The Alexander polynomial of a deform-spun knot in S^4 is symmetric

Ryan Budney,
Alexandra Mozgova

*Mathematics and Statistics, University of Victoria,
PO BOX 3045 STN CSC, Victoria, B.C., Canada V8W 3P4*
*IRMA Strasbourg,
7 rue René-Descartes, 67084 Strasbourg, France*

Email: rybu@rybu.org, mozgova@mpim-bonn.mpg.de

Abstract

This paper proves that if a co-dimension 2 knot in S^4 is deform-spun from a co-dimension 2 knot in S^3, then its Alexander polynomial is symmetric. Since there exist knots in S^4 with non-symmetric Alexander polynomials, this proves not all knots in S^4 are deform-spun. The proof of the main theorem uses nothing more than the definition of the Alexander polynomial, Poincare duality and elementary linear algebra.

AMS Classification numbers Primary: 57R40

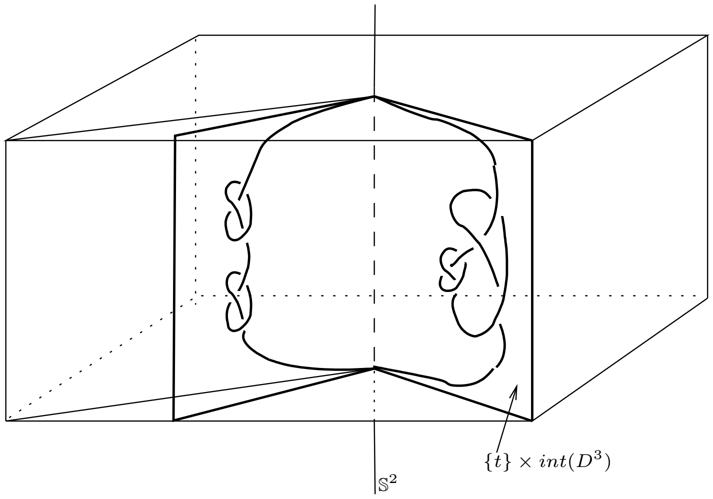
Secondary: 57R50, 57M25, 55Q45

Keywords: 1-knot, 2-knot, deform spun, Alexander polynomial

1 Introduction

In co-dimension 2 knot theory, typically the term 'n-knot' denotes a manifold pair (S^{n+2}, K) where K is the image of a smooth embedding $f : S^n \to S^{n+2}$. A 'long' n-knot is a pair (D^{n+2}, J) where J is the image of a smooth embedding $f : D^n \to D^{n+2}$ such that $f^{-1}(\partial D^{n+2}) = \partial D^n$ and such that f, when restricted to $\partial D^n = S^{n-1}$ is the standard inclusion, where we consider $D^n \subset D^{n+2}$ in the standard way. Every n-knot K is isotopic to a union $(S^{n+2}, K) = (D^{n+2}, J) \cup_\partial (D^{n+2}, D^n)$ for some unique isotopy class of long knot J provided we consider K to be oriented. Let $\mathrm{Diff}(D^{n+2}, J)$ denote the group of diffeomorphisms of D^{n+2} which restrict to the identity on $J \cup \partial D^{n+2}$. An $(n+1)$-knot (S^{n+3}, K') is deform-spun from (S^{n+2}, K) if there exists $g \in \mathrm{Diff}(D^{n+2}, J)$ such that the pair $\left((D^{n+2}, J) \times_g S^1\right) \cup_\partial \left((S^{n+1}, S^{n-1}) \times D^2\right)$ is diffeomorphic to the pair (S^{n+3}, K').

