibres to be endowed with Kähler metrics g^Z.

Our first result in [BGS] is

Theorem 1. The Quillen metric on λ^{KM} is smooth. The curvature of λ^{KM} is given by

$$\int_Z \mathrm{Td}(TZ)\mathrm{ch}\xi$$

where Td and ch are calculated with the holomorphic Hermitian connections on $T^{(1,0)}Z$ and ξ.

Our second series of result is related to exact sequences. Let

$$0 \to \xi_0 \underset{v}{\to} \cdots \underset{v}{\to} \xi_m \to 0$$

be an exact sequence of holomorphic Hermitian vector bundles on M. Let $\lambda_0^{KM}, \lambda_1^{KM} \ldots \lambda_m^{KM}$ be the corresponding determinants. Then by Knudsen-Mumford, $\lambda_0^{KM} \otimes (\lambda_1^{KM})^{-1} \otimes \ldots$ has a non zero holomorphic canonical section σ (which depends on v).

<u>Theorem 2</u> [BGS]: The following identity holds

$$\|\sigma\|^2 = \exp\left(\int_Z \mathrm{Td}(Z)(\tilde{\mathrm{ch}}\ \xi)\right)$$

where $\tilde{\mathrm{ch}}(\xi)$ is the Bott-Chern Secondary invariant associated with the exact sequence.

Assume now that (g^Z,h^ξ) and (g'^Z,h'^ξ) are two different cocycles of metrics on $(T^{(1.0)}Z,\xi)$ where g^Z, g'^Z are Kähler.

<u>Theorem 3</u> [BGS] If $\|\ \|$ and $\|\ \|'$ are the Quillen metrics associated with (g^Z,h^ξ) and (g'^Z,h'^ξ), then

$$\frac{\|\ \|'^2}{\|\ \|^2} = \exp \int_Z [\tilde{\mathrm{Td}}(g^Z,g'^Z)\mathrm{ch}\ \xi + \mathrm{Td}(g'^Z(\tilde{\mathrm{ch}}(\xi)]$$

where again ~ are secondary Bott-Chern classes.

The proofs rely on a formalism which combines Quillen's superconnections, Bott-Chern classes and Ray-Singer torsion. They are much in the spirit of the Atiyah-Singer Index Theorem, where one equals a quantity coming from analysis to a geometric object.

[BGS] Bismut-Gillet-Soule: Analytic torsion and holomorphic determiant bundles.

OPERATOR FORMALISM IN HIGHER GENUS SURFACES

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ABSTRACT

One of the most fruitful approaches to string theory has been the use of conformal field theory . The results of Belavin Polyakov and Zamolodchikov , and of Friedan Martinec and Shenker rely heavily on the use of operator methods on the twice puncture sphere (the Wick rotated version of $S^1 \times R$). The study of string perturbation theory however requires that one analyzes conformal field theories on Riemann surfaces of arbitrary genus . So far the methods employed use functional integration and algebraic geometry . In order to extend the operator method to this setting , we have found that the infinite dimensional grassmannian used in the study of the KP hierarchy [1] provides the natural framework to extend operator techniques to higher genus surfaces . The basic idea [2] involves working always on a punctured disc where at the center one places the standard vacuum for the field theory being considered , and on teh boundary circle , a state of the Fock space which carries all the geometrical information concerning the surface and the bundles used . This is just a Bogoliubov transformation of the standard vacuum . The grassmannian , through the Krichever map give a rather precise characterization of the Fock space states that convey all the geometrical information . So far this formalism has been used to obtain a proof of chiral bosonization on higher genus Riemann surfaces , and to understand the action of the Virasoro algebra in higher orders of string perturbation theory . The use of the operator formalism also provides a rather simple way of obtaining the Polyakov measure for bosonic strings in terms of the Mumford form and the decoupling of spurious states . Further details and references can be found in [3]

REFERENCES

[1] For details and references see : G. Segal and G. Wilson : Publ. Math. de l'I.H.E.S. 61 (1985) 1 .

[2] S. Saito : Tokyo Metropolitan Univ. Preprints TMUP-HEL-8613/8615/ 8701/8707 . N. Ishibashi , Y. Matsuo and H. Oogouri Mod. Phys. Lett. A2 (1987) 119 . L. Alvarez-Gaumé , C. Gomez and C. Reina : Phys. Lett. 190B (1987) 55 . C. Vafa : Phys. Lett. 190B (1987) 47 .

[3] L. Alvarez-Gaumé , C. Gomez , G. Moore and C. Vafa : String Theory in the Operator Formalism . HUTP, CERN , BUHEP , IASSNS preprint . In preparation .