To visualize the deform-spun knot, assume that the diffeomorphism $g \in \mathrm{Diff}(D^{n+2}, J)$ is isotopic to the identity when considered as a diffeomorphism of D^{n+2} (every deform-spun knot can be obtained using such a diffeomorphism, so this is no loss of generality [1]). Let g_t be the null-isotopy of g, ie: $g_0 = g$, $g_1 = Id_{D^{n+2}}$ and g_t is a diffeomorphism of D^{n+2} which restricts to the identity on ∂D^{n+2} for all $0 \le t \le 1$. Consider S^{n+3} to be the union of a great $(n+1)$-sphere S^{n+1} and a trivial vector bundle over S^1. Identify this trivial vector bundle over S^1 with $S^1 \times int(D^{n+2})$, and identify S^1 with $\mathbb{R}/\mathbb{Z}$. We assume that the inclusion $S^1 \times int(D^{n+2}) \to S^{n+3}$ extends to a map $S^1 \times D^{n+2} \to S^{n+3}$ such that the restriction $S^1 \times S^{n+1} \to S^{n+3}$ factors as projection onto the great sphere S^{n+1} followed by inclusion $S^{n+1} \to S^{n+3}$. Then the set $\{(t, x) \in S^1 \times int(D^{n+2}) : x = g_t(p), p \in int(J)\}$ is a subset of S^{n+3} whose closure is an $(n+1)$-knot. This is the deform-spun knot.



A connect sum of two trefoils, being deform-spun to produce a 2-knot in S^4

The main result of this paper is to show that not every 2-knot is deform-spun from a 1-knot. The obstruction is given by Theorem 2.4, which states that 2-knots with asymmetric Alexander polynomials are not deform-spun. The set of polynomials realisable as Alexander polynomials of 1-knots is known [5] to be

$$\{p(t) \in \mathbb{Z}[\mathbb{Z}] : p(1) = \pm 1, p(t^{-1}) = p(t)\}.$$

On the other hand, Kinoshita [7] has proved that the set of polynomials realisable as Alexander polynomials of 2-knots is

$$\{p(t) \in \mathbb{Z}[\mathbb{Z}] : p(1) = \pm 1\}.$$

Theorem 2.4 has as a consequence that the set of polynomials realizable as Alexander polynomials of deform-spun knots in S^4 are precisely the Alexander polynomials of knots in S^3.

Litherland's deform-spinning construction has its origin in a paper of Zeeman's. Zeeman proved that the complements of certain co-dimension two 'twist-spun' knots fiber over S^1 [10]. Litherland later went on to formulate a more general notion of spinning called 'deform-spinning,' further generalising Zeeman's theorem on when such knot complements fiber over S^1 [8]. Specifically, Litherland proved that if the diffeomorphism f preserves a Seifert surface for the knot, then the deform-spun knot associated to the diffeomorphism Mf fibers over S^1, provided $M : (D^n, J) \to (D^n, J)$ is a non-zero multiple of the meridional Dehn twist about J.

This paper was largely motivated by a result in 'high' co-dimension knot theory. Let $\mathcal{K}_{n,j}$ denote the space of smooth embeddings $f : D^j \to D^n$ such that $f^{-1}(\partial D^n) = \partial D^j$ and the restriction of f to ∂D^j is the standard inclusion. In a previous paper [1] the first author showed that Litherland's deform-spun knot construction generalises to 'graphing' map $\mathrm{gr}_1 : L\mathcal{K}_{n-1,j-1} \to \mathcal{K}_{n,j}$ where $L\mathcal{K}_{n-1,j-1}$ denotes the free loop space on $\mathcal{K}_{n-1,j-1}$, this is the space of smooth maps from S^1 to $\mathcal{K}_{n-1,j-1}$. A proof was given that the map $\pi_0 L\mathcal{K}_{n-1,j-1} \to \pi_0\mathcal{K}_{n,j}$ is onto provided $n - j > 2$. Further consider $\mathcal{K}_{n,j}$ to be a based-space with basepoint the unknot, then the graphing map gr_1 restricts to a map $\mathrm{gr}_1 : \Omega\mathcal{K}_{n-1,j-1} \to \mathcal{K}_{n,j}$. In [1] it was further shows that $\mathrm{gr}_{1*} : \pi_1\mathcal{K}_{n-1,j-1} \to \pi_0\mathcal{K}_{n,j}$ is onto. By iterating the graphing construction, one gets a map $\mathrm{gr}_i : \Omega^i\mathcal{K}_{n-i,j-i} \to \mathcal{K}_{n,j}$. Goodwillie's dissertation was applied to show that the induced map $\mathrm{gr}_{i*} : \pi_i\mathcal{K}_{n-i,j-i} \to \pi_0\mathcal{K}_{n,j}$ is onto provided $i \leq 2n - 2j - 4$. This result is frequently sharp: for example, $\mathrm{gr}_2 : \pi_2\mathcal{K}_{4,1} \to \pi_0\mathcal{K}_{6,3} \simeq \mathbb{Z}$ is an isomorphism. See [1] for a precise definition of gr_i and the above results.

The paper [2] gives a 'computation' of the groups $\pi_0\mathrm{Diff}(D^3, J)$. These groups turn out to be the fundamental groups of the components of $\mathcal{K}_{3,1}$, and are described in terms of the JSJ-decomposition of the knot complement [3]. The group structure of $\pi_0\mathrm{Diff}(D^3, J)$ is fairly involved. For example, the classifying space $B(\pi_0\mathrm{Diff}(D^3, J))$ has the homotopy-type of a compact manifold, which is a $K(\pi, 1)$. The dimension of this manifold is bounded below by the number of tori in the JSJ-decomposition of the complement of J in D^3. It was the complexity of the groups $\pi_0\mathrm{Diff}(D^3, J)$ that led the first author to think deform-spinning could be a way to produce many interesting higher-dimensional knots. The point of this paper is to say that, at least in S^4, deform-spinning does not produce all knots.

2 Asymmetry obstruction

Given a co-dimension 2 knot K in S^n, the complement of the knot, C_K is a homology S^1. Let $\tilde{C}_K$ denote the universal abelian cover of C_K, ie: the cover corresponding to the abelianization map $\pi_1 C_K \to \mathbb{Z}$, and consider $H_1(\tilde{C}_K; \mathbb{Q})$ to be a module over the group-ring of covering transformations $\mathbb{Q}[\mathbb{Z}]$. It's known that $H_1(\tilde{C}_K; \mathbb{Q})$ is a torsion $\mathbb{Q}[\mathbb{Z}]$-module [4], so $H_1(\tilde{C}_K; \mathbb{Q}) \simeq \bigoplus_i \mathbb{Q}[\mathbb{Z}]/p_i$ for some collection of polynomials p_i. The product $\prod_i p_i$ is called the Alexander polynomial of K, or the order ideal of $H_1(\tilde{C}_K; \mathbb{Q})$ (since $\mathbb{Q}[\mathbb{Z}]$ is a principal ideal domain, an ideal is the same thing as a polynomial up to a multiple of a unit). The Alexander polynomial can be defined directly in terms of the $\mathbb{Z}[\mathbb{Z}]$-module structure of $H_1(\tilde{C}_K;\mathbb{Z})$, and so the Alexander polynomial admits a canonical normalisation to an element of $\mathbb{Z}[\mathbb{Z}]$. This normalization is easy to compute from the $\mathbb{Q}[\mathbb{Z}]$ polynomial as the $\mathbb{Z}[\mathbb{Z}]$ polynomial satisfies $p(1) = \pm 1$. Given a finitely-generated torsion $\mathbb{Q}[\mathbb{Z}]$-module H, the order ideal will be denoted $\Delta_H(t)$, similarly the Alexander polynomial of K is denoted $\Delta_K(t) = \Delta_{H_1(\tilde{C}_K;\mathbb{Q})}(t)$.

Lemma 2.1 *[6] (7.2.7) Given a short exact sequence of finitely generated torsion $\mathbb{Q}[\mathbb{Z}]$-modules*

$$0 \to H_1 \to H \to H_2 \to 0$$

the order ideals satisfy $\Delta_{H_1}(t)\Delta_{H_2}(t) = \Delta_H(t)$.