CONCLUDING REMARKS

Michael Atiyah

1. General observations

This has been a very stimulating conference with a great deal of interaction between different subjects. Perhaps the following brief overview may be helpful. Most topics centered on a relation between two consecutive dimensions. Thus conformal field theory (and elliptic cohomology) dealt with circles (dim 1) on Riemann surfaces (dim 2). Vaughan Jones' work on knot polynomials produced invariants of knots (in dim 3) from a plane projection (dim 2). Moreover the 2-dimensional theories used by Jones were closely related (via statistical mechanical models etc.) to conformal field theories. Finally Donaldson related instanton invariants in dimension 4 to Floer homology in dimension 3. Again the 3-dimensional situations studied by Floer and Jones both centered on the fundamental group. An obvious challenge is to find a definition of the Jones polynomial which is directly 3-dimensional. One might reasonably speculate that this should come from a field theory approach relative to Floer's homology.

2. 2-dimensional Floer theory

It may be helpful to string theorists to explain a 2-dimensional counterpart of the Floer-Donaldson story. In fact this is a variant (different boundary conditions - closed strings) of Floer's main work on symplectic geometry.

Consider the supersymmetric non-linear $\mathbb{C}P_n$-model. Instantons in this theory are given by holomorphic (i.e. rational) maps of $\mathbb{C}P_1 = S^2$ to $\mathbb{C}P_n$. More generally we can consider rational maps of a Riemann surface of genus g to $\mathbb{C}P_n$. A classical question in algebraic geometry is to understand curves of genus g and degree k in $\mathbb{C}P_n$. Consider for simplicity those values of g and k for which this number is essentially finite (i.e. up to automorphisms of $\mathbb{C}P_n$). Computing this number, say $N(g,k)$, is the analogue of the problem Donaldson described of finding the number of instantons (Yang-Mills) on a given 4-manifold.

Consider now the free loop space $L(\mathbb{C}P_n)$. If X_1 is a Riemann surface with boundary the circle S^1 we get a subspace $\Sigma_1 \subset L(\mathbb{C}P_n)$ representing boundary values of holomorphic maps $X_1 \to \mathbb{C}P_n$. If X_2 has boundary $-S^1$ (i.e. with orientation reversed) we get similarly a subspace $\Sigma_2 \subset L(\mathbb{C}P_n)$. Clearly, if $X = X_1 \cup X_2$ is the closed surface defined by gluing X_1 and X_2 along S^1, then $\Sigma_1 \cap \Sigma_2$ is a set of points on L whose cardiality is $N(g,k)$, where $g = \text{genus } X$ and k the degree of the resulting map [Actually we have to be careful to pick the right points in $\Sigma_1 \cap \Sigma_2$ to give this degree].

Since the definition of $N(g,k)$ should be invariant under deformation of the conformal structure of X we look for a homology theory in which Σ_1 and Σ_2 represent homology classes. Now Σ_1 and Σ_2 are both infinite-dimensional but have so to-speak "half" the dimension of L. The right homology theory is the Floer version of the Morse theory built on the function $L \to U(1)$ given by the holonomy (of the standard line bundle on $\mathbb{C}P_n$) around a loop. The critical points of this function are just the point loops represented by $\mathbb{C}P_n \subset L(\mathbb{C}P_n)$. The situation here differs from that considered in Donaldson's lecture because the critical points are not isolated but is otherwise similar. The Floer homology should now be identified with the homology of $\mathbb{C}P_n$. Given a cycle σ in $\mathbb{C}P_n$ we can "grow" a Floer cycle $\Sigma \subset L$ by moving off the point loops in the "positive energy" or holomorphic directions, i.e. each point loop grows into a small holomorphic disc. An interesting and important feature of the Floer homology is that Σ will return and intersect L again in higher dimensional cycles after having wound the "hole" in L (representing the generator of $\pi_1(L) = \pi_2(\mathbb{C}P_n) = \mathbb{Z}$).

The Donaldson procedure for computing $\Sigma_1 \cap \Sigma_2$ consists in a degeneration process shrinking S^1 to a point in X so as to decompose it in the limit into $\bar{X}_1 \cup \bar{X}_2$ where $\bar{X}_i = X_i/S^1$ is a closed surface of genus g_i ,$(g_1 + g_2 = g)$. This corresponds to bringing Σ_i into the standard position of a Σ associated to $\sigma \subset \mathbb{C}P_n$ described above. Thus finally $N(g,k)$ is computed by an intersection number $\sigma_1 \cdot \sigma_2$ in $\mathbb{C}P_n$.

The conclusion of this process is now easily recognizable as the classical procedure to calculate such numbers in algebraic geometry by degenerating curves into simpler (reducible) curves, i.e. by going to the boundary of the moduli space $\mathfrak{M}_g$.

The point of describing this example was to explain, by analogy, the more difficult and interesting 4-dimensional Donaldson case. It also may have some relevance to string theory, but that I leave to the relevant experts. I just point out that the Hodge-deRham version of the Floer cohomology is just the supersymmetric Hamiltonian studied by Witten.