Notice that the dimension of H as a $\mathbb{Q}$-module is the degree of the polynomial $\Delta_H(t)$, where 'degree' is interpreted as the difference between the exponent of the highest and lowest order non-zero terms in the polynomial.

As context for the next lemma, let G be a finite abelian group. We briefly mention the construction of the duality pairing $G \times Ext_{\mathbb{Z}}(G,\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$. The idea is to start with a presentation

$$\mathbb{Z}^n \xrightarrow{M} \mathbb{Z}^n \xrightarrow{\pi_G} G$$

and the induced presentation of Ext

$$Hom(\mathbb{Z}^n,\mathbb{Z}) \xrightarrow{M^\perp} Hom(\mathbb{Z}^n,\mathbb{Z}) \xrightarrow{\pi^G} Ext_{\mathbb{Z}}(G,\mathbb{Z})$$

The duality pairing sends a pair $(\pi_G g, \pi^G f)$ to $\frac{\langle g',f\rangle}{|g|} = \frac{\langle g,f'\rangle}{|h|}$, where $|g|g = M(g')$ and $|h|h = M^\perp(h')$. This gives a natural identification $Ext_{\mathbb{Z}}(G,\mathbb{Z}) \simeq Hom_{\mathbb{Z}}(G,\mathbb{Q}/\mathbb{Z})$.

Lemma 2.2 *Let H be a finitely-generated torsion $\mathbb{Q}[\mathbb{Z}]$-module. Denote by $[\mathbb{Q}[\mathbb{Z}]]$ the field of fractions of $\mathbb{Q}[\mathbb{Z}]$. Consider $\mathbb{Q}[\mathbb{Z}]$ to be the submodule of $[\mathbb{Q}[\mathbb{Z}]]$ with denominator 1.*

There are canonical isomorphisms:

$$Ext_{\mathbb{Q}[\mathbb{Z}]}(H,\mathbb{Q}[\mathbb{Z}]) \simeq Hom_{\mathbb{Q}[\mathbb{Z}]}(H,[\mathbb{Q}[\mathbb{Z}]]/\mathbb{Q}[\mathbb{Z}]) \text{ and } Hom_{\mathbb{Q}[\mathbb{Z}]}(H,[\mathbb{Q}[\mathbb{Z}]]/\mathbb{Q}[\mathbb{Z}]) \simeq Hom_{\mathbb{Q}}(H,\mathbb{Q})$$

where the first isomorphism is an isomorphism of $\mathbb{Q}[\mathbb{Z}]$-modules, while the last is only an isomorphism of $\mathbb{Q}$-vector spaces.

Proof The idea of the first part of the proof is to construct a duality pairing

$$H \times Ext_{\mathbb{Q}[\mathbb{Z}]}(H,\mathbb{Q}[\mathbb{Z}]) \to [\mathbb{Q}[\mathbb{Z}]]/\mathbb{Q}[\mathbb{Z}]$$

as before. Start with a presentation

$$\mathbb{Q}[\mathbb{Z}]^n \xrightarrow{M} \mathbb{Q}[\mathbb{Z}]^n \xrightarrow{\pi_H} H$$

which gives a dual presentation

$$\mathbb{Q}[\mathbb{Z}]^n \xrightarrow{M^\perp} \mathbb{Q}[\mathbb{Z}]^n \xrightarrow{\pi^H} Ext_{\mathbb{Q}[\mathbb{Z}]}(H,\mathbb{Q}[\mathbb{Z}])$$

So given $(\pi_H h, \pi^H f) \in H \times Ext_{\mathbb{Q}[\mathbb{Z}]}(H, \mathbb{Q}[\mathbb{Z}])$, if $|h|h = M(h')$ and $|f|f = M^{\perp}(f')$ for some $|h|, |f| \in \mathbb{Q}[\mathbb{Z}]$ define

$$\langle \pi_H h, \pi^H f\rangle = \frac{\langle h', f\rangle}{|h|} = \frac{\langle h, f'\rangle}{|f|} \in [\mathbb{Q}[\mathbb{Z}]]/\mathbb{Q}[\mathbb{Z}].$$

For the second claim, consider a rational polynomial $\frac{p(t)}{q(t)} \in [\mathbb{Q}[\mathbb{Z}]]$. By the division algorithm $p(t) = s(t)q(t) + r(t)$ for unique Laurent polynomials $s(t), r(t) \in \mathbb{Q}[\mathbb{Z}]$ such that $r(t) \in \mathbb{Q}[t]$ and $deg(r(t)) < deg(q(t))$. Define a $\mathbb{Q}$-linear map $[\mathbb{Q}[\mathbb{Z}]]/\mathbb{Q}[\mathbb{Z}] \to \mathbb{Q}$ by sending $\frac{p(t)}{q(t)}$ to the constant coefficient of $r(t)$. This gives a $\mathbb{Q}$-linear map:

$$Hom_{\mathbb{Q}[\mathbb{Z}]}(H, [\mathbb{Q}[\mathbb{Z}]]/\mathbb{Q}[\mathbb{Z}]) \to Hom_{\mathbb{Q}}(H, \mathbb{Q})$$

which respects connect-sum decompositions of the domain H. Thus to verify that it is an isomorphism, we need to only check it on a torsion $\mathbb{Q}[\mathbb{Z}]$-module with one generator.

$$Hom_{\mathbb{Q}[\mathbb{Z}]}(\mathbb{Q}[\mathbb{Z}]/p, [\mathbb{Q}[\mathbb{Z}]]/\mathbb{Q}[\mathbb{Z}]) \to Hom_{\mathbb{Q}}(\mathbb{Q}[\mathbb{Z}]/p, \mathbb{Q}).$$

In this case the target space is free of rank $deg(p)$; the free generators are the dual classes to the polynomials t^i for $0 \le i < deg(p)$. The domain is a free $\mathbb{Q}$-module of rank $deg(p)$ generated by the homomorphisms that send 1 to t^i/p where $0 \le i < deg(p)$. Hence the map is a bijection between these basis vectors. □

Remark. As $[\mathbb{Q}[\mathbb{Z}]]$ is injective $\mathbb{Q}[\mathbb{Z}]$-module [9], the first part of the above proof can also be seen by applying $Hom(H, \star)$ to the short exact sequence $0 \to \mathbb{Q}[\mathbb{Z}] \to [\mathbb{Q}[\mathbb{Z}]] \to [\mathbb{Q}[\mathbb{Z}]]/\mathbb{Q}[\mathbb{Z}] \to 0$.

Lemma 2.3 *Let $g : H \to H$ be a $\mathbb{Q}[\mathbb{Z}]$-linear map, where H is a finitely-generated torsion $\mathbb{Q}[\mathbb{Z}]$-module. Let $g^* : Ext_{\mathbb{Q}[\mathbb{Z}]}(H, \mathbb{Q}[\mathbb{Z}]) \to Ext_{\mathbb{Q}[\mathbb{Z}]}(H, \mathbb{Q}[\mathbb{Z}])$ the Ext-dual of g. Then $ker(g)$ and $ker(g^*)$ have the same order ideals (Alexander polynomials).*

Proof The order ideal of H admits a prime factorisation, so let $P \subset \mathbb{Q}[\mathbb{Z}]$ be the set of primes used in the prime factorisation. Given $p(t) \in P$ let $H_{p(t)} \subset H$ be the sub-module of elements killed by a power of $p(t)$. Then there is a canonical isomorphism $\bigoplus_{p(t) \in P} H_{p(t)} \simeq H$. This splits g as a direct sum

$$g = \bigoplus_{p(t) \in P} g_{p(t)} : H_{p(t)} \to H_{p(t)}$$

Thus,

$$\Delta_{ker(g)}(t) = \prod_{p(t) \in P} \Delta_{ker(g_{p(t)})}(t).$$

Let $d_{p(t)} \in \mathbb{Z}$ be defined so that $\Delta_{ker(g_{p(t)})}(t) = p(t)^{d_{p(t)}}$. By Lemma 2.2, g and g^* can be thought of as the $Hom_{\mathbb{Q}}(\cdot, \mathbb{Q})$-duals of each other, thus $ker(g)$ and $ker(g^*)$ have the same dimension as $\mathbb{Q}$-vector spaces. But by the comments following Lemma 2.1, $dim_{\mathbb{Q}}(ker(g_{p(t)})) = deg(p(t))d_{p(t)}$. Thus, $\Delta_{ker(g_{p(t)})}(t)$ is determined by the rank of $ker(g_{p(t)})$ as a $\mathbb{Q}$-vector space. Hence $ker(g)$ and $ker(g^*)$ have the same order ideals. □

Remark. Although they have the same order ideals, in general the two kernels are not isomorphic as $\mathbb{Q}[\mathbb{Z}]$-modules. An example is given by $g : \mathbb{Q}[\mathbb{Z}]/p(t) \oplus \mathbb{Q}[\mathbb{Z}]/p(t)^2 \to \mathbb{Q}[\mathbb{Z}]/p(t) \oplus \mathbb{Q}[\mathbb{Z}]/p(t)^2$ defined by $g(a(t), b(t)) = (0, p(t)a(t))$. In this case, $ker(g) \simeq \mathbb{Q}[\mathbb{Z}]/p(t)^2$, while $ker(g^*) \simeq \bigoplus_2 \mathbb{Q}[\mathbb{Z}]/p(t)$.

Theorem 2.4 *Let K' be a 2-knot which is deform-spun, then $\Delta_{K'}(t^{-1}) = \Delta_{K'}(t)$.*

Proof We use the notation in the introduction. Let $C_{K'}$ be the complement of a tubular neighbourhood of K', and C_K the complement of a tubular neighbourhood of K. Let g be the diffeomorphism of C_K obtained by restricting the diffeomorphism in the definition of $C_{K'}$. There is a homeomorphism

$$C_{K'} \simeq (C_K \rtimes_g S^1) \cup_{\nu S^1 \times S^1} ((\nu S^1) \times D^2)$$

where νS^1 is a trivial I-bundle over S^1, considered to be a tubular neighbourhood of a meridian in ∂C_K. This gives a short exact sequence of Alexander modules

$$0 \to img(g_* - I) \to H_1(\tilde{C}_K; \mathbb{Q}) \to H_1(\tilde{C}_{K'}; \mathbb{Q}) \to 0.$$

where $g_* : H_1(\tilde{C}_K; \mathbb{Q}) \to H_1(\tilde{C}_K; \mathbb{Q})$ is the induced map of Alexander modules.

On the other hand, $g_* - I : H_1(\tilde{C}_K; \mathbb{Q}) \to H_1(\tilde{C}_K; \mathbb{Q})$ gives rise to a short exact sequence

$$0 \to ker(g_* - I) \to H_1(\tilde{C}_K; \mathbb{Q}) \to img(g_* - I) \to 0$$

Apply Lemma 2.1 to both short exact sequences, giving $\Delta_{K'}(t) = \Delta_{ker(g_*-I)}(t)$. This reduces the problem to showing that $\Delta_{ker(g_*-I)}(t)$ is a symmetric polynomial.

We reconsider the proof that $\Delta_K(t^{-1}) = \Delta_K(t)$ [4, 6] paying special attention to naturality with respect to diffeomorphisms $g \in \mathrm{Diff}(C_K)$.

(1) $H_1(\tilde{C}_K) \simeq H_1(\tilde{C}_K, \partial)$: this is a natural isomorphism coming from the long exact sequence of a pair.

(2) $\overline{H^2(\tilde{C}_K)}$ denotes $\mathbb{Q}[\mathbb{Z}]$-module $H^2(\tilde{C}_K)$ where the action of $\mathbb{Z}$ is replaced by the inverse action. We have $H_1(\tilde{C}_K, \partial) \simeq \overline{H^2(\tilde{C}_K)}$: this is the isomorphism coming from Poincaré duality; it is also natural although it reverses arrows.

(3) $H^2(\tilde{C}_K) \simeq Ext(H_1(\tilde{C}_K), \mathbb{Q}[\mathbb{Z}])$: this is a natural isomorphism coming from the universal coefficient theorem, since $Hom(H^2(\tilde{C}_K), \mathbb{Q}[\mathbb{Z}]) = 0$.

(4) $Ext(H_1(\tilde{C}_K), \mathbb{Q}[\mathbb{Z}]) \simeq H_1(\tilde{C}_K)$. This last result uses that both modules have a square presentation matrix, with one being the transpose of the other. Since $\mathbb{Q}[\mathbb{Z}]$ is a principal ideal domain, the presentation matrices are equivalent to the same diagonal matrices. This isomorphism is not natural.

Thus we have an isomorphism $H_1(\tilde{C}_K) \simeq \overline{H_1(\tilde{C}_K)}$ which gives the identity $\Delta_K(t^{-1}) = \Delta_K(t)$. Using the previous Lemma we get a commutative diagram where all the maps are $\mathbb{Q}[\mathbb{Z}]$-linear.

$$\begin{tikzcd}
\overline{H_1(\tilde{C}_K)} \arrow[r] \arrow[d, "g_*"] & \overline{H_1(\tilde{C}_K, \partial)} \arrow[r, "PD"] \arrow[d, "g_*"] & H^2(\tilde{C}_K) & Ext_{\mathbb{Q}[\mathbb{Z}]}\left(H_1(\tilde{C}_K), \mathbb{Q}[\mathbb{Z}]\right) \arrow[l, "UCT"'] \\
\overline{H_1(\tilde{C}_K)} \arrow[r] & \overline{H_1(\tilde{C}_K, \partial)} \arrow[r, "PD"] & H^2(\tilde{C}_K) \arrow[u, "g^*"] & Ext_{\mathbb{Q}[\mathbb{Z}]}\left(H_1(\tilde{C}_K), \mathbb{Q}[\mathbb{Z}]\right) \arrow[l, "UCT"'] \arrow[u, "(g_*)^*"]
\end{tikzcd}$$

This gives us an isomorphism of $\mathbb{Q}[\mathbb{Z}]$-modules $\overline{ker(I-g_*)} \simeq ker(I-(g_*^{-1})^*)$, so

$$\overline{ker(I-g_*)} \simeq ker(I-(g_*^{-1})^*) = ker(I-(g_*)^*).$$

By Lemma 2.3, $ker(I-(g_*)^*)$ and $ker(I-g_*)$ have the same Alexander polynomials. Thus, $\Delta_{K'}(t^{-1}) = \Delta_{K'}(t)$. □

3 Comments and questions

Alexander polynomials $p(t)$ of co-dimension 2 knots in S^n for $n \geq 4$ are known to only satisfy the restriction $p(1) = \pm 1$ [7], so there is no direct generalisation of Theorem 2.4 to higher dimensions.

Question 3.1 (1) *Is the asymmetry of the Alexander polynomial the only obstruction to a 2-knot being deform-spun?*

(2) *Are there any obstructions to an n-knot being deform-spun for $n > 2$?*

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Mathematics and Statistics, University of Victoria,
PO BOX 3045 STN CSC, Victoria, B.C., Canada V8W 3P4
IRMA Strasbourg,
7 rue René-Descartes, 67084 Strasbourg, France

Email: `rybu@rybu.org`, `mozgova@mpim-bonn.mpg.de`